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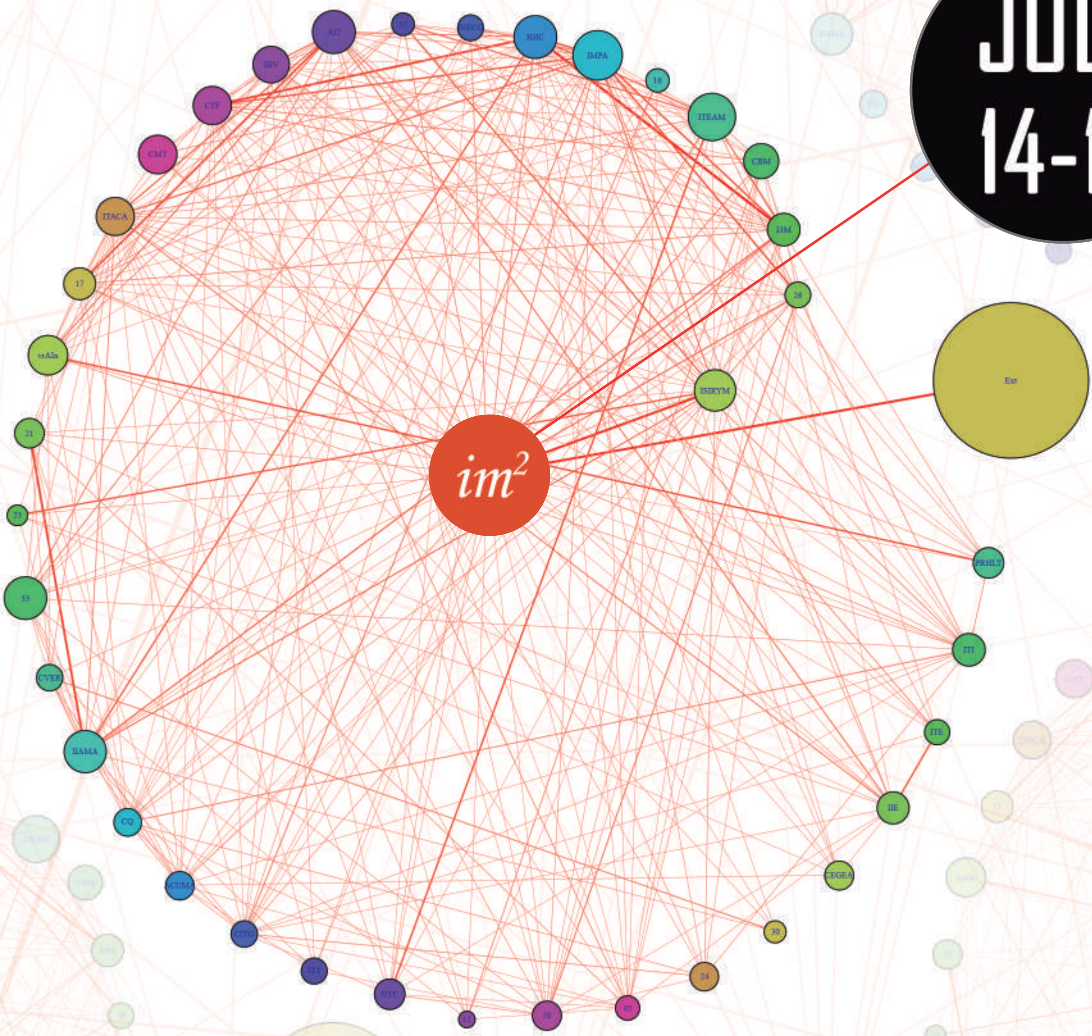

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Some advances in Relativistic Positioning Systems

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1 Introduction

For more than a decade our research team has been working in the framework of Relativistic Positioning Systems (RPS). A new algorithm was built to compute the positioning errors. This algorithm has been improved along the years so as to better compute this positioning. ESA Galileo Satellites Constellation has mainly been simulated in our computations. However our method can also be applied to other satellites simulations at different heights. In this extended abstract an overview of this work is made. Some conclusions and perspectives are also presented.

2 Methods

In 2012, [1] presented a simulation of the Galileo ESA and GPS NASA Satellites Constellations. They described the timelike geodesics of such satellites as circumferences centred at the same point as Earth. Schwarzschild metric was used to make such a description. These orbits are named the nominal orbits. Then a description of the null geodesics of photons emitted from such satellites were simulated in Minkowski space-time. The analytical solution of [2] was used to described the light trajectories from emission to reception inertial coordinates of any event. All these computations were implemented in a numerical algorithm. In that work, such code was applied and tested. The analytical study of [2] was there numerically performed. The so-called emission coordinate region and co-region, the bifurcation problem (double localization) in the positioning of the receiver satellite, among other analysis were then pointed out for the satellites taken there and discussed in detail.
Further on, in [3] the mentioned code was applied to Relativity Positioning Systems (RPS). Uncertainties in the satellite world lines lead to dominant positioning errors. A new approach was presented to compute some positioning errors. As the satellite orbits are not circular due to

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the perturbations they have, to achieve a more realistic description such perturbations need to be taken into account. In the study presented in 2014, perturbations of the nominal orbits were statistically simulated. Using the formula from [2] a user location is determined with the four satellites proper times that the user receives together with the satellite world lines. This formula can be used with any satellite description, although photons necessarily travel in a Minkowskian space-time. The difference of the user position determined with the nominal and the statistically perturbed satellite orbits was computed. This difference between nominal and perturbed orbits was defined as the U-error.

Later on, in [4] the positioning errors associated to the simplifying assumption that photons move in Minkowski space-time (S-errors) were estimated and compared with the U-errors. Both errors were computed for the same points and times to make comparisons possible. For a certain realistic modelling of the world line uncertainties, the estimated S-errors were proved to be smaller than the U-errors, which showed that the approach based on the assumption that the Earth's gravitational field produces negligible effects on photons might be used in a large region surrounding Earth. The applicability of this approach –which simplifies numerical calculations– to positioning problems, and the usefulness of our S-error maps, were pointed out. A better approach, based on the assumption that photons move in the Schwarzschild space-time governed by an idealized Earth, was also analysed. More accurate descriptions of photon propagation involving non symmetric space-time structures were demonstrated not to be necessary for ordinary positioning and spacecraft navigation around Earth. The assumptions point out in this work are taken on in our next works. Therefore, for our purposes, the computation of the photon geodesics in Minkowski space-time is sufficient.

At present, in [5], we have improved our previous computations. Now we compute the perturbed orbits of the satellites considering a General Relativity space-time metric. The metric considered so far takes into account the gravitational effects of the Earth, the Moon and the Sun, and also the Earth oblateness. Once this study has been made, more perturbations are being taken into account. However, the order of magnitude of every contribution depends on the distance to the Earth. Notice that here linear perturbation theory is not considered. A metric is used from the first step to describe the space-time. The time-like satellite geodesic equations are computed and a study of the satellite orbits in this new metric is first introduced. Once this study is being made, the approach presented in [3] is developed, by means of a new analysis of the U-errors inside a great region surrounding Earth. This analysis is performed by the comparison of the positions given with the Schwarzschild metric and the new metric introduced there. A Runge-Kutta algorithm is implemented to solve the geodesics ordinary differential equations. Although we are aware that other effects should be considered we decided to start with the three effects commented in this paragraph to implement our algorithm. The terms considered in the present work are sufficient to test the algorithm and prepare it for our new purposes in RPS. Some improvements in the computation of the U-errors using both metrics are introduced with respect to our previous works. We are now incorporating more PN contributions and other terms. Great accuracy in the digits is needed when considering all PN contributions.

3 Results

The geodesic ordinary differential equations (ODE) are solved using a Runge-Kutta algorithm. High accuracy is needed (10^{-18}) for our purposes. Moreover multiple precision (40 significant digits for each number) is taken. In order to solve the ODE suitable initial conditions have been determined. So as to precisely calculate the user location precise satellite geodesics are needed. Also, this precision is required to incorporate other small contributions to the satellite orbits perturbations. As it is well known, one of the greater contributions at the height of Galileo satellites is the Earth oblateness (about a few kilometres). The Moon and Sun gravitational effects are also important (about a few hundred Kilometres). Furthermore, the satellite position is shifted when the Moon is considered after one orbital period. This fact is due to the greater relative Moon motion. Our results are in accordance with other results known in the literature. Our algorithm is working properly because the order of magnitude calculated in previous researches with Newtonian GNSS (Global Navigation Satellite Systems) approximation is computed (see for instance [6], Chapter 3) with it. The RPS methods are more exact than GNSS classical computations, even when the last ones incorporate relativity corrections. Notice that we numerically solve the satellite geodesic equations (ODE) in our given space-time from the beginning.

We first simulated satellite world lines at Galileo Satellite Constellation height, but afterwards other satellite world lines further away from Earth were simulated. A study of the influence of the perturbations considered in our metric was then performed. At $5 \times 10^4 Km$ distance from the centre of the Earth, the Earth oblateness orbit perturbing contribution is of the same order of magnitude than that of Moon and Sun gravitational effect. As it is well known, when the satellite orbital height increases, the Earth oblateness effect decreases and Moon and Sun effects increase. That is what we obtain. The Earth oblateness effect is smaller than the Moon and the Sun contributions at $10^5 Km$. As it occurs at the Galileo satellites distances. At the further distances we are considering, the satellite position is also shifted when the Moon presence is considered. Now it is possible to appreciate the separate contribution of each perturbation in the RPS. This is an improvement with respect to our previous works.

After studying the satellite timelike world lines, we compute RPS with our metric. A similar procedure as in [3] has been carried out. But now, a metric taking into account the greater physical perturbations at Galileo Satellite Constellation is considered to compute Δ_d, the U-error, and not statistical perturbations. Notice that the same formula, [2], has been used to calculate the proper times that the user receives from the satellites. Besides, HEALPIx maps are considered to describe the U-errors as in [3].

The positioning errors values, Δ_d, are almost of the same order of magnitude as those of the perturbed satellite orbits (orbital perturbation effect). This is the same conclusion as in [3] although now a better accuracy is obtained. Here the highest Δ_d values correspond to having the maximum radial distance deviations of the satellite for the case of the four chosen satellites. Therefore, the value Δ_d depends directly on the satellite-Earth-Moon-Sun relative spatial configuration and it does so for each of the four satellites. Almost the same HEALPIx maps are recovered after a Galileo satellite orbital period. This is because the relative spatial configuration among satellite-Moon-Sun-Earth does not nearly change after 14.2 h (periodic effect), as the Moon and Sun hardly move after a Galileo orbital period. Here the Δ_d values are smaller than the ones obtained with the statistical procedure used in [3]. This fact is due to a more realistic representation of the satellite orbital perturbations by the use of metrics instead of statistical deviations. Therefore a more accurate computation of the U-errors is performed and so a more precise calculation of the user's positioning can be achieved. This represents an advance in our computations respect our previous works.

As in [3] when the radii of the spherical surfaces increase, the Δ_d values increase. This is because if the user is very far from the satellites, these are all in a small solid angle and the tetrahedron volume V_T is expected to be small. Moreover, on these maps, there are regions with $J \simeq 0$ when the radius is greater than about $20000 Km$. One solution when $J \simeq 0$, is to choose another combination of four satellites from Galileo or GPS constellation, whose user-satellites configuration is associated to values of J not close to zero. An alternative way to do that could be to locate the satellites in other orbits in the solar system, in a way that the user-satellites spatial configuration corresponds to V_T values not close to zero.

4 Conclusions and Perspectives

At the height of Galileo and GPS satellites, Our results show that the greatest effect comes from the Earth quadrupole, the second one from Moon and the third one from Sun. Those satellite world lines are needed to position a user in RPS.
The perturbations computed in [5], using metrics, improve our previous works based on statistical methods as: 1) A better description of the real satellite orbits is obtained. 2)The effect of each perturbing effect in the satellite trajectories can be analysed. 3) Also the combination of two of the three terms in the metric is performed and the three of them together. Therefore, according with the terms considered, the satellite world lines have slightly different descriptions. 4) So, a study of the contribution of each effect on the user's positioning can also be done. 5) The value of the U-errors is now smaller. 6) That means a more precise computation of the user's positioning. The main goal of our present research is to analyse the U-errors computed as the difference in RPS by using Schwarzschild and a more accurate metric to describe the satellite world lines. Our present work is an advance with respect to the study performed in the past. In [3] the satellite perturbations were statistically calculated. Nowadays we calculate such perturbations in a different way. First the effects of Moon, Sun and Earth quadrupole in the metric have been considered. At the height of Galileo or GPS constellation these are the most important effects perturbing satellite world lines. A deep exploration of the change in the Galileo satellites orbits description allows us to better understand the difference in the positioning, the U-errors. This analysis considers the contribution of such three terms by (i) taking just one of them, (ii) summing two of them and (iii) combining all three in the metric.
An improvement in our numerical procedure is currently being considered. The Newton-Raphson numerical method could be avoided, in such a way that analytical functions of the proper time are not used, since they imply the use of Schwarzschild world lines for the satellites. For example, to use the secant method with the satellite world lines (taking into account the contributions in the metric we consider). This change will probably improve the numerical code and results.
Another interesting improvement in projection is to create HEALPIx maps, but calculating the positioning error Δ_d on the geoid, instead of on the spherical surface of radius centred in the Earth. A better precision of orbits could be achieved. Data from Galileo Constellation, and other constellations, could be compared. These results should also be interesting for geodesic treatment. As stated ahead, we are now working in the consideration of other perturbations a part from those considered in [5] to compute the satellite world lines. The order of magnitude of such contributions depends on the satellite's height as it can be seen, for instance, in [7] (see, in particular, fig 3.1 at page 55). Therefore, this variation of the trajectories of the four satellites considered should contribute to the change in the calculations of RPS.
The use of our algorithm in space navigation is also being planned. To locate the emitters (four satellites) in the solar system it is better to use the Barycentric Celestial Reference System. Besides, the configurations of the user-satellites should be located in such a way U-errors are minimized. For instance, near the Moon, two emitters fixed on its surface (North and South poles) and two emitters from Galileo satellites. In such a case, the V_T value (see [1]) is not very small, in most of the cases. The positioning of a spacecraft that navigates in the solar system should be determined taking emitters in other appropriate places to be analysed.

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