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Fixed point index computations for multivalued mapping and application to the problem of positive eigenvalues in ordered space

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Abstract

In this paper, we present some results on fixed point index calculations for multivalued mappings and apply them to prove the existence of solutions to multivalued equations in ordered space, under flexible conditions for the positive eigenvalue.

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1. Introduction

The theory of the fixed point index for the compact single-valued mapping has achieved many brilliant achievements in studying the existence of solutions of equations, through the existence results on fixed points of operators (see e.g. [1, 9, 14–17]). This concept has been extended to multivalued mappings very early in [10] and the references therein. Up to now, this topic has always been interested by many mathematicians (see e.g. [2–8,12,18–23,28,30,31]). An impressive achievement when extending this theory to multivalued mappings can be found in [12], in which the authors established the concept of a fixed point index with respect to the cone for multivalued operators acting on convex Fréchet spaces with convex and closed values. This concept has remarkable

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properties including the fixed point index concept for compact single-valued mappings.

When studying multivalued equations, we may face several difficulties due to some strict properties appearing on the multivalued operator including possessing the value of a closed, convex set (which the single-valued mapping obviously owns). A natural problem generating is to find an alternative one, which is still feasible for the study on the existence of solutions. For example, we can find a selection function f satisfying the condition $f(x) \in T(x)$, where $T(x)$ is the multivalued operator. In this article, the strategy of finding alternative functions can be described as follows. We choose functions which can act as upper/lower bounds thanks to several relations between the two sets and possess natural properties including continuous linear. In addition, we also consider the condition which allows us to use a map with better properties in the case the neighborhood of the origin is sufficiently small or large enough.

Let us recall a well-known result on the relationship between the concept of spectral radius and the eigenvalues of a linear mapping, which is known as the Krein-Rutman Theorem [24].

Theorem 1.1. *Let E be a Banach space with the ordered by cone K and $\varphi : E \to E$ be a positive completely continuous with spectral radius $r(\varphi) > 0$. Then, $r(\varphi)$ is eigenvalue of φ with respect to eigenvector x_0. Further, if φ is strongly positive and* $\mathrm{int}K \neq \varnothing$*, then*

1. $x_0 \in \mathrm{int}K$,
2. $r(\varphi)$ is geometrically simple,
3. if $\lambda \neq r(\varphi)$ is the eigenvalue of φ, $|\lambda| \leq r(\varphi)$.

The above results have been extended to some non-strong positive mapping classes such as u_0-positive [31], non-decomposable maps, etc, in the works of Krasnoselskii and his students [25]. Recently, in the papers of Nussbaum [27], K.Chang [11], Mahadevan [26], Krein's theorem has been extended to the increasing, positively 1-homogeneous mapping class. Following these works, in [14], we have extended these concepts to positively 1-homogeneous positive-homogeneous multivalued mappings. In [29], we evaluate the range of eigenvalues for multivalued operators, find a sufficient condition for existence of eigenvalues for the dual operator of the multivalued mapping [30]. In this paper, we continue to demonstrate a result that looks like the spectral radius of a linear mapping.

We have structured our paper as follows. In the next section, we briefly recall some useful preliminaries. Section 3 is divided in to two subsections with two separate results. In Subsection 3.1, some results on the fixed point index of the multivalued operator are established. In Subsection 3.2, some existence results for the positive eigen-pair are stated.

2. Preliminaries

Let X be a Banach space and K be a cone in X, i.e, K is a closed convex subset of X such that $K + K \subset K$, $\lambda K \subset K$ for $\lambda \geq 0$ and $K \cap -K = \{\theta\}$ (θ

is the zero element of X). A partial order in X is defined by $a \leq b$ (or, $b \geq a$) if and only if $a - b \in -K$. For nonempty subsets A, B of X, we write $A \preceq_1 B$ (or, $B \succeq_1 A$) iff for every $a \in A$, there is $b \in B$ satisfying $a \leq b$ (or, $a \geq b$), and write $A \preceq_2 B$ (or, $B \succeq_2 A$) iff for every $b \in B$, there is $a \in A$ satisfying $a \leq b$ (or, $a \geq b$). A mapping $T : X \to 2^X \backslash \{\varnothing\}$ is said to be *positive* if $T(K) \subset K$.

Throughout this paper, we use the following notations if there is no appearance of special cases. Let $(X, K, \|.\|)$ be an ordered Banach space with cone K, X^* be the dual topology space of X, $\Omega \subset X$ be a convex neighbourhood of the origin θ, $cc(K)$ be the all nonempty closed convex subset of K,

$$\dot{K} = K \backslash \{\theta\},$$

$$\partial_K \Omega = K \cap \partial\Omega, \text{ where } \partial\Omega \text{ is boundary of } \Omega \text{ in } X,$$

$$\langle x \rangle_+ = \{\alpha x : \alpha > 0\}, \text{ where } x \in X,$$

$$B(x, r) = \{y \in X : \|x - y\| < r\}, \text{ where } x \in X, r > 0;$$

$$K^* = \{f \in X^* : f(x) \geq 0 \ \forall x \in K\},$$

$$S^*_+ = K^* \cap \{p \in X^* : \|p\| = 1\}$$

$$\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}; \ \dot{\mathbb{R}}_+ = \mathbb{R}_+ \backslash \{0\}.$$

A multivalued mapping $T : K \cap \overline{\Omega} \to 2^K \backslash \{\varnothing\}$ is said to be *compact* if $T(E)$ is relatively compact for any bounded subset E of $K \cap \overline{\Omega}$, where $T(E) = \cup_{x \in E} T(x)$ and $\overline{\Omega}$ is the closure of Ω in X. T is called an *upper semi-continuous* (in short, u.s.c.) if $\{x \in K \cap \overline{\Omega} : T(x) \subset W\}$ is open in $K \cap \overline{\Omega}$ for every open subset W of K. Further, if $x \notin T(x)$ for all $x \in \partial_K \Omega$, the fixed point index of T in Ω with respect to K is defined and we denote this integer index by $i_K(T, \Omega)$ (see e.g. [12]). T is said to be *convex* if its graph is convex subset in $(X \times X)$. Clearly, T is convex iff $(1 - \lambda)T(x) + \lambda T(y) \subset T((1 - \lambda)x + \lambda y)$ for all $\lambda \in [0, 1]$ and $x, y \in X$.

In what follows, we present some useful properties, which are of importance in constructing the main results in the next section.

Proposition 2.1 ([12]). *Let Ω be a bounded open and $T : K \cap \overline{\Omega} :\to cc(K)$ be an u.s.c compact satisfying $x \notin \partial_K \Omega$. Then*

1. *If $i_K(T, \Omega) \neq 0$, then T has a fixed point,*
2. *If $x_0 \in \Omega$, then $i_K(\hat{x_0}, \Omega) = 1$, where $\hat{x_0}$ is a constant mapping with $\hat{x_0}(x) = x_0$, $\forall x \in K \cap \overline{\Omega}$.*
3. *If $\Omega_1, \Omega_2 \subset \Omega$ are onpen with $\Omega_1 \cap \Omega_2 = \varnothing$ and $x \notin T(x)$ for all $x \in K \cap (\overline{\Omega} \backslash (\Omega_1 \cup \Omega_2))$, then*

$$i_K(T, \Omega_1) + i_K(T, \Omega_2) = i_K(T, \Omega).$$

4. *If $H : [0, 1] \times (K \cap \overline{\Omega}) :\to cc(K)$ is an u.s.c compact satisfying $x \notin H(\alpha, x)$ for all $(\alpha, x) \in [0, 1] \times \partial_K \Omega$, then*

$$i_K(H(1, .), \Omega) = i_K(H(0, .), \Omega)$$

Proposition 2.2 ([12, 14]). *Let Ω be a bounded open subset of X, and $T : K \cap \overline{\Omega} \to cc(K)$ be an u.s.c compact such that $x \notin T(x)$ for all $x \in \partial_K \Omega$. Then*

1. $i_K(T,\Omega) = 0$ *if there is* $u \in \dot{K}$ *such that*

$$x \notin T(x) + \lambda u \text{ for all } (\lambda, x) \in (0,\infty) \times \partial_K \Omega.$$

2. $i_K(T,\Omega) = 1$ *if*

$$\lambda x \notin T(x) \text{ for all } (\lambda, x) \in (1,\infty) \times \partial_K \Omega.$$

Let $L : X \to X$ be positive continuous linear operator, and $u_0 \in \dot{K}$. L is said to be u_0-*positive* if for every $x \in \dot{K}$, there are $\alpha > 0,\ \beta > 0$ and $n, m \in \mathbb{N}$ satisfying $\alpha u_0 \leq L^n x$ and $L^m x \leq \beta u_0$.

Proposition 2.3 ([31]). *Let* $L_1, L_2 : X \to X$ *be positive continuous linear operators, and one of them is* u_0*-positive. Assume that* $L_1 u \leq L_2 u$ *for all* $u \in K$ *and there exists* $(\lambda, x) \in \dot{\mathbb{R}}_+ \times \dot{K}$, $(\mu, y) \in \dot{\mathbb{R}}_+ \times \dot{K}$ *such that*

$$\lambda x \leq L_1 x \text{ and } L_2 y \leq \mu y.$$

Then, the following properties hold

1. $\lambda \leq \mu$,
2. $\langle x \rangle = \langle y \rangle$ *if* $\lambda = \mu$.

Proposition 2.4.

1. $x \in K$ *iff* $\langle f, x \rangle \geq 0$ $, \forall f \in K^*$.
2. *For* $x \in \dot{K}$*, there exists* $f \in K^*$ *such that* $\langle f, x \rangle > 0$.

Proposition 2.5 ([13]). *Let* X, Y *be Banach spaces,* $T : \Omega \subset X \to 2^Y \backslash \{\varnothing\}$ *be u.s.c. Assume that* $\{(x_n, y_n)\}$ *is a consequence in* $\operatorname{graph}(T)$ *satisfying* $\lim_{n\to\infty}(x_n, y_n) = (x, y)$. *Then, we have* $(x, y) \in \operatorname{graph}(T)$ *if* $T(x)$ *is closed subset of* Y*, where* $\operatorname{graph}(T) = \{(a, b) : a \in \Omega, b \in T(a)\}$.

3. Abstract results

3.1. **The fixed point index of the multivalued operator.** In this subsection, we present several results on the fixed point index for multivalued mappings by using some useful tools including some continuous linear operators and approximate mappings at the origin and the infinity.

Theorem 3.1. *Let* Ω *be a bounded open subset of* X, $A : K \cap \overline{\Omega} \to cc(K)$ *be an u.s.c and compact operator.*

1. $i_K(A,\Omega) = 1$ *if there exists a continuous linear operator* L *with the spectral radius* $r(L) \leq 1$ *such that*

$$A(u) \preceq_1 Lu \text{ and } u \notin A(u)\ \forall u \in \partial_K \Omega. \tag{3.1}$$

2. *Assume that* $X = K - K$ *and there exists a continuous linear mapping and* u_0*-positive* L *with the spectral radius* $r(L) \geq 1$ *such that*

$$Lu \preceq_2 A(u) \text{ and } u \notin A(u)\ \forall u \in K \cap \partial\Omega, \tag{3.2}$$

   then $i_K(A,\Omega) = 0$.

Proof. 1. To use Proposition 2.2, we aim at showing that

$$\lambda u \notin T(u) \text{ for all } (\lambda, u) \in (1, \infty) \times \partial_K \Omega. \tag{3.3}$$

Assume that it is not true, then we can find $(\lambda, u) \in (1, \infty) \times \partial_K \Omega$ satisfying $\lambda u \in T(u)$. From (3.1), we have $\lambda u \leq Lu$, which implies that $(I - \lambda^{-1} L)^{-1}$ is a positive continuous linear operaor. This gives $u \leq \theta$, which leads to $u = \theta$. A contradiction can be seen here obviously.

2. Let us choose $x_0 \in \dot{K}$, we will prove that

$$u \notin T(u) + \lambda x_0 \;, \forall (\lambda, u) \in (0, \infty) \times \partial_K \Omega. \tag{3.4}$$

Indeed, assume that (3.4) is not true, then $u \in T(u) + \lambda x_0$, for some $(\lambda, u) \in (0, \infty) \times \partial_K \Omega$. Then, from (3.2), one obtain $u \geq Lu$. By the Krein-Rutman theorem, we have $r(L)$ is the eigen vallue of L, i.e, there exists $y \in \dot{K}$ such that $Ly = r(L)y$. Using Proposition 2.3, we have $r(L) = 1$ and $u \in \langle y \rangle_+$. By setting $u = \alpha y$ $(\alpha > 0)$, one can see $Lu = u$ which implies that

$$u \geq Lu + \lambda x_0 = u + \lambda x_0.$$

This is impossible. By Proposition 2.2, we obtain $i_K(T, \Omega) = 0$. The proof is complete. □

Theorem 3.2. *Let Ω be a bounded open subset of X, $T : K \to cc(K)$ is an u.s.c compact convex satisfying $x \notin T(x)$ for all $x \in K$. Then*

1. *$i_K(T, \Omega) = 0$ if there exists $(\lambda_0, x_0) \in (1, \infty) \times \dot{K}$ such that $\lambda_0 x_0 \in T(x_0)$.*
2. *$i_K(T, \Omega) = 1$ if $\lambda x \in T(x)$ for all $(\lambda, x) \in (1, \infty) \times \dot{K}$.*

Proof. The second assertion can be seen as a consequence of Proposition 2.2. To prove the first assertion, we will show that

$$x \in T(x) + \lambda x_0 \ \forall (\lambda, x) \in (0, \infty) \times \partial_K \Omega. \tag{3.5}$$

Indeed, assume the contrary, namely, $x \notin T(x) + \lambda x_0$, for some $(\lambda, x) \in (0, \infty) \times \partial_K \Omega$. Then, there exists $y \in T(x), x = y + \lambda x_0$. For arbitrary positive numbers α, β we have

$$\alpha \lambda_0 x + \beta x_0 = \alpha \lambda_0 y + \left(\frac{\beta}{\lambda_0} + \alpha \lambda \right) \lambda_0 x_0.$$

Therefore, the following identity holds

$$\alpha \lambda_0 x + \beta x_0 \in \alpha \lambda_0 T(x) + \left(\frac{\beta}{\lambda_0} + \alpha \lambda \right) T(x_0). \tag{3.6}$$

Let us choose α as follows

$$\alpha = \left(\lambda_0 + \frac{\lambda \lambda_0}{\lambda_0 - 1} \right)^{-1} \text{ and } \beta = \frac{\alpha \lambda \lambda_0}{\lambda_0 - 1}.$$

Then, it is clear that β satisfies

$$\beta = \frac{\beta}{\lambda_0} + \alpha \lambda \text{ and } \alpha \lambda_0 + \frac{\beta}{\lambda_0} + \alpha \lambda = 1.$$

Now, we set $v = \alpha\lambda_0 x + \beta x_0$, $v \in \dot{K}$. Since T is convex, we have

$$\alpha\lambda_0 T(x) + \left(\frac{\beta}{\lambda_0} + \alpha\lambda\right) T(x_0) \subset T\left(\alpha\lambda_0 x + \left(\frac{\beta}{\lambda_0} + \alpha\lambda\right)\right)$$

This together with (3.6) yields $v \in T(v)$. This is a contradiction, hence $i_K(T, \Omega) = 0$. □

Let $F, \varphi : K \to 2^K \backslash \{\varnothing\}$. For every $x \in K$, we denote

$$\mathscr{D}(F(x), \varphi(x)) = \sup\{\|y - y'\| : y \in F(x), y' \in \varphi(x)\}.$$

We consider the following conditions for the pair (F, φ).

(C_0) : $\lim\limits_{x \in \dot{K}, \|x\| \to 0} \frac{\mathscr{D}(F(x), \varphi(x))}{\|x\|} = 0.$

(C_∞) : $\lim\limits_{x \in \dot{K}, \|x\| \to \infty} \frac{\mathscr{D}(F(x), \varphi(x))}{\|x\|} = 0.$

In what follows, we aim at giving several relations between the aforementioned conditions and the results on the fixed point index for multivalued mappings. First of all, we are interested in giving an answer for the natural question

"*When does the two aforementioned conditions for the pair (F, φ) are guaranteed?*"

by presenting some simple illustrations for such pair.

Example 3.3. Let $X = \mathbb{R}$, $K = \mathbb{R}_+$, $B = [0, 1]$.
1. Let $F, \varphi : K \to 2^K$ with

$$F(x) = x + x^2 B \text{ and } \varphi(x) = x.$$

Then, $\mathscr{D}(F(x), \varphi(x)) = x^2$; hence, the pair (F, φ) satisfies the condition (C_0).
2. We define $F, \varphi : K \to 2^K$ by

$$F(x) = \begin{cases} \{0\}, & x = 0, \\ x + B, & x \in (0, \infty) \end{cases}$$

and $\varphi(x) = x$. Then, for $x \neq 0$ we have

$$\begin{aligned} \mathscr{D}(F(x), \varphi(x)) &= \sup\{|y - x| : y = x + \alpha, \alpha \in B\} \\ &= 1. \end{aligned}$$

Therefore the pair (F, φ) satisfies the condition (C_∞).
3. Let $F : \mathbb{R} \to \mathbb{R}$ be a Fréchet differentiable function with $F(0) = 0$, φ be the Fréchet differentiable of F with recpect to K at 0 (at ∞, resp.). Then, the pair (F, φ) satisfies the condition (C_0) ((C_∞), resp.)

Theorem 3.4. *Let $F, \varphi : K \to cc(K)$ be u.s.c and compact with $x \notin \varphi(x)$, for all $x \in K$. Assume that $\varphi(\lambda x) = \lambda\varphi(x)$, for all $\lambda > 0, x \in K$ (in order words, A is positively 1-homogeneous). Then, it holds*

$$i_K(F, B(\theta, r)) = i_K(\varphi, B(\theta, r)),$$

in the case the following conditions hold

1. (F, φ) *satisfies the condition* (C_0) *if* r *is sufficiently small.*
2. (F, φ) *satisfies the condition* (C_∞) *if* r *is sufficiently large.*

Proof. For the sake of convenience, we denote

$$H(\alpha, x) = \alpha F(x) + (1-\alpha)\varphi(x),\ x \in \dot{K}, \alpha \in [0,1].$$

The operator $H(\alpha, .)$ is u.s.c compact with closed convex values. For $y \in F(x)$ and $y' \in \varphi(x)$ we have

$$\begin{aligned} \|x - \alpha y - (1-\alpha)y'\| &= \|x - y' - \alpha(y - y')\| \\ &\geq \|x - y'\| - \alpha\|y - y'\| \\ &\geq \|x - y'\| - \|y - y'\|. \end{aligned} \tag{3.7}$$

Set $b = \inf\{\|x-y\| : x \in K, \|x\| = 1, y \in \varphi(x)\}$. If $b = 0$, we can find sequences $\{x_n\} \subset K$, $\{y_n\} \subset K$ such that

$$\|x_n\| = 1,\ y_n \in \varphi(x_n) \text{ and } \|x_n - y_n\| \to 0.$$

Thanks to the compactness of $\varphi(x)$, we can assume that $\lim_{n\to\infty} y_n = y_0 \in K$, hence $\|y_0\| = 1$. Since φ is u.s.c, we have $y_0 \in \varphi(y_0)$ which is contradictory with the assumption. Thus, $b > 0$. Now, fix $x \in \dot{K}$, write $x = \lambda x'$ with $\lambda = \|x\|$, then $x' \in \dot{K}$ and $\|x'\| = 1$. It follows from the positively 1-homogeneous properties of φ that

$$\frac{\inf_{w\in\varphi(x)} \|x - w\|}{\|x\|} = \frac{\inf_{\frac{1}{\lambda}w\in\varphi(x')} \|x' - \frac{1}{\lambda}w\|}{\|x\|} \geq b.$$

This implies that

$$\inf_{w\in\varphi(x)} \|x - w\| \geq b\|x\|. \tag{3.8}$$

From (3.8) and (3.7), we have

$$\frac{\|x - \alpha y - (1-\alpha)y'\|}{\|x\|} \geq b - \frac{\mathscr{D}(F(x), \varphi(x))}{\|x\|}. \tag{3.9}$$

If (F, φ) satisfies (C_0), there exists $r > 0$ such that $b - \frac{\mathscr{D}(F(x),\varphi(x))}{\|x\|} > 0$ for all $x \in \dot{K}$ with $\|x\| \leq r$. From (3.9), it follows that

$$x \in H(\alpha, x) \text{ for all } x \in \partial_K B(\theta, 0).$$

Hence, we deduce that $i_K(F, B(0,r)) = i_K(\varphi, B(0,r))$. If (F, φ) satisfies (C_∞), we make the same argument as above. The proof is complete. □

3.2. **Existence of a positive eigen-pair.** In this section we present results on the existence of the eigenvalue for multivalued operator.

Theorem 3.5. *Let* $A : K \to cc(K)$ *be u.s.c compact. Assume that* $X = K - K$, Ω_1, Ω_2 *are bounded open subsets of* x, $\theta \in \Omega_1 \subsetneq \Omega_2$ *satisfy the following conditions*

1. *There exist completely continuous linear maps $P, Q : K \to K$ with spectral radius $r(P), r(Q)$, respectively, and P is u_0-positive such that either*

$$Px \preceq_2 A(x)\ \forall x \in \partial_K \Omega_1, \quad A(x) \preceq_1 Qx\ \forall x \in \partial_K \Omega_2 \tag{3.10}$$

or

$$Px \preceq_2 A(x)\ \forall x \in \partial_K \Omega_2, \quad A(x) \preceq_1 Qx\ \forall x \in \partial_K \Omega_1 \tag{3.11}$$

2. $0 < r(Q) < r(P)$.

Then, for $\lambda \in (r(Q), r(P))$, the inclusion $\lambda x \in A(x)$ has a positive solution.

Proof. We assume that (3.10) is satisfied and $x \notin \mu A(x)$, for all $x \in \partial\Omega_1 \cup \Omega_2$. Denote by $\mu = \lambda^{-1}$. Then, we have

$$\mu A(u) \preceq_1 \mu Q u\ , \forall u \in \partial_K \Omega_2.$$

Since $r(\mu Q) \leq 1$ by Theorem 3.1, we obtain $i_K(\mu A, \Omega_2) = 1$. Similarly, $i_K(\mu A, \Omega_1) = 0$. It follows that $i_K(\mu A, \Omega_2 \backslash \overline{\Omega_1}) = 1$ by Proposition 2.1. Hence, μA has a fixed point in $\Omega_2 \backslash \overline{\Omega_1}$. By a similar argument as in the previous one with the condition (3.11). □

Let $\varphi : K \to 2^K \backslash \{\varnothing\}$, we denote

$$r^*(\varphi) = \sup\left\{\lambda > 0 : \exists x \in \dot{K}, \lambda x \in \varphi(x)\right\}, \text{ define } \sup\varnothing = 0;$$

$$r_*(\varphi) = \inf\left\{\lambda > 0 : \exists x \in \dot{K}, \lambda x \in \varphi(x)\right\}, \text{ define } \inf\varnothing = \infty;$$

Theorem 3.6. *Let $A : K \to cc(K)$ be u.s.c compact. Assume that there exist positively 1-homogeneous convex operators $P, Q : K \to cc(K0)$ satisfying the following conditions*

1. *(A, P) satisfies (C_0) and (A, Q) satisfies (C_∞),*
2. *$0 < r^*(P) < r_*(Q) < \infty$ (or $0 < r^*(Q) < r_*(P) < \infty$, resp.)*

Then, if $\lambda \in (r^*(P), r_*(Q))$ (or $\lambda \in (r^*(Q), r_*(P))$ resp.), the equation $\lambda x \in A(x)$ has a solution in $\dot{K}$.

Proof. Denote by $\mu = \lambda^{-1}$, $F = \mu A$, $\varphi_1 = \mu P$, $\varphi_2 = \mu Q$. We first prove that there are $r_1 > 0, r_2 > 0$ $(r_1 < r_2)$ such that

$$i_K(F, \Omega_1) = i_K(\varphi_1, \Omega_1) \text{ and } i_K(F, \Omega_2) = i_K(\varphi_2, \Omega_2), \tag{3.12}$$

where $\Omega_1 = B(\theta, r_1)$, $\Omega_2 = B(\theta, r_2)$. Indeed, applying Theorem 3.4 for the pair (F, φ_1) we can find $r_1 > 0$ (small enough) such that $i_K(F, B(\theta, r_1)) = i_K(\varphi_1, B(\theta, r_1))$. Similarly, there exists $r_2 > 0$ (large enough) satisfying $i_K(F, B(\theta, r_2)) = i_K(\varphi_2, B(\theta, r_2))$. Now, we assume that $0 < r^*(P) < r_*(Q)$. By Theorem 3.2, $i_K(F, \Omega_2) = 0$ and $i_K(F, \Omega_1) = 1$, this leads to the assertion that needs to be proved. In the case $0 < r^*(Q) < r_*(P) < \infty$ the proof is analogous to the one above. □

Let A in a nonempty subset of K, for ervery $p \in K^*$ we define

$$\sigma(A,p) = \{\langle p, x\rangle : x \in A\},$$

where $\langle p, x\rangle$ is value of p at x. For $u \in \dot{K}$ we denote $S = u+K$. In the following lemma, we present the eigenvalue for the bounded multivalued operator.

Lemma 3.7. *Assume $F : S \to 2^K \backslash \{\varnothing\}$ satisfying the conditions following*

(i) $\sigma(F(x), p)$ for all $(p, x) \in S^*_+ \times S$,
(ii) $F(S)$ is relatively compact in X,
(iii) there is $(\alpha, v) \in (0, \infty) \times S$ such that $\alpha v \preceq_1 F(v)$.

Then

$$0 < \sup_{p\in S^*_+} \left(\inf_{p\in S} \frac{\langle p, x\rangle}{\sigma(F(x), p)} \right) < \infty. \tag{3.13}$$

Proof. Since the conditions (i) and (ii) are satisfied $0 < M := \sup\{\sigma(F(S), p) : p \in S^*_+\} < \infty$. By Proposition 2.4, there exists $p_0 \in S^*_+$ such that $\mu_0 := \langle p_0, u\rangle > 0$. For any $x \in S$ with $x = u + y, y \in K$, we have

$$\langle p_0, x\rangle = \langle p_0, u\rangle + \langle p_0, y\rangle \geq \mu_0.$$

Hence, $\frac{\langle p_0,x\rangle}{\sigma(F(x),p_0)} \geq \frac{\mu_0}{M}$ $\forall x \in S$, which gives

$$\inf_{x\in S} \frac{\langle p_0, x\rangle}{\sigma(F(x), p_0)} \geq \frac{\mu_0}{M} > 0.$$

This implies that

$$\sup_{p\in S^*_+} \left(\inf_{p\in S} \frac{\langle p, x\rangle}{\sigma(F(x), p)} \right) \geq \inf_{x\in S} \frac{\langle p_0, x\rangle}{\sigma(F(x), p_0)} \geq \frac{\mu_0}{M} > 0. \tag{3.14}$$

From the condition (iii), we can find $z \in F(v)$ such that $\alpha z \leq z$. Therefore

$$\langle p, \alpha v\rangle \leq \langle p, \alpha z\rangle \leq \sigma(F(v), p),$$

so $\frac{\langle p,v\rangle}{\sigma(F(v),p)} \leq \frac{1}{\alpha}$ for all $p \in S^*_+$. It follows that

$$\inf_{y\in S} \frac{\langle p, y\rangle}{\sigma(F(y), p)} \leq \frac{1}{\alpha} \text{ for all } p \in S^*_+.$$

This implies that

$$\sup_{p\in S^*_+} \left(\inf_{p\in S} \frac{\langle p, x\rangle}{\sigma(F(x), p)} \right) \leq \frac{1}{\alpha}. \tag{3.15}$$

The proof is complete. □

Theorem 3.8. *Let $F : S \to cc(K)$ be an u.s.c convex multivalued operator satisfying the conditions in Lemma 3.7. Then*

1. If λ_0 *is defined by*

$$\frac{1}{\lambda_0} = \sup_{p\in S_+^*}\left(\inf_{p\in S}\frac{\langle p,x\rangle}{\sigma(F(x),p)}\right),$$

there exists $x_0 \in S$ *such that* $\lambda_0 x_0 \in F(x_0)$ *and*

$$\frac{1}{\lambda_0} = \sup_{p\in S_+^*}\frac{\langle p,x_0\rangle}{\sigma(F(x_0),p)}.$$

2. Further, if $(\lambda, x) \in (0,\infty)\times S$ *with* $\lambda x \in F(x)$*, we have* $\lambda \leq \lambda_0$.

Proof. We first see that λ_0 is fine defined by Lemma 3.7.
We will prove the first assertion by steps following
Step 1. (Showing that $(F - \lambda_0 I)(S)$ is convex subset of K). Assmue that $z, z' \in (F-\lambda_0 I)(S)$ and $\alpha, \beta \in [0,1]$ with $\alpha+\beta = 1$. There is $(x,x') \in S\times S$ such that $z \in F(x) - \lambda_0 x$ and $z' \in F(x') - \lambda_0 x'$. We have

$$\alpha z + \beta x \in \alpha F(x) + \beta F(x') - \lambda_0(\alpha x + \beta x').$$

By the convexity of F, $\alpha F(x) + \beta F(x') \subset F(\alpha x + \beta x')$. Since S is convex, $\alpha x + \beta x' \in S$, hence $\alpha z + \beta z' \in (F-\lambda_0 I)(S)$.
Step 2. (Showing that $(F-\lambda_0 I)(S)$ is close subset of K). Assume $\{z_n\}_{n=1,2,...}$ is a sequence in $(F-\lambda_0 I)(S)$ with $\lim_{n\to\infty} z_n = z$. We can find a sequence $\{x_n\} \subset S$ and $\{y_n\}$ satisfying $z_n \in F(x_n) - \lambda_0 x_n$, $y_n \in F(x_n)$ and

$$z_n = y_n - \lambda_0 x_n. \tag{3.16}$$

Since $F(S)$ is relatively compact, we can assume $\lim_{n\to\infty} y_n = y$. Therefore, there exists $x \in S$, $\lim_{n\to\infty} x_n = x$. On the other hand, F is u.s.c and $F(x)$ is closed set, by Proposition 2.5 it follows that $y \in F(x)$. Letting $n \to \infty$ in (3.16) we obtain $z = y - \lambda_0 x$, thus $z \in (F - \lambda_0 I)(S)$.
Step 3. (Proving $\theta \in (F-\lambda_0 I)(S)$). Assume the contrary, that $\theta \notin (F - \lambda_0 I)(S)$. By applying separation Theorem for two sets $\{\theta\}$ and $(F-\lambda_0 I)(S)$ we can fine a number $\epsilon > 0$ and $p_1 \in X^*$ with $\|p_1\| = 1$ (for if not, we replace p_1 by $\frac{1}{\|p_1\|}p_1$) such that $\langle p_1, z\rangle < -\epsilon\ \forall z \in (F-\lambda_0 I)(S)$, i.e,

$$\langle p_1, y\rangle - \lambda_0\langle p_1, x\rangle < -\epsilon\ \forall (x,y) \in S\times F(x).$$

This implies that

$$\sigma(F(x),p_1) - \lambda_0\langle p_1,x\rangle \leq -\epsilon \text{ for all } x \in S. \tag{3.17}$$

We now will show that $p_1 \in S_+^*$. Indeed, if there exists $y \in K$ such that $\langle p_1, y\rangle < 0$, using (3.17) for $x = u + ny$, $(n = 1, 2, ...)$ we have

$$\sigma(F(x),p_1) - \lambda_0\langle p_1,u\rangle - n\lambda_0\langle p_1,y\rangle \leq -\epsilon. \tag{3.18}$$

Set $c = \sup\{\sigma(F(x),p_1) : x \in S\}$. Since $F(S)$ is relatively compact, $c \in (-\infty,\infty)$. Letting $n\to\infty$ in (3.17) we obtain a contradiction, hence $p_1 \in S_+^*$. From (3.17) it follows that

$$\frac{1}{\lambda_0} \leq \frac{\langle p_1,x\rangle}{\sigma(F(x),p_1)} - \frac{\epsilon}{\lambda_0 c} \text{ for all } x \in S.$$

Thus

$$\frac{1}{\lambda_0} \leq \inf_{x\in S} \frac{\langle p_1, x\rangle}{\sigma(F(x), p_1)} - \frac{\epsilon}{\lambda_0 c}.$$

On the other hand,

$$\inf_{x\in S} \frac{\langle p_1, x\rangle}{\sigma(F(x), p_1)} \leq \sup_{p\in S_+^*} \left(\inf_{x\in S} \frac{\langle p, x\rangle}{\sigma(F(x), p)}\right) = \frac{1}{\lambda_0}.$$

This implies $\frac{1}{\lambda_0} \leq \frac{1}{\lambda_0} - \frac{\epsilon}{\lambda_0 c}$. We have a contradiction. Hence $\theta \in (F - \lambda_0 I)(S)$. Therefore, there exists $x_0 \in S$ such that $\lambda_0 x_0 \in F(x_0)$.

Step 4. (Showing that $\frac{1}{\lambda_0} = \sup_{p\in S_+^*} \left(\frac{\langle p, x_0\rangle}{\sigma(F(x_0), p)}\right)$). For every $p \in S_+^*$, we have $\sigma(F(x_0), p) \geq \langle p, \lambda_0 x_0\rangle = \lambda_0 \langle p, x_0\rangle$. Thus,

$$\frac{1}{\lambda_0} \geq \frac{\langle p, x_0\rangle}{\sigma(F(x_0), p)} \text{ for all } p \in S_+^*.$$

On the other hand, we have

$$\begin{aligned} \frac{1}{\lambda_0} &\geq \frac{\langle p, x_0\rangle}{\sigma(F(x_0), p)} \\ &\geq \inf_{x\in S} \frac{\langle p, x\rangle}{\sigma(F(x), p)} \text{ for all } p \in S_+^*. \end{aligned}$$

This implies that

$$\frac{1}{\lambda_0} \geq \sup_{p\in S_+^*} \frac{\langle p, x_0\rangle}{\sigma(F(x_0), p)} \geq \sup_{p\in S_+^*} \left(\inf_{x\in S} \frac{\langle p, x\rangle}{\sigma(F(x), p)}\right) = \frac{1}{\lambda_0}.$$

We deduce $\frac{1}{\lambda_0} = \sup_{p\in S_+^*} \left(\frac{\langle p, x_0\rangle}{\sigma(F(x_0), p)}\right)$.

Now, we prove the second assertion. Assume that $\lambda x \in F(x)$ for some $(\lambda, x) \in (0, \infty) \times S$. Then, we have

$$\sigma(F(x), p) \geq \langle p, \lambda x\rangle = \lambda \langle p, x\rangle.$$

Thus,

$$\frac{1}{\lambda} \geq \frac{\langle p, x\rangle}{\sigma(F(x), p)} \geq \inf_{y\in S} \frac{\langle p, y\rangle}{\sigma(F(y), p)}.$$

It follows that

$$\frac{1}{\lambda} \geq \sup_{p\in S_+^*} \left(\inf_{x\in S} \frac{\langle p, x\rangle}{\sigma(F(x), p)}\right) = \frac{1}{\lambda_0}.$$

Hence $\lambda \leq \lambda_0$. The proof is complete. □

Remark 3.9.

1. In the proofs of our results, we have not used the cone condition with non-empty interior (which is called the solid cone).Therefore, the case $int(K) = \varnothing$ is just a special case of the results in this work. In Theorem 3.1 and Theorem 3.5 , we have used the condition that K is a reproducing cone. A solid cone is a reproducing cone. However, the opposite is not true.

For example, Let Ω be a bouned subset of $\mathbb{R}^{\mathbb{N}}$, $X = L^p(\Omega)$. The set of nonnegative functions K in X is a reproducing cone. However, it has empty interior.

2. Same as above, the normal cone condition has not been used.

4. Conclusion

This paper is a continuation of the series works [14, 29, 30] of extending the well-known result of Krein-Rutman Theorem. Initially, we investigate the fixed point index for multivalued mappings by using some useful tools including some continuous linear operators and approximate mappings at the origin and the infinity. Lastly, distinct results on the existence of solutions to the multivalued equations are constructed flexibly.

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