7

A new family for solving nonlinear systems based on weight functions Kalitkin-Ermankov type

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1 Introduction

The main problem to study is to find the solution of $F(x) = 0$, where $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is a univariate function for $n = 1$ or multivariate when $n > 1$, defined on a convex set D. The Newton's scheme is the most employed iterative procedure to solve these kind of problems, its iterative expression is

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}),\ k = 0, 1, \ldots$$

where $F'(x^{(k)})$ denotes the Jacobian matrix of nonlinear function F evaluated on the iterate $x^{(k)}$. For good convergence of Newton's scheme an initial estimate "close" to the solutions is necessary, but this condition is not satisfied in the modeling of most technical problems. To deal with this issue and enlarging the domain of convergence, a damped modification of Newton's method was proposed for equations in [1], [2] with the general form

$$x_{k+1} = x_k - \gamma_k \frac{f(x)}{f'(x_k)},\ k = 0, 1, \ldots$$

being γ_k a sequence of real numbers determined by certain expression or algorithm. One of the expressions that γ_k can take is the Kalitkin - Ermankov coefficient [1]

$$\gamma_k = \frac{\|f(x_k)\|^2}{\|f(x_k)\|^2 + \left\|f(x_k - \frac{f(x)}{f'(x_k)})\right\|^2},\ k = 0, 1, \ldots,$$

where $\|\cdot\|$ denotes any norm.

In this paper, a new three step family is proposed. This scheme has classical Newton's procedure as the first step, and the corrector steps are composed by weight functions that slightly resemble Kalitkin - Ermankov coefficient. The γ_k coefficient for equations (that is the weight function as well) of this family has the form

$$\gamma_k = \frac{\alpha f(x_k)}{\alpha f(x_k) - 2f\left(x_k - \frac{f(x)}{f'(x_k)}\right)}, \qquad k = 0, 1, \ldots$$

being α a parameter, and its extension to multivariate case is

$$\Gamma^{(k)} = \alpha \left[\alpha I - 2\left(I - [F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F]\right)\right]^{-1}. \tag{1}$$

2 Basic definitions

Let $\{x^{(k)}\}_{k\geq 0}$ be a sequence in $\mathbb{R}^n$ which converges to ξ, being ξ the solution of certain nonlinear system of equations, then the convergence is called of order p with $p \geq 1$, if there exists $M > 0$ ($0 < M < 1$ if $p = 1$) and k_0 such that

$$\|x^{(k+1)} - \xi\| \leq M\|x^{(k)} - \xi\|^p, \forall k \geq k_0,$$

or

$$\|e^{(k+1)}\| \leq M\|e^{(k)}\|^p, \forall k \geq k_0, \text{ where } e^{(k)} = x^{(k)} - \alpha.$$

Moreover, being ξ such that $F(\xi) = 0$ and supposing that $x^{(k-1)}, x^{(k)}, x^{(k+1)}$ are three consecutive iterations close to ξ, we introduce ρ as the *approximated computational order of convergence* that can be estimated using the expression

$$\rho \approx \frac{\ln(\|x^{(k-1)} - x^{(k)}\|/\|x^{(k)} - x^{(k-1)}\|)}{\ln(\|x^{(k)} - x^{(k-1)}\|/\|x^{(k-1)} - x^{(k-2)}\|)} \qquad k = 2, 3, \ldots \tag{2}$$

Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently Fréchet differentiable function in D, for $\xi + h \in \mathbb{R}^n$ lying in a neighborhood of a solution ξ of $F(x) = 0$, applying Taylor expansion and assuming that the Jacobian matrix $F'(\xi)$ is non singular, we have

$$F(\xi + h) = F'(\xi)\left[h + \sum_{q=2}^{p-1} C_q h^q\right] + O(h^p). \tag{3}$$

where $C_q = (1/q!)[F'(\xi)]^{-1}F^{(q)}(\xi)$, $q \geq 2$. We take into account that $C_q h^q \in \mathbb{R}^n$ since $F^{(q)}(\xi) \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$ and $[F'(\xi)]^{-1} \in \mathcal{L}(\mathbb{R}^n)$. Therefore, we can express F' as

$$F'(\xi + h) = F'(\xi)\left[I + \sum_{q=2}^{p-1} qC_q h^{q-1}\right] + O(h^{p-1}), \tag{4}$$

where I is the identity matrix and $qC_q h^{q-1} \in \mathcal{L}(\mathbb{R}^n)$.
On the other hand, in accordance with the notation defined by Artidiello et al. in [3], if $X = \mathbb{R}^{n\times n}$ denotes the Banach space of real square matrices of size $n \times n$, we can define a matrix function H, $H : X \to X$ such that the Frechet derivative satisfies:

(a) $H'(u)(v) = H_1 uv$, where $H' : X \to \mathcal{L}(X)$ and $H_1 \in \mathbb{R}$,

(b) $H''(u, v)(v) = H_2 uvw$, where $H'' : X \times X \to \mathcal{L}(X)$ and $H_2 \in \mathbb{R}$.

3 The proposed iterative family PM_{KE}

In [4] it was presented a three step iterative class based on weight functions

$$\begin{aligned} y^{(k)} &= x^{(k)} - F'(x^{(k)})^{-1}F(x^{(k)}), \\ z^{(k)} &= y^{(k)} - H(t^{(k)})F'(x^{(k)})^{-1}F(y^{(k)}), \\ x^{(k+1)} &= z^{(k)} - H(t^{(k)})F'(x^{(k)})^{-1}F(z^{(k)}),\ k \geq 0 \end{aligned} \tag{5}$$

being $t^{(k)} = I - [F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F]$. This class has six order of convergence, proven in the next result.

Theorem 10. *Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently Fréchet differentiable function in an open neighborhood D of $\xi \in \mathbb{R}^n$ such that $F(\xi) = 0$, and $H : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ a sufficiently Fréchet differentiable matrix function. Let us also assume that $F'(x)$ is non singular at ξ and $x^{(0)}$ is an initial value close enough to ξ. Then, sequence $\{x^{(k)}\}_{k\geq 0}$ obtained from class* (15) *converges to ξ with order 6 if $H_0 = I$, $H_1 = 2$ and $|H_2| < \infty$, where $H_0 = H(0)$ and I is the identity matrix, being its error equation*

$$\begin{aligned} e^{(k+1)} &= \frac{1}{4}\left[(H_2^2 - 22H_2 + 120)C_2^5 + (-24 + 2H_2)C_2^2C_3C_2 + (-20 + 2H_2)C_3C_2^3 + 4C_3^2C_2\right] e^{(k)^6} \\ &\quad + O(e^{(k)^7}), \end{aligned}$$

where $C_q = \frac{1}{q!}[F'(\xi)]^{-1}F^{(q)}(\xi)$, $q = 2, 3, \ldots$ and $e^{(k)} = x^{(k)} - \xi$.

Considering (1) the multivariate extension of the coefficient γ_k for equations, we can write the weight function as

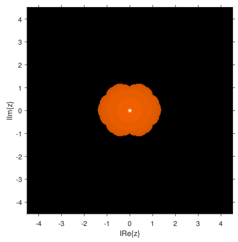
$$H(t^{(k)}) = \Gamma^{(k)} = \alpha\left[\alpha I - 2\left(I - [F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F]\right)\right]^{-1} = \alpha\left[\alpha I - 2t^{(k)}\right]^{-1} \quad (6)$$

that satisfies the conditions of Theorem 1 for $\alpha = 1$.
With the help of (6) PM_{KE} become

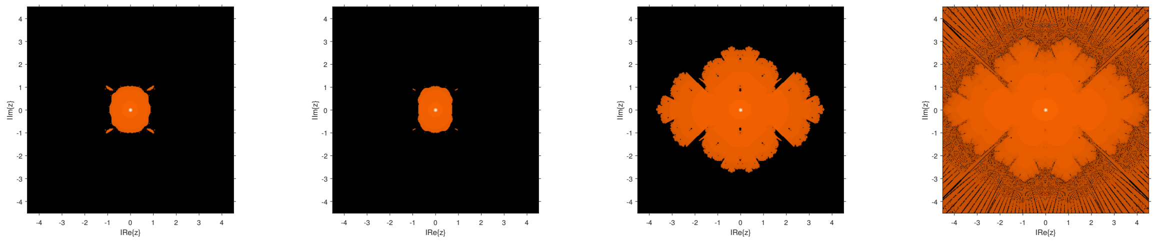
$$\begin{aligned} y^{(k)} &= x^{(k)} - F'(x^{(k)})^{-1}F(x^{(k)}), \\ z^{(k)} &= y^{(k)} - \alpha\left[\alpha I - 2t^{(k)}\right]^{-1} F'(x^{(k)})^{-1}F(y^{(k)}), \\ x^{(k+1)} &= z^{(k)} - \alpha\left[\alpha I - 2t^{(k)}\right]^{-1} F'(x^{(k)})^{-1}F(z^{(k)}),\ k \geq 0. \end{aligned} \quad (7)$$

4 Dynamical behavior comparison

In what follows, we construct and compare the dynamical planes of PM_{KE} and Newton's procedure acting on certain functions. The graphics are calculated following the routines described in [5]. A grid with 400-point per axis is constructed, every initial estimation is iterated a maximum of 50 times, checking its closeness to the root with a tolerance of 10^{-3}. The points in mesh are painted depending on the root reached, color is brighter when lesser are the iterations required for achieving the root. If all the iterations are completed and no convergence to any root is reached, then the point is painted in black.
For $n = 1$, the phase diagrams correspond to the equation $f(x) = \arctan(x)$, whose only root is $\xi = 0$.
In the vectorial case, the dynamical planes represented correspond to the function $F(\mathbf{x}) = \{x_1^2 + x_2^2 - 5,\ x_1x_2 - 2\}$, whose zeros are $\xi_1 = (-2, -1)$, $\xi_2 = (-1, -2)$, $\xi_3 = (2, 1)$ and $\xi_4 = (1, 2)$.
An analysis of Figure 1 reveals that in case of $f(x)$ the larger domain of convergence is roughly [-4.5, 4.5] belonging to PM_{KE} for $\alpha = 1.0$ and the shorter correspond to $\alpha = -1.57$ for real initial estimations. In the first case the performance of the Newton method has been improved.
In Figure 2 the dynamical planes corresponding to the function $F(\mathbf{x})$ for Newton and PM_{KE} are presented. Let us notice the good performance of the Newton procedure for functions of polynomial type. With respect to the PM_{KE} family represented in Figures 2b, 2c, 2d and 2e we can say that it is very stable except for the $\alpha = 0.85$ (Figure 2d) where it turns slightly chaotic. In general, the PM_{KE} sufficiently emulates Newton's procedure with higher order of convergence for the α values studied for the function of polynomial type $F(\mathbf{x})$.

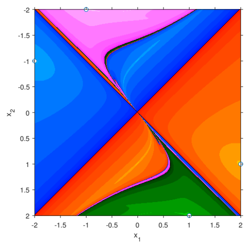


(a) Newton, $f(x)$

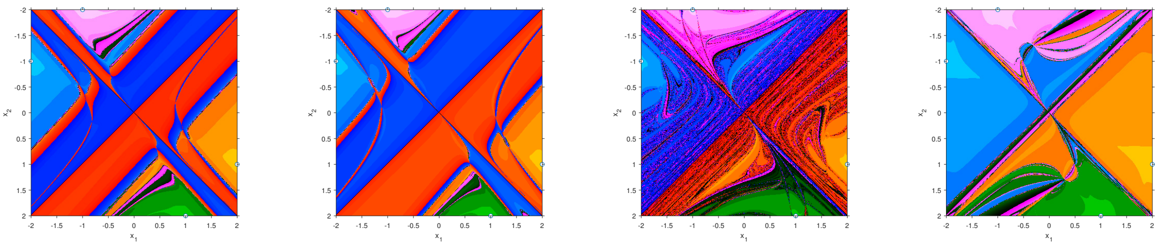


(b) $PM_{KE}, \alpha = -3.5$ (c) $PM_{KE}, \alpha = -1.57$ (d) $PM_{KE}, \alpha = -0.85$ (e) $PM_{KE}, \alpha = 1.0$

Figure 1: Phase diagrams of Newton and PM_{KE} procedures acting on $f(x)$



(a) Newton, $F(\mathbf{x})$



(b) $PM_{KE}, \alpha = -3.5$ (c) $PM_{KE}, \alpha = -1.57$ (d) $PM_{KE}, \alpha = 0.85$ (e) $PM_{KE}, \alpha = 1.0$

Figure 2: Phase diagrams of PM_{KE} and Newton procedures acting on $F(\mathbf{x})$

5 Numerical test

In this section, we compare the numerical performance of PM_{KE} and Newton methods. To make the comparative numerical experiments we use Matlab computer algebra system with 2000 digits of mantissa in variable precision arithmetics. The stopping criterion used is $\|x^{(k+1)} - x^{(k)}\| < 10^{-200}$ or $\|F(x^{(k+1)})\| < 10^{-200}$. The initial values employed and the searched solutions are symbolized as $x^{(0)}$ and ξ, respectively. For each nonlinear system one table is displayed with the results of the numerical experiments. The given information is organized as follows: $x^{(0)}$ is the initial approximation, k is the number of iteration needed to converge to the solution, the value of the stopping residuals $\|x^{(k+1)} - x^{(k)}\|$ or $\|F(x^{(k+1)})\|$ and the approximated computational order of convergence ρ (if the value of ρ for the last iterations is not stable, then '-' appears in the table). In this way, it can be checked if the convergence has reached the root ($\|F(x^{(k+1)})\| < 10^{-200}$ is achieved) or it is only a very slow convergence with a no significant difference between the two last iterates ($\|x^{(k+1)} - x^{(k)}\| < 10^{-200}$ but $\|F(x^{(k+1)})\| > 10^{-200}$), or both criteria are satisfied. The examples of nonlinear systems are the followings:

Example 1. The first nonlinear system is

$$\begin{cases} x_1 + 10\,x_2 = 0, \\ \sqrt{5}(x_3 - x_4) = 0, \\ (x_2 - 2\,x_3)^2 = 0, \\ \sqrt{10}(x_1 - x_4)^2 = 0, \end{cases}$$

with solution $\xi = (0,\, 0,\, 0)^T$ and initial estimation $x^{(0)} = (3, -1, 0, 1)$, the results are given in Table 1. Note the large number of iterations involved to reach the solutions and almost null of residuals $\|F(x^{(k+1)})\|$ for the given schemes. The minimum number of iterations in the performance corresponds to PM_{KE} for $\alpha = 0.85$. The computational approximation order of convergence is unstable in all the cases.

Table 1: Numerical results of the examined methods for the Example 1

$x^{(0)}$	Method	k	$\|x^{(k+1)} - x^{(k)}\|$	$\|F(x^{(k+1)})\|$	ρ
(3, -1, 0, 1)	N	335	35.0430×10^{-102}	2.5901×10^{-201}	-
	$\mathrm{PM}_{\mathrm{KE}\{\alpha=-3.50\}}$	318	56.6818×10^{-102}	6.1163×10^{-201}	-
	$\mathrm{PM}_{\mathrm{KE}\{\alpha=-1.57\}}$	274	6.6149×10^{-102}	3.7807×10^{-201}	-
	$\mathrm{PM}_{\mathrm{KE}\{\alpha=0.85\}}$	90	$221.9237 \times 10^{-102}$	$847.3685 \times 10^{-204}$	-
	$\mathrm{PM}_{\mathrm{KE}\{\alpha=1.00\}}$	141	$105.8336 \times 10^{-102}$	1.9240×10^{-201}	-

Example 2. Finally, we test the proposed methods with a nonlinear system of variable size. It is described as

$$\arctan(x_i) + 1 - 2\left(\sum_{j=1}^{n} x_j^2 - x_i^2\right) = 0,\ i = 1, 2, \ldots, n,$$

with $n = 20$ we employ the initial estimation $x^{(0)} = (0.75, \ldots, 0.75)^T$. In this case, the solution is $\xi \approx (0.1758, \ldots, 0.1758)^T$ and the obtained results can be found at Table 2.
In this case, the scheme PM_{KE} provide excellent results as well as Newton's with null residual $\|F(x^{(k+1)})\|$ for $\alpha = 0.85$ and lower number of iterations in all the cases. When $\alpha = 1$, the theoretical order of convergence is very close to 6 in correspondence with the Theorem 10.

6 Conclusions

In this work, we have presented a parametric family that reaches sixth-order of convergence for one value of the parameter equal to unity. The study carried out reveals good performance for the family that improves the Newton method for some functions and emulates Newton's method on polynomial functions with n=2. The numerical experiments reinforces the latest thesis.

Table 2: Numerical results of the examined methods for the Example 2

$x^{(0)}$	Method	k	$\|x^{(k+1)} - x^{(k)}\|$	$\|F(x^{(k+1)})\|$	ρ
$(0.75, \dots, 0.75)$	N	11	$174.2412 \times 10^{-201}$	0.0	2.0
	$PM_{KE1\{\alpha=-3.50\}}$	10	$279.9073 \times 10^{-126}$	$563.9181 \times 10^{-249}$	2.0
	$PM_{KE1\{\alpha=-1.57\}}$	9	$777.3794 \times 10^{-129}$	1.8215×10^{-252}	2.0
	$PM_{KE1\{\alpha=0.85\}}$	8	66.8781×10^{-201}	0.0	2.0
	$PM_{KE1\{\alpha=1.00\}}$	4	4.6966×10^{-36}	41.1062×10^{-213}	5.9493

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