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Representations of bornologies

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Abstract

Bornologies abstract the properties of bounded sets of a metric space. But there are unbounded bornologies on a metric space like $\mathcal{P}(\mathbb{R})$ with the Euclidean metric. We show that by replacing $[0, \infty)$ with a partially ordered monoid every bornology is the set of bounded subsets of a generalized metric mapped into a partially ordered monoid. We also prove that the set of bornologies on a set is the join completion of the equivalence classes of a relation on the power set of the set.

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1. Introduction

Let X be a topological space. A bornology on X is a family of subsets $\mathcal{A}$ of X which satisfies the following axioms:

- $\mathcal{A}$ is closed under finite union and $\bigcup \mathcal{A} = X$
- $A \in \mathcal{A}$ and $B \subseteq A$ implies $B \in \mathcal{A}$

A typical example of a bornology is the set of bounded subsets of a metric space. Another example is the set of finite subsets of a given set; it is denoted by $\mathcal{F}$. One can easily verify that every bornology on a set contains $\mathcal{F}$.

The bornology generated by a given subset H of a set X is the smallest bornology containing H and in [4] is called a principal bornology and it can be easily seen that it is $\downarrow H \cup \mathcal{F}$, where $\downarrow H = \{Y \subseteq X : Y \subseteq H\}$.

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Given bornologies $\mathcal{A}$ and $\mathcal{B}$ on the sets X and Y respectively, a function $f : X \to Y$ is called bounded map provided that $f(M) \in \mathcal{B}$ if $M \in \mathcal{A}$.

The theory of bornologies play an important role in functional analysis, see [5] and [7]. Bornologies also have been considered in the theory of locally convex spaces, see [11].

Bornologies on topological spaces are the generalization of the set of bounded sets of a metric space, the sets whose elements are within a fixed distance from each other. Metrics are useful when we talk about distance between the points and sets, closeness of sets and points, and more importantly one can talk about Cauchy sequences and convergence. But, some of these notions fail when we deal with topological spaces. These concepts are all related to boundedness. In this article we overcome this issue by relating bornologies with boundedness. In [3], it is proved that a bornology $\mathcal{A}$ on a non-empty set X is coincident with the set of bounded sets of a metric $d : X \times X \to [0, \infty)$ if and only if $\mathcal{A}$ has a countable base. Therefore, when one wants to relate a bornology with uncountable base with boundedness, the boundedness or the set of radii must come from a more general setting than the real numbers. Here we consider generalized metrics into partially ordered monoids, po-monoids. Po-monoids are monoids with an order which is cooperative with the binary relation. This will enable us to talk about distance between the points and sets and more importantly about bounded sets and bornologies.

In [12] Hu considered bornologies on topological spaces and he proved the following theorem:

Hu's Theorem: Let $\mathcal{B} \neq \mathcal{P}(X)$ be a bornology on a normal topological space X. Then there is a continuous function $f : X \to [0, \infty)$ such that $\mathcal{B} = \downarrow \{\{x \in X : f(x) < t\} : t \geq 0\}$ if and only if $\mathcal{B}$ has a countable base and every $x \in X$ is in the interior of some element $B \in \mathcal{B}$.

In the next section we prove a similar theorem for every bornology on a set X by allowing the range of the function be a lattice rather than $[0, \infty)$. In Section three we show that every bornology is the set of bounded sets of a generalized metric space. In the last section we show that the set of bornologies on a set X is a frame and we investigate the properties of this frame.

2. Induced bornologies by compatible functions

Let L be a poset, D be a directed subset of L, and $\alpha : X \to L$ be a function, for every $t \in D$ define $N_{t,\alpha} = \{x \in X : \alpha(x) < t\}$ and $\overline{N}_{t,\alpha} = \{x \in X : \alpha(x) \leq t\}$. We call $N_{t,\alpha}$ and $\overline{N}_{t,\alpha}$ respectively the strict sublevel and the sublevel of X with respect to α and D.

Define $\mathcal{B}_{\alpha,D} = \{A \subseteq X : \exists t \in D \text{ such that } A \subseteq N_{t,\alpha}\}$ and $\overline{\mathcal{B}}_{\alpha,D} = \{A \subseteq X : \exists t \in D \text{ such that } A \subseteq \overline{N}_{t,\alpha}\}$. One can easily see that $N_{t,\alpha} \subseteq \overline{N}_{t,\alpha}$ and therefore, $\mathcal{B}_{\alpha,D} \subseteq \overline{\mathcal{B}}_{\alpha,D}$.

We say α and D are compatible if for every $x \in X$ there is a $t \in D$ such that $\alpha(x) < t$.

Theorem 2.1. *Let $(L, \leq)$ be a poset and D be a directed subset of L. If $\alpha : X \to L$ is a function compatible with D, then $\mathcal{B}_{\alpha,D}$ and $\overline{\mathcal{B}}_{\alpha,D}$ are bornologies on X.*

Proof. We prove $\mathcal{B}_{\alpha,D}$ is a bornology and the proof that $\overline{\mathcal{B}}_{\alpha,D}$ is a bornology is similar and we will leave it to the reader. One can easily see that if $B \subseteq A \in \mathcal{B}_{\alpha,D}$ then $B \in \mathcal{B}_{\alpha,D}$. Next if $A \subseteq N_{t,\alpha}$ and $B \subseteq N_s$, then for every $x \in A \cup B$ either $\alpha(x) < t$ or $\alpha(x) < s$. Since D is directed, there is an $r \in D$ such that $s, t \leq r$. Since $t, s \leq r$, we have $\alpha(x) < r$. Thus, $A \cup B \subseteq N_r$ and therefore, $\mathcal{B}_{\alpha,D}$ is closed under finite union. Since α is a function compatible with D, one can see that $\bigcup \mathcal{B}_{\alpha,D} = X$ □

Proposition 2.2. *Let L and J be posets and D be a directed subset of L. If $\alpha : X \to L$ is a function compatible with D and $f : L \to J$ is an order preserving map, then $\mathcal{B}_{f\circ\alpha,f(D)}$ is a bornology on X such that $\mathcal{B}_{\alpha,D} \subseteq \bar{\mathcal{B}}_{f\circ\alpha,f(D)}$.*

Proof. First of all note that $f(D)$ is a directed subset of J because for every $f(a), f(b) \in f(D)$, we have $f(a), f(b) \leq f(r)$, where $r \in D$ with $a, b \leq r$. Next note that if $M \subseteq E \in \mathcal{B}_{f\circ\alpha,f(D)}$ then obviously, $M \in \mathcal{B}_{f\circ\alpha,f(D)}$. Also if $E, T \in \mathcal{B}_{f\circ\alpha,f(D)}$ then there are $f(r), f(s) \in f(D)$ such that $E \subseteq \{x \in X : (f \circ \alpha)(x) < f(r)\}$ and $T \subseteq \{x \in X : (f \circ \alpha)(x) < f(s)\}$. Consequently, $E \cup T \subseteq \{x \in X : (f \circ \alpha)(x) < f(p)\}$, where $p \in D$ and $r, s \leq p$ and so, $E \cup T \in \mathcal{B}_{f\circ\alpha,f(D)}$.

Also for every $x \in X$, since α is compatible with D, there is a $t \in D$ such that $\alpha(x) < t$ and therefore, $x \in \{x \in X : (f \circ \alpha)(x) < f(t)\}$.

For the second part, note that if $A \subseteq \{x : \alpha(x) < t\}$. Then, $A \subseteq \{x : f(\alpha(x)) \leq f(t)\}$. □

Proposition 2.3. *Let $(L, \leq)$ be a poset and D be a directed subset of L. If $\alpha : X \to L$ is a function compatible with D and $f : R \to X$ is a function, then $\mathcal{B}_{\alpha\circ f,D}$ is a bornology on R and $f : R \to X$ is a bounded map between $(R, \mathcal{B}_{\alpha\circ f,D})$ and $(X, \mathcal{B}_{\alpha,D})$.*

Proof. Note that if $M \subseteq E \in \mathcal{B}_{\alpha\circ f,D}$ then $M \in \mathcal{B}_{\alpha\circ f,D}$. Also if $E, J \in \mathcal{B}_{\alpha\circ f,D}$ then there are $r, s \in D$ such that $E \subseteq \{x \in X : (\alpha \circ f)(x) < r\}$ and $J \subseteq \{x \in X : (\alpha \circ f)(x) < s\}$. Consequently, $E \cup J \subseteq \{x \in X : (\alpha \circ f)(x) < p\}$ where $p \in D$ and $r, s \leq p$ and so, $E \cup J \in \mathcal{B}_{\alpha\circ f,D}$. Also for every $x \in R$ we have $f(x) \in X$. Since α is compatible with D, there is a $t \in D$ such that $\alpha(f(x)) < t$ and therefore, $x \in \{x \in X : (\alpha \circ f)(x) < t\}$.

For the second part, suppose $C \in \mathcal{B}_{\alpha\circ f,D}$. We show that $f(C) \in \mathcal{B}_{\alpha,D}$. Since $C \in \mathcal{B}_{\alpha\circ f,D}$, there is an $r \in D$ such that $C \subseteq \{x \in R : (\alpha \circ f)(x) < r\}$. Thus, for every $c \in C$ we have $\alpha(f(c)) < r$ and so, $f(c) \in \{x \in X : \alpha(x) < r\}$. Thus, $f(C) \subseteq \{x \in X : \alpha(x) < r\} \in \mathcal{B}_{\alpha,D}$.

□

In order to show the converse of Theorem 2.1, we need to employ a second relation on the partially ordered set that is compatible with the original order of the partial ordered set. In [16] and [20] a second relation was called a good relation , so we adopt the same name.

Definition 2.4. A *good* relation $\prec$ on a poset $(G, \leq)$ is a subset of $\leq$ such that for each $a, b, c, d \in G$:
(trans) $a \leq b \prec c \leq d \Rightarrow a \prec d$.

Example 2.5.

- On each poset , $\varnothing$ is a good relation.
- If $(L, \leq)$ is a partially ordered set, then $<$ is a good order on L
- Let $G = \mathbb{R} \times H$ with the lexicographic order, H any poset. Define $(r, h) \prec (s, k)$ if $r < s$; then $\prec$ is a good relation on G.

Theorem 2.6. *Let $\mathcal{A}$ be a bornology on a set X. There is a lattice $(L, \leq)$ equipped with a good relation $\prec$, a directed subset D of L, and a function $\alpha : X \to L$ compatible with D such that $\mathcal{A} = \mathcal{B}^*_{\alpha,D}$, where $\mathcal{B}^*_{\alpha,D} = \{A \subseteq X : \exists t \in D$ such that $A \subseteq \{x \in X : \alpha(x) \prec t\}\}$. Moreover, $\mathcal{A} = \{\{x \in X : \alpha(x) \prec t\} : t \in D\}$, the strict sublevel of X with respect to α and D.*

Proof. Suppose that $\mathcal{A}$ is a bornology on a set X. For each $K \in \mathcal{A}$ let $L_K = [0, 1] \times [0, 1]$ equipped with the lexicographic order: $(q, r) \leq (s, t)$ if $q < s$ or $q = s$ and $r \leq t$. Then L_K is a lattice. Let $L = \prod_{K \in \mathcal{A}} L_K$ and define a good relation on L as follows: if $(g_K)_{K\in\mathcal{A}}, (h_K)_{K\in\mathcal{A}} \in L$ then $(g_K)_{K\in\mathcal{A}} \prec (h_K)_{K\in\mathcal{A}}$ if and only if $g_K < h_K$ for every $K \in \mathcal{A}$. One can easily see that $\prec$ obeys the good order condition.

For each $K \in \mathcal{A}$ let $D_K = [0, 1] \times (0, 1]$ and
$D = \{r = ((r^1_K, r^2_K))_{K\in\mathcal{A}} \in L : \exists M \in \mathcal{A}$ such that $r^1_K = 0$ for every $K \supseteq M\}$

We prove that D is a directed subset of L. Let $r = ((r^1_K, r^2_K))_{K\in\mathcal{A}} \in D$ and $s = ((s^1_K, s^2_K))_{K\in\mathcal{A}} \in D$. By definition, there is an $M \in \mathcal{A}$ such that $r^1_K = 0$ for every $K \supseteq M$ and there is an $E \in \mathcal{A}$ such that $s^1_K = 0$ for every $K \supseteq E$. Now define

$$t^1_K = \begin{cases} 0, & \text{if } K \supseteq M \cup E; \\ r^1_K \vee s^1_K, & \text{otherwise} \end{cases}.$$

One can see that $t = ((t^1_K, t^2_K))_{K\in\mathcal{A}}$ where $t^2_K = r^2_K \vee s^2_K$ is in D and $r, s \leq t$. Therefore, D is a directed set.

For $K \in \mathcal{A}$ define $\varphi_K : X \to [0, 1]$ such that $\varphi_K(z) = \begin{cases} 0, & \text{if } z \in K \\ 1, & \text{if } z \notin K \end{cases}$. Now define the function $\alpha : X \to L$ by $(\alpha(x))_K = (0, \varphi_K(x))$.

We prove that $B^*_{\alpha,D} = \mathcal{A}$. First we show that $\mathcal{B}^*_{\alpha,D} \subseteq \mathcal{A}$. Suppose that $W \in \mathcal{B}^*_{\alpha,D}$. There is a $t \in D$ such that $W \subseteq \{x \in X : \alpha(x) \prec t\}$. Let $t = ((t^1_K, t^2_K))_K \in D$. For every $x \in W$ we have, $\alpha(x) \prec t$ and therefore for every $K \in \mathcal{A}$ we have $(0, \varphi_K(x)) < (t^1_K, t^2_K)$. Since $t \in D$, there is a $C \in \mathcal{A}$

such that $t_K^1 = 0$ for every $K \supseteq C$ and in particular, $t_C^1 = 0$. Now we show that $W \subseteq C$ and therefore $W \in \mathcal{A}$

By way of contradiction, suppose that $W \not\subseteq C$. Let $z \in W \setminus C$. Then $(\alpha(z))_C = (0, \varphi_C(z)) = (0,1) \not< (0, t_C^2) = (t_C^1, t_C^2)$ and therefore, $z \notin \{x \in X : \alpha(x) \prec t\}$ which is a contradiction. Consequently, $W \subseteq C$. Hence, $\mathcal{B}^*_{\alpha,D} \subseteq \mathcal{A}$.

Next , we prove that $\mathcal{A} \subseteq \mathcal{B}^*_{\alpha,D}$. Suppose $H \in \mathcal{A}$. We will prove that $H = \{x \in X : \alpha(x) \prec t_H\}$, where $t_H = (t_K^1, t_K^2) = \begin{cases} (0, 0.5), & \text{if } K \supseteq H \\ (1,1), & \text{if } K \not\supseteq H \end{cases}$.

Note that for every $x \in H$ we have $\alpha(x)_K = (0, \varphi_K(x)) = \begin{cases} (0,0), & \text{if } x \in K \\ (0,1), & \text{if } x \notin K \end{cases}$.

There are two cases either $x \in K$ or $x \notin K$. If $x \in K$, then $\alpha(x)_K = (0, \varphi_K(x)) = (0,0) < (t_K^1, t_K^2)$. Suppose now that $x \notin K$. In this case, $\alpha(x)_K = (0, \varphi_K(x)) = (0,1)$. From $x \notin K$ we have $K \not\supseteq H$, which implies that $(t_K^1, t_K^2) = (1,1)$. Hence, $\alpha(x)_K = (0, \varphi_K(x)) = (0,1) < (1,1) = (t_K^1, t_K^2)$. By cases, $\alpha(x) \prec t_H$. Therefore, $H \subseteq \{x \in X : \alpha(x) \prec t_H\}$. Therefore, $\mathcal{A} = \mathcal{B}^*_{\alpha,D}$.

Since H was an arbitrary element of $\mathcal{A}$ in the previous paragraph, showing $\{x \in X : \alpha(x) \prec t_H\} \subseteq H$ completes proving $\mathcal{A} = \{\{x \in X : \alpha(x) \prec t\} : t \in D\}$. Note that if $z \notin H$, then $\varphi_H(z) = 1$. Thus, $\alpha(z)_H = (0, \varphi_H(z)) = (0,1) \not< (t_H)_H = (0, 0.5)$. Therefore, $z \notin \{x \in X : \alpha(x) \prec t_H\}$ and so, $\{x \in X : \alpha(x) \prec t\} \subseteq H$. Consequently, $\mathcal{A} \subseteq \{\{x \in X : \alpha(x) \prec t\} : t \in D\} \subseteq \mathcal{B}^*_{\alpha,D} = \mathcal{A}$. Therefore, $\mathcal{A} = \{\{x \in X : \alpha(x) \prec t\} : t \in D\}$ and the proof is complete.

□

In the next theorem we will simplify the lattice in Theorem 2.6 and will obtain a similar result. Here we work with sublevels of X rather than strict sublevels.

Theorem 2.7. *Let $\mathcal{A}$ be a bornology on a set X. Then there is a poset $(L, \leq)$, a directed subset D of L, and a function $\alpha : X \to L$ compatible with D such that $\mathcal{A} = \overline{\mathcal{B}}_{\alpha,D} = \{\{x \in X : \alpha(x) \leq t\} : t \in D\}$.*

Proof. Suppose that $\mathcal{A}$ is a bornology on a set X. Let $\prod_{K \in \mathcal{A}}[0,1]$ be ordered coordinatewise and define $\alpha : X \to \prod_{K \in \mathcal{A}}[0,1]$ by $\alpha(x) = (\varphi_K(x))_{K \in \mathcal{A}}$ where $\varphi_K(x)$ is defined as Theorem 2.6. For every $H \in \mathcal{A}$ define $(S_H)_K = \begin{cases} 0, & \text{if } K \supseteq H; \\ 1, & \text{if } K \not\supseteq H. \end{cases}$.

Define $D = \{S_H : H \in \mathcal{A}\} \subseteq L$. One can see that if S_H and S_P are in D then, $S_H, S_P \leq S_{H \cup P}$ and therefore D is directed.

We show that $H = \overline{N}_{S_H}$ for every $H \in \mathcal{A}$ which implies $\mathcal{A} = \overline{\mathcal{B}}_{\alpha,D}$. First, we show that $H \subseteq \overline{N}_{S_H}$ by proving that if $a \in H$, then $\alpha(a)_K \leq (S_H)_K$ for every $K \in \mathcal{A}$. It is enough to show that $\alpha(a)_K = 1$ implies $(S_H)_K = 1$. Note that $\alpha(a)_K = 1$ implies $a \notin K$. So, $a \in H \setminus K$. Thus, $K \not\supseteq H$, which implies that $(S_H)_K = 1$. Therefore, $H \subseteq \overline{N}_{S_H}$.

Next suppose $a \in \overline{N}_{S_H}$. So, $\alpha(a)_K \leq (S_H)_K$ for every $K \in \mathcal{A}$. We prove that $a \in H$. By way of contradiction, suppose that $a \notin H$. In this case, $\alpha(a)_H = 1$. On the other hand, $(S_H)_H = 0$ which is a contradiction. Thus, $\overline{N}_{S_H} \subseteq H$. Therefore, $H = \overline{N}_{S_H}$. Consequently, $\mathcal{A} = \{\{x \in X : \alpha(x) \leq t\} : t \in D\}$. Thus, $\overline{\mathcal{B}}_{\alpha,D}$, the bornology generated by $\{\{x \in X : \alpha(x) \leq t\} : t \in D\}$, equals $\mathcal{A}$ as $\mathcal{A}$ is a bornology. Therefore, $\{\{x \in X : \alpha(x) \leq t\} : t \in D\} = \mathcal{A} = \overline{\mathcal{B}}_{\alpha,D}$. □

We conclude this section by representing bornologies on a set as special subsets of the power set of the set.

Let $\mathcal{A}$ be a bornology on a set X. Define:

$$\nu : \mathcal{A} \hookrightarrow \prod_{K \in \mathcal{P}(X)} \{0,1\} \text{ by } \nu(H)_K = \begin{cases} 0, & \text{if } K \in \mathcal{A} \text{ and } K \supseteq H; \\ 1, & \text{if } K \in \mathcal{A} \text{ and } K \not\supseteq H \\ 0, & \text{if } K \notin \mathcal{A} \end{cases}.$$

Lemma 2.8. *Let $\mathcal{A}$ be a bornology on a set X.*

- $\nu(\varnothing) = (0)_{K \in \mathcal{A}}$,
- $\nu(H \cup T) = \nu(H) \vee \nu(T)$ *and therefore, ν is order preserving.*
- *ν is one-to-one.*

Proof. We leave the proof of the first claim to the reader. For the second claim note that if $(\nu(H \cup T))_K = 0$ then $K \supseteq H \cup T$ and so, $K \supseteq H$ and $K \supseteq T$. Therefore, $(\nu(H))_K = (\nu(T))_K = 0$. On the other hand if $(\nu(H \cup T))_K = 1$ then $K \not\supseteq H \cup T$ and therefore, either $K \not\supseteq H$ or $K \not\supseteq T$. Thus, $(\nu(H))_K = 1$ or $(\nu(T))_K = 1$. Consequently, $\nu(H \cup T) = \nu(H) \vee \nu(T) = 1$

For the third claim note that if $A, B \in \mathcal{A}$ and $A \neq B$, either $A \setminus B \neq \varnothing$ or $B \setminus A \neq \varnothing$. Without lost of generality suppose that $A \setminus B \neq \varnothing$ and $z \in A \setminus B$. Then, $(\nu(A))_{(A \cup B)\setminus\{z\}} = 1$ but $(\nu(B))_{(A \cup B)\setminus\{z\}} = 0$ and therefore, $\nu(A) \neq \nu(B)$. Consequently, ν is one-to-one. □

Theorem 2.9. *Suppose $L \subseteq \prod_{K \in \mathcal{P}(X)} \{0,1\}$. Consider the following four conditions.*

(a) *For every $r \in L$ and $K \in \mathcal{P}(X)$ if $r_K = 0$, then $r_M = 0$ for every $M \supseteq K$.*
(b) *For every $r \in L$ and $K \in \mathcal{P}(X)$ if $r_K = 1$, then $r_M = 1$ for every $M \subseteq K$.*
(c) *For every $r, s \in L$ and $M, N \in \mathcal{P}(X)$ if $r_M = 1$ and $s_{N \setminus M} = 1$, then there is an $t \in L$ such that $t_{M \cup N} = 1$.*
(d) *If X has more than one element, then for every $x \in X$ there is an $t \in L$ such that $t_{\{x\}} = 1$.*

If L satisfies conditions (a)-(d), then $\mathcal{A}_L = \{M \subseteq X : \exists r \in L$ such that $r_M = 1\}$ is a bornology on X. Conversely, if $\mathcal{A} \neq \mathcal{P}(X)$ is a bornology on the set X, then $\nu(\mathcal{A}) \subseteq \prod_{K \in \mathcal{P}(X)} \{0,1\}$ satisfies conditions (a)-(d). Moreover, $\mathcal{A}_{\nu(\mathcal{A})} = \mathcal{A}$.

Proof. Suppose that $L \subseteq \prod_{K\in\mathcal{P}(X)}\{0,1\}$ satisfies conditions (a)-(d). We prove that $\mathcal{A}_L = \{M \subseteq X : \exists r \in L \text{ such that } r_M = 1\}$ is a bornology on X. Suppose $T \subseteq M \in \mathcal{A}_L$. There exists an $r \in L$ such that $r_M = 1$. So, $r_T = 1$ by property (b). Thus, $T \in \mathcal{A}_L$. Next we show that $M, N \in \mathcal{A}_L$ implies $M \cup N \in \mathcal{A}_L$. Since $M, N \in \mathcal{A}_L$, there are $s, t \in L$ such that $s_M = 1$ and $t_N = 1$. Since $t_N = 1$, by property (b) we have $t_{N\setminus M} = 1$. So, by property (c) there is an $r \in L$ such that $r_{M\cup N} = 1$. Therefore, $M \cup N \in \mathcal{A}_L$. Finally, since for every $x \in X$ there is an $r \in L$ such that $r_{\{x\}} = 1$, we have $\{x\} \in \mathcal{A}_L$. Therefore, $\bigcup \mathcal{A}_L = X$.

For the converse suppose $\mathcal{A}$ is a bornology on the set X.

We first show that conditions (a) and (b) are satisfied. To this end let $r = \nu(H) \in \mathcal{A}_{\nu(\mathcal{A})}$, where $H \in \mathcal{A}$. For condition (a) assume that $r_K = 0$ and $K \in \mathcal{P}(X)$ and $M \supseteq K$. If $M \notin \mathcal{A}$ then $r_M = 0$ by definition of ν. If $M \in \mathcal{A}$ then $K \in \mathcal{A}$ and hence $K \supseteq H$. Consequently, $M \supseteq H$ and therefore, $r_M = 0$. So, $\nu(\mathcal{A})$ satisfies condition (a). For condition (b), assume $r_K = 1$. We show that $r_M = 0$ for $M \subseteq K$. Since $r_K = 1$, we have $K \in \mathcal{A}$ and $K \not\supseteq H$. Thus, $M \subseteq K$ implies $M \not\supseteq H$. Therefore, $r_M = 1$. So, $\nu(\mathcal{A})$ satisfies condition (b).

For condition (c) assume that $M, N \in \mathcal{P}(X)$ and $r, s \in \nu(\mathcal{A})$ are such that $r_M = 1$, and $s_{N\setminus M} = 1$. Let $H, J \in \mathcal{A}$ be such that $r = \nu(H)$ and $s = \nu(J)$. Notice that $M \in \mathcal{A}$, $M \not\supseteq H$, $N \setminus M \in \mathcal{A}$, and $N \setminus M \not\supseteq J$. Thus, $M \cup N \in \mathcal{A}$. Now, let $t = \nu(M \cup N \cup \{z\})$ where $z \notin M \cup N$. Therefore, $r_{M\cup N} = 1$. So, $\nu(\mathcal{A})$ satisfies condition (c).

For condition (d) note that for every $y \neq x$ we have, $\nu(\{y\})_{\{x\}} = 1$.

At last we prove $\mathcal{A}_{\nu(\mathcal{A})} = \mathcal{A}$. Let $B \in \mathcal{A}$. Since $\mathcal{A} \neq \mathcal{P}(X)$, there is a $z \in X \setminus B$. Now, $\nu(B \cup \{z\})_B = 1$ and so, $B \in \mathcal{A}_{\nu(\mathcal{A})}$. Therefore, $\mathcal{A} \subseteq \mathcal{A}_{\nu(\mathcal{A})}$. On the other hand, if $B \notin \mathcal{A}$, then by definition of ν we have $\nu(H)_B = 0$ for every $H \in \mathcal{A}$ and therefore $B \notin \mathcal{A}_{\nu(\mathcal{A})}$. Consequently, $\mathcal{A}_{\nu(\mathcal{A})} = \mathcal{A}$. □

Definition 2.10. Let $L \subseteq \prod_{K\in\mathcal{P}(X)}\{0,1\}$. By the kernel of L we mean $ker(L) = \{K \subseteq X : \forall r \in L, r_K = 0\}$ and $ker^C(L)$, the co-kernel of L, will be defined by $\mathcal{P}(X) \setminus ker(L)$

Definition 2.11. We say $L \subseteq \prod_{K\in\mathcal{P}(X)}\{0,1\}$ is called a ν-subset of $\prod_{K\in\mathcal{P}(X)}\{0,1\}$ if for every $r = (r_K)_{K\in\mathcal{P}(X)} \in L$ there is an $H \subseteq X$ such that for every $K \in ker^C(L)$ we have $r_K = 1$ if and only if $K \not\supseteq H$.

Remark 2.12. By what we learned for every ν-subset of $\prod_{K\in\mathcal{P}(X)}\{0,1\}$ we can define a bornology on X and conversely we can assign a ν-set to every bornology.

3. Generalized metrics

The goal of this section is to show that for every bornology $\mathcal{A}$ on a set X we can find an algebraic structure M and a generalized metric $d : X \times X \to M$ so that $\mathcal{A}$ is the set of bounded subsets of X. Since d is a metric, at minimum M must be equipped with a binary operation $+$ and have a 0. Also, there should be an order on M that is compatible with $+$.

There is a long tradition of allowing metrics to take values in structures more general than the non-negative reals or to satisfy weaker axioms. Kelley, in [13], lists several references going back as far as [18] in 1941. For more recent related work see [8], [9], [10], [15], [14], [17], [19], and [20]. Here we study generalized metrics that satisfy all of the axioms of metrics except that their values are in a po-monoid equipped with a partial order rather than $[0, \infty)$.

The motivation of this paper comes from our some previous work in generalized metrics. In [17], topologies induced by positive filters on abelian lattice ordered groups were defined and used to show that some of the topologies on $C(X)$ such as the m-topology or the uniform topology are obtained that way. Later in [20], k-metrics, another generalization of metrics, were defined and topologies induced by positive filters on general ℓ-groups were studied and the methods developed there were used later on in [16] to show that even though non-T_0 completely regular spaces cannot be subspaces of powers of $[0, 1]$ similar results can be obtained by replacing $\mathbb{R}$ with a non-Archimedean partially ordered group, which can be given a natural Euclidean-like bitopological structure.

To pursue our goal here it seems that partially ordered monoids, po-monoids for short, are the natural candidates for the values of the generalized metric we are after.

We bring the following definitions from [6].

Definition 3.1. The structure $(M, +, 0, \leq)$ is an (abelian) *po-monoid* if $M \neq \{0\}$ and $(M, +, 0)$ is an (abelian) monoid which is equipped with a partial order $\leq$ such that for every $a, x, y \in M$, $x \leq y$ implies $a+x \leq a+y$ and $x+a \leq y+a$. We call the po-monoid M abelian, if $x + y = y + x$ for every $x, y \in M$.

A po-monoid is a good candidate for the values of a metric and can be used as a set of radii for inducing topology as it is equipped with an order and a binary operation.

Example 3.2. Both $(\mathbb{R}, +, \leq)$ and $([0, 1], \oplus, \leq)$, where $\oplus$ is the truncated sum, are abelian po-monoids. Another example is $G = \mathbb{R} \times M$ with the lexicographic order and coordinatewise addition, where M is any po-monoid.

Let M to be the set of order preserving maps on a poset P with composition of functions. For $f, g \in M$ define $f \leq g$ provided $f(x) \leq g(x)$ for every $x \in P$. Then M with the composition of functions and $\leq$ is a non-abelian po-monoid.

Definition 3.3. Let $(M, +, 0, \leq)$ be po-monoid such that $0 \leq m$ for every $m \in M$. We define an *M-metric* on a set X to be a map $d : X \times X \to M$ satisfying the axioms for a metric, except that it maps into M rather than $[0, \infty)$.

Metrics on a set induce a topology on the set. Here we define a topology induced by a generalized metric into a monoid M. By M^+ we mean $\{r \in M : r > 0\}$.

Definition 3.4. Let $(M, +, 0, \leq)$ be a po-monoid with 0 the smallest element, E be a subset of M^+, and $d : X \times X \to M$ be an M-metric. For every $r \in E$ let $N_r(x) = \{y \in X : d(x, y) < r\}$ and $\overline{N}_r(x) = \{y \in X : d(x, y) \leq r\}$. We call sets of the form $N_r(x)$ open balls and sets of the form $\overline{N}_r(x)$ closed balls. Define $\tau_E = \{T \subseteq X : \forall x \in T \exists r \in E$ such that $N_r(x) \subseteq T\} \cup \{X\}$.

We say a set A is E-bounded if there is an $r \in E$ such that $A \times A \subseteq \{(x, y) : d(x, y) < r\}$. We call A bounded if A is M^+-bounded.

Theorem 3.5. *Let $(M, +, 0, \leq)$ be a po-monoid with 0 the smallest element, $E \subseteq M^+$, and $d : X \times X \to M$ be an M-metric on the set X. Then*

- *If E is a down directed set, τ_E is a topology on X.*
- *If $M = \downarrow\!\!\downarrow E = \{x : \exists e \in E$ such that $x < e\}$ then the set of E-bounded subsets of X form a bornology.*

Proof. For the first part suppose E is a down directed set. We show that τ_E is a topology on X. It is straightforward to show that τ_E is closed under union and $X, \varnothing \in \tau_E$. We show that τ_E is closed under intersection. Let $A, B \in \tau_E$ and $x \in A \cap B$, then there are $r, s \in E$ such that $N_r(x) \subseteq A$ and $N_s(x) \subseteq B$. Since E is a down directed set, there is a $t \in E$ such that $t \leq r, s$. Now one can easily see that $N_t(x) \subseteq A \cap B$. Hence, τ_E is a topology.

We now show the second part. Assume $M = \downarrow\!\!\downarrow E = \{x : \exists e \in E$ such that $x < e\}$. We show the collection of E-bounded sets is a bornology.

Suppose that A and B are bounded. We prove that $A \cup B$ is bounded. Since A and B are bounded, there are $r, s \in E$ such that $d(x, y) < r$ for every $x, y \in A$ and $d(x, y) < s$ for every $x, y \in B$. Let a be a fixed element of A and b be a fixed element of B. Thus, for every $x \in A$ and $y \in B$, we have $0 \leq d(x, y) \leq d(x, a) + d(a, b) + d(b, y) \leq r + d(a, b) + d(b, y) \leq r + d(a, b) + s$. Since $M = \downarrow\!\!\downarrow E$, there is an $e \in E$ such that $r + d(a, b) + s < e$. Thus, for every $x, y \in A \cup B$ we have $d(x, y) < e$ and therefore, $A \cup B$ is E-bounded.

It is obvious that if A is E-bounded and $H \subseteq A$ then H is E-bounded. Finally note that the union of all E-bounded sets is X as every singleton set is E-bounded. □

Corollary 3.6. *Let $(M, +, 0, \leq)$ be a po-monoid with 0 the smallest element, $E \subseteq M^+$, and $d : X \times X \to M$ be an M-metric on the set X. If $M^+ = \downarrow\!\!\downarrow M^+ = \{x : \exists e > 0$ such that $x < e\}$ then the set of bounded subsets of X form a bornology.*

Theorem 3.7. *Let $\mathcal{A}$ be a bornology on a set X. Then there is an abelian po-monoid $(M, +, 0, \leq)$ with 0 the smallest element and an M-metric d on X such that $\mathcal{A}$ is the set of bounded subsets of d.*

Proof. Suppose $\mathcal{A}$ is a bornology on a set X. For each $K \in \mathcal{A}$ let $S_K = [0, \infty) \times [0, \infty)$. Define $+_K : S_K \times S_K \to S_K$ coordinatewise and let $\leq_K$ be the lexicographic order.

Let $S = \prod_{K \in \mathcal{A}} S_K$. Then S with the coordinatewise addition and coordinatewise order forms a po-monoid. Define $M \subseteq S$ by,

$M = \{((x_K^1, x_K^2))_{K\in\mathcal{A}} \in S : \exists C \in \mathcal{A}$ such that $x_K^2 = 0$ for every $K \supseteq C\}$.

We show that M with inherited addition and order from S is a po-monoid. It is enough to show that M is a monoid. Note that $(0,0)_{K\in\mathcal{A}} \in M$ because by letting $C = \varnothing$ we have $x_K^2 = 0$ for every K and obviously it is the smallest element of M. Suppose $((x_K^1, x_K^2))_{K\in\mathcal{A}}$ and $((y_K^1, y_K^2))_{K\in\mathcal{A}}$ are elements of M. Then, by definition of M there are $C, D \in \mathcal{A}$ such that $x_K^2 = 0$ for every $K \supseteq C$ and $y_K^2 = 0$ for every $K \supseteq D$. It is straightforward to verify that $x_K^2 + y_K^2 = 0$ for every $K \supseteq C \cup D \in \mathcal{A}$ and we leave it to the reader. Therefore, $((x_K^1, x_K^2))_{K\in\mathcal{A}} + ((y_K^1, y_K^2))_{K\in\mathcal{A}} \in M$. Thus, M is a monoid.

Define $d : X \times X \to M$ by $(d(x,y))_K = (0, |\varphi_K(x) - \varphi_K(y)|)$, where φ is defined as in Theorem 2.6. First we show that d is well defined. Note that whenever $K \supseteq \{x,y\}$ we have $\varphi_K(x) = \varphi_K(y) = 0$ and therefore, $|\varphi_K(x) - \varphi_K(y)| = 0$. Consequently $d(x,y) \in M$. Next we prove that d is an M-metric. One can easily see that for every $x, y \in X$, $d(x,x) = 0$ and $d(x,y) = d(y,x)$. We show that $d(x,z) \le d(x,y) + d(y,z)$ for every $x,y,z \in X$. Due to the definition of d, it is enough to show that for every $K \in \mathcal{A}$ the case of $d(x,z)_K^2 = 1$ and $d(x,y)_K^2 = d(y,z)_K^2 = 0$ is impossible. Note that $d(x,z)_K^2 = 1$ implies that one of x and z belongs to K and the other one does not belong. Without loss of generality, suppose that $x \in K$ and $z \notin K$. Then $d(x,y)_K^2 = 0$ and $x \in K$ implies $y \in K$. On the other hand, $d(y,z)_K^2 = 0$ and $z \notin K$ implies $y \notin K$, which is in contradiction with $y \in K$. Thus, the case $d(x,z)_K^2 = 1$ and $d(x,y)_K^2 = d(y,z)_K^2 = 0$ is not possible and therefore the triangularity condition holds. Next, we show that $d(x,y) = ((0,0))_{K\in\mathcal{A}}$ implies $x = y$. Note that if $x \ne y$ then $(d(x,y))_{\{x\}} = (0, |\varphi_{\{x\}}(x) - \varphi_{\{x\}}(y)|) = (0, |0-1|) \ne (0,0)$ and therefore $d(x,y) \ne ((0,0))_{K\in\mathcal{A}}$. Consequently, d is an M-metric.

Next we show that the set of bounded subsets of X equals $\mathcal{A}$. Suppose $H \in \mathcal{A}$. We show that $H \times H \subseteq \{(x,y) : d(x,y) < t_H\}$, where $(t_H)_K = \begin{cases} (0,1), & \text{if } K \supseteq H; \\ (1,1), & \text{if } K \not\supseteq H. \end{cases}$.

Note that for every $x, y \in H$, $d_K(x,y) = (0, |\varphi_K(x) - \varphi_K(y)|) < (1,1) = (t_H)_K$ when $K \not\supseteq H$. On the other hand $x, y \in H$ implies $x, y \in K$ when $K \supseteq H$. Therefore, in this case $d_K(x,y) = (0, |\varphi_K(x) - \varphi_K(y)|) = (0,0) < (t_H)_K$. Thus, $d(x,y) < t_H$ for every $x, y \in H$ and therefore H is bounded.

For the reverse inclusion suppose that A is a bounded subset of X. Thus, there is a $t = ((t_K^1, t_K^2))_{K\in\mathcal{A}} \in M$ such that $d(x,y) < (t_K^1, t_K^2)$ for every $x, y \in A$. By definition there is a $Q \in \mathcal{A}$ such that $t_K^2 = 0$ for every $K \supseteq Q$. We prove that $A \subseteq Q$. Assuming the contrary, if $A \not\subseteq Q$ then there is a $z \in A \setminus Q$. If $A = \{z\}$ obviously $A \in \mathcal{A}$ and we are done. If $A \ne \{z\}$, consider $y \in A \setminus \{z\}$. Then, $R = Q \cup \{y\} \in \mathcal{A}$. Then since $R \supseteq Q$, we have $(t_R^1, t_R^2) = (0,0)$. Now, $d_R(y,z) = (0, |\varphi_R(y) - \varphi_R(z)|) = (0, |0-1|) \not< (0,0)$, which is a contradiction. Thus, $\mathcal{A}$ equals the set of bounded sets of the metric d.

□

In the previous theorem for every A in the bornology $\mathcal{A}$ define $B_{t_A} = \{P \subseteq X : \forall x, y \in P, d(x,y) < t_A\}$, where $(t_A)_K = \begin{cases} (0,1), & \text{if } K \supseteq A; \\ (1,1), & \text{if } K \not\supseteq A., \end{cases}$ then

Proposition 3.8. *Let $\mathcal{A}$ be a bornology on the set X then $B_{t_A} = \downarrow A = \{P : P \subseteq A\}$ for every $A \in \mathcal{A}$.*

Remark 3.9. For a bornology $\mathcal{A}$ on a set X let $S = \prod_{K\in\mathcal{A}}[0,\infty)$. Then S with the coordinatewise addition and coordinatewise order forms a po-monoid. Define $M \subseteq S$ by,

$M = \{(x_K)_{K\in\mathcal{A}} \in S : \exists C \in \mathcal{A}$ such that $x_K = 0$ for every $K \supseteq C\}$.

Similarly, it can be shown that M with inherited addition and order from S is a po-monoid and $(0)_{K\in\mathcal{A}}$ is in M. Next define $d : X \times X \to M$ by $(d(x,y))_K = |\varphi_K(x) - \varphi_K(y)|$. Similarly, d is an M-metric on X.

Then by an argument similar to the one in the previous theorem one can verify that for every $A \in \mathcal{A}$ we have, $A \times A \subseteq \{(x,y) : d(x,y) \leq t_A\}$, where $t_{A_K} = \begin{cases} 0, & \text{if } K \supseteq A; \\ 1, & \text{if } K \not\supseteq A. \end{cases}$

4. The lattice of bornologies on a set

In this section we consider, $\mathcal{B}_X$, the set of bornologies on a set X. We prove some of the properties of it as a lattice. The ultimate goal of this section is to prove that $\mathcal{B}_X$ is the join-completion of $\mathcal{P}(X)$ modulo finite sets.

The set of bornologies on a set X, $\mathcal{B}_X$, forms a complete lattice, where $\mathcal{F}$ is the smallest element and $\mathcal{P}(X)$ is the largest element; for $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{B}_X$ we have $\mathcal{B}_1 \wedge \mathcal{B}_2 = \mathcal{B}_1 \cap \mathcal{B}_2$ and $\mathcal{B}_1 \vee \mathcal{B}_2 = \{B_1 \cup B_2 : B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$. So, one can see that if $\mathcal{S} \subseteq \mathcal{B}_X$, then $\bigvee \mathcal{S} = \{B_1 \cup \cdots \cup B_n : B_i \in \mathcal{B}_i \in \mathcal{S}\}$; see [4].

A complete lattice L is a frame if $a \wedge \bigvee S = \bigvee_{s\in S}(a \wedge s)$ for every $a \in L$ and $S \subseteq L$.

A frame X is normal if $x \vee y = 1$ implies the existence of u and v in X satisfying $u \wedge v = 0$ and $x \vee u = y \vee v = 1$.

Theorem 4.1. *The set of bornologies on a set X is a normal frame.*

Proof. It is enough to show that $\mathcal{A} \cap \bigvee \mathcal{S} \subseteq \bigvee\{\mathcal{A} \cap \mathcal{B} : \mathcal{B} \in \mathcal{S}\}$. Let $H \in \mathcal{A} \cap \bigvee \mathcal{S}$. There are $B_i \in \mathcal{B}_i \in \mathcal{S}$, $i = 1, \cdots, n$ such that $H = B_1 \cup \cdots \cup B_n$. So, $H = (H \cap B_1) \cup \cdots \cup (H \cap B_n)$. Since $H \cap B_i \in \mathcal{A} \cap \mathcal{B}_i$, we conclude that $H \in \bigvee\{\mathcal{A} \cap \mathcal{B} : \mathcal{B} \in \mathcal{S}\}$.

For normality, suppose that $\mathcal{A} \vee \mathcal{B} = \mathcal{P}(X)$. If $\mathcal{A} = \mathcal{F}$ or $\mathcal{B} = \mathcal{F}$ then we are done. So, assume that $\mathcal{A} \neq \mathcal{F}$ and $\mathcal{B} \neq \mathcal{F}$. Since $\mathcal{A} \vee \mathcal{B} = \mathcal{P}(X)$, there are $R \in \mathcal{A}$ and $S \in \mathcal{B}$ such that $R \cup S = X$. Now, let $\mathcal{U} = \mathcal{F} \cup \{S\}$ and $\mathcal{V} = \mathcal{F} \cup \{R \setminus S\}$. Obviously, $\mathcal{U} \cap \mathcal{V} = \mathcal{F}$, $\mathcal{A} \vee \mathcal{U} = \mathcal{P}(X)$, and $\mathcal{B} \vee \mathcal{V} = \mathcal{P}(X)$. □

In [1] they define more general structures which they call L-bornologies and they show that the set of L-bornologies is a frame. However, they do not state that it is a frame. They just prove the distributivity of meet over arbitrary join.

Consider $\mathcal{P}(X)$ and define the relation $\lhd$ on $\mathcal{P}(X)$ by $A \lhd B$ if $(A \setminus B) \cup (B \setminus A)$ is finite. Let $[\mathcal{P}(X)]$ be the set of equivalence classes of the relation $\lhd$. Define $\sqsubseteq$ on $[\mathcal{P}(X)]$ by $[H] \sqsubseteq [K]$ if and only if $H \setminus K$ is finite.

Lemma 4.2. *Let X be a set. Then $[\mathcal{P}(X)]$ with $\sqsubseteq$ forms a lattice.*

Proof. It is straightforward to show that $\sqsubseteq$ is well defined and we will leave it to the reader. Obviously, for every $[H]$ we have $[H] = [H]$. We leave to the reader to prove that $\sqsubseteq$ is antisymmetric. We show that $\sqsubseteq$ is transitive. Assume $[G] \sqsubseteq [H]$ and $[H] \sqsubseteq [K]$ then $[G] \sqsubseteq [K]$. Note that since $G \setminus K \subseteq (G \setminus H) \cup (H \setminus K)$ and both $G \setminus H$ and $H \setminus K$ are finite, we have $G \setminus K$ is finite. Thus, $[G] \sqsubseteq [K]$. Therefore, $\sqsubseteq$ is transitive.

Next we show that $[H \cup K] = [H] \vee [K]$ and $[H \cap K] = [H] \wedge [K]$ for every $[H], [K] \in [\mathcal{P}(X)]$. Note that $H \subseteq K$ implies $[H] \sqsubseteq [K]$. Thus, $[H \cap K] \sqsubseteq [H] \wedge [K] \sqsubseteq [H] \vee [K] \sqsubseteq [H \cup K]$ for every $[H], [K] \in [\mathcal{P}(X)]$. We now show that $[H \cup K] \sqsubseteq [H] \vee [K]$. Let $[M] \in [\mathcal{P}(X)]$ be such that let $[H], [K] \sqsubseteq [M]$. Since $H \setminus M$ is finite and $K \setminus M$ is finite, $(H \cup K) \setminus M$ is finite. So, $[H \cup K] \sqsubseteq [M]$. Similarly, if $[P] \in \mathcal{P}(X)$ is such that $[P] \sqsubseteq [H], [K]$ then $P \setminus H$ is finite and $P \setminus K$ is finite. So, $P \setminus (H \cap K)$ is finite. Thus, $[P] \sqsubseteq [H \cap K]$.

□

Let $\mathcal{B}_X$ be the set of bornologies on the set X. Define $\theta : [\mathcal{P}(X)] \to \mathcal{B}_X$ by $\theta([H]) = \downarrow H \cup \mathcal{F}$, where $\mathcal{F}$ is the bornology of finite subsets of X and $\downarrow H = \{Y : Y \subseteq H\}$. Recall that bornologies of $\downarrow H \cup \mathcal{F}$ form are called principal bornologies.

Lemma 4.3. *The map $\theta : [\mathcal{P}(X)] \to \mathcal{B}_X$ is an embedding.*

Proof. It is straightforward to show that θ is well defined and order preserving and we will leave it to the reader. Note that if $\theta([H]) \subseteq \theta([K])$ then $H \in \theta([K])$ and therefore, $H = Z \cup E$ where $Z \subseteq K$ and E is a finite subsets of X. Thus, $H \setminus K$ is finite and therefore, $[H] \sqsubseteq [K]$.

□

Corollary 4.4. *If $I(\mathcal{B}_X)$ is the set of principal bornologies of a set X, then $\theta : [\mathcal{P}(X)] \to I(\mathcal{B}_X)$ is an isomorphism.*

Definition 4.5. Let L be a lattice and C be a complete lattice. We say C is a join-completion of L if L is a sublattice of C and every element of C is join of elements of L. The concept of a meet-completion is defined dually.

The join-completions and meet-completions were introduced by Banaschewski in [2] and were extensively studied and extended by Schimidt in [21] and [22].

Theorem 4.6. *For every set X, the frame $\mathcal{B}_X$ is a join-completion of $I(\mathcal{B}_X)$; it is a meet-completion if and only if X is finite. Further, if L is another complete lattice which is join-completion of $I(\mathcal{B}_X)$, there is a unique isomorphism from $\mathcal{B}_X$ to L fixing $I(\mathcal{B}_X)$ and therefore, L must be a frame.*

Proof. Let $\mathcal{A}$ be a bornology on a set X. Then, $\mathcal{A} = \bigvee_{H \in \mathcal{A}} (\downarrow H \cup \mathcal{F})$ and therefore, $I(\mathcal{B}_X)$ is join-dense in L.

Next we show that if X is infinite then $I(\mathcal{B}_X)$ is not meet-dense in $\mathcal{B}_X$. It is enough to show that $I(\mathcal{B}(\mathbb{Z}))$ is not meet-dense in $\mathcal{B}_{\mathbb{Z}}$. Let $\mathcal{A} = \bigvee_{p \in P} (\downarrow p\mathbb{Z} \cup \mathcal{F})$, where P is the set of prime numbers. We show that $\mathcal{A}$ cannot be the infimum of principal bornologies. Suppose to the contrary that $\mathcal{A} = \bigwedge_{i \in I} (\downarrow H_i \cup \mathcal{F}) = \bigcap_{i \in I} (\downarrow H_i \cup \mathcal{F})$. Then, $p\mathbb{Z} \in (\downarrow H_i \cup \mathcal{F})$ for every $p \in P$ and every $i \in I$. Thus, $\bigcup_{p \in P} p\mathbb{Z} \in (\downarrow H_i \cup \mathcal{F})$ for every $i \in I$. Therefore, $\bigcup_{p \in P} p\mathbb{Z} \in \bigcap_{i \in I} (\downarrow H_i \cup \mathcal{F})$. But $\bigcup_{p \in P} p\mathbb{Z} \notin \mathcal{A}$ which is a contradiction.

If $i : I(\mathcal{B}_X) \to L$ is an embedding, define $f : \mathcal{B}_X \to L$ by $f(\mathcal{A}) = \bigvee_{H \in \mathcal{A}} i(\downarrow H \cup \mathcal{F})$. Then f is an isomorphism. □

Corollary 4.7. *The set of bornologies on a set X is the join-completion of $[\mathcal{P}(X)]$.*

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References

[1] M. Abel and A. Šostak, Towards the theory of L-bornological spaces, Iran. J. Fuzzy Syst. 8, no. 1 (2011), 19–28.

[2] B. Banaschewski, Hüllensysteme und Erweiterung von Quasi-Ordnungen, Z. Math. Logik Grundlagen Math. 2 (1956), 35–46.

[3] G. Beer, On metric boundedness structures, Set-valued Anal 7 (1999), 195–208.

[4] G. Beer and S. Levi, Total boundedness and bornologies, Topology and its Applications 156 (2009), 1271–1288.

[5] G. Beer and S. Levi, Strong uniform continuity, J. Math. Anal. Appl. 350, no. 2 (2009), 568–589.

[6] G. Birkhoff, Lattice Theory, 3rd Edition, American Math. Society Colloquium Publications, Volume 25, (1967), Providence, RI.

[7] J. Cao and A. H. Tomita, Bornologies, topological games and function spaces, Topology Appl. 184 (2015), 16–28.

[8] A. Chand and I. Weiss, Completion of continuity spaces with uniformly vanishing asymmetry, Topology Appl. 183 (2015), 130–140.

[9] R. C. Flagg, Quantales and continuity spaces, Algebra Universalis 37 (1997), 257–276.

[10] R. C. Flagg and R. Kopperman, Continuity spaces: Reconciling domains and metric spaces, Chic. J. Theoret. Comput. Sci. 77 (1977), 111–138.

[11] H. Hogbe-Nlend, Bornologies and functional analysis, North-Holland, Amsterdam-New York-Oxford, 1977.

[12] S. T. Hu, Boundedness in a topological space, J. Math. Pures Appl. 228 (1949), 287–320.

[13] J. L. Kelley, General Topology, D. van Nostrand, 1955.

[14] R. Kopperman, All topologies come from generalized metrics, Am. Math. Monthly 95 (1988), 89–97.

[15] R. Kopperman, S. Matthews and H. Pajoohesh, Partial metrizability in value quantales, Applied General Topology 5, no. 1 (2004), 115–127.

[16] R. Kopperman and H. Pajoohesh, Representing topologies using partially ordered semigroups, Topology and its applications 249 (2018), 135–149.

[17] R. Kopperman, H. Pajoohesh and T. Richmond, Topologies arising from metrics valued in abelian ℓ-groups, Algebra Universalis 65 (2011), 315–330.

[18] J. P. Lasalle, Topology based upon the concept of pseudo-norm, Proc. Nat. Acad. Sci. 27 (1941), 448–451.

[19] S. G. Matthews, Partial metric topology, in: Proc. 8th Summer Conference on Topology and its Applications, ed S. Andima et al., New York Academy of Sciences Annals 728 (1994), 183–197.

[20] H. Pajoohesh, k-metric spaces, Algebra Universalis 69, no. 1 (2013), 27–43.

[21] J. Schmidt, Universal and internal properties of some completions of k-join-semilattices and k-join-distributive partially ordered sets, J. Reine Angew. Math. 255 (1972), 8–22.

[22] J. Schmidt, Each join-completion of a partially ordered set is the solution of a universal problem, J. Austral. Math. Soc. 17 (1974), 406–413.