Weighted composition operators on spaces of analytic funcions

Esther Gómez Orts

March 2022

Advisers
José Bonet Solves
Pablo Galindo Pastor

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

Agraïments

Vull agraïr en primer lloc a Pablo i a Pepe, per tot el temps i la confiança que m'han dedicat. Gràcies Pepe per donar-me aquesta oportunitat, per estar sempre preparat per a resoldre'm qualsevol dubte i fer que la tesi avançara com toca. A tu, Pablo, t'agraïsc la paciència que has tingut amb mi, amb totes les hores tan aclaridores que hem passat al teu despatx. Gràcies per haver-me repetit arguments "trivials" una i mil voltes, era absolutament necessari.

A la resta de persones vos agraïsc el vostre pas per aquest viatge amb analogia amb la meua trilogia preferida:

Papà, Gandalf, per haver-me donat l'espentonet per a què em ficara en aquest projecte i no deixar-me sola en cap moment. Mamà, Aragorn, pel que hauràs patit per mi, per quedar-te sempre ben a prop i donar-me eixe suport tant necessari. Maria, Galadriel, per tindre les coses tant clares, tindre sempre present que el final estava ahí, i donar-me llum per a què arribara. Anna, Pippin, perquè eres família, i encara que el viatge no haja sigut teu, has viatjat amb mí. Óscar, Merry, perquè mai has dubtat de mí i has fet possibles molts passatges d'aquest trajecte. Alberto, Legolas, per ajudar-me a desxifrar el mapa amb només donar-li una ullada als núvols. Erik, Gimli, perquè malgrat de vindre d'una guerra distinta, has lluitat al meu costat en aquesta quan ha fet falta.

I a tu, Dani, perquè Frodo no haguera arribat tant lluny sense Sam.

Resum

L'objectiu d'aquesta tesi és estudiar distintes propietats dels operadors de composició ponderats en diferents espais ponderats de funcions analítiques.

Donat un pes v estrictament positiu i continu en el disc $\mathbb{D}$ del pla complex, considerem els espais de Banach de funcions analítiques H_v^∞ i H_v^0 en el disc complex. Aquests espais són, respectivament, els conjunts de les funcions holomorfes $f \in H(\mathbb{D})$ tals que $\sup_{|z|<1} v(z)|f(z)| < \infty$ i les funcions que compleixen que $v(z)|f(z)|$ tendeix a zero quan $|z|$ s'apropa a 1.

Per a cada $\alpha \geq 0$ i la successió de pesos $v_{\alpha_n}(z) := (1-|z|)^{\alpha+\frac{1}{n}}$, $z \in \mathbb{D}$, considerem l'espai de Fréchet $A_+^{-\alpha}$ com el límit projectiu de la successió $(H_{\alpha_n}^\infty := H_{v_{\alpha_n}}^\infty)_n$. Aquest espai està proveït de la topologia del límit projectiu, és a dir, la topologia de Fréchet induïda per les normes $\|\cdot\|_{\alpha_n}$. En canvi, si prenim $0 < \alpha \leq \infty$ i la successió de pesos $v_{\alpha_n}(z) := (1-|z|)^{\alpha+\frac{1}{n}}$, podem definir l'espai LB $A_-^{-\alpha}$ com el límit inductiu de la successió $(H_{\alpha_n}^\infty \equiv H_{v_{\alpha_n}}^\infty)_n$, amb la topologia del límit inductiu. Quan $\alpha_n = n$, obtenim l'espai de Korenblum $A^{-\infty}$ com el límit inductiu dels espais H_n^∞.

Estudiem la continuïtat, compacitat i invertibilitat de l'operador de composició pesat $W_{\psi,\varphi} := M_\psi C_\varphi$ on M_ψ és l'operador de multiplicació i C_φ el de composició, en els espais de tipus Korenblum $A_+^{-\alpha}$, $A_-^{-\alpha}$ i $A^{-\infty}$ definits dalt. També estudiem algunes propietats del seu espectre i del seu espectre puntual.

En el Capítol 1 compilem alguns preliminars. En el Capítol 2 estudiem la continuïtat, compacitat i invertibilitat de $W_{\psi,\varphi}$ en els espais de tipus Korenblum $A_+^{-\alpha}$, $A_-^{-\alpha}$ i $A^{-\infty}$. Al Capítol 3 ens centrem en l'estudi de l'espectre de $W_{\psi,\varphi}$ en els mateixos espais i també obtenim alguns resultats sobre l'espectre i l'espectre puntual dels operadors de multiplicació i composició. En el Capítol 4 investiguem l'espectre de certs operadors de composició, el símbol dels quals admet una extensió analítica a un entorn obert de $\overline{\mathbb{D}}$. Finalment, al Capítol 5 estudiem algunes propietats de $W_{\psi,\varphi}$ en límits projectius i inductius d'espais de Banach ponderats de funcions analítiques amb valors en un espai de Banach.

Resumen

El objetivo de esta tesis es estudiar distintas propiedades de los operadores de composición ponderados en varios espacios ponderados de funciones analíticas.

Dado un peso v estrictamente positivo y continuo en el disco $\mathbb{D}$ del plano complejo, consideramos los espacios de Banach de funciones analíticas H_v^∞ y H_v^0 en el disco complejo. Estos espacios son, respectivamente, los conjuntos de las funciones holomorfas $f \in H(\mathbb{D})$ tales que $\sup_{|z|<1} v(z)|f(z)| < \infty$ y las funciones que cumplen que $v(z)|f(z)|$ tiende a cero cuando $|z|$ se acerca a 1.

Para cada $\alpha \geq 0$ y la sucesión de pesos $v_{\alpha_n}(z) := (1-|z|)^{\alpha+\frac{1}{n}}$, $z \in \mathbb{D}$, consideramos el espacio de Fréchet $A_+^{-\alpha}$ como el límite proyectivo de la sucesión $(H_{\alpha_n}^\infty := H_{v_{\alpha_n}}^\infty)_n$. Este espacio está equipado con la topología del límite proyectivo, es decir, la topología de Fréchet inducida por las normas $\|\cdot\|_{\alpha_n}$. En cambio, si tomamos $0 < \alpha \leq \infty$ y la sucesión de pesos $v_{\alpha_n}(z) := (1-|z|)^{\alpha-\frac{1}{n}}$, podemos definir el espacio LB $A_-^{-\alpha}$ como el límite inductivo de la sucesión $(H_{\alpha_n}^\infty \equiv H_{v_{\alpha_n}}^\infty)_n$, con la topología del límite inductivo. Cuando $\alpha_n = n$, obtenemos el espacio de Korenblum $A^{-\infty}$ como el límite inductivo de los espacios H_n^∞.

Estudiamos la continuidad, compacidad e invertibilidad del operador de composición pesado $W_{\psi,\varphi} := M_\psi C_\varphi$ donde M_ψ es el operador de multiplicación y C_φ el de composición, en los espacios de tipo Korenblum $A_+^{-\alpha}$, $A_-^{-\alpha}$ y $A^{-\infty}$ definidos arriba. También estudiamos algunas propiedades de su espectro y de su espectro puntual.

En el Capítulo 1 recopilamos algunos preliminares. En el Capítulo 2 estudiamos la continuidad, compacidad e invertibilidad de $W_{\psi,\varphi}$ en los espacios de tipo Korenblum $A_+^{-\alpha}$, $A_-^{-\alpha}$ y $A^{-\infty}$. En el Capítulo 3 nos centramos en el estudio del espectro de $W_{\psi,\varphi}$ en los mismos espacios y obtenemos también algunos resultados sobre el espectro y el espectro puntual de los operadores de multiplicación y composición. En el Capítulo 4 investigamos el espectro de ciertos operadores de composición cuyos símbolos admiten una extensión analítica a un entorno abierto de $\overline{\mathbb{D}}$. Finalmente, en el Capítulo 5 estudiamos algunas propiedades de $W_{\psi,\varphi}$ en límites proyectivos e inductivos de espacios de Banach ponderados de funciones analíticas con valores en un espacio de Banach.

Summary

The aim of this thesis is to study different properties of weighted composition operators on several weighted spaces of analytic functions.

Given a strictly positive continuous weight v on the unit disc $\mathbb{D}$ of the complex plane, we consider the weighted Banach spaces of analytic functions H_v^∞ and H_v^0 on the complex disc. These spaces are, respectively, the sets of the holomorphic functions $f \in H(\mathbb{D})$ such that $\sup_{|z|<1} v(z)|f(z)| < \infty$ and the functions such that $v(z)|f(z)|$ tends to zero as $|z|$ goes to 1.

For each $\alpha \geq 0$ and the sequence of weights $v_{\alpha_n}(z) := (1-|z|)^{\alpha+\frac{1}{n}}$, $z \in \mathbb{D}$, we consider the Fréchet space $A_+^{-\alpha}$ as the projective limit of the sequence $(H_{\alpha_n}^\infty := H_{v_{\alpha_n}}^\infty)_n$. This space is endowed with the projective limit topology, that is, the Fréchet topology induced by the norms $\|\cdot\|_{\alpha_n}$. If, instead, we take some $0 < \alpha \leq \infty$ and the sequence of weights $v_{\alpha_n}(z) := (1-|z|)^{\alpha-\frac{1}{n}}$, we can define the LB-space $A_-^{-\alpha}$ as the inductive limit of the sequence $(H_{\alpha_n}^\infty \equiv H_{v_{\alpha_n}}^\infty)_n$, endowed with the inductive limit topology. When $\alpha_n = n$, we obtain the Korenblum space $A^{-\infty}$ as the inductive limit of the spaces H_n^∞.

The continuity, compactness and invertibility of the weighted composition operator $W_{\psi,\varphi} := M_\psi C_\varphi$, where M_ψ is the multiplication operator and C_φ is the composition operator, is studied in the Korenblum type spaces $A_+^{-\alpha}$, $A_-^{-\alpha}$ and $A^{-\infty}$ defined above. Also we study some properties of its spectrum and point spectrum.

In Chapter 1 we collect some preliminaries. In Chapter 2 we study the continuity, compactness and invertibility of $W_{\psi,\varphi}$ on the Korenblum type spaces $A_+^{-\alpha}$, $A_-^{-\alpha}$ and $A^{-\infty}$. In Chapter 3 we focus on the study of the spectrum of $W_{\psi,\varphi}$ on the same spaces and we obtain some results about the spectrum and point spectrum of the multiplication and composition operators. In Chapter 4 we investigate the spectrum of composition operators whose symbols admit an analytic extension to an open neighbourhood of $\overline{\mathbb{D}}$. Finally, in Chapter 5 we study some properties of $W_{\psi,\varphi}$ on projective and inductive limits of weighted Banach spaces of analytic functions with values in a Banach space.

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Introduction

The aim of this thesis is to study different properties of weighted composition operators on several weighted spaces of analytic functions.

A weight on the unit disc $\mathbb{D}$ of the complex plane is a continuous strictly positive function. Given a weight v, we consider the weighted Banach spaces of analytic functions H_v^∞ and H_v^0 on the complex disc. These spaces are, respectively, the set of the holomorphic functions $f \in H(\mathbb{D})$ such that $v(z)|f(z)| < \infty$ and the functions such that $v(z)|f(z)|$ tends to zero as $|z|$ goes to 1.

For each $\alpha \geq 0$ and the set of the weights $v_{\alpha_n}(z) := (1-|z|)^{\alpha+\frac{1}{n}}$, $z \in \mathbb{D}$, we consider the Fréchet space $A_+^{-\alpha}$ as the projective limit of the sequence $(H_{\alpha_n}^\infty := H_{v_{\alpha_n}}^\infty)_n$,

$$A_+^{-\alpha} := \bigcap_{n\in\mathbb{N}} H_{\alpha_n}^\infty = \text{proj}_n H_{\alpha_n}^\infty,$$

which is endowed with the projective limit topology, that is, the Fréchet topology induced by the norms $\|\cdot\|_{\alpha_n}$. If, instead, we take some $0 < \alpha < \infty$ and the sequence of weights $v_{\alpha_n}(z) := (1-|z|)^{\alpha-\frac{1}{n}}$, we can define the LB-space $A_-^{-\alpha}$ as the inductive limit of the sequence $(H_{\alpha_n}^\infty \equiv H_{v_{\alpha_n}}^\infty)_n$,

$$A_-^{-\alpha} := \bigcup_{n\in\mathbb{N}} H_{\alpha_n}^\infty = \text{ind}_n H_{\alpha_n}^\infty,$$

endowed with the inductive limit topology. When we take the weights $v_n(z) := (1-|z|)^n$ instead of the weights v_{α_n}, we obtain the Korenblum space $A^{-\infty}$ as the inductive limit of the spaces H_n^∞.

The continuity, compactness and invertibility of the weighted composition operator $W_{\psi,\varphi} := M_\psi C_\varphi$, where M_ψ is the multiplication operator and C_φ is the composition operator, is studied in the Korenblum type spaces $A_+^{-\alpha}$, $A_-^{-\alpha}$ and $A^{-\infty}$ defined above. Also we study some properties of the spectrum and point spectrum.

Weighted Banach spaces of analytic functions on the disc have been extensively studied by many autors, like Bierstedt and Summers [9] and Bierstedt, Bonet and Taskinen [13]. Weighted composition operators defined on spaces of functions of one variable have been extensively studied, and there is a huge related literature. We refer to the books of Shapiro [43] and Cowen, MacCluer [27]. Continuity and compactness of weighted composition operators between spaces of type H_α^∞ were described by Contreras and Hernández-Díaz in [26] and Montes-Rodríguez [38] even in a more general context. Previously, the composition operators on these Banach spaces were studied by Bonet, Domański, Lindström and Taskinen [19]. Bourdon investigated the invertibility of weighted composition operators in [23]. Continuity of linear operators between projective and inductive limits of Banach spaces was characterised by Albanese, Bonet and Ricker in [4] and [5]. The spectrum and point spectrum of linear operators on projective and inductive limits of Banach spaces were studied by Albanese, Bonet and Ricker in [2] and [3]. Kamowitz investigated the spectrum and point spectrum of weighted composition operators on spaces of holomorphic functions that contain the polynomials [34], and also determined the spectra of composition operators in the case where the symbol is analytic on an open region containing $\overline{\mathbb{D}}$ [33]. Aron and Lindström studied the spectrum of weighted composition operators on weighted Banach spaces of analytic functions [7] and, recently, Bonet investigated the spectrum of composition operators induced by a rotation [16].

Some properties of weighted composition operators on projective and inductive limits of weighted Banach spaces of vector-valued analytic functions are also studied in this work. If E is a complex Banach space and v a weight on the unit disc, we define the weighted Banach spaces of vector-valued analytic functions $H_v^\infty(\mathbb{D}, E)$ and $H_v^0(\mathbb{D}, E)$. If, instead of a weight v we consider a decreasing sequence of weights $V = (v_n)_n$, we can define the LB-space $VH(\mathbb{D}, E)$ as the inductive limit of the Banach spaces $H_{v_n}^\infty(\mathbb{D}, E)$,

$$VH(\mathbb{D}, E) := \underset{n}{\operatorname{ind}} H_{v_n}^\infty(\mathbb{D}, E).$$

Or, if we take an increasing sequence of weights $W = (w_n)_n$, we can define the Fréchet space $HW(\mathbb{D}, E)$ as the projective limit of the Banach spaces $H_{w_n}^\infty(\mathbb{D}, E)$,

$$HW(\mathbb{D}, E) := \underset{n}{\operatorname{proj}} H_{v_n}^\infty(\mathbb{D}, E).$$

In particular, for the increasing and decreasing sequences of the weights v_{α_n}, we obtain the Korenblum type spaces for the vector-valued case, $A_-^{-\alpha}(E)$, $A_+^{-\alpha}(E)$ and $A^{-\infty}(E)$.

In Chapter 2 we introduce the weighted composition operator, $W_{\psi,\varphi}$, for a symbol $\varphi(\mathbb{D}) \subset \mathbb{D}$ analytic and a weight $\psi \in H(\mathbb{D})$. Based on some results of [26], in Section 2.1 we give a sufficient condition for the weighted composition operator to be continuous on $A_+^{-\alpha}$ and $A_-^{-\alpha}$, and a characterization of the continuity on $A^{-\infty}$. In Section 2.2, we characterize the compactness of weighted composition operators on each of the three Korenblum type spaces. Further, as corollaries, we obtain a necessary condition where the compactness of $W_{\psi,\varphi}$ on these spaces implies its compactness on some Banach spaces H_β^0, and that if the symbol lies in a closed disc centered in zero inside the unit disc then $W_{\psi,\varphi}$ is compact whenever it is continuous on any of the Korenblum type spaces. As a consequence of two results of Bourdon, we characterize in Section 2.3 the invertibility of $W_{\psi,\varphi}$ on the spaces $A_+^{-\alpha}$, $A_-^{-\alpha}$ and $A^{-\infty}$.

In Chapter 3 we study the spectrum and point spectrum of the weighted composition operators on the spaces $A_+^{-\alpha}$, $A_-^{-\alpha}$, $A^{-\infty}$. Theorem 3.3.2 in Section 3.3 shows that the spectrum of $W_{\psi,\varphi}$ contains the set of points $\{0\} \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^\infty$ and that it is contained in $\overline{B}(0, \lim_k r_e(W_{\psi,\varphi}, H_{\alpha_k}^\infty)) \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^\infty$ for the spaces $A_+^{-\alpha}$ and $A_-^{-\alpha}$ and in $\overline{B}(0, \lim_k r_e(W_{\psi,\varphi}, H_k^\infty)) \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^\infty$ for the Korenblum space $A^{-\infty}$. Here $r_e(T, X)$ denotes the essential spectral radius of the operator T in the Banach space X. In Theorem 3.3.5 we prove that the point spectrum of $W_{\psi,\varphi}$ in $A_+^{-\alpha}$ and in $A_-^{-\alpha}$ contains the set of points $\{\psi(0)\varphi'(0)^n\}_{n=0}^\infty \setminus \overline{B}(0, r_e(W_{\psi,\varphi}, H_\alpha^\infty))$ and it is contained in the set of points $\{\psi(0)\varphi'(0)^n\}_{n=0}^\infty$, and for the space $A^{-\infty}$ we obtain that $\sigma_p(W_{\psi,\varphi}, A^{-\infty}) \subseteq \{\psi(0)\varphi'(0)^n\}_{n=0}^\infty$. In Section 3.4 we investigate the spectrum and point spectrum of composition operators in the three Korenblum type spaces. As a corollary of Theorem 3.3.2, we deduce that the spectrum of C_φ in the spaces $A_+^{-\alpha}$ and $A_-^{-\alpha}$ is contained in $\overline{B}(0, r_e(C_\varphi, H_\alpha^\infty)) \cup \{\varphi'(0)^n\}_{n=0}^\infty$ and contains the set of points $\{0\} \cup \{\varphi'(0)^n\}_{n=0}^{\infty}$. For the Korenblum space we prove in Theorems 3.4.2 and 3.4.4 that $\sigma(C_\varphi, A^{-\infty}) = \{0\} \cup \{\varphi'(0)^n\}_{n=0}^{\infty}$ and that $\sigma_p(C_\varphi, A^{-\infty}) = \{\varphi'(0)^n\}_{n=0}^{\infty}$. In Section 3.5 we study the spectrum of the multiplication operator. Corollary 3.5.2 shows that, in any of the spaces $A_+^{-\alpha}$, $A_-^{-\alpha}$ or $A^{-\infty}$, the point spectrum of M_ψ is empty and the spectrum contains the image of the weight, $\psi(\mathbb{D})$, and it is contained in its closure. Moreover, in Section 3.6 we investigate the spectrum and point spectrum of the composition operator in the case where the symbol is a rotation. In comparison with [16, Theorems 1 and 2], for the composition operator in the Korenblum space $C_\varphi : A^{-\infty} \to A^{-\infty}$ with $\varphi(z) = cz$ where $|c| = 1$ and it is not a root of unity, we obtain that a complex number $\lambda \neq 1$, $|\lambda| = 1$ belongs to the resolvent set, that is, the complemetary set of the spectrum, if, and only if, there are $s \geq 1$ and $\varepsilon > 0$ such that $|c^n - \lambda| \geq \varepsilon n^{-s}$ for each $n \in \mathbb{N}$. We also obtain a characterization of the number 1 belonging to the resolvent set in relation with Diophantine numbers.

Chapter 4 is devoted to the study of the spectrum of the composition operators whose symbols admit an analytic extension to an open neighbourhood of the closed unit disc $\mathbb{D}$ of the complex plane. We follow the argument of Kamowitz in [33, Theorem 3.4]. In Section 4.1 we introduce some Lemmas used in the proof of the main results. Theorems 4.2.1 and 4.2.4 in Section 4.2 show that for $\varphi : \mathbb{D} \to \mathbb{D}$ analytic, which is analytic on a neighbourhood of the closed unit disc, with an interior fixed point and a repelling fixed point z_0 in the circle, the spectrum of the composition operator C_φ on $A_-^{-\alpha}$ and $A_+^{-\alpha}$ contains the closed ball $\overline{B}(0, |\varphi'(z_0)|^{-\alpha})$. This enlarges the knowledge of the size of $\sigma(C_\varphi)$ that it is known to be a subset of $\overline{B}(0, r_e(C_\varphi, H_\alpha^\infty)) \cup \{\varphi'(0)^n\}_{n=0}^{\infty}$ according to Corollary 3.4.1.

In Chapter 5 we investigate different properties of weighted composition operators on projective and inductive limits of weighted Banach spaces of vector-valued analytic fucntions. In Section 5.1 we study the continuity, compactness and weak compactness of $W_{\psi,\varphi}$ between two Banach spaces of vector-valued functions, in comparison with their equivalents in the scalar case. That is, the operators $W_{\psi,\varphi} : H_v^\infty(\mathbb{D}, E) \to H_w^\infty(\mathbb{D}, E)$ and $W_{\psi,\varphi} : H_v^0(\mathbb{D}, E) \to H_w^0(\mathbb{D}, E)$ in comparison with $W_{\psi,\varphi} : H_v^\infty \to H_w^\infty$ and $W_{\psi,\varphi} : H_v^0 \to H_w^0$. We obtain that the operators in the vector case are continuous if, and only if, they are so in the scalar case. Proposition 5.1.11 shows that $W_{\psi,\varphi} : H_v^\infty(\mathbb{D}, E) \to H_w^\infty(\mathbb{D}, E)$ being compact is equivalent to $W_{\psi,\varphi} : H_v^0(\mathbb{D}, E) \to H_w^0(\mathbb{D}, E)$ being compact and that it is also equivalent to the compactness in the scalar cases and that E has finite dimension. In Proposition 5.1.14 we have the same equivalences for the weak compactness but in there the space E has to be reflexive instead of finite dimensional. In Proposition 5.2.10 of Section 5.2 we characterize when linear operators between LB-spaces are bounded, Montel, reflexive, compact or weakly compact. Then, in Section 5.2.1, we apply Proposition 5.2.10 to characterize when weighted composition operators $W_{\psi,\varphi}$ on the space $VH(\mathbb{D}, E)$ are bounded, Montel, reflexive, compact and weakly compact. As a consequence, in Section 5.2.2, we obtain the characterizations of the same properties in the cases when weighted composition operators act on the LB-spaces $A_-^{-\alpha}(E)$ and $A^{-\infty}(E)$. In Section 5.3.1 we apply [17, Proposition 4.2] to give some characterizations of the $W_{\psi,\varphi}$ being bounded, Montel, reflexive, compact and weakly compact, on the projective limits $HW(\mathbb{D}, E)$. These results are gathered in Proposition 5.3.11, which should be compared with [17, Theorem 4.3]. In Section 5.3.2 we obtain the characterizations of the same properties for weighted composition operators acting on the projective limit $A_+^{-\alpha}(E)$, as consequences of Proposition 5.3.11.

Chapter 1

Preliminaries

1.1 Previous definitions and useful results

Let X be a topological space. The space X is a *Hausdorff* space if for each pair of distinct points x, y there are respective neighbourhoods U_x, U_y such that $U_x \cap U_y = \emptyset$. We say that X is *regular* if it is Hausdorff and each point possesses a base of closed neighbourhoods. Moreover, every Hausdorff locally convex space is a topological regular space.

Let X and Y be topological spaces. A bijection b of X onto Y such that $b(A)$ is open in Y if, and only if, A is open in X, is called a *homeomorphism*. The spaces X and Y are *homeomorphic* if there exists a homeomorphism of X onto Y. Two topological vector spaces X, Y over the same field K are called *isomorphic* if there exists a bijective linear map u of X onto Y which is a homeomorphism; u is called an *isomorphism* of X onto Y. We say that u is an *automorphism* if u is an isomorphism of X onto itself. (See [42, pp. 4, 13]). A linear operator $T\colon X \to Y$ between two normed spaces is called an *isometry* (or, more precisely, *linear isometry*) if $\|Tx\| = \|x\|$ for each $x \in X$. Two normed spaces are said to be *isometric* if there exists a surjective linear isometry from one space onto the other. (See [1, p. 2]).

An operator $T\colon X \to Y$ is said to be *continuous* at $a \in X$ if for each neighbourhood V of $T(a)$ there exists a neighbourhood U of a such that $T(U) \subseteq V$. We say that T is *continuous* if it is continuous at all points in X. The operator T is continuous if, and only if, the preimage of every open set is open.

A continuous linear operator $T\colon E \to F$ between Hausdorff locally convex spaces E and F is called:

- *bounded* if there exists a 0-neighbourhood U in E such that $T(U)$ is bounded in F;

- *Montel* if for every bounded subset B of E, $T(B)$ is relatively compact in F;

- *reflexive* if for every bounded subset B of E, $T(B)$ is weakly relatively compact in F;

- *compact* if there exists a 0-neighbourhood U in E such that $T(U)$ is relatively compact in F;

- *weakly compact* if there exists a 0-neighbourhood U in E such that $T(U)$ is weakly relatively compact in F.

Let X be a vector space over a non-discrete field $\mathbb{K}$. Let A and B be two subsets of X. We say that A is *circled* if $\lambda A \subseteq A$ for each $|\lambda| \leq 1$. A circled, convex subset is called *absolutely convex.* We say that A *absorbs* B if there exists $\lambda_0 \in \mathbb{K}$ such that $B \subseteq \lambda A$ whenever $|\lambda| \geq |\lambda_0|$. The subset A is called *absorbent* if absorbs every finite subset of X.

A *barrel* in a topological vector space E is a subset which is absorbent, absolutely convex and closed. A locally convex space E is *barreled* if each barrel in E is a neighbourhood of 0. Every Banach space and every Fréchet space is barreled (see [42, p. 60]). A locally convex space E for which $E = E''$ is called *semi-reflexive*. Every semi-reflexive space E is *reflexive* if, and only if, it is barreled. Every semi-reflexive normed space is a reflexive Banach space (see [42, p. 145]). A reflexive locally convex space in which every closed, bounded subset is compact, is called a *Montel* space. Observe that in Montel spaces an operator is compact if, and only if, it is bounded. A normed space is Montel if, and only if, it is finite dimensional ([37, Exercise 7 p. 293]). Morevoer, since the unit ball of a Banach space is bounded and it is a 0-neighbourhood, then every operator on a Banach space is reflexive if, and only if, it is weakly compact. For more information about barreled, reflexive or Montel spaces, see [37] and [42].

Proposition 1.1.1 ([37], Theorem 23.18)**.** *A locally convex space E is semi-reflexive if, and only if, every bounded set in E is relatively weakly compact.*

Note that this proposition implies that if F is a semi-reflexive locally convex space and $T : E \to F$ an operator, then T is weakly compact if, and only if, it is bounded.

Let X be a topological vector space. A *fundamental system* of bounded sets of X is a family $\mathcal{B}$ of bounded sets such that every bounded subset of X is contained in a suitable member of $\mathcal{B}$.

Let E be a locally convex space. A subset $M \subseteq E$ is said to be *bornivorous* if for each bounded set B in E there exists a $\lambda > 0$ such that $B \subseteq \lambda M$. The space E is called *DF-space* if it has the following properties:

1. E has a countable fundamental system of bounded sets.
2. If $V \subseteq E$ is bornivorous and is the intersection of a sequence of absolutely convex zero neighbourhoods, then V is itself a zero neighbourhood.

Every normed space is a DF-space. For more information about DF-spaces see [37, Chapter 25].

Along all the work the set of all holomorphic functions on the complex disc $\mathbb{D}$ is denoted by $H(\mathbb{D})$ and the set of all bounded holomorphic functions on $\mathbb{D}$ is written H^∞. The topology of the uniform convergence on compact sets is denoted by τ_{co}. The weak topology in a topological space X is denoted by $\sigma(X, X')$. The weak* topology (or, the pointwise topology) in the dual space X' is written $\sigma(X', X)$ or simply w^*-topology.

1.2 Inductive and projective limits

A $\mathbb{K}-$vector space E together with a family of locally convex spaces $(E_i)_{i\in I}$ and linear maps $j_i \colon E_i \to E$, $i \in I$, is called an *inductive system* if $\bigcup_{i\in I} j_i(E_i) = E$. If there exists a finest locally convex topology on E for which all the maps j_i are continuous, then it is called the *inductive topology* of the system.

Proposition 1.2.1 ([37], Proposition 24.7)**.** *Let the locally convex space E have the inductive topology of the system $(j_i\colon E_i \to E)_{i\in I}$. A linear map $A\colon E \to F$ into a locally convex space F is continuous if, and only if, $A \circ j_i$ is continuous for all $i \in I$.*

A countable inductive system $(j_n\colon E_n \to E)_{n\in\mathbb{N}}$ is called an *imbedding spectrum* if the following holds for all $n \in \mathbb{N}$:

1. E_n is a linear subspace of E and j_n is the inclusion.

2. E_n is contained in E_{n+1} and the inclusion $E_n \hookrightarrow E_{n+1}$ is continuous.

If the inductive topology τ of the system exists, then we refer to $E = (E, \tau)$ as its *inductive limit*, and we write $E = \text{ind}_n E_n$. We say that a locally convex inductive limit $E = \text{ind}_n E_n$ is *regular* if every bounded subset of E is contained and bounded in a step E_n. The inductive limit of an increasing sequence of Banach spaces is called an *LB-space*. Every LB-space is barreled ([42, Corollary 2, p. 61]). For more information about inductive limits and LB-spaces see [14].

Lemma 1.2.2 ([5], Lemma 4.1)**.** *Let $X = ind_n X_n$ and $Y = ind_m Y_m$ the inductive limits given by two increasing unions of Banach spaces $X = \cup_{n=1}^{\infty} X_n$ and $Y = \cup_{m=1}^{\infty} Y_m$. Let $T\colon X \to Y$ be a linear map.*

(i) T is continuous if, and only if, for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $T(X_n) \subseteq Y_m$ and the restriction $T\colon X_n \to Y_m$ is continuous.

(ii) Assume that Y is a regular LB-space. Then T is bounded if, and only if, there exists $m \in \mathbb{N}$ such that $T(X_n) \subseteq Y_m$ and $T\colon X_n \to Y_m$ is continuous for all $n \geq m$.

A $\mathbb{K}-$vector space E together with a family of locally convex spaces $(E_i)_{i\in I}$ and linear maps $p_i\colon E \to E_i$, $i \in I$, is called a *projective system* if, for each $x \in E$, $x \neq 0$, there is an $i \in I$ with $p_i(x) \neq 0$. The *projective topology* on E is the coarsest topology on E for which each of the mappings p_i is continuous. We write $E = \bigcap_{i\in I} p_i^{-1}(E_i)$.

Lemma 1.2.3 ([4], Lemma 25)**.** *Let $E := proj_m E_m$ and $F := proj_n F_n$ be Fréchet spaces such that $E = \cap_{m\in\mathbb{N}} E_m$ with each $(E_m, \|\cdot\|_m)$ a Banach space (resp. $F = \cap_{n\in\mathbb{N}} F_n$ with each $(F_n, \|\cdot\|_n)$ a Banach space). Moreover, it is assumed that E is dense in E_m and that $E_{m+1} \subseteq E_m$ with a continuous inclusion for each $m \in \mathbb{N}$ (resp. $F_{n+1} \subseteq F_n$ with a continuous inclusion for each $n \in \mathbb{N}$). Let $T\colon E \to F$ be a linear operator.*

(i) T is continuous if, and only if, for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that T has a unique continuous linear extension $T_{m,n}\colon E_m \to F_n$.

(ii) Assume that T is continuous. Then T is bounded if, and only if, there exists $m \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$, the operator T has a unique continuous linear extension $T_{m,n}\colon E_m \to F_n$.

1.3 About the spectrum

The space of all continuous linear operators between two locally convex spaces X and Y is denoted by $\mathcal{L}(X, Y)$. If $X = Y$ then we write $\mathcal{L}(X)$.

Let $T\colon X \to X$ be a continuous linear operator on a locally convex space X. The *resolvent set* $\rho(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (T - \lambda I)^{-1}$ is a continuous linear operator, that is, $T - \lambda I\colon X \to X$ is bijective and has a continuous inverse. Here I stands for the identity operator on X. The set $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T. The *point spectrum* $\sigma_p(T)$ of T is the set consisting of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective. In other words, the point spectrum is the set of all eigenvalues of T. If we need to stress the space X, then we write $\sigma(T, X)$, $\sigma_p(T, X)$ and $\rho(T, X)$. Unlike for Banach spaces, it may happen that $\rho(T) = \emptyset$ or that $\rho(T)$ is not open.

Proposition 1.3.1 ([32], Theorem 4 p. 204)**.** *If T is a compact operator in a locally convex topological vector space, then $\sigma(T)$ is either finite or it is formed by 0 and the points of a sequence that converges to 0.*

1.4 Weighted Banach spaces

If $v\colon \mathbb{D} \to \mathbb{R}^+$ is a bounded continuous (strictly) positive funcion, we say that v is a *weight*.

Let $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$, which is endowed with the Fréchet topology of uniform convergence on compact sets. We are interested in the following weighted Banach spaces:

$$H_v^\infty = H_v^\infty(\mathbb{D}) := \{f \in H(\mathbb{D}) :\ ||f||_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\}\ ,$$
$$H_v^0 = H_v^0(\mathbb{D}) := \{f \in H(\mathbb{D}) :\ \lim_{|z|\to 1^-} v(z)|f(z)| = 0\}\ .$$

Those spaces are Banach, endowed with the norm $|| \cdot ||_v$.

We say that v is *radial* whenever $v(z) = v(|z|)$ for all $z \in \mathbb{D}$. Any radial, positive continuous funcion $v\colon \mathbb{D} \to \mathbb{R}^+$, which is non-increasing with respect to $|z|$ and is such that $\lim_{|z|\to 1^-} v(z) = 0$, is called a *typical weight*. Through the document, we just consider v to be radial and typical.

To each weight v corresponds the so-called *growth condition* $u\colon \mathbb{D} \to \mathbb{R}^+$, $u = 1/v$, and $B_v := \{f \in H(\mathbb{D}) :\ |f| \le u\}$. A new function $\tilde{u}\colon \mathbb{D} \to \mathbb{R}^+$ is defined by

$$\tilde{u}(z) := \sup_{f\in B_v} |f(z)|,$$

and the weight *associated* with v is defined by $\tilde{v} := 1/\tilde{u}$. (See some of their properties in [13]). In [19, Proposition 2.1] it is shown that if v is typical then $\tilde{v}$ is typical too and by [19, Proposition 2.3] we have that $H_v^\infty = H_{\tilde{v}}^\infty$ isometrically and, if $\lim_{|z|\longrightarrow 1^-} v(z) = 0$ then $H_v^0 = H_{\tilde{v}}^0$ isometrically as well. Moreover, in [13, Observation 1.12] it is proved that $H_v^\infty = H_{\tilde{v}}^\infty$ and the norms $\| \|_v$ and $\| \|_{\tilde{v}}$ coincide. A weight v is called *essential* if there exists a constant $C > 0$ such that $v(z) \le \tilde{v}(z) \le Cv(z)$, for all $z \in \mathbb{D}$.

1.4.1 Our weights

Given $\alpha > 0$, we define $v_\alpha(z) := (1 - |z|)^\alpha$, $z \in \mathbb{D}$. From now on, in order to abreviate the notation, we write H_α^∞ and H_α^0 instead of $H_{v_\alpha}^\infty$ and $H_{v_\alpha}^0$, and $\| \cdot \|_\alpha$ instead of $\| \cdot \|_{v_\alpha}$. For every α, $H_\alpha^0 \subset H_\alpha^\infty$.

Since $1 - |z| \le 1 - |z|^2 \le 2(1 - |z|)$, the weights $(1 - |z|^2)^\alpha$ define the same space. The v_α weights are called *standard weights*. They are also essential, with constant $C = 1$ (see Remark 1.4.1).

Remark 1.4.1. For every $\alpha > 0$, $v_\alpha(z) = \tilde{v}_\alpha(z)$, for all $z \in \mathbb{D}$.

Indeed, it is clear that $v_\alpha \leq \tilde{v}_\alpha$. On the other hand, if we fix $z_0 \in \mathbb{D}$ and prove that there exists some $f \in H(\mathbb{D})$ such that $|f(z)| \leq 1/v_\alpha(z)$ for every $z \in \mathbb{D}$ with $f(z_0) = 1/v_\alpha(z_0)$, we get the result. When $z_0 = 0$, we take $f \equiv 1$. Otherwise, if $z_0 \in \mathbb{D} \setminus \{0\}$, we can write $z_0 = |z_0|e^{i\theta}$. Define the function $f \in H(\mathbb{D})$ as follows:

$$f(z) := \frac{1}{(1-e^{-i\theta}z)^\alpha}, \ \forall z \in \mathbb{D}.$$

Thus,

$$\begin{aligned} |f(z)| &= \frac{1}{|(1-e^{-i\theta}z)^\alpha|} = \frac{1}{|1-e^{-i\theta}z|^\alpha} \leq \frac{1}{(1-|z|)^\alpha} = 1/v_\alpha(z), \\ f(z_0) &= \frac{1}{(1-e^{-i\theta}z_0)^\alpha} = \frac{1}{(1-|z_0|)^\alpha}. \end{aligned}$$

Lemma 1.4.2. *The space H_α^0 is a closed subspace of H_α^∞ and coincides with the closure of the polynomials on H_α^∞.*

Proof. We prove that for each function in H_α^0, there is a sequence of polynomials that tends to the function in the norm $\| \ \|_\alpha$.

Fix $f \in H_\alpha^0$, which can be written as $f(z) = \sum_{k=0}^\infty a_k z^k$. The Cesàro means of the partial sums of the Taylor series of f about zero are denoted by $C_n(f)$, $n = 0, 1, 2, \ldots$; that is,

$$(C_n(f))(z) = \frac{1}{n+1} \sum_{i=0}^{n} \left(\sum_{k=0}^{i} a_k z^k \right), \ \ z \in \mathbb{D}\,.$$

Each $C_n(f)$ si a polynomial of degree less than or equal to n and $C_n(f) \longrightarrow f$ uniformly on every compact subset of $\mathbb{D}$. Applying [12, Lemma 1.1] and taking into account that $(1-|z|)^\alpha$ is a radial weight we obtain

$$\sup_{z\in\mathbb{D}} (1-|z|)^\alpha |(C_n(f))(z)| \leq \sup_{z\in\mathbb{D}} (1-|z|)^\alpha (\max_{|\lambda|=1} |f(\lambda z)|) = \sup_{z\in\mathbb{D}} (1-|z|)^\alpha |f(z)|\,.$$

Now, similarly to the proof of [12, Proposition 1.2 (e)], we prove that $C_n(f) \longrightarrow f$ with the norm $\| \ \|_\alpha$.

For each $\varepsilon > 0$ there exists $0 < r < 1$ such that $(1-|z|)^\alpha |f(z)| \leq \varepsilon/2$ for all $z \in \mathbb{D} \setminus \overline{B}(0,r)$. On the other hand, since $C_n(f) \to f$ uniformly on $K = \overline{B}(0,r)$ (because it is a compact set), we can choose $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $\max_{z\in K} |f(z) - (C_n(f)(z))| \leq \varepsilon$. Thus, for any $n \geq n_0$ we have that

$$\begin{aligned} &\sup_{z\in\mathbb{D}} (1-|z|)^\alpha |f(z) - (C_n(f))(z)| \leq \\ &\qquad \leq \max\left\{ \max_{z\in K} |f(z) - (C_n(f)(z))|, \sup_{z\in\mathbb{D}\setminus K} (1-|z|)^\alpha |f(z) - (C_n(f))(z)| \right\} \\ &\qquad \leq \max\left\{ \varepsilon, \frac{\varepsilon}{2} + \sup_{z\in\mathbb{D}\setminus K} (1-|z|)^\alpha |(C_n(f))(z)| \right\} \\ &\qquad \leq \max\left\{ \varepsilon, \frac{\varepsilon}{2} + \sup_{z\in\mathbb{D}\setminus K} (1-|z|)^\alpha (\max_{|\lambda|=1} |f(\lambda z)|) \right\} \\ &\qquad \leq \max\left\{ \varepsilon, \frac{\varepsilon}{2} + \sup_{z\in\mathbb{D}\setminus K} (1-|z|)^\alpha |f(z)| \right\} \leq \varepsilon\,. \end{aligned}$$

Here we have used that if $z \in \mathbb{D} \setminus K$ then $\lambda z \in \mathbb{D} \setminus K$ for each $|\lambda| = 1$. □

Lemma 1.4.3. *For every $\beta > \alpha > 0$ the inclusion $H_\alpha^\infty \hookrightarrow H_\beta^0$ is continuous and $\|f\|_\beta \leq \|f\|_\alpha$ for all $f \in H_\alpha^\infty$.*

Proof. Let $f \in H_\alpha^\infty$. Let us call $M := \sup_{z\in\mathbb{D}} v_\alpha(z)|f(z)| < \infty$. For every $\varepsilon > 0$, take $\delta := (\varepsilon/M)^{\frac{1}{\beta-\alpha}} > 0$. Assume $1 - |z| < \delta$. Then,

$$v_\beta(z)|f(z)| < \delta^{\beta-\alpha}(1-|z|)^\alpha|f(z)| = \delta^{\beta-\alpha}v_\alpha(z)|f(z)| \leq \delta^{\beta-\alpha}M = \varepsilon.$$

That is, $f \in H_\beta^0$. Morevover, since $\beta > \alpha$, $v_\beta(z) \leq v_\alpha(z)$ for all $z \in \mathbb{D}$ and so, for every $f \in H_\alpha^\infty$,

$$\|f\|_\beta = \sup_{z\in\mathbb{D}} v_\beta(z)|f(z)| \leq \sup_{z\in\mathbb{D}} v_\alpha(z)|f(z)| = \|f\|_\alpha.$$

Therefore, the inclusion is continuous. □

Lemma 1.4.4. *For every $\beta > \alpha > 0$, the inclusion $H_\alpha^\infty \hookrightarrow H_\beta^\infty$ is compact.*

Proof. Observe that $\beta - \alpha > 0$ since $\beta > \alpha > 0$. So,

$$\lim_{|z|\longrightarrow 1^-} \frac{v_\beta(z)}{v_\alpha(z)} = \lim_{|z|\longrightarrow 1^-} (1-|z|)^{\beta-\alpha} = 0.$$

We need to see that the closed unit ball $\overline{B}_\alpha$ of H_α^∞ is compact in H_β^∞. That is, we want that for every sequence $(f_k)_k \subset \overline{B}_\alpha$ exists a subsequence $(f_{k_j})_j$ such that $\lim_j \|f_{k_j} - f_0\|_\beta = 0$ for some $f_0 \in H_\beta^\infty$.

The ball $\overline{B}_\alpha$ is bounded and closed in $(H(\mathbb{D}), \tau_{co})$. By Montel's Theorem, $\overline{B}_\alpha$ is τ_{co}-compact. Or, what it is the same, there exists a subsequence $(f_{k_j})_j$ τ_{co}-convergent to some f_0. We prove now that such subsequence is also convergent in H_β^∞.

In order to do this, let us denote $g_j := f_{k_j} - f_0$. So, $(g_j)_j$ converges to 0 uniformly on compact sets of $\mathbb{D}$. Also, $\|g_j\|_\alpha \leq 2$, $\forall j$.

Now, since $\lim_{|z|\to 1^-} \frac{v_\beta(z)}{v_\alpha(z)} = 0$, for every $\varepsilon > 0$, there exists $r_0 \in]0,1[$ such that

$$\frac{v_\beta(z)}{v_\alpha(z)} < \varepsilon, \text{ for all } |z| \geq r_0.$$

We call $M := \max_{z\in\mathbb{D}} v_\beta(z)$ (such maximum exists because v_β is continuous, positive, and $\lim_{|z|\to 1^-} v_\beta(z) = 0$). The set $K := \{z \in \mathbb{D} : |z| \leq r_0\}$ is compact. There exists j_0 such that, if $j \geq j_0$ then $\sup_{z\in K} |g_j(z)| < \frac{\varepsilon}{M}$. Let $z \in \mathbb{D}$ and $j \geq j_0$. So, when $z \in K$,

$$v_\beta(z)|g_j(z)| \leq M\frac{\varepsilon}{M} = \varepsilon.$$

If $z \in \mathbb{D} \setminus K$,

$$v_\beta(z)|g_j(z)| = \frac{v_\beta(z)}{v_\alpha(z)} v_\alpha(z)|g_j(z)| < 2\varepsilon.$$

That is, $(g_j)_j$ converges to 0 in H_β^∞. □

Remark 1.4.5. Lemma 1.4.4 can be seen as a consequence of [19, Theorem 3.3] by taking the symbol φ the identity. This way, the composition operator is the inclusion.

Lemma 1.4.6. *For each $\alpha > 0$, H_α^∞ is isometrically isomorphic to the bidual Banach space $(H_\alpha^0)''$.*

Proof. By [9, Corollary 1.2], it is enough to prove that $\overline{B}_{H^0_\alpha}$ is τ_{co}-dense on $\overline{B}_{H^\infty_\alpha}$. So, we prove this by following the argument of [9, Example 2.1].

If we fix $f \in \overline{B}_{H^\infty_\alpha}$, for each $0 < r < 1$, the function $f_r(z) := f(rz)$ belongs to H^0_α. Indeed, since $r\mathbb{D}$ is a relatively compact subset of $\mathbb{D}$, and f is analytic on $\mathbb{D}$, we have that

$$\sup_{z\in\mathbb{D}} |f_r(z)| = \sup_{z\in r\mathbb{D}} |f(z)| < \infty,$$

what implies $\lim_{|z|\longrightarrow 1^-}(1-|z|)^\alpha |f_r(z)| = 0$, for all $0 < r < 1$.

Moreover, given $z_0 \in \mathbb{D}$, the Maximum Modulus Principle yields $\lambda \in \mathbb{C}$, $|\lambda| = 1$ such that $|f(rz_0)| \leq \sup_{z\in B(0,|z_0|)} |f(z)| = |f(\lambda z_0)|$ and

$$v_\alpha(z_0)|f_r(z_0)| \leq v_\alpha(z_0)|f(\lambda z_0)| \leq \|f\|_\alpha \leq 1 \ ,$$

that is, $f_r \in \overline{B}_{H^0_\alpha}$ for every $0 < r < 1$. Since $(f_r)_r$ is a bounded set for the norm topology, then it is bounded in the τ_{co} topology, that is, it is uniformly bounded on compact sets. Then, by Montel's Theorem, there exists a subsequence uniformly convergent on compact sets. And, since $\lim_{r\longrightarrow 1^-} f(rz) = f(z)$ for all $z \in \mathbb{D}$ (because f is continuous on each $z \in \mathbb{D}$), then the subsequence of $(f_r)_r$ tends to f uniformly on compact sets of $\mathbb{D}$ as $r \longrightarrow 1^-$. Thus, $\overline{B}_{H^0_\alpha}$ is τ_{co}-dense on $\overline{B}_{H^\infty_\alpha}$. □

The preceding remark explains why the spaces H^∞_α and $(H^0_\alpha)''$ are isometrically isomorphic. However, by [9, Theorem 2.3] it is enough to have that the space H^0_α contains the polynomials.

1.5 Korenblum type spaces

For every $n \in \mathbb{N}$, set $v_n(z) := (1-|z|)^n$, $z \in \mathbb{D}$. Clearly, $v_n \geq v_{n+1}$. Thus, $H^\infty_n \subset H^\infty_{n+1}$, with $||\cdot||_{n+1} \leq ||\cdot||_n$.

The Korenblum space (see [35]) is defined as

$$A^{-\infty} := \bigcup_{n\in\mathbb{N}} H^\infty_n \ .$$

It is endowed with the inductive limit topology: $A^{-\infty} = \operatorname{ind}_n H^\infty_n$. It is the finest locally convex topology on $\bigcup_n H^\infty_n$ such that the inclusion $H^\infty_n \hookrightarrow A^{-\infty}$ is continuous for each $n \in \mathbb{N}$. The space $A^{-\infty}$ is a regular LB-space (see Proposition 5.2.2).

Remark 1.5.1. By Lemmas 1.4.3 and 1.4.4, taking $\alpha = n$ and $\beta = n+1$, we have $H^\infty_n \hookrightarrow H^0_{n+1}$ is continuous and $H^\infty_n \hookrightarrow H^\infty_{n+1}$ is compact.

By [37, Proposition 25.20], the space $A^{-\infty}$ is a DFS-space. This means that it is a DF-space which is also Schwarz. A locally convex space is called *Schwarz* if for every absolutely convex zero neighbourhood U there exists a zero neighbourhood V so that for each $\varepsilon > 0$ exist $f_1, \ldots, f_n \in V$ such that $V \subseteq \bigcup_{j=1}^n (f_j + \varepsilon U)$. In particular, $A^{-\infty}$ is a Montel space (see [14, pp. 61–62]).

Remark 1.5.2. The Korenblum space is a locally convex algebra. In fact, $A^{-\infty}$ is a locally convex space and an algebra because it is a vector space over the field $\mathbb{C}$ which has defined a product fg for each $f, g \in A^{-\infty}$ that verifies:

- $(fg)h = f(gh)$,

- $(f+g)h = fh + gh, \quad f(g+h) = fg + fh,$
- $(fg)\alpha = f(g\alpha) = (f\alpha)g,$

for any $f, g, h \in A^{-\infty}$, $\alpha \in \mathbb{C}$. In order to see that $A^{-\infty}$ is a locally convex algebra, one has to check that the multiplication is separately continuous, because, then, due to the topological properties of the space, it will be continuous (see [32] and [42, pp. 202]).

Multiplication is separately continous, because if we fix $f \in A^{-\infty}$, there exists $k \in \mathbb{N}$ with $f \in H_k^\infty$. For any $g \in H_n^\infty$,

$$\|fg\|_{n+k} = \sup_{z\in\mathbb{D}}(1-|z|)^{n+k}|fg(z)| = \sup_{z\in\mathbb{D}}(1-|z|)^{n}|g(z)|(1-|z|)^{k}|f(z)| < \infty\ .$$

That is, for each $n \in \mathbb{N}$, there exist $m(= n+k) \in \mathbb{N}$ such that the multiplication with f is a continuous operator from H_n^∞ to H_m^∞ and, then, it is continuous on $A^{-\infty}$. Analogously, one can prove the continuity with the second element fixed.

1.5.1 The $A_+^{-\alpha}$ space

Fix $\alpha \geq 0$. Take $\alpha_n := \alpha + \frac{1}{n}$. We set

$$A_+^{-\alpha} := \bigcap_{n\in\mathbb{N}} H_{\alpha_n}^\infty\ .$$

The space $A_+^{-\alpha}$ is an intersection of Banach spaces, and it is also a Fréchet space when endowed with the locally convex topology generated by the increasing sequence of (semi)norms

$$||f||_{\alpha_n} := \sup_{z\in\mathbb{D}}(1-|z|)^{\alpha_n}|f(z)|\,.$$

This space is also a projective limit, endowed with the projective limit topology: $\mathrm{proj}_n H_{\alpha_n}^\infty$. It is the coarsest topology for which the inclusion $A_+^{-\alpha} \hookrightarrow H_{\alpha_n}^\infty$ is continuous for all $n \in \mathbb{N}$.

Remark 1.5.3. Notice that $\bigcap_{n\in\mathbb{N}} H_{\alpha_n}^\infty = \bigcap_{n\in\mathbb{N}} H_{\alpha_n}^0$. This occurs because the inclusion $H_{\alpha_{n+1}}^\infty \subseteq H_{\alpha_n}^0$ holds for each $n \in \mathbb{N}$ (see Proposition 1.4.3).

Proposition 1.5.4. *For every $\alpha \geq 0$, $n \in \mathbb{N}$, $A_+^{-\alpha}$ is dense in $H_{\alpha_n}^0$.*

Proof. If $\mathcal{P} = \mathcal{P}(\mathbb{D})$ denotes the space of the polynomials over $\mathbb{D}$, we know that $\mathcal{P} \subset H^\infty \subset H_{\alpha_n}^\infty$, for every $\alpha_n = \alpha + \frac{1}{n}$, where $\alpha \geq 0$ and $n \in \mathbb{N}$, since for all $z \in \mathbb{D}$, $(1-|z|)^{\alpha_n}|f(z)| \leq ||f||_\infty$, when $f \in H^\infty$. Thus,

$$\mathcal{P} \subset \bigcap_{n\in\mathbb{N}} H_{\alpha_n}^\infty \ \left(= \bigcap_{n\in\mathbb{N}} H_{\alpha_n}^0 \right)\ .$$

By Lemma 1.4.2, $\mathcal{P}$ is dense in $H_{\alpha_n}^0$ for every $n \in \mathbb{N}$. Now, as $A_+^{-\alpha}$ is Fréchet and $\mathcal{P} \subset A_+^{-\alpha} \subset H_{\alpha_n}^0$, it follows that $\mathcal{P}$ is dense in $A_+^{-\alpha}$. So, $A_+^{-\alpha}$ is dense in $H_{\alpha_n}^0$. ☐

1.5.2 The $A_{-}^{-\alpha}$ space

Fix $\alpha > 0$. Take $\alpha_n := \alpha - \frac{1}{n}$, where $n \geq n_0$ such that $\alpha - \frac{1}{n_0} > 0$. We set

$$A_{-}^{-\alpha} := \bigcup_{n \in \mathbb{N}} H_{\alpha_n}^{\infty} .$$

The space $A_{-}^{-\alpha}$, endowed with the inductive limit topology, that is, the finest locally convex topology on $\bigcup_n H_{\alpha_n}^{\infty}$ such that the inclusion $H_{\alpha_n}^{\infty} \hookrightarrow A_{-}^{-\alpha}$ is continuous, is a regular LB-space (see Proposition 5.2.2). So,

$$A_{-}^{-\alpha} := \operatorname*{ind}_{n} H_{\alpha_n}^{\infty} .$$

Notice that when $\alpha = \infty$, $A_{-}^{-\alpha} = A^{-\infty}$.

Remark 1.5.5. Notice that $\bigcup_{n \in \mathbb{N}} H_{\alpha_n}^{\infty} = \bigcup_{n \in \mathbb{N}} H_{\alpha_n}^{0}$. This occurs because the inclusions $H_{\alpha_n}^{0} \subseteq H_{\alpha_n}^{\infty} \subseteq H_{\alpha_{n+1}}^{0}$ hold for all $n \in \mathbb{N}$, as we have seen in Lemma 1.4.3.

We prevent the reader to pay attention to the implicit signification of the notation α_n, which depends on the context. The exponents α_n of this section are different from those in Section 1.5.1.

Chapter 2

Weighted composition operators

Consider an analytic map $\varphi\colon \mathbb{D} \to \mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$, and $\psi \in H(\mathbb{D})$. The *weighted composition operator* is defined by

$$W_{\psi,\varphi} f(z) := \psi(z) f(\varphi(z)), \quad z \in \mathbb{D} .$$

In this definition, φ is called the *symbol* and ψ the *weight*. Observe that $W_{\psi,\varphi} = M_\psi C_\varphi$ where M_ψ is the multiplication operator defined by $M_\psi(f) := \psi \cdot f$ and C_φ is the composition operator defined by $C_\varphi(f) := f \circ \varphi$. All over the text, φ and ψ denote as stated above.

In this chapter we investigate the properties of continuity, compactness and invertibility of the weighted composition operator on the Korenblum space $A^{-\infty}$ and the Korenblum type spaces $A_+^{-\alpha}$ and $A_-^{-\alpha}$. The main results of this chapter are collected in [31, Section 2].

The continuity and compactness of the operator $W_{\psi,\varphi}$ on weighted Banach spaces has been deeply studied by Contreras and Hernández-Díaz in [26] even in a more general context; see also [19] and [38]. Below we state some characterizations of continuity and compactness described in [26].

Proposition 2.0.1 ([26], Proposition 3.1)**.** *Let v and w be weights. Then the operator $W_{\psi,\varphi}\colon H_v^\infty \to H_w^\infty$ is continuous if, and only if,* $\sup_{z\in\mathbb{D}} |\psi(z)|w(z)/\tilde{v}(\varphi(z)) < \infty$.

If v is essential, then the operator $W_{\psi,\varphi}\colon H_v^\infty \to H_w^\infty$ is continuous if, and only if, $\sup_{z\in\mathbb{D}} |\psi(z)|w(z)/v(\varphi(z)) < \infty$.

Proposition 2.0.2 ([26], Proposition 3.2)**.** *Let v and w be typical weights. Then the operator $W_{\psi,\varphi}\colon H_v^0 \to H_w^0$ is continuous if, and only if, $\psi \in H_w^0$ and* $\sup_{z\in\mathbb{D}} |\psi(z)|w(z)/\tilde{v}(\varphi(z)) < \infty$.

If v is essential, then the operator $W_{\psi,\varphi}\colon H_v^0 \to H_w^0$ is continuous if, and only if, $\psi \in H_w^0$ and $\sup_{z\in\mathbb{D}} |\psi(z)|w(z)/v(\varphi(z)) < \infty$.

Proposition 2.0.3 ([26], Corollary 4.3)**.** *Let v and w be weights. Then the operator $W_{\psi,\varphi}\colon H_v^\infty \to H_w^\infty$ is compact if, and only if, $\psi \in H_w^\infty$ and*

$$\lim_{r\longrightarrow 1^-} \sup_{|\varphi(z)|>r} \frac{|\psi(z)|w(z)}{\tilde{v}(\varphi(z))} = 0.$$

When v is essential, $\tilde{v}$ can be replaced by v.

Proposition 2.0.4 ([26], Corollary 4.5)**.** *Let v and w be typical weights. Then the operator $W_{\psi,\varphi}\colon H_v^0 \to H_w^0$ is compact if, and only if,*

$$\limsup_{|z|\longrightarrow 1^-} \frac{|\psi(z)|w(z)}{\tilde{v}(\varphi(z))} = 0.$$

When v is essential, $\tilde{v}$ can be replaced by v.

2.1 Continuous weighted composition operators

This section is focused in the study of the continuity of $W_{\psi,\varphi}$ in the Korenblum type spaces. For the spaces $A_+^{-\alpha}$ and $A_-^{-\alpha}$ we obtain sufficient conditions for the continuity and for the Korenblum space we get a characterization. Recall that every bounded operator is continuous.

Rewriting Propositions 2.0.1 and 2.0.2 for our weights v_α (which satisfy $v_\alpha = \tilde{v}_\alpha$, that is, they are essential), we get the following results.

Proposition 2.1.1. *Let $\alpha, \beta \geq 0$. Then the operator $W_{\psi,\varphi} \colon H_\alpha^\infty \to H_\beta^\infty$ is bounded if, and only if,*

$$\sup_{z \in \mathbb{D}} \frac{|\psi(z)| v_\beta(z)}{v_\alpha(\varphi(z))} < \infty \ .$$

Proposition 2.1.2. *Let $\alpha, \beta \geq 0$. Then the operator $W_{\psi,\varphi} \colon H_\alpha^0 \to H_\beta^0$ is bounded if, and only if, $\psi \in H_\beta^0$ and*

$$\sup_{z \in \mathbb{D}} \frac{|\psi(z)| v_\beta(z)}{v_\alpha(\varphi(z))} < \infty \ .$$

2.1.1 Continuity on $A_+^{-\alpha}$

The following theorem characterizes the continuity of the operator $W_{\psi,\varphi} \colon A_+^{-\alpha} \to A_+^{-\alpha}$. In order to state the theorem, we have applied Proposition 2.1.2 and Lemma 1.2.3. This lemma can be used here because the space $A_+^{-\alpha}$ is defined as the projective limit of the Banach spaces $H_{\alpha_n}^\infty$, that is, the projective limit of the Banach spaces $H_{\alpha_n}^0$, with the inclusion $H_{\alpha_{n+1}}^0 \hookrightarrow H_{\alpha_n}^0$ continuous, and holding that the space $A_+^{-\alpha}$ is dense in $H_{\alpha_n}^0$ for all $n \geq 0$ (Proposition 1.5.4).

Theorem 2.1.3. *Let $\alpha \geq 0$, $\psi, \varphi \in H(\mathbb{D})$ and $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. The operator $W_{\psi,\varphi} \colon A_+^{-\alpha} \to A_+^{-\alpha}$ is continuous if, and only if, $\psi \in A_+^{-\alpha}$ and for each $n \in \mathbb{N}$ there exists $m > n$ such that*

$$\sup_{z \in \mathbb{D}} \frac{|\psi(z)| v_{\alpha_n}(z)}{v_{\alpha_m}(\varphi(z))} < \infty \ .$$

Proof. Applying Lemma 1.2.3 we have that $W_{\psi,\varphi} \colon A_+^{-\alpha} \to A_+^{-\alpha}$ is continuous if, and only if, for every $n \in \mathbb{N}$ there exists $m > n$ such that $W_{\psi,\varphi}$ has a unique continuous extension $\tilde{W} \colon H_{\alpha_m}^0 \to H_{\alpha_n}^0$. Since $A_+^{-\alpha}$ is dense in $H_{\alpha_m}^0$, for each $f \in H_{\alpha_m}^0$ there exists a sequence $(f_i)_i \subseteq A_+^{-\alpha}$ such that $f_i \longrightarrow f$. Then, by the continuity of $\tilde{W}$, we have that $\tilde{W}(f_i) = \psi(f_i \circ \varphi)$ tends to $\tilde{W}(f) \in H_{\alpha_n}^0$. Thus, $\psi(z) f_j(\varphi(z)) \longrightarrow \tilde{W} f(z)$ for all $z \in \mathbb{D}$, and $\psi(z) f_j(\varphi(z)) \longrightarrow \psi(f \circ \varphi)(z)$ for all $z \in \mathbb{D}$. This means that $\tilde{W} = W_{\psi,\varphi}$. Now, applying Proposition 2.1.2, the preceding holds if, and only if, $\psi \in H_{\alpha_n}^0$ for all $n \in \mathbb{N}$, that is, $\psi \in A_+^{-\alpha}$; and $\sup_{z \in \mathbb{D}} |\psi(z)| v_{\alpha_n}(z) / v_{\alpha_m}(\varphi(z)) < \infty$. □

Notice that we have used Proposition 2.1.2 instead of Proposition 2.1.1 because one of the conditions in Lemma 1.2.3 is the density, and we have that $A_+^{-\alpha}$ is dense in $H_{\alpha_n}^0$ (Proposition 1.5.4), but not in $H_{\alpha_n}^\infty$.

Proposition 2.1.4. *Consider $\psi \in H(\mathbb{D})$ and $\alpha \geq 0$. The operator M_ψ is continuous on $A_+^{-\alpha}$ if, and only if, $\psi \in A_+^{-0}$.*

Proof. Applying Theorem 2.1.3 to M_ψ we have that $M_\psi \in \mathcal{L}(A_+^{-\alpha})$ if, and only if, for every $n \in \mathbb{N}$ there exists $m > n$ such that $\psi \in H^\infty_{\frac{1}{n}-\frac{1}{m}}$.

Assume $\psi \neq 0$. Notice $\frac{1}{n} - \frac{1}{m} < \frac{1}{n}$. By Lemma 1.4.3, $H^0_{\frac{1}{n}-\frac{1}{m}} \subseteq H^0_{\frac{1}{n}}$. So, if for every $n \in \mathbb{N}$ there is some $m \in \mathbb{N}$ with $\psi \in H^0_{\frac{1}{n}-\frac{1}{m}}$ then for all $n \in \mathbb{N}$, $\psi \in H^0_{\frac{1}{n}}$, that is, $\psi \in A_+^{-0}$.

On the other hand, if $\psi \in A_+^{-0}$, for every $n \in \mathbb{N}$ exists $m > n$ such that $\psi \in H^0_{\frac{1}{nm}}$, and $\frac{1}{nm} \leq \frac{m-n}{nm} = \frac{1}{n} - \frac{1}{m}$ so, $\psi \in H^0_{\frac{1}{n}-\frac{1}{m}}$. Thus, $M_\psi \in \mathcal{L}(A_+^{-\alpha})$ if, and only if, $\psi \in A_+^{-0}$. □

The following theorem shows a necessary condition for the continuity of $W_{\psi,\varphi}$ on $A_+^{-\alpha}$.

Theorem 2.1.5. *Let $\alpha \geq 0$, $\psi, \varphi \in H(\mathbb{D})$ and $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. If $\psi \in A_+^{-0}$, then $W_{\psi,\varphi} \in \mathcal{L}(A_+^{-\alpha})$.*

Proof. As for each $n \in \mathbb{N}$, v_{α_n} is typical, by [19, Theorem 2.3] we have $C_\varphi \colon H^\infty_{\alpha_n} \to H^\infty_{\alpha_n}$ and $C_\varphi \colon H^0_{\alpha_n} \to H^0_{\alpha_n}$ are continuous for all $n \in \mathbb{N}$. Also, by Lemma 1.4.3, for every $n, m \in \mathbb{N}$ with $n < m$ (and so, $\alpha_m < \alpha_n$), $C_\varphi \colon H^\infty_{\alpha_m} \longrightarrow H^\infty_{\alpha_n}$ and $C_\varphi \colon H^0_{\alpha_m} \longrightarrow H^0_{\alpha_n}$ are continuous. Now, as for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ ($n < m$) such that the composition operator has a unique linear continuous extension $C_\varphi \colon H^0_{\alpha_m} \longrightarrow H^0_{\alpha_n}$, by Lemma 1.2.3, we get $C_\varphi \in \mathcal{L}(A_+^{-\alpha})$ for any $\alpha \geq 0$.

Finally, since the composition of two continuous operators is also continuous, by applying Proposition 2.1.4 we get the result. □

The next example shows that the converse of Theorem 2.1.5 does not hold.

Example 2.1.6. Set $\alpha > 0$. If we take $\varphi(z) = z/2$, for all $z \in \mathbb{D}$, and some $\psi \in A_+^{-\alpha} \setminus A_+^{-0}$, it holds that for every $n \in \mathbb{N}$ there exists some $m > n$ such that

$$\sup_{z\in\mathbb{D}} |\psi(z)| \frac{(1-|z|)^{\alpha+\frac{1}{n}}}{(1-|z|/2)^{\alpha+\frac{1}{m}}} \leq 2^{\alpha+\frac{1}{m}} \sup_{z\in\mathbb{D}} |\psi(z)|(1-|z|)^{\alpha+\frac{1}{n}} < \infty\ .$$

This way, by Theorem 2.1.3, $W_{\psi,\varphi}$, with such symbol and weight, is continuous on $A_+^{-\alpha}$ and $\psi \notin A_+^{-0}$.

2.1.2 Continuity on $A_-^{-\alpha}$ and $A^{-\infty}$

The following theorems characterize the continuity of the weighted composition operators on $A_-^{-\alpha}$ and $A^{-\infty}$. We have stated the theorems using Proposition 2.1.1 and Lemma 1.2.2. This lemma can be used in both cases because they are inductive limits of Banach spaces.

Theorem 2.1.7. *Let $\alpha > 0$, $\psi, \varphi \in H(\mathbb{D})$ and $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. The operator $W_{\psi,\varphi} \colon A_-^{-\alpha} \to A_-^{-\alpha}$ is continuous if, and only if, for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that*

$$\sup_{z\in\mathbb{D}} \frac{|\psi(z)| v_{\alpha_m}(z)}{v_{\alpha_n}(\varphi(z))} < \infty\ .$$

Theorem 2.1.8. *Let $\psi, \varphi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. The operator $W_{\psi,\varphi} \colon A^{-\infty} \to A^{-\infty}$ is continuous if, and only if, for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that*

$$\sup_{z\in\mathbb{D}} \frac{|\psi(z)| v_m(z)}{v_n(\varphi(z))} < \infty\ .$$

The following propositions are used in the proofs of the two theorems below.

Proposition 2.1.9. *Let $\varphi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Then $C_\varphi \in \mathcal{L}(A^{-\infty})$ and $C_\varphi \in \mathcal{L}(A_-^{-\alpha})$ for every $\alpha > 0$.*

Proof. By [19, Theorem 2.3] and Lemma 1.4.3, we obtain that for every $n \in \mathbb{N}$ exists $m \in \mathbb{N}$, $(m \geq n)$ such that $C_\varphi \colon H^\infty_{\alpha_n} \to H^\infty_{\alpha_m}$ and $C_\varphi \colon H^\infty_n \to H^\infty_m$ are continuous. Now, if we take $\psi \equiv 1$, by using Proposition 2.1.1, Theorem 2.1.7 and Theorem 2.1.8 we get $C_\varphi \colon A_-^{-\alpha} \to A_-^{-\alpha}$ and $C_\varphi \colon A^{-\infty} \to A^{-\infty}$ are both continuous. □

Proposition 2.1.10. *Consider $\psi \in H(\mathbb{D})$. The operator $M_\psi \in \mathcal{L}(A_-^{-\alpha})$ if, and only if, $\psi \in A_+^{-0}$.*

Proof. If we apply Theorem 2.1.7 to M_ψ we have $M_\psi \in \mathcal{L}(A_-^{-\alpha})$ iff for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\psi \in H^0_{\frac{1}{n}-\frac{1}{m}}$. The rest of the proof is analogous to the proof of Proposition 2.1.4. □

Theorem 2.1.11. *Let $0 < \alpha < \infty$, $\varphi, \psi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. If $\psi \in A_+^{-0}$ then $W_{\psi,\varphi} \in \mathcal{L}(A_-^{-\alpha})$.*

Proof. Recall that the operator $W_{\psi,\varphi}$ is the composition of the operators M_ψ and C_φ. Then, if C_φ and M_ψ are both continuous and well defined, the operator $W_{\psi,\varphi}$ will be continuous and well defined as well.

Proposition 2.1.9 sets that $C_\varphi \in \mathcal{L}(A_-^{-\alpha})$ and, by Theorem 2.1.10, we have that if $\psi \in A_+^{-0}$ then M_ψ is also continuous and well defined. Therefore, when $\psi \in A_+^{-0}$ the operator $W_{\psi,\varphi} \in \mathcal{L}(A_-^{-\alpha})$. □

The converse is not true. It can be easily checked taking $\varphi(z) = z/2$, as in Example 2.1.6.

Theorem 2.1.12. *Let $\psi, \varphi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Then $W_{\psi,\varphi} \in \mathcal{L}(A^{-\infty})$ if, and only if, $\psi \in A^{-\infty}$.*

Proof. Assume $W_{\psi,\varphi}$ is continuous on $A^{-\infty}$. The constant function $\mathbf{1} \in A^{-\infty}$ so, $W_{\psi,\varphi}(\mathbf{1})(z) = \psi(z)$ for all $z \in \mathbb{D}$. Then $\psi \in A^{-\infty}$.

Conversely, assume now $\psi \in A^{-\infty}$. Thus, there exists some $k \in \mathbb{N}$ such that $\psi \in H^\infty_j$ for any $j \geq k$. So, for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ $(m \geq n)$ such that $\psi \in H^\infty_{m-n}$. (It suffices to take m with $m - n \geq k$). Applying Proposition 2.1.1 and Theorem 2.1.8 to M_ψ we obtain M_ψ is continuous on $A^{-\infty}$. Therefore, by Proposition 2.1.9, the result is obtained. □

2.2 Compact weighted composition operators

In this section we study the compactness of $W_{\psi,\varphi}$ in the Korenblum type spaces. We obtain a necessary condition where the compactness of $W_{\psi,\varphi}$ on these spaces implies its compactness on some Banach spaces H^0_β. Also, we prove that if the symbol satisfies that its image is contained in a closed disc centered in zero inside the unit disc then $W_{\psi,\varphi}$ is compact whenever it is continuous on any of the Korenblum type spaces.

Recall that for all Montel spaces E and $T \in \mathcal{L}(E)$, T is compact if, and only if, it is bounded. Taking this into account we can write the following characterizations.

First, combining Lemma 1.2.3 with Proposition 2.1.2, we obtain the next characterization of the compactness of $W_{\psi,\varphi} \colon A_+^{-\alpha} \to A_+^{-\alpha}$.

Theorem 2.2.1. *Let $\alpha \geq 0$, $\psi, \varphi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. The operator $W_{\psi,\varphi} \colon A_+^{-\alpha} \to A_+^{-\alpha}$ is compact if, and only if, it is continuous and there exists $m \in \mathbb{N}$ such that for each $n \in \mathbb{N}$*

$$\sup_{z\in\mathbb{D}} \frac{|\psi(z)| v_{\alpha_n}(z)}{v_{\alpha_m}(\varphi(z))} < \infty \ .$$

On the other hand, combining Lemma 1.2.2 with Proposition 2.1.1, we can state the two following results, which are a characterization of the compactness of the operators $W_{\psi,\varphi} \colon A_-^{-\alpha} \to A_-^{-\alpha}$ and $W_{\psi,\varphi} \colon A^{-\infty} \to A^{-\infty}$.

Theorem 2.2.2. *Let $\alpha > 0$, $\psi, \varphi \in H(\mathbb{D})$ and $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. The operator $W_{\psi,\varphi} \colon A_-^{-\alpha} \to A_-^{-\alpha}$ is compact if, and only if, it is continuous and there exists some $m \in \mathbb{N}$ such that for every $n \geq m$*

$$\sup_{z\in\mathbb{D}} \frac{|\psi(z)| v_{\alpha_m}(z)}{v_{\alpha_n}(\varphi(z))} < \infty \ .$$

Theorem 2.2.3. *Let $\psi, \varphi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Then $W_{\psi,\varphi} \colon A^{-\infty} \to A^{-\infty}$ is compact if, and only if, it is continuous and there exists $m \in \mathbb{N}$ such that for all $n \geq m$*

$$\sup_{z\in\mathbb{D}} \frac{|\psi(z)| v_m(z)}{v_n(\varphi(z))} < \infty \ .$$

Corollary 2.2.4. *Let $\psi, \varphi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$.*

(i) Let $\alpha \geq 0$. If $W_{\psi,\varphi} \colon A_+^{-\alpha} \to A_+^{-\alpha}$ is compact, then there exists $n \in \mathbb{N}$ such that $W_{\psi,\varphi} \colon H^0_{\alpha+\frac{1}{n}} \to H^0_{\alpha+\frac{1}{n}}$ is compact.

(ii) Let $\alpha > 0$. If $W_{\psi,\varphi} \colon A_-^{-\alpha} \to A_-^{-\alpha}$ is compact, then there exists $n \in \mathbb{N}$ such that $W_{\psi,\varphi} \colon H^0_{\alpha-\frac{1}{n}} \to H^0_{\alpha-\frac{1}{n}}$ is compact.

(iii) If $W_{\psi,\varphi} \colon A^{-\infty} \to A^{-\infty}$ is compact, then there exists $n \in \mathbb{N}$ such that $W_{\psi,\varphi} \colon H^0_n \to H^0_n$ is compact.

Proof. This is a direct consequence of Theorems 2.2.1, 2.2.2 and 2.2.3, and Proposition 2.0.4. □

Corollary 2.2.5. *Assume that there exists an r, $0 < r < 1$, such that $|\varphi(z)| \leq r$ for all $z \in \mathbb{D}$.*

(i) Let $\alpha \geq 0$. If $W_{\psi,\varphi} \colon A_+^{-\alpha} \to A_+^{-\alpha}$ is continuous, then it is compact.

(ii) Let $\alpha > 0$. If $W_{\psi,\varphi} \colon A_-^{-\alpha} \to A_-^{-\alpha}$ is continuous, then it is compact.

(iii) If $W_{\psi,\varphi} \colon A^{-\infty} \to A^{-\infty}$ is continuous, then it is compact.

Corollary 2.2.6. *Assume that M_ψ is continuous on $A_+^{-\alpha}$, $A_-^{-\alpha}$ or $A^{-\infty}$. If M_ψ is compact, then $\psi \equiv 0$.*

Proof. For $A_+^{-\alpha}$ and $A_-^{-\alpha}$ it is already shown in proof of Propositon 2.1.4 and 2.1.10.

In the case of $A^{-\infty}$, we have that if M_ψ is compact, by Theorem 2.2.3 there exists $m \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq m$, $\sup_{z\in\mathbb{D}} |\psi(z)|(1-|z|)^{m-n} < \infty$. But, for all $z \in \mathbb{D}$, $\lim_n (1-|z|)^{m-n} = \infty$. □

Example 2.2.7 ([19], Corollary 3.2)**.** Let v and w be weights. If there exists an r, $0 < r < 1$, such that $|\varphi(z)| \leq r$ for all $z \in \mathbb{D}$, then $C_\varphi \colon H_v^\infty \to H_w^\infty$ is compact.

2.3 Invertible weighted composition operators

In this section we study the invertibility of weighted composition operators on $A_+^{-\alpha}$, $A_-^{-\alpha}$ and $A^{-\infty}$ as a consequence of the following results due to Bourdon.

Theorem 2.3.1 ([23], Theorem 2.2)**.** *Suppose that X is a set of analytic functions on $\mathbb{D}$ such that*

(i) $W_{\psi,\varphi}$ maps X to X,

(ii) X contains a nonzero constant function,

(iii) X contains a function of the form $z \mapsto z + c$ for some constant c,

(iv) there is a dense subset S of the unit circle such that for each point in S there is a function in X that does not extend analytically to a neighbourhood of that point.

If $W_{\psi,\varphi}\colon X \to X$ is invertible, then φ is an automorphism of $\mathbb{D}$.

Corollary 2.3.2 ([23], Corollary 2.3)**.** *If X, ψ and φ satisfy the hypotheses of Theorem 2.3.1 and X is automorphism invariant, i.e., $f \circ \phi \in X$ for all automorphism ϕ of $\mathbb{D}$, then $W_{\psi,\varphi}$ is invertible on X if, and only if, φ is an automorphism of $\mathbb{D}$ and ψ as well as $1/\psi$ are multipliers of X.*

Remark 2.3.3. That the space $H^\infty_{\alpha_n}$ or H^∞_n is automorphism invariant is equivalent to the composition operator $C_\phi\colon H^\infty_{\alpha_n} \to H^\infty_{\alpha_n}$ or $C_\phi\colon H^\infty_n \to H^\infty_n$ being well defined for all $\phi \in \mathrm{Aut}(\mathbb{D})$. By [19, Theorem 2.3] the composition operators $C_\phi\colon H^\infty_{\alpha_n} \to H^\infty_{\alpha_n}$ and $C_\phi\colon H^\infty_n \to H^\infty_n$ are well defined for all $\phi \in \mathrm{Aut}(\mathbb{D})$. Now, since $H^\infty_{\alpha_n}$ and H^∞_n are automorphism invariant for all $n \in \mathbb{N}$, then $A_+^{-\alpha}$, $A_-^{-\alpha}$ and $A^{-\infty}$ are too.

Recall that a function g is a *multiplier* of a set X provided that $gf \in X$ whenever $f \in X$. As a consequence of Closed Graph Theorem, if X is a Fréchet space or an LB-space, g is a multiplier if, and only if, the multiplication operator M_g is continuous on X. In our cases this holds because $A_+^{-\alpha}$ is a Fréchet space and $A_-^{-\alpha}$ and $A^{-\infty}$ are LB-spaces.

Lemma 2.3.4. *Let $\alpha > 0$, $a \in \partial\mathbb{D}$. Consider the function $g_{\alpha,a}(z) := 1/(a-z)^\alpha$. Then $g_{\alpha,a} \in H^\infty_\alpha$ and does not extend analytically to any neighbourhood of a.*

Proof. Since $|a - z| \geq 1 - |z|$ for each $z \in \mathbb{D}$, it follows that $\sup_{z\in\mathbb{D}}(1-|z|)^\alpha |g_{\alpha,a}(z)| \leq 1$. Besides, notice $g_{\alpha,a}$ has a pole at a, so it does not extend analytically to a neighbourhood of a. □

Now, we can state the characterizations of the invertibility of the weighted composition operator.

Theorem 2.3.5. *Assume $\psi, \varphi \in H(\mathbb{D})$ and $\varphi(\mathbb{D}) \subseteq \mathbb{D}$.*

(i) Let $\alpha \geq 0$. The operator $W_{\psi,\varphi}$ is invertible on $A_+^{-\alpha}$ if, and only if, φ is an automorphism of $\mathbb{D}$ and $\psi, 1/\psi \in A_+^{-0}$.

(ii) Let $\alpha > 0$. The operator $W_{\psi,\varphi}$ is invertible on $A_-^{-\alpha}$ if, and only if, φ is an automorphism of $\mathbb{D}$ and $\psi, 1/\psi \in A_+^{-0}$.

(iii) The operator $W_{\psi,\varphi}$ is invertible on $A^{-\infty}$ if, and only if, φ is an automorphism of $\mathbb{D}$ and $\psi, 1/\psi \in A^{-\infty}$.

Proof. The operator $W_{\psi,\varphi}$ satisfies hypotesis *(i)* of Theorem 2.3.1 whenever it is continuous, and our spaces $A_+^{-\alpha}$, $A_-^{-\alpha}$ and $A^{-\infty}$ verify hypothesis *(ii)* of Theorem 2.3.1. Moreover, they are linear spaces which contain the constants and the polynomials, thus Theorem 2.3.1*(iii)* is equally satisfied. Therefore, just the last hypothesis is left.

If $\alpha > 0$ then $H_\alpha^\infty \subset H_{\alpha_n}^\infty$ for all $n \in \mathbb{N}$, where $\alpha_n = \alpha + \frac{1}{n}$, we get that the function $g_{\alpha,a}$ of Lemma 2.3.4 belongs to $A_+^{-\alpha}$. Also, applying Lemma 2.3.4 to some $\alpha - \frac{1}{n}$, $n \in \mathbb{N}$, we obtain $g_{\alpha,a} \in A_-^{-\alpha}$. And, for any $\alpha \geq 0$ there exists $n \in \mathbb{N}$, $n \geq \alpha$ such that $H_\alpha^\infty \subseteq H_n^\infty \subset A^{-\infty}$. Therefore, by Lemma 2.3.4 hypothesis *(iv)* of Theorem 2.3.1 is satisfied. On the other hand, if we take $\alpha = 0$, we can work with a suitable branch of $\log(a - z)$ instead of $g_{\alpha,a}$. The function defined by each branch of $\log(a - z)$ is in the Bloch space of $\mathbb{D}$, $\mathcal{B}$, and it holds that $\mathcal{B} \subset A_+^{-\alpha}, A_-^{-\alpha}, A^{-\infty}$ (see [45, p. 82]). With this, hypothesis *(iv)* of Theorem 2.3.1 would be verified too.

Now, by Corollary 2.3.2, $W_{\psi,\varphi}$ is invertible if, and only if, φ is an automorphism of $\mathbb{D}$ and ψ and $1/\psi$ are multipliers of $A_+^{-\alpha}$ or, what is the same, operators M_ψ and $M_{1/\psi}$ are continuous on $A_+^{-\alpha}$. Now, applying Proposition 2.1.4 we get the result.

Analogously, we can prove the cases of $A_-^{-\alpha}$ and $A^{-\infty}$, by applying Proposition 2.1.10 and Theorem 2.1.12. □

Remark 2.3.6. Observe that in Theorem 2.3.5 (iii), ψ and $1/\psi$ must be in $A^{-\infty}$ instead of A_+^{-0}, because in that case, the multiplication operator M_g is continuous whenever $g \in A^{-\infty}$. Nevertheless, although it is enough that $\psi, 1/\psi \in H^\infty$, the next example shows a multiplier, ψ, which is in A_+^{-0} and $A^{-\infty}$, but $\psi \notin H^\infty$ and $1/\psi \in H^\infty$.

Example 2.3.7. Consider the funcion $\psi(z) = \mathrm{Log}(z+1) - 5$ where Log denotes the principal branch of the logarithm. We know ψ is not bounded on $\mathbb{D}$, but $\psi \in \mathcal{B}$, so $\psi \in A_+^{-0}$ and $\psi \in A^{-\infty}$. Now, we see that $1/\psi$ is also in those spaces, that is, ψ satisfies the condition of Theorem 2.3.5.

If we call $z = a + bi$, with $a, b \in [0, 1[$, we have

$$\begin{aligned} |\mathrm{Log}(a + bi + 1) - 5| &= \left|\log\sqrt{(a+1)^2 + b^2} - 5 + i\mathrm{Arg}(a + bi)\right| \\ &= \sqrt{\left(\log\sqrt{(a+1)^2 + b^2} - 5\right)^2 + \mathrm{Arg}^2(a + bi + 1)} \\ &\geq \left|\log\sqrt{(a+1)^2 + b^2} - 5\right| . \end{aligned}$$

Case 1: if we assume $\log\sqrt{(a+1)^2 + b^2} < 0$, then $\left|\log\sqrt{(a+1)^2 + b^2} - 5\right| > 5$.

Case 2: suppose $\log\sqrt{(a+1)^2 + b^2} \geq 0$. It is clear that $|a + 1| \leq |a| + 1 < 2$ so, $(a + 1)^2 + b^2 < 5$. Thus, $0 \leq \log\sqrt{(a+1)^2 + b^2} < \log\sqrt{5} < 2$ (because log is increasing). Therefore, $\left|\log\sqrt{(a+1)^2 + b^2} - 5\right| > 3$.

In conclusion, $|\mathrm{Log}(z + 1) - 5| \geq 3$ so, $\frac{1}{|\mathrm{Log}(z+1)-5|} \leq \frac{1}{3}$ for all $z \in \mathbb{D}$. Then, $1/\psi \in H^\infty$.

Chapter 3

Spectrum

This chapter is focused in the study of the spectrum of the weighted composition operators on the spaces $A_+^{-\alpha}$, $A_-^{-\alpha}$ and $A^{-\infty}$. Thanks to two general versions of two lemmas due to Kamowitz in [34] and two lemmas of Albanese, Bonet and Ricker in [2] and [3], we are able to obtain some information about the spectrum and the point spectrum of $W_{\psi,\varphi}$ in the three Korenblum type spaces. Moreover, we investigate the spectrum of the multiplication operator and the composition operator in the case when the symbol is a rotation. For this last part we use a recent paper of Bonet [16]. The main results of this chapter are presented in [31, Section 3].

3.1 Useful general results

In this section we collect some results that we use along the chapter.

Lemma 3.1.1 ([2], Lemma 2.1). *Let $X = \cap_{n\in\mathbb{N}} X_n$ be a Fréchet space which is the intersection of a sequence of Banach spaces $((X_n, ||\cdot||_n))_{n\in\mathbb{N}}$ satisfying $X_{n+1} \subseteq X_n$ with $||x||_n \le ||x||_{n+1}$ for each $n \in \mathbb{N}$ and $x \in X_{n+1}$. Let $T \in \mathcal{L}(X)$ satisfy that for each $n \in \mathbb{N}$ there exists $T_n \in \mathcal{L}(X_n)$ such that the restriction of T_n to X (resp. of T_n to X_{n+1}) coincides with T (resp. with T_{n+1}). Then, $\sigma(T, X) \subseteq \cup_{n\in\mathbb{N}} \sigma(T_n, X_n)$.*

Lemma 3.1.2 ([3], Lemma 5.2). *Let $E = ind_n(E_n, ||\cdot||_n)$ be a Hausdorff inductive limit of Banach spaces. Let $T \in \mathcal{L}(E)$ satisfy that for each $n \in \mathbb{N}$ the restriction T_n of T to E_n maps E_n into itself and $T_n \in \mathcal{L}(E_n)$. Then, the following properties are satisfied.*

(i) $\sigma_p(T, E) = \cup_{n\in\mathbb{N}} \sigma_p(T_n, E_n)$.

(ii) $\sigma(T, E) \subseteq \cap_{m\in\mathbb{N}} (\cup_{n=m}^{\infty} \sigma(T_n, E_n))$.

We also need a general version of two lemmas due to Kamowitz [34, Lemmas 2.3 and 2.4]. The first one is proved as in the original version. In the second one, we had to modify some aspects.

Lemma 3.1.3 (General version of [34], Lemma 2.3). *Let E be a space of holomorphic functions containing the polynomials, and being the inclusion $E \subseteq H(\mathbb{D})$ continuous. Consider $\varphi, \psi \in H(\mathbb{D})$, with $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi(0) = 0$ such that $W_{\psi,\varphi}$ acts continuously from E into itself. Then, $\psi(0) \in \sigma(W_{\psi,\varphi})$ and $\psi(0)\varphi'(0)^n \in \sigma(W_{\psi,\varphi})$ for all $n \in \mathbb{N}$.*

Proof. (1) Suppose $\psi(0) \notin \sigma(W_{\psi,\varphi})$. That is, the operator $\psi(0)I - W_{\psi,\varphi}$ is invertible. Then, it is surjective. So, since $1 \in E$, there exists one $f \in E$ such that $\psi(0)f(z) - \psi(z)f(\varphi(z)) = 1$ for all $z \in \mathbb{D}$. Now, evaluating at $z = 0$, we get a contradiction.

(2) If $\varphi'(0) = 0$, then $0 \in \sigma(W_{\psi,\varphi})$. Suppose not. Then $W_{\psi,\varphi}$ is invertible. Let $f \in E$ be such that $W_{\psi,\varphi}(f)(z) = z$ for all $z \in \mathbb{D}$. Then, $\psi(0)f(0) = 0$ and, after differentiating, $\psi'(0)f(0) = 1$. This leads to $\psi(0) = 0$, which means $\psi(0) \notin \sigma(W_{\psi,\varphi})$ against the just proved statement (1).

Therefore, when $\varphi'(0) = 0$, $\psi(0)\varphi'(0)^n = 0 \in \sigma(W_{\psi,\varphi})$ for every positive integer n.

(3) If $\psi(0) = 0$, by (1) we get $W_{\psi,\varphi}$ is not invertible and, $\psi(0)\varphi'(0)^n = 0 \in \sigma(W_{\psi,\varphi})$ for every positive integer n.

(4) Finally, assume $\psi(0)\varphi'(0) \neq 0$. Suppose that for some positive integer n the operator $\psi(0)\varphi'(0)^nI - W_{\psi,\varphi}$ is surjective. So, there exists $f \in E$ with $\psi(0)\varphi'(0)^nf(z) - \psi(z)f(\varphi(z)) = z^n$ for all $z \in \mathbb{D}$.

Write $f(z) = z^m f_0(z)$, where $f_0 \in H(\mathbb{D})$ and $f_0(0) \neq 0$. Then, $f_0(z) = f_0(0) + \mathcal{O}(z)$. Also, let $\psi(z) = \psi(0) + \mathcal{O}(z)$ and $\varphi(z) = \varphi'(0)z + \mathcal{O}(z^2)$.

Then,

$$\psi(0)\varphi'(0)^n f(z) - \psi(z)f(\varphi(z)) = z^n$$

is equivalent to

$$\psi(0)\varphi'(0)^n z^m(f_0(0) + \mathcal{O}(z)) - (\psi(0) + \mathcal{O}(z))(\varphi'(0)^m z^m + \mathcal{O}(z^{m+1}))(f_0(0) + \mathcal{O}(z)) = z^n$$

or

$$\underbrace{\Big(\psi(0)\varphi'(0)^n f_0(0) - \psi(0)\varphi'(0)^m f_0(0)\Big)}_{(*)} z^m + \mathcal{O}(z^{m+1}) = z^n$$

Now, if $m \neq n$, the left side has order m and the right one has order n, a contradiction. On the other hand, if $m = n$, $(*)$ vanishes, and we get that the left side has order $n + 1$ and the right side has order n, which is again a contradiction.

Hence, for each positive integer n, $\psi(0)\varphi'(0)^n \in \sigma(W_{\psi,\varphi})$.

□

Lemma 3.1.4 (General version of [34], Lemma 2.4)**.** *Let E be a space of holomorphic functions containing the polynomials, and being the inclusion $E \subseteq H(\mathbb{D})$ continuous. Consider $\varphi, \psi \in H(\mathbb{D})$, $\psi \not\equiv 0$, $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, $\varphi(0) = 0$ and that φ is not a constant function. If λ is an eigenvalue of $W_{\psi,\varphi}\colon E \to E$, then $\lambda \in \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} \setminus \{0\}$.*

Proof. Suppose λ is an eigenvalue of $W_{\psi,\varphi}$ with $f \in E$ as corresponding eigenvector.

If $\lambda = 0$, $\psi(z)f(\varphi(z)) = 0$ for all $z \in \mathbb{D}$. Define $G := \{z \in \mathbb{D} : \psi(z) \neq 0\}$, which is a non empty open set because $\psi \not\equiv 0$. Then, $f \circ \varphi \equiv 0$ on G. Now, φ cannot be constant on G because if it were, it would be necessarily constant on $\mathbb{D}$, by the Identity Principle, which contradicts the hypothesis. So, φ is not constant on G, which means $\varphi(G)$ is open and non empty. If $f \equiv 0$ on $\varphi(G)$, $f \equiv 0$, which is again a contradiction. Hence, $\lambda \neq 0$.

Write $f(z) = az^m + \mathcal{O}(z^{m+1})$, $m \geq 0$, $\psi(z) = bz^r + \mathcal{O}(z^{r+1})$, $r \geq 0$ and $\varphi(z) = cz^s + \mathcal{O}(z^{s+1})$, $s \geq 0$, where $abc \neq 0$.

Then, $\lambda f(z) = \psi(z)f(\varphi(z))$ becomes

$$\lambda(az^m + \mathcal{O}(z^{m+1})) = (bz^r + \mathcal{O}(z^{r+1}))(a(cz^s + \mathcal{O}(z^{s+1})^m + \mathcal{O}(z^{ms+1})))$$

or

$$a\lambda z^m + \mathcal{O}(z^{m+1}) = abc^m z^{r+ms} + \mathcal{O}(z^{r+ms+1}) \ .$$

Equating powers, we get $m = r + ms$ and $a\lambda = abc^m$.

Since r and m are non negative integers and s is a positive integer, $m = r + ms$ implies that $r = m = 0$ or that $r = 0$ and $s = 1$. In the first case, $b = \psi(0)$ and so, $a\lambda = abc^m$ implies $\lambda = \psi(0)$. On the other hand, if $r = 0$ and $s = 1$, then $b = \psi(0)$, $c = \varphi'(0)$ and $a\lambda = abc^m$ implies $\lambda = \psi(0)\varphi'(0)^m$ for some positive integer m. □

3.2 Essential norm and essential spectral radius

The *spectral radius* of an operator T on a Banach space X is defined as $r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}$. Let us denote by $\mathcal{K}(X) \subseteq \mathcal{L}(X)$ the space of all compact operators on X. The quotient vector space $\mathcal{L}(X)/\mathcal{K}(X)$ is a unital algebra called Calkin algebra. The canonical projection of an operator T to the Calkin algebra is denoted by $\pi(T)$. The *essential spectrum* $\sigma_e(T)$ is the spectrum of $\pi(T)$ in the Calkin algebra. The *essential norm* of an operator $T \in \mathcal{L}(X)$ is defined as $\|T\|_e := \inf\{\|T-K\| : K \in \mathcal{K}(X)\}$, that is, the distance of the operator to the set of compact operators on X. Notice that $\|T\|_e = 0$ if, and only if, T is compact. The essential norm is indeed a norm in the Calkin algebra. The *essential spectral radius*, denoted by $r_e(T)$, is the spectral radius of $\pi(T)$, that is, $r_e(T) = r(\pi(T)) = \max\{|\lambda| : \lambda \in \sigma_e(T)\}$. We write $\sigma_e(T, X)$ and $r_e(T, X)$ if we need to stress the space X. For more information about the essential norm and essential spectral radius, we refer the reader to the book [1, Section 7.5].

Montes-Rodríguez in [38] studied the essential norm of weighted composition operators on weighted Banach spaces of analytic functions and gave formula in terms of the weights and the symbols. We recall the following results of such investigation.

Proposition 3.2.1 ([38], Theorem 2.1)**.** *Let v and w be weights and let $W_{\psi,\varphi}\colon H_v^\infty \to H_w^\infty$ be a weighted composition operator. Then, either $\psi \notin H_w^\infty$ and $\|W_{\psi,\varphi}\|_e = \infty$ or, $\psi \in H_w^\infty$ and*

$$\|W_{\psi,\varphi}\|_e = \lim_{r\longrightarrow 1^-} \sup_{|\varphi(z)|>r} \frac{w(z)}{\tilde{v}(\varphi(z))}|\psi(z)|.$$

Proposition 3.2.2 ([38], Theorem 2.2)**.** *Let v and w be weights and let $W_{\psi,\varphi}\colon H_v^0 \to H_w^0$ be a weighted composition operator. Then, either $\psi \notin H_w^0$ and $\|W_{\psi,\varphi}\|_e = \infty$ or, $\psi \in H_w^0$ and*

$$\|W_{\psi,\varphi}\|_e = \limsup_{|z|\longrightarrow 1^-} \frac{w(z)}{\tilde{v}(\varphi(z))}|\psi(z)|.$$

Proposition 3.2.3. *Let v and w be two typical weights, $\varphi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\psi \in H_w^0$. Then,*

$$\lim_{r\longrightarrow 1^-} \sup_{|\varphi(z)|>r} \frac{w(z)}{\tilde{v}(\varphi(z))}|\psi(z)| = \limsup_{|z|\longrightarrow 1^-} \frac{w(z)}{\tilde{v}(\varphi(z))}|\psi(z)| \ .$$

The equation in Proposition 3.2.3 is exactly the equation (4) that appears in the proof of [38, Theorem 2.2]. The proof can be seen in there.

Taking into account Propositions 3.2.1, 3.2.2 and 3.2.3, we get that the essential norm of the weighted composition operator on the Banach space H_p^∞ or H_p^0 for some $p > 0$, when $\psi \in H_p^0$, is

$$\|W_{\psi,\varphi}\|_e = \limsup_{|z| \longrightarrow 1^-} \frac{(1-|z|)^p}{(1-|\varphi(z)|)^p} |\psi(z)| \ . \tag{3.2.1}$$

Moreover, by [1, Definition 7.46 and Theorem 6.12], its essential spectral radius is the following limit:

$$r_e(W_{\psi,\varphi}, H_p^\infty) = \lim_k ||W_{\psi,\varphi}^k||_e^{\frac{1}{k}} \ , \tag{3.2.2}$$

where

$$\|W_{\psi,\varphi}^k\|_e = \limsup_{|z| \longrightarrow 1^-} \frac{|\psi(z)| \ldots |\psi(\varphi_{k-1}(z))|(1-|z|)^p}{(1-|\varphi_k(z)|)^p} \ ,$$

and $\varphi_k(z)$ denotes the n-th iterate of the symbol: $\varphi_k(z) = (\varphi \overset{(k \text{ times})}{\circ \ldots \circ} \varphi)(z)$.

Remark 3.2.4. In the particular case of the composition operators the constant function **1** belongs to H_p^0, hence we have

$$r_e(C_\varphi, H_p^\infty) = \lim_k \|C_\varphi^k\|_e^{\frac{1}{k}} = \lim_k \|C_{\varphi_k}\|_e^{\frac{1}{k}} = \lim_k \left(\limsup_{|z| \longrightarrow 1^-} \left(\frac{1-|z|}{1-|\varphi_k(z)|} \right)^p \right)^{\frac{1}{k}} .$$

Lemma 3.2.5. *Let $\varphi, \psi \in H(\mathbb{D})$, $\varphi(0) = 0$, φ not constant. Then,*

$$r_e(W_{\psi,\varphi}, H_\beta^\infty) \le r_e(W_{\psi,\varphi}, H_\alpha^\infty)$$

whenever $0 < \alpha \le \beta < \infty$.

Proof. By Schwarz's Lemma, $|\varphi(z)| \le |z|$ for all $z \in \mathbb{D}$. Then, since $0 < \alpha \le \beta < \infty$,

$$\left(\frac{1-|z|}{1-|\varphi(z)|} \right)^\beta \le \left(\frac{1-|z|}{1-|\varphi(z)|} \right)^\alpha$$

for every $z \in \mathbb{D}$.

Thus, considering (3.2.1) and (3.2.2), and taking into account that for any $n \in \mathbb{N}$, φ_n has the same properties than φ, we get that $||W_{\psi,\varphi}^n||_{e,H_\beta^\infty} \le ||W_{\psi,\varphi}^n||_{e,H_\alpha^\infty}$ for all $n \in \mathbb{N}$. □

The following lemma is useful in Section 3.4.

Lemma 3.2.6. *Let $\varphi \in H(\mathbb{D})$ not a rotation with $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi(0) = 0$. Then, there is $r_0 \in]0, 1[$ such that $r_e(C_\varphi, H_p^\infty) < r_0^p$, for each $p > 0$. In particular, $\lim_{p \longrightarrow \infty} r_e(C_\varphi, H_p^\infty) = 0$.*

Proof. If for some $n \in \mathbb{N}$, $|\varphi_n(z)| < 1/2$ for all $z \in \mathbb{D}$, then $|\varphi_j(z)| < 1/2$ for every $z \in \mathbb{D}$, $j \geq n$. Appealing to Remark 3.2.4, we get that

$$r_e(C_\varphi, H_p^\infty) = \lim_k \left(\limsup_{|z| \longrightarrow 1^-} \left(\frac{1-|z|}{1-|\varphi_k(z)|} \right)^p \right)^{\frac{1}{k}} < \lim_k \left(\limsup_{|z| \longrightarrow 1^-} 2(1-|z|)^p \right)^{\frac{1}{k}} = 0,$$

and the statement is clear.

Now, suppose that for every $n \in \mathbb{N}$, φ_n satisfies $|\varphi_n(z)| \geq 1/2$ for some $z \in \mathbb{D}$. Then, by Schwarz Lemma, $|\varphi_j(z)| \geq 1/2$ for every $j = 0, \ldots, n-1$.

By [27, Lemma 7.33] there exists $r_0 \in]0,1[$ such that

$$\frac{1-|z|}{1-|\varphi(z)|} < r_0 \quad \text{for all } |z| \geq 1/2.$$

Thus,

$$\frac{1-|z|}{1-|\varphi_n(z)|} = \prod_{j=0}^{n-1} \frac{1-|\varphi_j(z)|}{1-|\varphi_{j+1}(z)|} \leq r_0^n < r_0 \ .$$

So, for all $n \in \mathbb{N}$,

$$\sup_{|\varphi_n(z)| \geq \frac{1}{2}} \frac{1-|z|}{1-|\varphi_n(z)|} \leq r_0^n \ ,$$

and

$$\sup_{|\varphi_n(z)| \geq \frac{1}{2}} \left(\frac{1-|z|}{1-|\varphi_n(z)|} \right)^p \leq r_0^{np} \ .$$

Now, for $\frac{1}{2} \leq s < 1$, it holds that for all $n \in \mathbb{N}$,

$$\sup_{|\varphi_n(z)| \geq s} \left(\frac{1-|z|}{1-|\varphi_n(z)|} \right)^p \leq r_0^{np} \ .$$

Furthermore,

$$||C_{\varphi_n}||_{e,H_p^\infty} = \inf_{s \in]0,1[} \sup_{|\varphi_n(z)| \geq s} \left(\frac{1-|z|}{1-|\varphi_n(z)|} \right)^p = \lim_{s \to 1^-} \sup_{|\varphi_n(z)| \geq s} \left(\frac{1-|z|}{1-|\varphi_n(z)|} \right)^p \leq r_0^{np} \ ,$$

and

$$||C_{\varphi_n}||_{e,H_p^\infty}^{\frac{1}{n}} \leq r_0^p.$$

□

As a Corollary of Lemma 3.2.6, we obtain the following result, which can be also found in [6].

Corollary 3.2.7 ([6], Theorem 5.1)**.** *Let $\varphi \in H(\mathbb{D})$, with $\varphi(\mathbb{D}) \subset \mathbb{D}$, $0 < |\varphi'(0)| < 1$ and $\varphi(0) = 0$. Then, $r_e(C_\varphi, H_p^\infty) < 1$ for all $p > 0$.*

Proposition 3.2.8. *Let $\varphi \in H(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi(0) = 0$, $0 < |\varphi'(0)| < 1$. Then, $r_e(C_\varphi, H_p^\infty) = r_e(C_\varphi, H_1^\infty)^p$ for each $p > 0$. Moreover, the function $R\colon]0, +\infty[\to [0,1[$ defined by $R(p) = r_e(C_\varphi, H_p^\infty)$ is continuous.*

Proof. Consider the function R defined by:

$$R(p) := \lim_k \left(\limsup_{|z| \longrightarrow 1^-} \left(\frac{1-|z|}{1-|\varphi_k(z)|} \right)^p \right)^{\frac{1}{k}} .$$

Observe that we can rewrite R as:

$$R(p) = \left(\lim_k \left(\limsup_{|z| \longrightarrow 1^-} \frac{1-|z|}{1-|\varphi_k(z)|} \right)^{\frac{1}{k}} \right)^p .$$

Then, since both limits exist, the function R is continuous because

$$\lim_k \left(\limsup_{|z| \longrightarrow 1^-} \frac{1-|z|}{1-|\varphi_k(z)|} \right)^{\frac{1}{k}} = r_e(C_\varphi, H_1^\infty) .$$

□

Corollary 3.2.9. *Let $\varphi \in H(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi(0) = 0$, $0 < |\varphi'(0)| < 1$. Denote the essential spectral radius $r_e(C_\varphi, H_p^\infty) = r_{e,p}$.*

a) If $\alpha > 0$, $\alpha_n := \alpha + \frac{1}{n}$, then $\lim_n r_{e,\alpha_n} = r_{e,\alpha}$.

b) If $\alpha \geq 1$, $\alpha_n := \alpha - \frac{1}{n}$, then $\lim_n r_{e,\alpha_n} = r_{e,\alpha}$.

Proof. The function R in Proposition 3.2.8 can be written as $R(p) = (r_{e,1})^p$ and is continuous. Since $(\alpha_n)_n$ tends to α in both cases, then $R(\alpha_n)$ tends to $R(\alpha)$. In other words, $\lim_n r_{e,\alpha_n} = r_{e,\alpha}$. □

3.3 Spectra of $W_{\psi,\varphi}$

In this section we study the spectrum of the weighted composition operators on the spaces $A_+^{-\alpha}$, $A_+^{-\alpha}$ and $A^{-\infty}$. From the next theorem of Aron and Lindström for the weighted Banach spaces $H_{\alpha_n}^\infty$ we deduce some of the results for the Korenblum type spaces.

Theorem 3.3.1 ([7], Theorem 7)**.** *Let $p > 0$ and suppose φ, not an automorphism, has fixed point $a \in \mathbb{D}$ and $W_{\psi,\varphi} \colon H_p^\infty \to H_p^\infty$ is bounded. Then*

$$\sigma(W_{\psi,\varphi}, H_p^\infty) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_e(W_{\psi,\varphi}, H_p^\infty)\} \cup \{\psi(a)\varphi'(a)^n\}_{n=0}^\infty .$$

The next theorem gives some information about the spectrum of $W_{\psi,\varphi}$ on the Korenblum type spaces.

Theorem 3.3.2. *Let $\alpha \geq 0$, $\psi, \varphi \in H(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi(0) = 0$, $0 < |\varphi'(0)| < 1$. Then,*

(i) if $\alpha_k := \alpha + \frac{1}{k}$ and $W_{\psi,\varphi} \in \mathcal{L}(A_+^{-\alpha})$,

$$\{0\} \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^\infty \subseteq \sigma(W_{\psi,\varphi}, A_+^{-\alpha}) \subseteq \overline{B}(0, \lim_k r_e(W_{\psi,\varphi}, H_{\alpha_k}^\infty)) \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^\infty,$$

(ii) if $\alpha_k := \alpha - \frac{1}{k}$ and $W_{\psi,\varphi} \in \mathcal{L}(A_-^{-\alpha})$,

$$\{0\} \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^\infty \subseteq \sigma(W_{\psi,\varphi}, A_-^{-\alpha}) \subseteq \overline{B}(0, \lim_k r_e(W_{\psi,\varphi}, H_{\alpha_k}^\infty)) \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^\infty,$$

(iii) if $W_{\psi,\varphi} \in \mathcal{L}(A^{-\infty})$*,*

$$\{0\} \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} \subseteq \sigma(W_{\psi,\varphi}, A^{-\infty}) \subseteq \overline{B}(0, \lim_k r_e(W_{\psi,\varphi}, H_k^{\infty})) \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} .$$

Proof. (i) First inclusion: If $0 \notin \sigma(W_{\psi,\varphi}, A_+^{-\alpha})$, the operator $W_{\psi,\varphi} \colon A_+^{-\alpha} \to A_+^{-\alpha}$ would be invertible and, by Theorem 2.3.5 (i), φ would be an automorphism, which is a contradiction. Also, $\{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} \subset \sigma(W_{\psi,\varphi}, A_+^{-\alpha})$ (see Lemma 3.1.3).

Second inclusion: by Lemma 3.1.1 and Theorem 3.3.1,

$$\sigma(W_{\psi,\varphi}, A_+^{-\alpha}) \subseteq \bigcup_{k\in\mathbb{N}} \left(\overline{B}(0, r_e(W_{\psi,\varphi}, H_{\alpha_k}^{\infty}) \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} \right).$$

If $\lambda \in \cup_k \overline{B}(0, r_e(W_{\psi,\varphi}, H_{\alpha_k}^{\infty}))$, there exists $k_0 \in \mathbb{N}$ such that $|\lambda| \leq r_e(W_{\psi,\varphi}, H_{\alpha_{k_0}}^{\infty}) \leq \sup_k r_e(W_{\psi,\varphi}, H_{\alpha_k}^{\infty})$. Taking into account $(r_e(W_{\psi,\varphi}, H_{\alpha_k}^{\infty}))_k$ is an increasing sequence of positive real numbers bounded by $r_e(W_{\psi,\varphi}, H_{\alpha}^{\infty})$, it follows that $\lim_k r_e(W_{\psi,\varphi}, H_{\alpha_k}^{\infty}) = \sup_k r_e(W_{\psi,\varphi}, H_{\alpha_k}^{\infty})$.

(ii) Analogously to (i), the first inclusion is obtained directly from Theorem 2.3.5 (ii) and Lemma 3.1.3.

Second inclusion: by Lemma 3.1.2 and Theorem 3.3.1,

$$\sigma(W_{\psi,\varphi}, A_-^{-\alpha}) \subseteq \bigcap_{k\in\mathbb{N}} \left(\bigcup_{m=k}^{\infty} \left(\overline{B}(0, r_e(W_{\psi,\varphi}, H_{\alpha_m}^{\infty}) \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} \right) \right).$$

Moreover, the sequence $(r_e(W_{\psi,\varphi}, H_{\alpha_m}^{\infty}))_m$ is decreasing. Thus,

$$\bigcup_{m=k}^{\infty} \overline{B}(0, r_e(W_{\psi,\varphi}, H_{\alpha_m}^{\infty})) = \overline{B}(0, r_e(W_{\psi,\varphi}, H_{\alpha_k}^{\infty})) .$$

Then,

$$\begin{aligned}\sigma(W_{\psi,\varphi}, A_-^{-\alpha}) &\subseteq \bigcap_{k\in\mathbb{N}} \overline{B}(0, r_e(W_{\psi,\varphi}, H_{\alpha_k}^{\infty})) \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} \\ &= \overline{B}(0, \inf_k r_e(W_{\psi,\varphi}, H_{\alpha_k}^{\infty})) \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} \\ &= \overline{B}(0, \lim_k r_e(W_{\psi,\varphi}, H_{\alpha_k}^{\infty})) \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} .\end{aligned}$$

(iii) From Theorem 2.3.5 (iii), we obtain $0 \in \sigma(W_{\psi,\varphi}, A^{-\infty})$. Also, using Lemma 3.1.3 we obtain that $\{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} \subseteq \sigma(W_{\psi,\varphi}, A^{-\infty})$.

On the other hand, applying Lemma 3.1.2 and Theorem 3.3.1, we get

$$\sigma(W_{\psi,\varphi}, A^{-\infty}) \subseteq \bigcap_{k\in\mathbb{N}} \left(\bigcup_{m\geq k} \overline{B}(0, r_e(W_{\psi,\varphi}, H_m^{\infty}) \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} \right)$$

Moreover, since the sequence $(r_e(W_{\psi,\varphi}, H_m^{\infty}))_m$ is decreasing, $\cup_{m\geq k} \overline{B}(0, r_e(W_{\psi,\varphi}, H_m^{\infty}) = \overline{B}(0, r_e(W_{\psi,\varphi}, H_k^{\infty}))$. Then,

$$\begin{aligned}\sigma(W_{\psi,\varphi}, A^{-\infty}) &\subseteq \bigcap_{k\in\mathbb{N}} \overline{B}(0, r_e(W_{\psi,\varphi}, H_k^{\infty})) \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} = \\ &= \overline{B}(0, \lim_k r_e(W_{\psi,\varphi}, H_k^{\infty})) \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} .\end{aligned}$$

□

Remark 3.3.3. Recall that if T is a compact operator, then $\|T\|_e = 0$ and then, $r_e(T) = 0$ as well. From Theorem 3.3.2 and Proposition 1.3.1, we have that if $W_{\psi,\varphi}$ is compact on $A_+^{-\alpha}$, $A_-^{-\alpha}$or $A^{-\infty}$ then $\sigma(W_{\psi,\varphi}) = \{0\} \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty}$.

Remark 3.3.4. When $\varphi'(0) = 0$, $\{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} = \{0, \psi(0)\}$. Then, we have that

$$\{0, \psi(0)\} \subseteq \sigma(W_{\psi,\varphi}, A_+^{-\alpha}) \subseteq \bigcup_n \sigma(W_{\psi,\varphi}, H_{\alpha_n}^{\infty}),$$

$$\{0, \psi(0)\} \subseteq \sigma(W_{\psi,\varphi}, A_-^{-\alpha}) \subseteq \bigcap_{m\in\mathbb{N}} \Big(\bigcup_{n=m}^{\infty} \sigma(W_{\psi,\varphi}, H_{\alpha_n}^{\infty})\Big), \text{ and}$$

$$\{0, \psi(0)\} \subseteq \sigma(W_{\psi,\varphi}, A_-^{-\alpha}) \subseteq \bigcap_{m\in\mathbb{N}} \Big(\bigcup_{n=m}^{\infty} \sigma(W_{\psi,\varphi}, H_n^{\infty})\Big).$$

The first inclusion is clear in all the cases applying Lemma 3.1.3. In the second one we use Lemmas 3.1.1 and 3.1.2.

In the case when φ is not an automorphisim, applying Theorem 3.3.1 and reasoning as in Theorem 3.3.2 we obtain:

$$\{0, \psi(0)\} \subseteq \sigma(W_{\psi,\varphi}, A_+^{-\alpha}) \subseteq \overline{B}(0, \lim_n r_{e,\alpha_n}) \cup \{0, \psi(0)\},$$

$$\{0, \psi(0)\} \subseteq \sigma(W_{\psi,\varphi}, A_-^{-\alpha}) \subseteq \overline{B}(0, \lim_n r_{e,\alpha_n}) \cup \{0, \psi(0)\}, \text{ and}$$

$$\{0, \psi(0)\} \subseteq \sigma(W_{\psi,\varphi}, A^{-\infty}) \subseteq \overline{B}(0, \lim_n r_{e,n}) \cup \{0, \psi(0)\}.$$

The following Theorem collects some information about the point spectrum of the weighted composition operator on the spaces $A_+^{-\alpha}$, $A_-^{-\alpha}$ and $A^{-\infty}$.

Theorem 3.3.5. *Let $\varphi, \psi \in H(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi(0) = 0$, $0 < |\varphi'(0)| < 1$.*

(i) If $0 \le \alpha < \infty$ and $W_{\psi,\varphi}$ is continuous on $A_+^{-\alpha}$, then

$$\{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} \setminus \overline{B}(0, r_e(W_{\psi,\varphi}, H_\alpha^{\infty})) \subseteq \sigma_p(W_{\psi,\varphi}, A_+^{-\alpha}) \subseteq \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty}\ .$$

(ii) If $0 < \alpha < \infty$ and $W_{\psi,\varphi}$ is continuous on $A_-^{-\alpha}$, then $\sigma_p(W_{\psi,\varphi}, A_-^{-\alpha}) \subseteq \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty}$.

(iii) If $W_{\psi,\varphi} \in \mathcal{L}(A^{-\infty})$, then $\sigma_p(W_{\psi,\varphi}, A^{-\infty}) \subseteq \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty}$.

Proof. (i) By Lemma 3.1.4, $\sigma_p(W, A_+^{-\alpha}) \subseteq \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty}$. On the other hand, fix $n \in \mathbb{N}$. If $|\psi(0)\varphi'(0)^n| > r_e(W_{\psi,\varphi}, H_\alpha^{\infty})$ then $\psi(0)\varphi'(0)^n \in \sigma_p(W_{\psi,\varphi}, H_\alpha^0)$ (see [1, Theorem 7.44]). That is, there exists $f \in H_\alpha^0$ such that $W_{\psi,\varphi} f = \psi(0)\varphi'(0)^n f$. But $H_\alpha^0 \subset A_+^{-\alpha}$, thus $\psi(0)\varphi'(0)^n \in \sigma_p(W_{\psi,\varphi}, A_+^{-\alpha})$.

The inclusions of (ii) and (iii) follow again from Lemma 3.1.4. □

3.4 Spectra of C_φ

From Theorem 3.3.2 in the preceding section we can obtain, as a corollary, the following result about the spectrum of the composition operators on $A_+^{-\alpha}$ and $A_-^{-\alpha}$ spaces. The spectrum and point spectrum of C_φ on the Korenblum space, $A^{-\infty}$, are studied in Theorems 3.4.2 and 3.4.4.

Corollary 3.4.1. *Let* $\varphi \in H(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi(0) = 0$, $0 < |\varphi'(0)| < 1$. *Then,*

(i) for any $\alpha \geq 0$,

$$\{0\} \cup \{\varphi'(0)^n\}_{n=0}^{\infty} \subseteq \sigma(C_\varphi, A_+^{-\alpha}) \subseteq \overline{B}(0, r_e(C_\varphi, H_\alpha^\infty)) \cup \{\varphi'(0)^n\}_{n=0}^{\infty} .$$

(ii) for any $0 < \alpha < \infty$,

$$\{0\} \cup \{\varphi'(0)^n\}_{n=0}^{\infty} \subseteq \sigma(C_\varphi, A_-^{-\alpha}) \subseteq \overline{B}(0, r_e(C_\varphi, H_\alpha^\infty)) \cup \{\varphi'(0)^n\}_{n=0}^{\infty} .$$

Proof. This is a direct consequence of Theorem 3.3.2 and Corollary 3.2.9. □

Theorem 3.4.2. *Let* $\varphi \in H(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi(0) = 0$ *and* $0 < |\varphi'(0)| < 1$. *Then,*

$$\sigma(C_\varphi, A^{-\infty}) = \{0\} \cup \{\varphi'(0)^n\}_{n=0}^{\infty} .$$

Proof. The operator $C_\varphi \colon A^{-\infty} \to A^{-\infty}$ is continuous (because it is continuous on each H_n^0 (or H_n^∞)).

By Lemma 3.1.3, $\varphi'(0)^n \in \sigma(C_\varphi, A^{-\infty})$ for each $n \in \mathbb{N}$.

Assume $0 \notin \sigma(C_\varphi, A^{-\infty})$, then $C_\varphi \colon A^{-\infty} \to A^{-\infty}$ is a surjective isomorphism. Since $A^{-\infty}$ satisfies the assumptions on X in [23, Theorem 2.1], then φ would be an automorphism, but this is not the case. We then have $\{0\} \cup \{\varphi'(0)^n : n \in \mathbb{N}\} \subseteq \sigma(C_\varphi, A^{-\infty})$.

Now, denote $T_k = C_\varphi \colon H_k^\infty \to H_k^\infty$. By Theorem 3.3.1,

$$\sigma(C_\varphi, H_k^\infty) = \sigma(T_k, H_k^\infty) = \overline{B}(0, r_e(C_\varphi, H_k^\infty)) \cup \{\varphi'(0)^n\}_{n=0}^{\infty} .$$

We know (by Lemma 3.2.5) that , since $k < k+1$, $r_e(C_\varphi, H_{k+1}^\infty) \leq r_e(C_\varphi, H_k^\infty)$. This implies

$$\bigcup_{j \geq k} \sigma(T_j) = \overline{B(0, r_e(T_k))} \cup \{\varphi'(0)^n\}_{n=0}^{\infty} .$$

But Lemma 3.1.2 yields

$$\sigma(C_\varphi, A^{-\infty}) \subseteq \bigcap_k \bigcup_{j \geq k} \sigma(T_j) = \left(\bigcap_k \overline{B(0, r_e(T_k))} \right) \cup \{\varphi'(0)^n\}_{n=0}^{\infty} .$$

Moreover, $r_e(T_k)$ tends to 0 when k goes to infinity (see Lemma 3.2.6). Therefore,

$$\sigma(C_\varphi, A^{-\infty}) \subseteq \{0\} \cup \{\varphi'(0)^n\}_{n=0}^{\infty} .$$

□

Corollary 3.4.3. *Let* $\varphi \in H(\mathbb{D})$ *with* $\varphi(\mathbb{D}) \subseteq \mathbb{D}, \varphi(0) = 0$ *and* $0 < |\varphi'(0)| < 1$. *Then,* *(i) for any* $0 \leq \alpha < \infty$,

$$\{\varphi'(0)^n\}_{n=0}^{\infty} \setminus \overline{B}(0, r_e(C_\varphi, H_\alpha^\infty)) \subseteq \sigma_p(C_\varphi, A_+^{-\alpha}) \subseteq \{\varphi'(0)^n\}_{n=0}^{\infty}.$$

(ii) for any $0 < \alpha < \infty$,

$$\{\varphi'(0)^n\}_{n=0}^{\infty} \setminus \overline{B}(0, r_e(C_\varphi, H_\alpha^\infty)) \subseteq \sigma_p(C_\varphi, A_-^{-\alpha}) \subseteq \{\varphi'(0)^n\}_{n=0}^{\infty}.$$

Proof. By Theorem 3.3.5 we obtain (i) and the second inclusion of (ii). For the first inclusion of (ii), assume $|\varphi'(0)^n| > r_e(C_\varphi, H_\alpha^\infty)$. According to Corollary 3.2.9 there exists $k \in \mathbb{N}$ such that $|\varphi'(0)^n| > r_e(C_\varphi, H_{\alpha_k}^\infty)$, where $\alpha_k = \alpha - \frac{1}{k}$. Then, by [1, Theorem 7.44], $\varphi'(0)^n \in \sigma_p(C_\varphi, H_{\alpha_k}^\infty)$, that is, there exists $f \in H_{\alpha_k}^\infty$ such that $C_\varphi(f) = \varphi'(0)^n f$. Finally, we obtain that $\varphi'(0)^n \in \sigma_p(C_\varphi, A_-^{-\alpha})$, since $H_{\alpha_k}^\infty \subseteq A_-^{-\alpha}$. ☐

The eigenfunction equation for the composition operator C_φ, $f \circ \varphi = \lambda f$, is called *Schröder's equation.* For holomorphic maps φ with an interior fixed point a, the eigenvalues of Schröder's equation lie among the numbers $\{\varphi'(a)^n\}$. Königs showed that if $0 < |\varphi'(a)| < 1$ then there is a solution σ for $\varphi'(a)$ and, σ^n is the solution for $\varphi'(a)^n$. The symbol σ denotes the unique eigenfunction of Schröder's equation for $\lambda = \varphi'(a)$ that has $\sigma'(a) = 1$. This function is called the *Königs function* of φ. For more information about the Königs function, see [43, pp. 90].

Theorem 3.4.4. *Consider* $C_\varphi\colon A^{-\infty} \to A^{-\infty}$ *where* $\varphi \in H(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi(0) = 0$, $0 < |\varphi'(0)| < 1$. *Then,* $\sigma_p(C_\varphi, A^{-\infty}) = \{\varphi'(0)^n\}_{n=0}^{\infty}$.

Proof. By Theorem 3.4.2, we know $\sigma(C_\varphi, A^{-\infty}) = \{0\} \cup \{\varphi'(0)^n\}_{n=0}^{\infty}$. If we denote $r_e(C_\varphi, H_n^\infty)$ by $r_{e,n}$ then we have that $(r_{e,n})_n$ is a decreasing sequence that tends to 0 when n tends to infinity (see Lemma 3.2.6). Thus, for a certain $n_0 \in \mathbb{N}$, $|\varphi'(0)| > r_{e,n_0}$. But, Bourdon proved in [22] that the Königs eigenfunction $\sigma \in H_{n_0}^0$ if, and only if, $|\varphi'(0)| > r_{e,n_0}$. So, we have $\sigma \in H_{n_0}^0$. That is, $\varphi'(0)$ is an eigenvalue. Therefore, since $A^{-\infty}$ is an algebra and the operator C_φ is an algebra homomorphism, $\varphi'(0)^n$ is also an eigenvalue for all $n \in \mathbb{N}$ and the proof is finished. ☐

3.5 Spectra of M_ψ

Given $\psi \in H(\mathbb{D})$, the multiplication operator M_ψ is a weighted composition operator for the selfmap $\varphi(z) = z$. In this section we study the spectrum of multiplication operators.

Proposition 3.5.1. *Let E be a space which is continuously included in $H(\mathbb{D})$, containing the polynomials and such that for any $\eta \in H^\infty$ the multiplication operator $M_\eta\colon E \to E$ is continuous. Let $\psi \in H(\mathbb{D})$ be non constant. If $M_\psi\colon E \to E$ is continuous, then $\sigma_p(M_\psi) = \emptyset$ and $\psi(\mathbb{D}) \subseteq \sigma(M_\psi) \subseteq \overline{\psi(\mathbb{D})}$.*

Proof. Point spectrum: suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of M_ψ, that is, there exists $f \in E$, $f \not\equiv 0$ sucht that $\psi(z)f(z) = \lambda f(z)$ for every $z \in \mathbb{D}$. However, since $f \not\equiv 0$, the set U of the points where f does not vanish is an open set. Then, $\psi(z) = \lambda$ for all $z \in U$, which implies ψ is a constant function by the Identity Principle. This contradicts the hypothesis. Accordingly, $\sigma_p(M_\psi) = \emptyset$.

Now we study the spectrum. On one hand, if $\lambda \notin \overline{\psi(\mathbb{D})}$, then there exists $\varepsilon > 0$ with $|\psi(z) - \lambda| \geq \varepsilon$ for all $z \in \mathbb{D}$. Thus, the function $\eta(z) := \frac{1}{\psi(z) - \lambda} \in H^\infty$ and M_η is continuous on E by assumption. This implies that $M_\psi - \lambda I$ is a surjective operator. Indeed, for any $g \in E$, function $f(z) := M_\eta g(z) = \frac{g(z)}{\psi(z)-\lambda}$ verifies $(M_\psi - \lambda I)f = g$ and $f \in E$. Moreover, $M_\psi - \lambda I$ is injective because $\sigma_p(M_\psi) = \emptyset$. Thus, $\lambda \notin \sigma(M_\psi)$.

On the other hand, if $\lambda \in \psi(\mathbb{D})$ then there exists $z_0 \in \mathbb{D}$ such that $\psi(z_0) = \lambda$. Every function in $(M_\psi - \lambda I)(E)$ vanishes at z_0. Indeed, $(M_\psi - \lambda I)f(z_0) = \psi(z_0)f(z_0) - \lambda f(z_0) = 0$. Thus, $M_\psi - \lambda I$ is not a surjective operator because since all functions in the range vanish at the same point, we can not have in the range constant functions different from the 0 function.

Therefore, $\psi(\mathbb{D}) \subseteq \sigma(M_\psi, E) \subseteq \overline{\psi(\mathbb{D})}$. □

Corollary 3.5.2. *If M_ψ is continuous on any of the spaces $A_+^{-\alpha}$, $\alpha \geq 0$, $A_-^{-\alpha}$, $0 < \alpha \leq \infty$ for some non-constant function $\psi \in H(\mathbb{D})$ then, $\sigma_p(M_\psi) = \emptyset$ and $\psi(\mathbb{D}) \subseteq \sigma(M_\psi) \subseteq \overline{\psi(\mathbb{D})}$.*

Proof. This is a direct consequence of Proposition 3.5.1 since, by Propositions 2.1.4, 2.1.10 and Theorem 2.1.12 its assumptions are satisfied. □

Remark 3.5.3. Notice that if we apply Proposition 3.5.1 to H_p^∞ for any $p > 0$, we obtain that $\sigma(M_\psi, H_p^\infty) = \overline{\psi(\mathbb{D})}$, since the spectrum of any operator on a Banach space is a compact set.

Unlike in Banach spaces, the spectrum of M_ψ is not necessarily a closed set. The following example shows that the spectrum may not coincide with $\overline{\psi(\mathbb{D})}$.

Example 3.5.4. The analytic function $\psi(z) := \frac{1}{1-z}$ belongs to $A^{-\infty}$. Thus, the multiplication operator M_ψ is continuous on $A^{-\infty}$ (see Theorem 2.1.12). Observe $\frac{1}{2} = \psi(-1) \in \overline{\psi(\mathbb{D})}$. But, $\frac{1}{2} \in \rho(M_\psi, A^{-\infty})$. In fact, the inverse of $M_\psi - \frac{1}{2}I = M_{\psi - \frac{1}{2}}$ is the operator $M_{\frac{1}{\psi - \frac{1}{2}}}$. And, for each $n > 1$,

$$\begin{aligned}\sup_{z\in\mathbb{D}}(1-|z|)^n \frac{1}{|\psi(z) - \frac{1}{2}|} &= \sup_{z\in\mathbb{D}}(1-|z|)^n \frac{2|1-z|}{|1+z|} \\ &\leq \sup_{z\in\mathbb{D}}(1-|z|)^n \frac{2|1-z|}{1-|z|} = \sup_{z\in\mathbb{D}} 2(1-|z|)^{n-1}|1-z| < \infty \ .\end{aligned}$$

Therefore, since $\frac{1}{\psi - \frac{1}{2}} \in A^{-\infty}$ then, $M_{\frac{1}{\psi - \frac{1}{2}}} \in \mathcal{L}(A^{-\infty})$ and $\frac{1}{2} \in \rho(M_\psi, A^{-\infty})$.

3.6 Spectra of C_φ whose symbol is a rotation

If φ is an automorphism of the disc such that $\varphi(0) = 0$, then it is a rotation. That is, there is $c \in \mathbb{C}$ with $|c| = 1$ such that $\varphi(z) = cz$ for all $z \in \mathbb{D}$. In this section we present a few results about the spectrum of composition operators on Korenblum type spaces when the symbol is a rotation. The first lemma shows the spectrum and point spectrum for the weighted Banach spaces.

Lemma 3.6.1. *Let $\alpha > 0$, $\varphi \in H(\mathbb{D})$, $\varphi(z) = cz$ for all $z \in \mathbb{D}$, with $|c| = 1$. Then*

(i) $\sigma_p(C_\varphi, H_\alpha^\infty) = \{c^n\}_{n=0}^\infty$.

(ii) If c is a root of unity, then $\sigma(C_\varphi, H_\alpha^\infty) = \sigma_p(C_\varphi, H_\alpha^\infty) = \{c^n\}_{n=0}^\infty$.

(iii) If c is not a root of unity, then $\sigma(C_\varphi, H_\alpha^\infty) = \partial\mathbb{D}$.

Proof. (i) Lemma 3.1.4 implies $\sigma_p(C_\varphi, H_\alpha^\infty) \subseteq \{c^n\}_{n=0}^\infty$. For each $m = 0, 1, 2, \dots$ the function $f_m(z) := z^m$ belongs to H_α^∞. Moreover, for each m, $f_m(\varphi(z)) = c^m f_m(z) = c^m z^m$ for all $z \in \mathbb{D}$. So, every c^m is an eigenvalue of C_φ with eigenvector f_m.

(ii) If c is a root of unity, then there exists $m \in \mathbb{N}$ with $\varphi^m(z) = z$ for every $z \in \mathbb{D}$. That is, $C_\varphi^m = C_{\varphi^m} = I$. By applying the Spectral Mapping Theorem [1, Theorem 6.31], we obtain that $(\sigma(C_\varphi))^m = \sigma(C_\varphi^m) = \sigma(I) = \{1\}$. Then $\sigma(C_\varphi) \subseteq \{\lambda : \lambda^m = 1\}$. But, we already had that $\{c^n\}_{n=0}^\infty \subseteq \sigma(C_\varphi)$, then $\{c^n\}_{n=0}^\infty = \{\lambda : \lambda^m = 1\}$ and we get the result.

(iii) Suppose c is not a root of unity. If $|\lambda| > 1 = ||C_\varphi|| = ||C_\varphi^{-1}||$, then $\lambda \in \rho(C_\varphi)$ and $\lambda \in \rho(C_\varphi^{-1})$ by [1, Theorem 6.3]. It is easy to check that $-\lambda C_\varphi^{-1}(\lambda I - C_\varphi^{-1})^{-1}$ is the inverse of $\frac{1}{\lambda} I - C_\varphi$, which implies $1/\lambda \in \rho(C_\varphi)$. This shows that $\{c^n\}_{n=0}^\infty \subset \sigma(C_\varphi, H_\alpha^\infty) \subset \partial\mathbb{D}$. Since c is not a root of unity, Kronecker's Theorem [40, Theorem 2.2.4] implies that $\{c^n\}_{n=0}^\infty$ is dense in $\partial\mathbb{D}$. Since the spectrum of an operator on a Banach space is compact, this completes the proof of part (iii). □

Corollary 3.6.2. *Let $\varphi \in H(\mathbb{D})$, $\varphi(z) = cz$ for all $z \in \mathbb{D}$, with $|c| = 1$. Let E be any of the spaces $A_+^{-\alpha}, \alpha \geq 0$, or $A_-^{-\alpha}, 0 < \alpha < \infty$. Then*

(i) $\sigma_p(C_\varphi, E) = \{c^n\}_{n=0}^\infty$.

(ii) If c is a root of unity, then $\sigma(C_\varphi, E) = \sigma_p(C_\varphi, E) = \{c^n\}_{n=0}^\infty$.

(iii) If c is not a root of unity, then $\{c^n\}_{n=0}^\infty \subset \sigma(C_\varphi, E) \subset \partial\mathbb{D}$.

Proof. The point spectrum is obtained with the same argument as in Lemma 3.6.1. And, for the spectrum, both cases follow from Lemmas 3.1.2 and 3.6.1. □

In the case of the Korenblum space we can characterize which points of the unit circle belong to the spectrum of C_φ when $\varphi(z) = cz$ for all $z \in \mathbb{D}$ and $c \in \partial\mathbb{D}$ is not a root of unity. Theorem 3.6.5 is a similar result to [16, Theorem 1]. We first need the following known characterization of the functions in the Korenblum space in terms of their Taylor expansion, Lemma 3.6.4. We use Lemma 3.6.3 to prove it.

Lemma 3.6.3. *Let $k > 0$, $n \in \mathbb{N} \cup \{0\}$, $\varphi(r) = (1-r)^k r^n$ with $\varphi(0) = \varphi(1) = 0$ when $n > 0$.*

(i) If $n = 0$,

$$\max_{r \in [0,1]} \varphi(r) = \max_{r \in [0,1]} (1-r)^k = 1.$$

(ii) If $n \in \mathbb{N}$,

$$\max_{r \in [0,1]} \varphi(r) = \frac{k^k n^n}{(n+k)^{n+k}}.$$

Proof. (i) It is clear. (ii) For $n \in \mathbb{N}$, we have that

$$\varphi'(r) = n(1-r)^k r^{n-1} - k(1-r)^{k-1} r^n = r^{n-1}(1-r)^{k-1}(n(1-r) - kr).$$

Since the differentiable function φ has a maximum at the interior point r, φ' vanishes at r. Thus, if we solve $\varphi'(r) = 0$, we have just three options: $r = 0$, $r = 1$ or $r = n/(n+k)$. We discard the first two because $\varphi(0) = \varphi(1) = 0$ by hypothesis. Then,

$$\max_{r \in [0,1]} \varphi(r) = \varphi\left(\frac{n}{n+k}\right) = \frac{k^k n^n}{(n+k)^{n+k}}.$$

□

Lemma 3.6.4. *A function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the unit disc $\mathbb{D}$ belongs to $A^{-\infty}$ if, and only if, there is $k \in \mathbb{N}$ such that $\sup_n n^{-k}|a_n| < \infty$.*

Proof. Since $\lim_n n^k (n+k)^{-k} = 1$, it is enough to prove that $f \in A^{-\infty}$ if, and only if, there is $k \in \mathbb{N}$ such that $\sup_n |a_n|(n+k)^{-k} < \infty$.

Suppose that $f \in A^{-\infty}$, that is, there is $k \in \mathbb{N}$ such that $\sup_{z\in\mathbb{D}}(1-|z|)^k|f(z)| < \infty$. Fix $0 < r < 1$. By Cauchy estimates [30, p. 118],

$$|a_n| \leq \frac{1}{r^n} \max_{|z|=r} |f(z)| \leq \|f\|_k \frac{1}{r^n(1-r)^k}.$$

From this, we can deduce that for all $n \in \mathbb{N}$,

$$|a_n| \max_{r\in[0,1]} r^n(1-r)^k \leq \|f\|_k.$$

Applying Lemma 3.6.3, we obtain that

$$\frac{|a_n|}{(n+k)^k} \leq \frac{(n+k)^n}{n^n} \frac{\|f\|_k}{k^k}.$$

Since $\lim_n (n+k)^n n^{-n} = e^k$, it follows that

$$\sup_n \frac{|a_n|}{(n+k)^k} < \infty.$$

On the other hand, assume now that there is a certain $k \in \mathbb{N}$ satisfying $\sup_n |a_n|(n+k)^{-k} < \infty$. That is, there exists $M > 0$ such that $|a_n| \leq M(n+k)^k$ for all $n \in \mathbb{N}$. For $z \in \mathbb{D}$ we have that

$$(1-|z|)^{k+2}|f(z)| \leq \sum_{n=0}^{\infty} |a_n||z|^n(1-|z|)^{k+2} \leq C \sum_{n=0}^{\infty} \frac{1}{(n+k+2)^2} < \infty,$$

for some positive constant C. Therefore, $f \in H^\infty_{k+2} \subset A^{-\infty}$. □

Theorem 3.6.5. *Let $c \in \mathbb{C}$ be an element of the unit circle $|c| = 1$ which is not a root of unity. Let $C_\varphi \colon A^{-\infty} \to A^{-\infty}$ be the composition operator with symbol $\varphi(z) = cz$, $z \in \mathbb{D}$. A complex number $\lambda \neq 1$, $|\lambda| = 1$, belongs to the resolvent set $\rho(C_\varphi, A^{-\infty})$ if, and only if, there are $s \geq 1$ and $\varepsilon > 0$ such that $|c^n - \lambda| \geq \varepsilon n^{-s}$ for each $n \in \mathbb{N}$.*

Proof. First, assume that there are $s \geq 1$ and $\varepsilon > 0$ such that $|c^n - \lambda| \geq \varepsilon n^{-s}$ for each $n \in \mathbb{N}$. In particular, $C_\varphi - \lambda I$ is injective by Corollary 3.6.2 (i). We prove that it is also surjective. Given $g(z) = \sum_{n=0}^{\infty} a_n z^n \in A^{-\infty}$, we define

$$f(z) := \sum_{n=0}^{\infty} \frac{a_n}{c^n - \lambda} z^n, \quad z \in \mathbb{D}.$$

It is easy to check that $(C_\varphi - \lambda I)f = g$. To conclude the proof of this implication it is enough to show that $f \in A^{-\infty}$. Since $g \in A^{-\infty}$, we apply Lemma 3.6.4 to find some $k \in \mathbb{N}$ and $M > 0$ such that $n^{-k}|a_n| \leq M$ for each $n = 0, 1, 2, \ldots$. Hence, for each $n = 0, 1, 2, \ldots$, we get that

$$\frac{|a_n|}{|c^n - \lambda|} \leq \frac{M n^k n^s}{\varepsilon} = \frac{M}{\varepsilon} n^{k+s},$$

which implies that f is analytic and belongs to $A^{-\infty}$ by Lemma 3.6.4.

Now, suppose that $\lambda \neq 1, |\lambda| = 1$, belongs to $\rho(C_\varphi, A^{-\infty})$. Then, the inverse operator $(C_\varphi - \lambda I)^{-1} \colon A^{-\infty} \to A^{-\infty}$ exists, is continuous and necessarily has the form

$$(C_\varphi - \lambda I)^{-1}(\sum_{n=0}^{\infty} a_n z^n) = \sum_{n=0}^{\infty} \frac{a_n}{c^n - \lambda} z^n.$$

The continuity of this inverse implies that for $m = 1$ there are $k > 1$ and $M > 0$ such that for each $\sum_{n=0}^{\infty} a_n z^n \in H_1^\infty$ we have

$$\sup_{z\in\mathbb{D}}(1 - |z|)^k \left|\sum_{n=0}^{\infty} \frac{a_n}{c^n - \lambda} z^n\right| \leq M \sup_{z\in\mathbb{D}}(1 - |z|) \left|\sum_{n=0}^{\infty} a_n z^n\right|.$$

Evaluating this inequality for each monomial $z^n,\ n = 0, 1, 2...$ we get, for each $n = 0, 1, 2, ...,$

$$\sup_{z\in\mathbb{D}}(1 - |z|)^k \frac{|z^n|}{|c^n - \lambda|} \leq M \sup_{z\in\mathbb{D}}(1 - |z|)|z^n|.$$

Therefore, evaluating the maximum of $r^n(1 - r)^k$, we get for each $n = 0, 1, 2, ...$

$$\frac{k^k n^n}{(n + k)^{n+k}} \leq M|c^n - \lambda| \frac{n^n}{(n + 1)^{n+1}}.$$

This implies, for each $n = 0, 1, 2, ...,$

$$|c^n - \lambda| \geq \frac{1}{M} \left(\frac{n + 1}{n + k}\right)^{n+1} \frac{1}{(n + k)^{k-1}},$$

which yields the desired inequality. □

A real number $x \in \mathbb{R}$ is called *Diophantine* if there are $\delta \geq 1$ and $d(x) > 0$ such that

$$\left|x - \frac{p}{q}\right| \geq \frac{d(x)}{q^{1+\delta}}$$

for all rational numbers p/q, $p, q \in \mathbb{Z}$. As we can see in [24, p. 43], this occurs if, and only if, $\lambda = e^{2\pi i x}$ satisfies

$$|\lambda^n - 1| \geq d(x)n^{-\delta}, \quad n \geq 1.$$

In the next proposition, a characterization of the complex number 1 belonging to the resolvent set in relation with Diophantine numbers is stated. In this result, $A_0^{-\infty}$ denotes the space of all functions $f \in A^{-\infty}$ such that $f(0) = 0$. This proposition should be compared with [16, Theorem 2].

Proposition 3.6.6. *Let $\varphi(z) = cz$, $z \in \mathbb{D}$, where $|c| = 1$ and c is not a root of unity. Let $C_\varphi \colon A_0^{-\infty} \to A_0^{-\infty}$ be the composition operator with symbol φ. Then $1 \in \rho(C_\varphi, A_0^{-\infty})$ if, and only if, $c = e^{2\pi i x}$, where x is a Diophantine number.*

Proof. Notice $1 \notin \sigma_p(C_\varphi)$ because if it were true, it would exist $f(z) = \sum_{n=1}^{\infty} a_n z^n \in A_0^{-\infty}$, $f \not\equiv 0$ such that

$$\sum_{n=1}^{\infty} a_n z^n = \sum_{n=1}^{\infty} a_n c^n z^n, \quad z \in \mathbb{D}.$$

However, since $f \not\equiv 0$, there exists $k \in \mathbb{N}$ with $a_k \neq 0$, what implies $c^k = 1$, which is a contradiction.

Now, suppose $c = e^{2\pi i x}$ with x Diophantine. Then, by [24, pp. 43] there exist $s \geq 1$ and $\varepsilon > 0$ such that $|c^n - 1| \geq \varepsilon n^{-s}$ for each $n \in \mathbb{N}$. Since $1 \notin \sigma_p(C_\varphi)$, to see $1 \notin \sigma(C_\varphi)$ it only remains to show $C_\varphi - I$ is surjective.

Given $g(z) = \sum_{n=1}^{\infty} a_n z^n$, $z \in \mathbb{D}$, define

$$f(z) := \sum_{n=1}^{\infty} \frac{a_n}{c^n - 1} z^n, \qquad z \in \mathbb{D}.$$

Clearly, $g = (C_\varphi - I)f$ and $f(0) = 0$. We want to check Lemma 3.6.4 for the function f. Since $g \in A^{-\infty}$, for some $k \in \mathbb{N}$, $M > 0$, $|a_n| \leq M n^k$ for all $n \in \mathbb{N}$. And, by hypothesis, $\frac{1}{|c^n - 1|} \leq \frac{n^s}{\varepsilon}$. Thus,

$$\frac{|a_n|}{|c^n - 1|} \leq \frac{M}{\varepsilon} n^{k+s} \ .$$

Therefore, $f \in A^{-\infty}$.

The converse follows as in the proof of Theorem 3.6.5, taking into account [24, p. 43]. $\square$

Chapter 4

The spectrum of some composition operators on Korenblum type spaces

In this chapter we consider composition operators C_φ whose symbol $\varphi\colon \mathbb{D} \to \mathbb{D}$ admits an analytic extension to an open neighbourhood of the closed unit disc $\overline{\mathbb{D}}$ of the complex plane. We prove that for a family of symbols singled out by H. Kamowitz in [33], the spectrum contains a closed ball of positive radius.

A new insight in the approach of Kamowitz study of the spectrum of some composition operators on H^p spaces ([33, Theorem 3.4]) allows us to use his method to prove that for $\varphi\colon \mathbb{D} \to \mathbb{D}$ analytic on an open neighbourhood of $\overline{\mathbb{D}}$, with an interior fixed point and a repelling fixed point z_0 in $\partial\mathbb{D}$, the spectrum of the composition operator C_φ on $A_-^{-\alpha}$ and $A_+^{-\alpha}$, contains the closed ball $\overline{B}(0, |\varphi'(z_0)|^{-\alpha})$. This enlarges the knowledge of the size of $\sigma(C_\varphi)$ that it is known to be a subset of $\overline{B}(0, r_e(C_\varphi, H_\alpha^\infty)) \cup \{\varphi'(0)^n\}_{n=0}^\infty$ according to Corollary 3.4.1.

The investigation described in this chapter is collected in [29].

4.1 Preliminaries

For a positive integer m and $\alpha > 0$, let $H_{\alpha,m}^\infty$ denote the closed subspace of H_α^∞ given by

$$H_{\alpha,m}^\infty := \{f \in H_\alpha^\infty : f \ \text{ has a zero of at least order } m \text{ at } 0\}.$$

Lemma 4.1.1. *Let $0 < p < \infty$. For any positive integer $m \in \mathbb{N}$, the map $F \in H_p^\infty \to z^m F \in H_{p,m}^\infty$ is an isomorphism.*

Proof. First, we see it is continuous. The norm in $H_{p,m}^\infty$ is the same as in H_p^∞. Let $F \in H_p^\infty$.

$$\|z^m F\|_p = \sup_{z\in\mathbb{D}} (1-|z|)^p |z|^m |F(z)| \le \sup_{z\in\mathbb{D}} (1-|z|)^p |F(z)| = \|F\|_p\,.$$

On the other hand, any $f \in H_{p,m}^\infty$ can be written as $f = z^m F$, for some holomorphic function F that indeed belongs to H_p^∞. Take $0 < r < 1$ and so,

$$\sup_{z\in\mathbb{D}} (1-|z|)^p |F(z)| = \max\Big\{ \sup_{z\in\overline{B}(0,r)} (1-|z|)^p |F(z)|, \sup_{z\in\mathbb{D}\setminus\overline{B}(0,r)} (1-|z|)^p |F(z)| \Big\}.$$

Since F is holomorphic and $\overline{B}(0,r)$ compact, there is $M > 0$ such that

$$\sup_{z\in\overline{B}(0,r)} (1-|z|)^p|F(z)| = M < \infty.$$

Moreover, since $f \in H_p^\infty$,

$$\begin{aligned}\sup_{z\in\mathbb{D}\setminus\overline{B}(0,r)} (1-|z|)^p|F(z)| &= \sup_{z\in\mathbb{D}\setminus\overline{B}(0,r)} (1-|z|)^p\frac{|z^mF(z)|}{|z|^m}\\ &\leq \sup_{z\in\mathbb{D}\setminus\overline{B}(0,r)} (1-|z|)^p\frac{|z^mF(z)|}{r^m} = \frac{\|f\|_p}{r^m} < \infty.\end{aligned}$$

Thus, the map is surjective and in addition it is clearly one-to-one, hence an isomorphism by the Open Mapping Theorem. □

We include a number of lemmas needed in the sequel.

Recall that a sequence $(z_k) \subset \mathbb{D}$ is an *iteration sequence* for φ if $\varphi(z_k) = z_{k+1}$ for all k. We need the following crucial lemmas due to Cowen and MacCluer.

Lemma 4.1.2 ([27], Lemma 7.34)**.** *If φ is an analytic map, not an automorphism, of the unit disk into itself and $\varphi(0) = 0$. For a given $0 < r < 1$, there exists $1 \leq M < \infty$ such that if $(z_k)_{k=-K}^{\infty}$ is an iteration sequence with $|z_n| \geq r$ for some non-negative integer n and if $(w_k)_{k=-K}^{n}$ are arbitrary numbers, then there is $f \in H^\infty$ such that*

$$f(z_k) = w_k, \quad -K \leq k \leq n$$

and

$$||f||_\infty \leq M \sup\{|w_k| : -K \leq k \leq n\}.$$

Lemma 4.1.3 ([27], Lemma 7.35)**.** *Let φ be as in the previous lemma. For any iteration sequence $(z_k)_k$ there exists $c < 1$ such that*

$$\frac{|z_{k+1}|}{|z_k|} \leq c$$

whenever $|z_k| \leq 1/2$.

Lemma 4.1.4. *Let $1 \leq p < \infty$, $m \in \mathbb{N}$. Suppose $\varphi\colon \mathbb{D} \to \mathbb{D}$ is analytic on $\mathbb{D}$ with $\varphi(0) = 0$. Consider $\lambda \neq \varphi'(0)^n$ for all non-negative integers n and $g \in H_{p,m}^\infty$. If there is an analytic function $f \in H_p^\infty$ with $g = \lambda f - f \circ \varphi$, then f also belongs to $H_{p,m}^\infty$.*

Proof. Observe that $f(0) = 0$ since $0 = g(0) = \lambda f(0) - f(0)$. If we differentiate the expression $g(z) = \lambda f(z) - f(\varphi(z))$ we obtain:

$$\begin{aligned}
g'(z) &= \lambda f'(z) - f'(\varphi(z))\varphi'(z)\\
g''(z) &= \lambda f''(z) - f''(\varphi(z))\varphi'(z)^2 - f'(\varphi(z))\varphi''(z)\\
g'''(z) &= \lambda f'''(z) - f'''(\varphi(z))\varphi'(z)^3 - f''(\varphi(z))2\varphi''(z) - f''(\varphi(z))\varphi'(z)\varphi''(z) - f'(\varphi(z))\varphi'''(z)\\
&\;\;\vdots
\end{aligned}$$

In particular, for $z = 0$,

$$
\begin{aligned}
0 &= g'(0) = \lambda f'(0) - f'(0)\varphi'(0)\\
0 &= g''(0) = \lambda f''(0) - f''(0)\varphi'(0)^2 - f'(0)\varphi''(0)\\
0 &= g'''(0) = \lambda f'''(0) - f'''(0)\varphi'(0)^3 - f''(0)2\varphi''(0) - f''(0)\varphi'(0)\varphi''(0) - f'(0)\varphi'''(0)\\
&\vdots
\end{aligned}
$$

Since $\lambda \neq \varphi'(0)^n$ for all $n \in \mathbb{N}$, from the first equation we get that $f'(0) = 0$, from the second one, that $f''(0) = 0$, and so on, we get that

$$f(0) = f'(0) = f''(0) = \ldots = f^{m-1)}(0) = 0.$$

Thus, $f(z) = z^m F(z)$ for some holomorphic function F and from Lemma 4.1.1, $F \in H_p^\infty$. Therefore $f \in H_{p,m}^\infty$. □

Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be analytic and $(x_k)_{k=-\infty}^{+\infty}$ be an iteration sequence, that is, $x_{k+1} = \varphi(x_k)$ for all integers k. For $f, g \in A_-^{-\alpha}$ and a complex number $\lambda \neq 0$, satisfying $\lambda f - f \circ \varphi = g$, one can check inductively that

$$\lambda^k f(x_{-k}) = f(x_k)\lambda^{-k} + \lambda^{-1} \sum_{i=-k}^{k-1} g(x_i)\lambda^{-i} \quad \text{for each positive integer } k. \tag{4.1.1}$$

Lemma 4.1.5 ([33], Theorem 2.5)**.** *Suppose φ is analytic in a neighbourhood of a fixed point z_1 and $c = \varphi'(z_1)$, $0 < |c| < 1$. Then there is a function A, analytic at z_1 such that $((\varphi_n(z) - z_1)/c^n) \to A(z)$ uniformly near z_1. In fact, $\varphi_n(z) = z_1 + c^n A(z) + \mathcal{O}(|c^n A(z)|^2)$. Further, if $\varphi\colon \mathbb{D} \to \mathbb{D}$ is analytic on $\overline{\mathbb{D}}$ and z_1 is a fixed point of φ with $|z_1| = 1$ and $\varphi'(z_1) = c < 1$, then for each $z \in \mathbb{D}$, z near z_1, we have $A_0(z) = \mathrm{Re}A(z) > 0$ and $|\varphi_n(z)| = 1 + c^n A_0(z) + \mathcal{O}(|c^n A(z)|^2)$.*

4.2 Results

Recall that for an operator T on a locally convex Hausdorff space, unlike for Banach spaces, the resolvent $\rho(T)$ might be the empty set or even not an open set.

The following theorem provides the same conclusion as [33, Theorem 3.4], but for the space $A_-^{-\alpha}$.

Theorem 4.2.1. *Consider $A_-^{-\alpha}$ with $\alpha > 0$. Suppose that $\varphi\colon \mathbb{D} \to \mathbb{D}$ has an analytic extension to an open neighbourhood of $\overline{\mathbb{D}}$ and that has a fixed point $a \in \mathbb{D}$. Suppose that there is a positive integer N such that φ_N has, at least, a fixed point z_0 in the unit circle and that $|\varphi'(z_0)| > 1$. Then $\sigma(C_\varphi, A_-^{-\alpha}) \supseteq \{\lambda : |\lambda| \leq \varphi_N'(z_0)^{-\alpha/N}\}$.*

Proof. Since $A_-^{-\alpha}$ is automorphism invariant (Remark 2.3.3), without loss of generality we assume that $a = 0$. Notice that φ cannot be an automorphism of $\mathbb{D}$ since then φ_N would be as well an automorphism and its fixed point structure prevents it. Thus $0 \in \sigma(C_\varphi)$ (see Theorem 2.3.1).

Lemma 4.1.5 can be applied to φ_N^{-1} because it exists locally near its fixed point z_0 and $(\varphi_N^{-1})'(z_0) = \frac{1}{(\varphi_N)'(z_0)} < 1$. Here we have applied the chain rule: $(\varphi_N^{-1} \circ \varphi_N)'(z) = z' = 1$ if, and only if, $(\varphi_N^{-1})'(\varphi_N(z))\varphi_N'(z) = 1$ and, for the fixed point z_0, we obtain $(\varphi_N^{-1})'(\varphi_N(z_0)) = (\varphi_N^{-1})'(z_0) = 1/\varphi_N'(z_0)$. Moreover, by [33, Lemma 1.1], $\varphi_N'(z_0) > 0$.

Thus we may choose $x_0 \in \mathbb{D}$ with $\lim_n \varphi_N^{-n}(x_0) = z_0$. Relying on it we construct an iteration sequence $(x_k)_{k=-\infty}^{+\infty}$ as follows. Define

$$x_k := \begin{cases} \varphi_k(x_0) & \text{if } k > 0, \\ \varphi_N^{-n}(x_0) & \text{if } k = -nN \text{ with } n > 0, \\ \varphi_p(x_{-nN}) & \text{if } k = -nN + p \text{ with } p = 1, \dots, N-1 \text{ and } n > 0. \end{cases}$$

Then, for all integers k, $\varphi(x_k) = x_{k+1}$.

Again by Lemma 4.1.5, if $n > 0$, we have

$$z_0 - x_{-nN} = z_0 - \varphi_N^{-n}(x_0) \sim \varphi_N'(z_0)^{-n} A(x_0) \tag{4.2.1}$$

and

$$1 - |x_{-nN}| \sim \varphi_N'(z_0)^{-n} A_0(x_0), \tag{4.2.2}$$

where $A(x_0)$ and $A_0(x_0) = \mathrm{Re}A(x_0)$ are not zero.

Since the point x_0 is chosen in a neighbourhood of z_0, and $|z_0| = 1$, we can assume that $|x_0| > \frac{1}{2}$. Let $m_0 := \max\{k : |x_k| \geq 1/4\}$. Observe that this maximum exists because the sequence $(x_k)_k$ has decreasing norms (Schwarz Lemma) and, since $\lim_n x_{-n} = z_0$, the sequence $(x_n)_n$ tends to z_0 when n goes to $-\infty$, and the norms tend to 1. Then, the set $\{k : |x_k| \geq 1/4\}$ can have so many negative integers, but just a finite number of positive ones, because when n goes to $+\infty$, $(x_n)_n$ tends to 0 (see [30, Exercise 8, p. 261]).

Then $m_0 \geq 0$ and $|x_k| < 1/4$ for $k > m_0$. By Lemma 4.1.3 there is b with $1/2 \leq b < 1$ for which $|x_{k+1}/x_k| \leq b$ for all $k \geq m_0$. This implies that

$$|x_k| \leq b^{k-m_0}|x_{m_0}|, \text{ for } k \geq m_0 . \tag{4.2.3}$$

Denote $c := \varphi_N'(z_0) = (\varphi'(z_0))^N$. Thus $c > 1$.

Fix λ so that $0 < |\lambda| \leq c^{-\alpha/N}$. Suppose for a contradiction that $\lambda \notin \sigma(C_\varphi)$.

Choose n_0 so large that

$$\frac{b^n}{|\lambda|} < 1 \ \forall n \geq n_0. \tag{4.2.4}$$

Fix $m \in \mathbb{N}$, $m > n_0$ such that $|\varphi'(0)|^m < |\lambda|$. Given $f \in A_-^{-\alpha}$, there exists $0 < \beta < \alpha$ such that $f \in H_\beta^\infty$, so $|f(x_{-nN})| \leq \|f\|_\beta(1 - |x_{-nN}|)^{-\beta}$. Therefore bearing in mind (4.2.2),

$$|\lambda^{nN} f(x_{-nN})| \lesssim \|f\|_\beta \frac{1}{A_0(x_0)^\beta} \left(\frac{|\lambda|^N}{c^{-\beta}}\right)^n. \tag{4.2.5}$$

Consequently, taking into account that $|\lambda| < c^{-\alpha/N}$ implies $|\lambda|^N/c^{-\alpha} < 1$ and that $|\lambda|^N/c^{-\beta} < |\lambda|^N/c^{-\alpha} < 1$, for all $f \in A_-^{-\alpha}$ we have

$$\lim_n |\lambda^{nN} f(x_{-nN})| = 0. \tag{4.2.6}$$

Let us denote by $A_{-,m}^{-\alpha}$ the inductive limit

$$A_{-,m}^{-\alpha} := \bigcup_{0<\beta<\alpha} H_{\beta,m}^\infty.$$

We claim that $\big(C_\varphi - \lambda I\big)(A^{-\alpha}_{-,m}) = A^{-\alpha}_{-,m}$. Let $g \in A^{-\alpha}_{-,m}$. Since $C_\varphi - \lambda I$ is onto, there is $f \in A^{-\alpha}_{-}$ such that $g = (C_\varphi - \lambda I)(f)$. Thus there is $0 < \beta < \alpha$ such that $f \in H^\infty_\beta$, and so also $g \in H^\infty_\beta$, hence $g \in H^\infty_{\beta,m}$. According to Lemma 4.1.4, f belongs as well to $H^\infty_{\beta,m}$, as claimed. In addition, $f(z) = z^m F(z)$ for some $F \in H^\infty_\beta$, as pointed out by Lemma 4.1.1, therefore $|f(x_{nN})| = |\varphi_{nN}(x_0)^m F(\varphi_{nN}(x_0))|$. Now applying [33, Lemma 2.6], we obtain for such an f,

$$\begin{aligned}\overline{\lim_n}\, |\lambda|^{-nN}|f(x_{nN})| &= \overline{\lim_n}\, |\varphi_{nN}(x_0)|^m|\lambda|^{-nN}|F(\varphi_{nN}(x_0))| \\ &\le \overline{\lim_n}\, |\varphi'(0)|^{mnN}|\lambda|^{-nN}|F(\varphi_{nN}(x_0))| = 0. \end{aligned} \tag{4.2.7}$$

From (4.1.1), we have

$$\lambda^{nN} f(x_{-nN}) = f(x_{nN})\lambda^{-nN} + \lambda^{-1}\sum_{i=-nN}^{Nn-1} g(x_i)\lambda^{-i}, \tag{4.2.8}$$

which together with limits (4.2.6) and (4.2.7) above show that if $g \in (C_\varphi - \lambda I)(A^{-\alpha}_{-,m})$, then

$$\sum_{k=-\infty}^{+\infty} g(x_k)\lambda^{-k} = 0. \tag{4.2.9}$$

Next, given the iteration sequence $(x_k)_{k=-K}^{+\infty}$, define the linear functionals L_K on $A^{-\alpha}_{-,m}$ by

$$L_K(f) := \sum_{k=-K}^{\infty} \frac{f(x_k)}{\lambda^k}\,. \tag{4.2.10}$$

If we denote the topological dual of $A^{-\alpha}_{-,m}$ with the inductive limit topology by $(A^{-\alpha}_{-,m})'$, then the functionals $L_K \in (A^{-\alpha}_{-,m})'$. In order to prove this, recall that $L_K\colon A^{-\alpha}_{-,m} \to \mathbb{C}$ is continuous if $L_K\colon H^\infty_{\beta,m} \to \mathbb{C}$ is continuous for all $\beta < \alpha$. That is, if for all $\beta < \alpha$ there exists a positive constant C such that $|L_K(f)| \le C\|f\|_\beta$ for all $f \in H^\infty_{\beta,m}$. Fix $\beta < \alpha$. For each $f \in H^\infty_{\beta,m}$ there is $F \in H^\infty_\beta$ such that $f(z) = z^m F(z)$ for all $z \in \mathbb{D}$ and, applying equation (4.2.3) we obtain:

$$\begin{aligned}|L_K(f)| &\le \sum_{k=-K}^{\infty} \frac{(1-|x_k|)^\beta |f(x_k)|}{(1-|x_k|)^\beta|\lambda|^k} \le \sum_{k=-K}^{\infty} \frac{(1-|x_k|)^\beta|x_k|^m |F(x_k)|}{(1-|x_k|)^\beta|\lambda|^k} \le \sum_{k=-K}^{\infty} \frac{|x_k|^m\|F\|_\beta}{(1-|x_k|)^\beta|\lambda|^k} \\
&\le \|F\|_\beta \sum_{k=-K}^{m_0} \frac{|x_k|^m}{(1-|x_k|)^\beta|\lambda|^k} + \|F\|_\beta \sum_{k=m_0+1}^{\infty} \frac{|x_k|^m}{(1-|x_k|)^\beta|\lambda|^k} \\
&\le \|F\|_\beta \sum_{k=-K}^{m_0} \frac{|x_k|^m}{(1-|x_k|)^\beta|\lambda|^k} + \|F\|_\beta \left(\frac{4}{3}\right)^\beta \sum_{k=m_0+1}^{\infty} \frac{|x_k|^m}{|\lambda|^k} \\
&= \|F\|_\beta \sum_{k=-K}^{m_0} \frac{|x_k|^m}{(1-|x_k|)^\beta|\lambda|^k} + \|F\|_\beta \left(\frac{4}{3}\right)^\beta \sum_{i=1}^{\infty} \frac{|x_{m_0+i}|^m}{|\lambda|^{m_0+i}} \\
&\le \|F\|_\beta \sum_{k=-K}^{m_0} \frac{|x_k|^m}{(1-|x_k|)^\beta|\lambda|^k} + \|F\|_\beta \left(\frac{4}{3}\right)^\beta \sum_{i=1}^{\infty} \frac{b^{im}|x_{m_0}|^m}{|\lambda|^{m_0+i}} \\
&= \|F\|_\beta \sum_{k=-K}^{m_0} \frac{|x_k|^m}{(1-|x_k|)^\beta|\lambda|^k} + \|F\|_\beta \left(\frac{4}{3}\right)^\beta \frac{|x_{m_0}|^m}{|\lambda|^{m_0}} \sum_{i=1}^{\infty} \frac{b^{im}}{|\lambda|^i}\,.\end{aligned}$$

But, since $m > n_0$, applying estimate (4.2.4) we obtain $b^m/|\lambda| < 1$ and so,

$$|L_K(f)| \leq C\|F\|_\beta \,,$$

for some positive constant C.

Moreover, the map $F \in H_\beta^\infty \to z^m F \in H_{\beta,m}^\infty$ is an isomorphism (see Lemma 4.1.1), which implies that has a continuous inverse. Therefore,

$$\|F\|_\beta \leq M\|z^m F\|_\beta = M\|f\|_\beta$$

for certain $M > 0$. Thus, $|L_K(f)| \leq CM\|f\|_\beta$, that is, L_K is continuous.

Furthermore, as proved previously, the operator $C_\varphi - \lambda I|_{A_{-,m}^{-\alpha}}$ is surjective and so, by (4.2.9), $\lim_K L_K(f) = 0$ for all $f \in A_{-,m}^{-\alpha}$. That is, the sequence (L_K) converges to 0 in the weak* topology $\sigma\big((A_{-,m}^{-\alpha})', A_{-,m}^{-\alpha}\big)$.

Fix $0 < \beta < \alpha$, then there exists $f_{x_0} \in H_\beta^\infty \subset A_-^{-\alpha}$ such that $||f_{x_0}||_\beta \leq 1$ and $|f_{x_0}(x_0)| = 1/(1-|x_0|)^\beta$. Let $1 \leq M < \infty$ be the constant in Lemma 4.1.2 for $r = 1/4$. Then there is $f_K \in H^\infty$ with $\|f_K\|_\infty \leq M,\ \ |f_K(x_0)| = 1$ and satisfying

$$x_0^m f_K(x_0) f_{x_0}(x_0) > 0 \ \text{ and } \ f_K(x_k) = 0 \text{ for } -K \leq k \leq m_0,\, k \neq 0.$$

Now, the function $g_K(x) := x^m f_K(x) f_{x_0}(x)$ belongs to $H_{\beta,m}^\infty$ and $\|g_K\|_\beta \leq M$. Observe that the constant M does not depend on m. Further,

$$L_K(g_K) = x_0^m f_K(x_0) f_{x_0}(x_0) + \sum_{k=m_0+1}^{\infty} \lambda^{-k} x_k^m f_K(x_k) f_{x_0}(x_k).$$

If, in addition, we choose m so that

$$M\frac{1}{|\lambda|^{m_0}}\frac{1}{v_\beta(x_{m_0})}\frac{b^m}{|\lambda|-b^m} < \frac{1}{2v_\beta(x_0)} \quad ,$$

and use again (4.2.3) and (4.2.4), we obtain

$$\Big|\sum_{k=m_0+1}^{+\infty} \lambda^{-k} g_K(x_k)\Big| \leq \frac{|x_{m_0}|^m}{2v_\beta(x_0)} \leq \frac{|x_0|^m}{2v_\beta(x_0)}.$$

And then,

$$|L_K(g_K)| \geq \frac{|x_0|^m}{2v_\beta(x_0)}. \tag{4.2.11}$$

To conclude, recall that the embedding $H^\infty \hookrightarrow H_\beta^\infty$ is a compact operator and the multiplication operator $M_{z^m f_{x_0}}$ is a continuous self map of H_β^∞. Hence $(g_K)_K$ is a relatively compact subset of $H_{\beta,m}^\infty \subset A_{-,m}^{-\alpha}$ because $g_K = M_{z^m f_{x_0}}(f_K)$. That is, since $\|f_K\|_\infty \leq M$, and compact operators transform bounded sets in relatively compact sets, $(f_K)_K$ is relatively compact. Additionally, since continuous operators transform relatively compact sets in relatively compact, $(g_K)_K$ is relatively compact in $H_{\beta,m}^\infty$.

This way we are led into a contradiction with (4.2.11) since we must have $\lim_K L_K(g_K) = 0$ because the pointwise bounded set $(L_K)_K$ is an equicontinuous set in the dual of the barreled space $A_{-,m}^{-\alpha}$ (Banach-Steinhauss Theorem) and, in this case, the topolgy of uniform convergence on compact sets coincides with the weak* topology. This contradiction proves that λ belongs to the spectrum. $\square$

Remark 4.2.2. If φ satisfies the assumptions of Theorem 4.2.1 with $N=1$ and $\varphi'(0)\neq 0$, then we can complete the information about the spectrum of C_φ by using Corollary 3.4.1 to obtain

$$\{\varphi'(0)^n\}_{n=0}^{\infty}\cup\{\lambda:|\lambda|\leq\varphi'(z_0)^{-\alpha}\}\subseteq\sigma(C_\varphi,A_-^{-\alpha})\subseteq\overline{B}(0,r_e(C_\varphi,H_\alpha^\infty))\cup\{\varphi'(0)^n\}_{n=0}^{\infty}.$$

Example 4.2.3. Consider the symbol $\varphi(z)=\frac{z}{2-z},\quad z\in\mathbb{D}$. It is analytic on $\overline{\mathbb{D}}$ and 0 is an interior fixed point, while $z_0=1$ is a boundary fixed point with $\varphi'(1)=2>1$. Then, the hypotheses of Theorem 4.2.1 hold and, for a given $\alpha>0$, the composition operator $C_\varphi\colon A_-^{-\alpha}\to A_-^{-\alpha}$ verifies $\sigma(C_\varphi,A_-^{-\alpha})\supseteq\{\lambda:|\lambda|\leq 2^{-\alpha}\}$. In [21, Example 1] it is noticed that $\varphi_n(z)=\frac{z}{2^n-(2^n-1)z}$.

Recall that the essential norm of $C_\varphi\colon H_\alpha^\infty\to H_\alpha^\infty$ can be computed according to $\|C_\varphi\|_e=\limsup_{|z|\to 1}\frac{(1-|z|)^\alpha}{(1-|\varphi(z)|)^\alpha}$ (see Section 3.2). Moreover, from [27, Proposition 2.46] we obtain that $\|C_\varphi\|_e=\max_{|\xi|=1}|\varphi'(\xi)|^{-\alpha}$. For the iterates φ^n, this maximum is achieved at $\xi=1$, with value $(\varphi^n)'(1)=2^n$. So $\|C_\varphi^n\|_e=(2^n)^{-\alpha}$, from where it follows that $r_e(C_\varphi,H_\alpha^\infty)=(\frac{1}{2})^\alpha$. And $\sigma(C_\varphi,H_\alpha^\infty)=\{0\}\cup\{\varphi'(0)^n:n\in\mathbb{N}\}\cup\overline{B}(0,r_e(C_\varphi,H_\alpha^\infty))$ as proved in [7, Theorem 8]. Realize that in this example $\sigma(C_\varphi,H_\alpha^\infty)=\sigma(C_\varphi,A_-^{-\alpha})$.

The following theorem is analogous to Theorem 4.2.1 for the Fréchet space $A_+^{-\alpha}$. Since the proof is quite similar we simply mention the claims that have been proved in Theorem 4.2.1 and detail the differences. The main difference is the choice of the function f_{x_0}.

Theorem 4.2.4. *Consider $A_+^{-\alpha}$, with $\alpha\geq 0$. Suppose that $\varphi\colon\mathbb{D}\to\mathbb{D}$ has an analytic extension to an open neighbourhood of $\overline{\mathbb{D}}$ and that has a fixed point $a\in\mathbb{D}$. Suppose that there is a positive integer N such that φ_N has, at least, a fixed point z_0 in the unit sphere and that $\varphi'(z_0)>1$. Then $\sigma(C_\varphi,A_+^{-\alpha})\supseteq\{\lambda:|\lambda|\leq\varphi_N'(z_0)^{-\alpha/N}\}$.*

Proof. Without loss of generality, we assume that $a=0$. Notice that φ cannot be an automorphism of $\mathbb{D}$ since then φ_N would be as well an automorphism and its fixed point structure prevents it. Thus $0\in\sigma(C_\varphi)$. Construct the sequence $(x_k)_k$ as in Theorem 4.2.1, which verifies the statements (4.2.1), (4.2.2) and (4.2.3). Denote $c:=\varphi_N'(z_0)$ then $c>1$.

Let $|\lambda|\leq c^{-\alpha/N}$ and suppose it does not belong to $\sigma(C_\varphi)$. We now choose n_0 so large that (4.2.4) holds.

Fix $m\in\mathbb{N}$, $m>n_0$ such that $|\varphi'(0)|^m<|\lambda|$. Each $f\in A_+^{-\alpha}$ satisfies $|f(z)|\leq\|f\|_{\alpha+\varepsilon}(1-|z|)^{-(\alpha+\varepsilon)}$ for all $z\in\mathbb{D}$, $\varepsilon>0$ and so,

$$|\lambda^{nN}f(x_{-nN})|\leq\|f\|_{\alpha+\varepsilon}\frac{1}{A_0(x_0)^{\alpha+\varepsilon}}\left(\frac{|\lambda|^N}{c^{-(\alpha+\varepsilon)}}\right)^n.$$

This implies that (4.2.6) holds for all $f\in A_+^{-\alpha}$.

Consider $f,g\in A_+^{-\alpha}$ with $\lambda f-f\circ\varphi=g$. Then they satisfy equation (4.2.8).

Let us denote by $A_{+,m}^{-\alpha}$ the projective limit

$$A_{+,m}^{-\alpha}:=\bigcap_{n=1}^{+\infty}H_{\alpha+\frac{1}{n},m}^\infty=\bigcap_{\varepsilon>0}H_{\alpha+\varepsilon,m}^\infty.$$

We claim that $\big(C_\varphi-\lambda I\big)(A_{+,m}^{-\alpha})=A_{+,m}^{-\alpha}$. Let $g\in A_{+,m}^{-\alpha}$. Since $C_\varphi-\lambda I$ is onto, there is $f\in A_+^{-\alpha}$ such that $g=(C_\varphi-\lambda I)(f)$. Thus, for all $\varepsilon>0$, $f\in H_{\alpha+\varepsilon}^\infty$, and so also $g\in H_{\alpha+\varepsilon}^\infty$, hence $g\in H_{\alpha+\varepsilon,m}^\infty$. According to Lemma 4.1.4, f belongs as well to $H_{\alpha+\varepsilon,m}^\infty$, as claimed and (4.2.7) holds.

Therefore, we deduce that if $g \in (C_\varphi - \lambda I)(A^{-\alpha}_{+,m})$ and $|\varphi'(0)|^m < |\lambda|$ then,

$$\sum_{k=-\infty}^{+\infty} g(x_k)\lambda^{-k} = 0.$$

The functionals L_K are defined on $A^{-\alpha}_{+,m}$ according to the same expression as in (4.2.10). They are continuous and the sequence (L_K) is $\sigma\big((A^{-\alpha}_{+,m})', A^{-\alpha}_{+,m}\big)$-null.

In order to obtain an inequality as (4.2.11) we have to deal separately with the cases $\alpha > 0$ and $\alpha = 0$. Let's begin with $\alpha > 0$. There exists $f_{x_0} \in H^\infty_\alpha \subset A^{-\alpha}_+$ such that $||f_{x_0}||_\alpha \leq 1$ and $|f_{x_0}(x_0)| = 1/(1-|x_0|)^\alpha$. Let $1 \leq M < \infty$ be the constant in Lemma 4.1.2 for $r = 1/4$. Then, there is $f_K \in H^\infty$ with $\|f_K\|_\infty \leq M$, $|f_K(x_0)| = 1$ and satisfying

$$x_0^m f_K(x_0) f_{x_0}(x_0) > 0 \text{ and } f_K(x_k) = 0 \text{ for } -K \leq k \leq m_0,\ k \neq 0.$$

Now, the function $g_K(x) := x^m f_K(x) f_{x_0}(x)$ belongs to $H^\infty_{\alpha,m}$ and $\|g_K\|_\alpha \leq M$. Further,

$$L_K(g_K) = x_0^m f_K(x_0) f_{x_0}(x_0) + \sum_{k=m_0+1}^{+\infty} \lambda^{-k} x_k^m f_K(x_k) f_{x_0}(x_k).$$

If, in addition, we choose m so that

$$M \frac{1}{|\lambda|^{m_0}} \frac{1}{v_\alpha(x_{m_0})} \frac{b^m}{|\lambda| - b^m} < \frac{1}{2v_\alpha(x_0)} \ ,$$

and use again (4.2.3) and (4.2.4), we obtain

$$|L_K(g_K)| \geq \frac{|x_0|^m}{2v_\alpha(x_0)}. \tag{4.2.12}$$

And now the case $\alpha = 0$. The function

$$\rho(r) := \begin{cases} 1, & r \in [0, 1-\frac{1}{e}[\\ -\log(1-r), & r \in [1-\frac{1}{e}, 1[\end{cases}$$

is non-decreasing, continuous and $\lim_{r\to 1^-} \rho(r) = +\infty$. We define $v(z) := \frac{1}{\rho(|z|)}$, $z \in \mathbb{D}$. For each $0 < \varepsilon < 1$, it can be seen that there exists C_ε such that

$$(1-|z|)^\varepsilon \leq C_\varepsilon v(z), \quad \text{for all } z \in \mathbb{D}.$$

In other words, the space H^∞_v is contained in A^{-0}_+. We can take $f_{x_0} \in H^\infty_v \subset A^{-0}_+$ such that $\|f_{x_0}\|_v \leq 1$ and $|f_{x_0}(x_0)| = 1/v(x_0)$. Let $1 \leq M < \infty$ be the constant in Lemma 4.1.2 for $r = 1/4$. Then there is $f_K \in H^\infty$ with $\|f_K\|_\infty \leq M$ and $|f_K(x_0)| = 1$, satisfying

$$x_0^m f_K(x_0) f_{x_0}(x_0) > 0 \text{ and } f_K(x_k) = 0 \text{ for } -K \leq k \leq m_0,\ k \neq 0.$$

Now, the function $g_K(x) := x^m f_K(x) f_{x_0}(x)$ belongs to $H^\infty_{v,m}$ and $\|g_K\|_v \leq M$. Further,

$$L_K(g_K) = x_0^m f_K(x_0) f_{x_0}(x_0) + \sum_{k=m_0+1}^{+\infty} \lambda^{-k} x_k^m f_K(x_k) f_{x_0}(x_k).$$

If, in addition, we choose m so that

$$M\frac{1}{|\lambda|^{m_0}}\frac{1}{v(x_{m_0})}\frac{b^m}{|\lambda|-b^m} < \frac{1}{2v(x_0)} \quad ,$$

and use again (4.2.3) and (4.2.4), we obtain

$$|L_K(g_K)| \geq \frac{|x_0|^m}{2v(x_0)}. \tag{4.2.13}$$

However, since $(g_K)_K$ is a relatively compact subset of $A^{-\alpha}_{+,m}$ and the sequence $(L_K)_K$ is weak*-null and equicontinuous, we would have that $\lim_K L_K(g_K) = 0$, which contradicts (4.2.12) and (4.2.13). This contradiction means that λ must belong to the spectrum, as wanted. □

Chapter 5

Weighted composition operators on projective and inductive limits of weighted Banach spaces of vector-valued analytic functions

In this chapter we characterize several properties of weighted composition operators when acting between weighted spaces of analytic functions with values on a Banach space. These results are applied to operators between weighted inductive and projective limits of spaces of analytic functions. The case of vector-valued Korenblum type spaces is also considered. The main results of this chapter are collected in [18].

5.1 The operator $W_{\psi,\varphi}$ on weighted Banach spaces of vector-valued functions

In this section we study the continuity, compactness and weak compactness of the weighted composition operators between two weighted Banach spaces of vector-valued functions, in comparison with their equivalents in the scalar valued case.

Definition 5.1.1. Let E be a complex Banach space. Let $F\colon \mathbb{D} \to E$ be a holomorphic mapping and $v(z)$ a weight on the unit disc $\mathbb{D}$. Then,

- $F \in H_v^\infty(\mathbb{D}, E)$ if $\sup\limits_{z\in\mathbb{D}} \|F(z)\|v(z) < \infty$.
- $F \in H_v^0(\mathbb{D}, E)$ if $\lim\limits_{|z|\to 1^-} \|F(z)\|v(z) = 0$.

For the norm $\|F\|_v = \sup_{z\in\mathbb{D}} \|F(z)\|v(z)$, the spaces $H_v^\infty(\mathbb{D}, E)$ and $H_v^0(\mathbb{D}, E)$ are Banach spaces. As in the scalar case, we also have that $H_v^\infty(\mathbb{D}, E) = H_{\tilde{v}}^\infty(\mathbb{D}, E)$ and the norms $\| \,\|_v$ and $\| \,\|_{\tilde{v}}$ coincide. To prove this it is enough to consider, by the Hahn-Banach Theorem, for every $F(z) \in E$, an element $u' \in E'$ with $\|u'\| \leq 1$ such that $\|F(z)\| = u'(F(z))$ and then, the scalar function $u' \circ F$.

A subset A for a Hausdorff locally convex space E is *precompact* if for every absolutely convex 0-neighbourhood $U \subseteq E$ there exist $x_1, \dots, x_s \in A$ such that

$$A \subseteq \bigcup_{i=1}^{s} (x_i + U).$$

Every precompact set is bounded. If $\alpha \in \mathbb{K}$ and A is precompact, then αA is precompact. Moreover, a finite union of precompact sets is precompact.

Remark 5.1.2 ([37], Remark 4.4(a))**.** If for every absolutely convex 0-neighbourhood U in E there exist $y_1, \dots, y_k \in E$ such that $A \subseteq \bigcup_{i=1}^{k}(y_i + U)$, then A is precompact.

Proposition 5.1.3. *Let $F\colon \mathbb{D} \to E$ be a mapping and v a weight. Then $F \in H_v^\infty(\mathbb{D}, E)$ if, and only if, $u' \circ F \in H_v^\infty$ for all $u' \in E'$. Further, $F \in H_v^0(\mathbb{D}, E)$ if, and only if, $\{u' \circ F : \|u'\| \le 1\}$ is a precompact subset of H_v^0.*

Proof. Consider the subset $A := \{v(z)F(z) : z \in \mathbb{D}\}$ of E. Then, $u'(A) = \{u'(v(z)F(z)) : z \in \mathbb{D}\} = \{v(z)u'(F(z)) : z \in \mathbb{D}\}$, for all $u' \in E'$. Recall that A is bounded if, and only if, it is weakly bounded (see [37, Proposition 8.11]). Since, in addition F is holomorphic if, and only if, it is weakly holomorphic ([39, Theorem 8.12]), we get the first asertion.

Suppose $F \in H_v^0(\mathbb{D}, E)$. Clearly, $u' \circ F \in H_v^0$ for all $u' \in B_{E'}$. We define the mapping $\tilde{F}\colon \overline{\mathbb{D}} \to E$ by $\tilde{F}(z) := v(z)F(z)$ whenever $|z| < 1$ and $\tilde{F}(z) = 0$ when $|z| = 1$. The mapping $\tilde{F}$ is continuous in $\overline{\mathbb{D}}$ and the set $\tilde{F}(\overline{\mathbb{D}})$ is compact since it is the image of a compact set. The set $L := \{v(z)F(z) : z \in \mathbb{D}\} = \tilde{F}(\mathbb{D})$ is relatively compact because it is a subset of the compact set $\tilde{F}(\overline{\mathbb{D}})$. Since the closed unit ball $\overline{B}_{E'}$ is an equicontinuous and w^*-compact set (Banach-Alaoglu's Theorem), by Arzelà-Ascoli Theorem it is also a compact set for the topology of uniform convergence on compact subsets of E. In particular, $\overline{B}_{E'}$ is compact with the norm $\|u'\|_{\overline{L}} := \sup_{x \in \overline{L}} |u'(x)|$. Hence, given $\varepsilon > 0$, there are $u'_1, \dots, u'_n \in B_{E'}$ such that for every $u' \in B_{E'}$, there is u'_k such that

$$\sup_{z \in \mathbb{D}} | < u' - u'_k, v(z)F(z) > | < \varepsilon.$$

That is,

$$\varepsilon > \sup_{z \in \mathbb{D}} |v(z)u'(F(z)) - v(z)u'_k(F(z))| = \|u' \circ F - u'_k \circ F\|_v.$$

Or, in other words,

$$\{u' \circ F : \|u'\| \le 1\} \subset \{u'_1 \circ F, \dots, u'_n \circ F\} + B_{H_v^0}(0, \varepsilon), \tag{5.1.1}$$

which proves the precompactness of $\{u' \circ F : \|u'\| \le 1\}$.

Converserly, if $\{u' \circ F : \|u'\| \le 1\}$ is a precompact subset of H_v^0, given $\varepsilon > 0$, formula (5.1.1) holds and, since $u'_k \circ F \in H_v^0$, there is $0 < r < 1$ such that

$$\sup_{|z|>r} |v(z)(u'_k \circ F)(z)| \le \varepsilon, \quad \text{for all } k \in \{1, \dots, n\}.$$

Hence, for all $u' \in B_{E'}$, we have that $\sup_{|z|>r} |v(z)(u' \circ F)(z)| \le 2\varepsilon$. From where it follows that $F \in H_v^0(\mathbb{D}, E)$ because

$$\sup_{|z|>r} |v(z)| \|F(z)\| = \sup_{|z|>r} \sup_{u' \in B_{E'}} |v(z)(u' \circ F)(z)| \le 2\varepsilon.$$

□

We recall that for an analytic map $\varphi\colon \mathbb{D} \to \mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$, and $\psi \in H(\mathbb{D})$, the weighted composition operator is defined by $W_{\psi,\varphi} f(z) := \psi(z) f(\varphi(z))$, for all $z \in \mathbb{D}$.

Remark 5.1.4. Applying the Closed Graph Theorem we observe that if $W_{\psi,\varphi}\colon H_v^\infty(\mathbb{D}, E) \to H_w^\infty(\mathbb{D}, E)$ is well defined, then it is continuous. Indeed, we see that if $W_{\psi,\varphi}$ is well defined, then $\mathrm{Graph}(W_{\psi,\varphi})$ is closed.

Take a sequence $(F_n, W_{\psi,\varphi}(F_n))_n \subset \mathrm{Graph}(W_{\psi,\varphi})$ convergent to a certain $(F, G) \in H_v^\infty(\mathbb{D}, E) \times H_w^\infty(\mathbb{D}, E)$. On one hand, since the evaluation functionals $\delta_z\colon H_v^\infty(\mathbb{D}, E) \to E$ defined by $\delta_z(F) := F(z)$ are continuous for all $z \in \mathbb{D}$, we have that $F_n(z) \longrightarrow F(z)$ and $F_n(\varphi(z)) \longrightarrow F(\varphi(z))$ for all $z \in \mathbb{D}$. On the other hand, $W_{\psi,\varphi}(F_n(z)) \longrightarrow G(z)$ for every $z \in \mathbb{D}$. But, $W_{\psi,\varphi}(F_n(z)) = \psi(z) F_n(\varphi(z))$ and $\psi(z) F_n(\varphi(z)) \longrightarrow \psi(z) F(\varphi(z))$ for all $z \in \mathbb{D}$. Thus, $\psi \cdot F \circ \varphi = G$. In conclusion, every convergent sequence on $\mathrm{Graph}(W_{\psi,\varphi})$ converges to an element of $\mathrm{Graph}(W_{\psi,\varphi})$. That is, $\mathrm{Graph}(W_{\psi,\varphi})$ is closed.

Remark 5.1.4 gives us that the operator $W_{\psi,\varphi}\colon H_v^\infty(\mathbb{D}, E) \to H_w^\infty(\mathbb{D}, E)$ is well defined if, and only if, it is continuous. The argument also works for the space $H_v^0(\mathbb{D}, E)$ thus, we also have this characterization for $W_{\psi,\varphi}\colon H_v^0(\mathbb{D}, E) \to H_w^0(\mathbb{D}, E)$.

Remark 5.1.5. From Remark 5.1.4 and Propositions 2.0.1 and 2.0.2, we have that if $W_{\psi,\varphi}\colon H_v^0 \to H_w^0$ is well defined then $W_{\psi,\varphi}\colon H_v^\infty \to H_w^\infty$ is also well defined.

Recall that any radial, positive continuous funcion $v\colon \mathbb{D} \to \mathbb{R}^+$, which is non-increasing with respect to $|z|$ and is such that $\lim_{|z|\to 1^-} v(z) = 0$, is called a *typical weight*. To each weight v corresponds the growth condition $u\colon \mathbb{D} \to \mathbb{R}^+$, $u = 1/v$, and $B_v := \{f \in H(\mathbb{D}) : |f| \leq u\}$. A new function $\tilde{u}\colon \mathbb{D} \to \mathbb{R}^+$ is defined by $\tilde{u}(z) := \sup_{f \in B_v} |f(z)|$, and the weight associated with v is defined by $\tilde{v} := 1/\tilde{u}$. A weight v is called *essential* if there exists a constant $C > 0$ such that $v(z) \leq \tilde{v}(z) \leq C v(z)$, for all $z \in \mathbb{D}$.

Proposition 5.1.6. *Suppose that $\varphi, \psi \in H(\mathbb{D})$, $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, that v, w are two typical weights and that E is a Banach space. The weighted composition operator $W_{\psi,\varphi}\colon H_v^\infty(\mathbb{D}, E) \to H_w^\infty(\mathbb{D}, E)$ is continuous if, and only if, $W_{\psi,\varphi}\colon H_v^\infty \to H_w^\infty$ is continuous.*

Proof. First, suppose $W_{\psi,\varphi}\colon H_v^\infty \to H_w^\infty$ is continuous. By Proposition 2.0.1, we have that

$$\sup_{z \in \mathbb{D}} |\psi(z)| \frac{w(z)}{\tilde{v}(\varphi(z))} =: M < \infty.$$

That is, $|\psi(z)| w(z) \leq M \tilde{v}(\varphi(z))$ for all $z \in \mathbb{D}$. Then, for each $F \in H_v^\infty(\mathbb{D}, E)$ we obtain that:

$$\begin{aligned}\|W_{\psi,\varphi}(F)\|_w &= \sup_{z \in \mathbb{D}} \|\psi(z) F(\varphi(z))\| w(z) = \sup_{z \in \mathbb{D}} |\psi(z)| \|F(\varphi(z))\| w(z) \leq \sup_{z \in \mathbb{D}} M \|F(\varphi(z))\| \tilde{v}(\varphi(z)) \\ &= M \sup_{\xi \in \varphi(\mathbb{D})} \|F(\xi)\| \tilde{v}(\xi) \leq M \sup_{z \in \mathbb{D}} \|F(z)\| \tilde{v}(z) = M \|F\|_{\tilde{v}} = M \|F\|_v.\end{aligned}$$

On the other hand, suppose now that $W_{\psi,\varphi}\colon H_v^\infty(\mathbb{D}, E) \to H_w^\infty(\mathbb{D}, E)$ is continuous. Choose $x_0 \in E$ and $u_0' \in E'$ such that $u_0' x_0 = 1$. Consider the following diagram:

$$\begin{array}{ccc} H_v^\infty(\mathbb{D}, E) & \xrightarrow{W_{\psi,\varphi}} & H_w^\infty(\mathbb{D}, E) \\ {\scriptstyle S}\big\uparrow & & \big\downarrow{\scriptstyle T} \\ H_v^\infty & \dashrightarrow & H_w^\infty \end{array}$$

where $S(f(z)) := f(z)x_0$ for all $f \in H_v^\infty$, $z \in \mathbb{D}$ and, $T(F) := u_0' \circ F$ for all $F \in H_w^\infty(\mathbb{D}, E)$. The operators S and T are continuous since

$$\|S(f)\|_v = \sup_{z\in\mathbb{D}} v(z)\|S(f(z))\| = \sup_{z\in\mathbb{D}} v(z)\|f(z)x_0\| \leq \sup_{z\in\mathbb{D}} v(z)\|x_0\||f(z)| = \|x_0\|\|f\|_v$$

and

$$\|T(F)\| = \sup_{z\in\mathbb{D}} w(z)|T(F)(z)| = \sup_{z\in\mathbb{D}} w(z)|(u_0' \circ F)(z)| \leq \sup_{z\in\mathbb{D}} w(z)\|u_0'\|\|F(z)\| \leq \|u_0'\|\|F\|_w.$$

Now, observe that $T \circ W_{\psi,\varphi} \circ S$ is exactly the weighted composition operator $W_{\psi,\varphi}$ for the scalar case. In fact, for each $f \in H_v^\infty$ and $z \in \mathbb{D}$ we have

$$\begin{aligned}(T \circ W_{\psi,\varphi} \circ S)(f)(z) &= (T \circ W_{\psi,\varphi})(f(z)x_0) = T(\psi(z)f(\varphi(z))x_0) \\ &= u_0'(\psi(z)f(\varphi(z))x_0) = \psi(z)f(\varphi(z))u_0'x_0 = W_{\psi,\varphi}(f)(z).\end{aligned}$$

Therefore, $W_{\psi,\varphi} \colon H_v^\infty \to H_w^\infty$ is continuous since it is the composition of continuous operators. □

Proposition 5.1.7. *Suppose that $\varphi, \psi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Let v and w be two typical weights and let E be a Banach space. The weighted composition operator $W_{\psi,\varphi} \colon H_v^0(\mathbb{D}, E) \to H_w^0(\mathbb{D}, E)$ is continuous if, and only if, $W_{\psi,\varphi} \colon H_v^0 \to H_w^0$ is continuous.*

Proof. First, suppose $W_{\psi,\varphi} \colon H_v^0 \to H_w^0$ is continuous. By Proposition 2.0.2,

$$\sup_{z\in\mathbb{D}} |\psi(z)| \frac{w(z)}{\tilde{v}(\varphi(z))} =: M < \infty.$$

That is, $|\psi(z)|w(z) \leq M\tilde{v}(\varphi(z))$ for all $z \in \mathbb{D}$. By the proof of the Proposition 5.1.6, we know that for every $F \in H_v^0(\mathbb{D}, E)$, $\|W_{\psi,\varphi}(F)\|_w \leq c\|F\|_v$. It remains to see that for any $F \in H_v^0(\mathbb{D}, E)$, the function $W_{\psi,\varphi}(F)$ belongs to $H_w^0(\mathbb{D}, E)$.

By Proposition 5.1.3 we know that $\{u' \circ F : \|u'\| \leq 1\}$ is a precompact subset of H_v^0. Since for every $u' \in E'$ and $z \in \mathbb{D}$

$$(u' \circ W_{\psi,\varphi}(F))(z) = \psi(z)u'(F(\varphi(z))) = \psi(z)(u' \circ F)(\varphi(z)) = W_{\psi,\varphi}(u' \circ F)(z),$$

then $u' \circ W_{\psi,\varphi}(F) = W_{\psi,\varphi}(u' \circ F)$ and the set $\{u' \circ W_{\psi,\varphi}(F) : \|u'\| \leq 1\}$ is the image of the precompact set $\{u' \circ F : \|u'\| \leq 1\}$ by the linear and continuous map $W_{\psi,\varphi} \colon H_v^0 \to H_w^0$. This proves that for every $u' \in E$ with $\|u'\| \leq 1$ the operator $u' \circ W_{\psi,\varphi}(F)$ belongs to H_w^0 and applying again Proposition 5.1.3 we obtain that $W_{\psi,\varphi}(F) \in H_w^0(\mathbb{D}, E)$.

By using the operators $S \colon H_v^0 \to H_v^0(\mathbb{D}, E)$ and $T \colon H_w^0(\mathbb{D}, E) \to H_w^0$ defined as in Propositon 5.1.6, the converse is obtained analogously. □

Remark 5.1.8. Suppose $\dim E = N$, then $H(\mathbb{D}, E)$ is canonically isomorphic to $H(\mathbb{D})^N$. To see this, we prove that there is a canonical isomorphism between $H(\mathbb{D}, E)$ and $H(\mathbb{D}, \mathbb{C}^N)$ and also that $H(\mathbb{D}, \mathbb{C}^N) = H(\mathbb{D})^N$.

Take $F \in H(\mathbb{D}, E)$. If $i \colon E \to \mathbb{C}^N$ is a canonical isomorphism,

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{F} & E \\ & \searrow^{i\circ F} & \downarrow i \\ & & \mathbb{C}^N \end{array}$$

then $i \circ F \in H(\mathbb{D}, \mathbb{C}^N)$. The canonical isomorphism is $F \in H(\mathbb{D}, E) \longmapsto i \circ F \in H(\mathbb{D}, \mathbb{C}^N)$.

On the other hand, if we take the canonical projections we obtain that $p_j \circ (i \circ F) \in H(\mathbb{D})$ for all $1 \leq j \leq N$, because the composition of holomorphic funcions is holomorphic. In other words, $i \circ F \in H(\mathbb{D})^N$. With this, we have that $H(\mathbb{D}, \mathbb{C}^N) = H(\mathbb{D})^N$.

Remark 5.1.9. If $\dim E = N$, the space $H(\mathbb{D}, E)$ is Fréchet Montel. Recall that $H(\mathbb{D}, E) \simeq H(\mathbb{D})^N$ (Remark 5.1.8). The space $H(\mathbb{D})^N$ is Fréchet because $H(\mathbb{D})$ is. Further, $H(\mathbb{D}, E)$ is Montel. In fact, if we take a bounded sequence $(F_n)_n \subset H(\mathbb{D}, E)$, the sequence $(i \circ F_n)_n \subset H(\mathbb{D}, \mathbb{C}^N) = H(\mathbb{D})^N$ is also bounded. Since $H(\mathbb{D})$ is Montel, for the first coordinate we have that $(p_1 \circ (i \circ F_n))_n$ has a convergent subsequence denoted by $(p_1 \circ (i \circ F_{k_1}))_{k_1}$. For the second coordinate, we obtain that $(p_2 \circ (i \circ F_{k_1}))_{k_1}$ is bounded too and has a convergent subsequence, $(p_2 \circ (i \circ F_{k_2}))_{k_2}$. Repeating this procedure, we get that the sequence $(i \circ F_n)_n$ has the subsequence $(i \circ F_{k_N})_{k_N}$, which is convergent. Therefore, the subsequence $(F_{k_N})_{k_N}$ of $(F_n)_n$ is convergent too.

Another way to see it is applying that the product of Fréchet Montel spaces is also Fréchet Montel.

Remark 5.1.10. Observe that, if the operator $W_{\psi,\varphi} \colon H_v^0(\mathbb{D}, E) \to H_w^0(\mathbb{D}, E)$ is continuous, then the operators $W_{\psi,\varphi} \colon H_v^0 \to H_w^0$, $W_{\psi,\varphi} \colon H_v^\infty \to H_w^\infty$ and $W_{\psi,\varphi} \colon H_v^\infty(\mathbb{D}, E) \to H_w^\infty(\mathbb{D}, E)$ are also continuous. See, respectively, Proposition 5.1.7, Propositions 2.0.1 and 2.0.2, and Proposition 5.1.6.

Proposition 5.1.11. *Let v, w be two typical weights, $\varphi, \psi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and $\psi \in H_w^0$, and let E be a Banach space. Suppose $W_{\psi,\varphi} \colon H_v^0(\mathbb{D}, E) \to H_w^0(\mathbb{D}, E)$ is continuous. The following statements are equivalent:*

i) $W_{\psi,\varphi} \colon H_v^\infty(\mathbb{D}, E) \to H_w^\infty(\mathbb{D}, E)$ is compact,

ii) $W_{\psi,\varphi} \colon H_v^0(\mathbb{D}, E) \to H_w^0(\mathbb{D}, E)$ is compact,

iii) $W_{\psi,\varphi} \colon H_v^\infty \to H_w^\infty$ is compact and E has finite dimesion,

iv) $W_{\psi,\varphi} \colon H_v^0 \to H_w^0$ is compact and E has finite dimension.

Proof. First of all, recall that, since $W_{\psi,\varphi} \colon H_v^0(\mathbb{D}, E) \to H_w^0(\mathbb{D}, E)$ is continuous, the operator $W_{\psi,\varphi}$ is also continuous in all the other spaces used in this proposition (see Remark 5.1.10).

$\boxed{i) \Rightarrow ii)}$ Assume $W_{\psi,\varphi} \colon H_v^\infty(\mathbb{D}, E) \to H_w^\infty(\mathbb{D}, E)$ is compact. Then, $W_{\psi,\varphi} \colon H_v^0(\mathbb{D}, E) \to H_w^0(\mathbb{D}, E)$ is compact because it is the restriction of a compact operator.

$\boxed{iii) \Rightarrow i)}$ Suppose $W_{\psi,\varphi} \colon H_v^\infty \to H_w^\infty$ is compact. Since $H_v^\infty(\mathbb{D}, E)$ and $H_w^\infty(\mathbb{D}, E)$ are Banach spaces, it is enough to see that for each bounded sequence $(F_n)_n \subset H_v^\infty(\mathbb{D}, E)$, the sequence $(W_{\psi,\varphi}(F_n))_n \subset H_w^\infty(\mathbb{D}, E)$ is a relatively compact set.

Let $(F_n)_n \subset H_v^\infty(\mathbb{D}, E)$ be a sequence such that there is $M > 0$ with

$$\sup_{z \in \mathbb{D}} \|F_n(z)\| \tilde{v}(z) = \sup_{z \in \mathbb{D}} \|F_n(z)\| v(z) < M, \quad \text{for all } n \in \mathbb{N}. \tag{5.1.2}$$

Since $H_v^\infty(\mathbb{D}, E)$ is a subspace of $H(\mathbb{D}, E)$ and $(H(\mathbb{D}, E), \tau_{co})$ is a Fréchet Montel space because $\dim E < \infty$ (Remark 5.1.9), then each τ_{co}-bounded sequence has a τ_{co}-convergent subsequence. Here, τ_{co} denotes the topology of the uniform convergence on the compact subsets of $\mathbb{D}$. The sequence $(F_n)_n$ is norm bounded then, in particular, it is τ_{co}-bounded. Thus, $(F_n)_n$ has a subsequence that we denote in the same way which is τ_{co}-convergent to a certain $F \in H(\mathbb{D}, E)$. Moreover, since $\psi, \varphi \in H(\mathbb{D})$, $W_{\psi,\varphi}\colon H(\mathbb{D}, E) \to H(\mathbb{D}, E)$ is well defined and, by Closed Graph Theorem, $W_{\psi,\varphi}$ is $\tau_{co}-$continuous (analogous to Remark 5.1.4). Thus, $(W_{\psi,\varphi}(F_n))_n \underset{\tau_{co}}{\longrightarrow} W_{\psi,\varphi}(F)$. It only remains to prove that $(W_{\psi,\varphi}(F_n))_n$ converges to $W_{\psi,\varphi}(F)$ in norm, and that $W_{\psi,\varphi}(F) \in H_w^\infty(\mathbb{D}, E)$.

According to Proposition 2.0.3, the compactness of $W_{\psi,\varphi}\colon H_v^\infty \to H_w^\infty$ shows that

$$\lim_{r\to 1} \sup_{|\varphi(z)|>r} |\psi(z)| \frac{w(z)}{\tilde{v}(\varphi(z))} = 0. \tag{5.1.3}$$

Furthermore, by Proposition 3.2.3 we have that for all $\varepsilon > 0$ there is $0 < r < 1$ such that

$$\sup_{z\in\mathbb{D}\setminus\overline{B}(0,r)} |\psi(z)| \frac{w(z)}{\tilde{v}(\varphi(z))} < \varepsilon. \tag{5.1.4}$$

Now, applying (5.1.4) and (5.1.2) we obtain that there exists $m_0 \in \mathbb{N}$ such that if $n \geq m_0$ then

$$\begin{aligned}
\sup_{z\in\mathbb{D}\setminus\overline{B}(0,r)} \|(W_{\psi,\varphi}(F_n) - W_{\psi,\varphi}(F))(z)\| w(z) &= \sup_{z\in\mathbb{D}\setminus\overline{B}(0,r)} \|\psi(z)(F_n - F)(\varphi(z))\| w(z) \\
&\leq \sup_{z\in\mathbb{D}\setminus\overline{B}(0,r)} |\psi(z)| \|(F_n - F)(\varphi(z))\| \tilde{v}(\varphi(z)) \frac{w(z)}{\tilde{v}(\varphi(z))} \\
&\leq 2M \sup_{z\in\mathbb{D}\setminus\overline{B}(0,r)} |\psi(z)| \frac{w(z)}{\tilde{v}(\varphi(z))} < 2M\varepsilon.
\end{aligned}$$

Observe that, since w is a weight, there exists a positive constant C such that $w(z) \leq C$ for every $z \in \mathbb{D}$. In addition, if we apply the τ_{co}-convergence, we get that there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then

$$\sup_{z\in\overline{B}(0,r)} \|(W_{\psi,\varphi}(F_n) - W_{\psi,\varphi}(F))(z)\| w(z) \leq C \sup_{z\in\overline{B}(0,r)} \|(W_{\psi,\varphi}(F_n) - W_{\psi,\varphi}(F))(z)\| < C\varepsilon.$$

Consequently, for all $n \geq \max\{m_0, n_0\}$,

$$\begin{aligned}
\|W_{\psi,\varphi}(F_n) - W_{\psi,\varphi}(F)\|_w &= \sup_{z\in\mathbb{D}} \|(W_{\psi,\varphi}(F_n) - W_{\psi,\varphi}(F))(z)\| w(z) \\
&\leq \max\Bigg\{ \sup_{z\in\overline{B}(0,r)} \|(W_{\psi,\varphi}(F_n) - W_{\psi,\varphi}(F))(z)\| w(z), \\
&\qquad \sup_{z\in\mathbb{D}\setminus\overline{B}(0,r)} \|(W_{\psi,\varphi}(F_n) - W_{\psi,\varphi}(F))(z)\| w(z) \Bigg\} \\
&\leq \max\{C\varepsilon, 2M\varepsilon\}.
\end{aligned}$$

Then, since ε is arbitrary, these computations show that $W_{\psi,\varphi}(F_n) - W_{\psi,\varphi}(F) \in H_w^\infty(\mathbb{D}, E)$. Hence, $W_{\psi,\varphi}(F) \in H_w^\infty(\mathbb{D}, E)$ and $W_{\psi,\varphi}(F_n) \longrightarrow W_{\psi,\varphi}(F)$ since $W_{\psi,\varphi}(F) = W_{\psi,\varphi}(F) - W_{\psi,\varphi}(F_n) + W_{\psi,\varphi}(F_n)$.

$\boxed{i) \Rightarrow iii)}$ Consider the diagram of Proposition 5.1.6, with the same operators S and T,

$$\begin{array}{ccc}
H_v^\infty(\mathbb{D}, E) & \xrightarrow{W_{\psi,\varphi}} & H_w^\infty(\mathbb{D}, E) \\
{\scriptstyle S}\uparrow & & \downarrow{\scriptstyle T} \\
H_v^\infty & \xrightarrow[W_{\psi,\varphi}]{} & H_w^\infty
\end{array}$$

Since $T \circ W_{\psi,\varphi} \circ S = W_{\psi,\varphi} \colon H_v^\infty \to H_w^\infty$, is the composition of compact and continuous operators, it is also compact. On the other hand, we construct the next diagram

$$\begin{array}{ccc} H_v^\infty(\mathbb{D},E) & \xrightarrow{W_{\psi,\varphi}} & H_w^\infty(\mathbb{D},E) \\ {\scriptstyle P}\uparrow & & \downarrow{\scriptstyle Q} \\ E & \dashrightarrow & E \end{array}$$

where $P(x) = f_x$ such that $f_x \colon \mathbb{D} \to E$, $f_x(z) := x$ for all $z \in \mathbb{D}$ and, $Q \colon H_w^\infty(\mathbb{D},E) \to E$ is defined by $Q(F) = F(z_0)/\psi(z_0)$, for some $z_0 \in \mathbb{D}$ with $\psi(z_0) \neq 0$. We can take $\psi(z_0) \neq 0$ because $\psi \not\equiv 0$. Both mappings P and Q are well defined, linear and continuous. Following the diagram, we can observe that

$$\begin{aligned} Q \circ W_{\psi,\varphi} \circ P(x) &:= Q(W_{\psi,\varphi}(f_x)) = W_{\psi,\varphi}(f_x)(z_0)/\psi(z_0) = \\ &= \psi(z_0) f_x(\varphi(z_0))/\psi(z_0) = f_x(\varphi(z_0)) = x. \end{aligned}$$

Thus, $Q \circ W_{\psi,\varphi} \circ P = I$ and I is compact because it is the composition of compact and continuous operators. Moreover, since the identity operator in a Banach space $I \colon E \to E$ is compact if, and only if, the space E is finite dimensional ([28, Theorem 1.24]), we obtain the result.

iii) $\Leftrightarrow$ *iv)* This follows from Propositions 2.0.1, 2.0.2, 2.0.3, 2.0.4 and 3.2.3, since $\psi \in H_w^0$.

ii) $\Leftrightarrow$ *iv)* Consider the following diagram

$$\begin{array}{ccc} H_v^0(\mathbb{D},E) & \xrightarrow{W_{\psi,\varphi}} & H_w^0(\mathbb{D},E) \\ {\scriptstyle S}\uparrow & & \downarrow{\scriptstyle T} \\ H_v^0 & \dashrightarrow & H_w^0 \end{array}$$

where operators S and T are defined as in Proposition 5.1.6. Both are well defined, indeed, for any $f \in H_v^0$ and $F \in H_w^0(\mathbb{D},E)$,

$$\lim_{|z|\to 1} v(z)\|S(f)(z)\| = \lim_{|z|\to 1} v(z)\|f(z)x_0\| = \|x_0\| \lim_{|z|\to 1} v(z)|f(z)| = 0,$$

$$\lim_{|z|\to 1} w(z)|T(F)(z)| = \lim_{|z|\to 1} w(z)|(u_0' \circ F)(z)| \leq \|u_0'\| \lim_{|z|\to 1} w(z)\|F(z)\| = 0.$$

Moreover, consider now the diagram

$$\begin{array}{ccc} H_v^0(\mathbb{D},E) & \xrightarrow{W_{\psi,\varphi}} & H_w^0(\mathbb{D},E) \\ {\scriptstyle P}\uparrow & & \downarrow{\scriptstyle Q} \\ E & \dashrightarrow & E \end{array}$$

where operators P and Q are the same as in *i)* $\Leftrightarrow$ *iii)*. Since for every $x \in E$,

$$\lim_{|z|\to 1} v(z)\|P(x)(z)\| = \lim_{|z|\to 1} v(z)\|f_x(z)\| = \lim_{|z|\to 1} v(z)\|x\| = \|x\| \lim_{|z|\to 1} v(z) = 0,$$

operators P and Q are also well defined. The rest of the proof is analogous to *i)* $\Leftrightarrow$ *iii)*. □

Lemma 5.1.12. *Let E be a Banach space and $T \colon E \to E$ an isomorphism. If T is weakly compact, then E is reflexive.*

Proof. By [37, Proposition 23.18] it is enough to show that $\overline{B}_E$ is weakly compact. Since T is open and surjective, there is $\lambda > 0$ such that $\lambda \overline{B}_E \subseteq T(\overline{B}_E)$. The set $T(\overline{B}_E)$ is relatively weakly compact by assumption, hence so is $\lambda \overline{B}_E$, and then $\overline{B}_E$. □

Lemma 5.1.13. *Let v, w be two typical weights, E be a Banach space and $\varphi, \psi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Assume that the operator $W_{\psi,\varphi} \colon H_v^\infty \to H_w^\infty$ is compact. If E is a reflexive space, then $W_{\psi,\varphi} \colon H_v^\infty(\mathbb{D}, E) \to H_w^\infty(\mathbb{D}, E)$ is weakly compact.*

Proof. Consider the transpose operator $W_{\psi,\varphi}^t \colon (H_w^\infty)' \to (H_v^\infty)'$. The predual of H_v^∞ is defined and denoted by ${}'H_v^\infty := \{u \in (H_v^\infty)' : u|_{\overline{B}_{H_v^\infty}} \text{ is } \tau_{co}\text{-continuous}\}$, as in [20, 5(c)]. Applying [20, 5(d)] to our space, we have that

$$ {}'H_v^\infty = \overline{\text{span}\{\delta_z : z \in \mathbb{D}\}}^{(H_v^\infty)'}, \quad {}'H_w^\infty = \overline{\text{span}\{\delta_z : z \in \mathbb{D}\}}^{(H_w^\infty)'}. \tag{5.1.5} $$

Since $W_{\psi,\varphi}^t(\delta_z) = \psi(z)\delta_{\varphi(z)}$ is contained in $\text{span}\{\delta_z : z \in \mathbb{D}\}$ because it is the product of the scalar $\psi(z)$ and the functional $\delta_{\varphi(z)}$, then

$$ W_{\psi,\varphi}^t(\text{span}\{\delta_z : z \in \mathbb{D}\}) \subseteq \text{span}\{\delta_z : z \in \mathbb{D}\}. \tag{5.1.6} $$

Now, by applying 5.1.5, 5.1.6 and the continuity of $W_{\psi,\varphi}^t$, we obtain that

$$ W_{\psi,\varphi}^t({}'H_w^\infty) = W_{\psi,\varphi}^t(\overline{\text{span}\{\delta_z : z \in \mathbb{D}\}}) \subseteq \overline{W_{\psi,\varphi}^t(\text{span}\{\delta_z : z \in \mathbb{D}\})} \subseteq \overline{\text{span}\{\delta_z : z \in \mathbb{D}\}} = {}'H_v^\infty. $$

Thus, the restricted operator $W_{\psi,\varphi}^t|_{{}'H_w^\infty} \colon {}'H_w^\infty \to {}'H_v^\infty$ is well defined.

On the other hand, consider the following diagram:

$$ \begin{array}{ccc} H_v^\infty(\mathbb{D}, E) & \xrightarrow{\quad W_{\psi,\varphi} \quad} & H_w^\infty(\mathbb{D}, E) \\ \phi \big\downarrow & & \big\uparrow \chi \\ \mathcal{L}({}'H_v^\infty, E) & \xrightarrow[W_{\psi,\varphi}^t|_{{}'H_w^\infty} \wedge I_E]{} & \mathcal{L}({}'H_w^\infty, E) \end{array} $$

where operators χ and ϕ are, respectively, the operators χ and ψ of [20, Lemma 10]. The map $\chi \colon \mathcal{L}({}'H_w^\infty, E) \to H_w^\infty(\mathbb{D}, E)$ is defined by $\chi(T) := T \circ \Delta$, where $\Delta \colon \mathbb{D} \to {}'H_w^\infty$ is given by $\Delta(z) = \delta_z$. For a fixed $F \in H_v^\infty(\mathbb{D}, E)$, the map $\phi \colon H_v^\infty(\mathbb{D}, E) \to \mathcal{L}({}'H_v^\infty, E)$ is defined by $(\phi(F)(g))(u') := g(u' \circ F)$ for all $g \in {}'H_v^\infty$ and $u' \in E'$. Both operators are well defined, linear, continuous and their norms are less or equal to 1. The wedge operator $W_{\psi,\varphi}^t|_{{}'H_w^\infty} \wedge I_E \colon \mathcal{L}({}'H_v^\infty, E) \to \mathcal{L}({}'H_w^\infty, E)$ maps each operator $X \in \mathcal{L}({}'H_v^\infty, E)$ to the composed operator $I_E \circ X \circ W_{\psi,\varphi}^t|_{{}'H_w^\infty}$, that is,

$$ {}'H_w^\infty \xrightarrow{W_{\psi,\varphi}^t|_{{}'H_w^\infty}} {}'H_v^\infty \xrightarrow{\quad X \quad} E \xrightarrow{\quad I_E \quad} E, \qquad {}'H_w^\infty \xrightarrow{I_E \circ X \circ W_{\psi,\varphi}^t|_{{}'H_w^\infty}} E $$

Now, since for every $F \in H_v^\infty(\mathbb{D}, E)$

$$ \begin{aligned} (\chi \circ (W_{\psi,\varphi}^t|_{{}'H_w^\infty} \wedge I_E) \circ \phi)(F)(z) &= \chi \circ (I_E \circ \phi(F) \circ W_{\psi,\varphi}^t|_{{}'H_w^\infty})(z) = \\ &= (I_E \circ \phi(F) \circ W_{\psi,\varphi}^t|_{{}'H_w^\infty}) \circ \Delta(z) = \\ &= I_E \circ \phi(F) \circ (W_{\psi,\varphi}^t|_{{}'H_w^\infty}(\delta_z)) = \phi(F)(\psi(z)\delta_{\varphi(z)}) = \\ &= \psi(z)\phi(F)(\delta_{\varphi(z)}) = \psi(z)F(\varphi(z)) = W_{\psi,\varphi}(F)(z) \end{aligned} $$

for all $z \in \mathbb{D}$, then $W_{\psi,\varphi} = \chi \circ (W^t_{\psi,\varphi}|_{'H^\infty_w} \wedge I_E) \circ \phi$. Moreover, taking into account that in Banach spaces an operator is reflexive in the sense of [17] if, and only if, it is weakly compact, we can apply [17, Corollary 2.11] (or [41, Theorem 2.9] applied to four different spaces, as it is said in the comment below the theorem) and obtain that $W_{\psi,\varphi}\colon H^\infty_v(\mathbb{D},E) \to H^\infty_w(\mathbb{D},E)$ is weakly compact, since $W^t_{\psi,\varphi}|_{'H^\infty_w}$ is compact, I_E is weakly compact and χ and ϕ are continuous. □

Proposition 5.1.14. *Let $\varphi, \psi \in H(\mathbb{D})$, v, w be two typical weights and let E be a Banach space. Suppose $W_{\psi,\varphi}\colon H^0_v(\mathbb{D},E) \to H^0_w(\mathbb{D},E)$ is continuous. The following statements are equivalent:*

i) $W_{\psi,\varphi}\colon H^\infty_v(\mathbb{D},E) \to H^\infty_w(\mathbb{D},E)$ is weakly compact,

ii) $W_{\psi,\varphi}\colon H^0_v(\mathbb{D},E) \to H^0_w(\mathbb{D},E)$ is weakly compact,

iii) $W_{\psi,\varphi}\colon H^\infty_v \to H^\infty_w$ is compact and E is reflexive,

iv) $W_{\psi,\varphi}\colon H^0_v \to H^0_w$ is compact and E is reflexive.

Proof. $\boxed{i) \Rightarrow ii)}$ Assume $W_{\psi,\varphi}\colon H^\infty_v(\mathbb{D},E) \to H^\infty_w(\mathbb{D},E)$ is weakly compact. Since $W_{\psi,\varphi}\colon H^0_v(\mathbb{D},E) \to H^0_w(\mathbb{D},E)$ is well defined (because it is continuous by hypothesis) then it is weakly compact because it is the restriction of a weakly compact operator.

$\boxed{ii) \Rightarrow iv)}$ Consider the following diagram

$$\begin{array}{ccc} H^0_v(\mathbb{D},E) & \xrightarrow{W_{\psi,\varphi}} & H^0_w(\mathbb{D},E) \\ S\big\uparrow & & \big\downarrow T \\ H^0_v & \dashrightarrow & H^0_w \end{array}$$

where the operators S and T are the same as in Proposition 5.1.6. As we have seen in the proof of Proposition 5.1.11, both operators are well defined. Thus, since $T \circ W_{\psi,\varphi} \circ S = W_{\psi,\varphi}\colon H^0_v \to H^0_w$, is the composition of weakly compact and continuous operators, it is also weakly compact. If it was not compact, by [26, Theorem 5.1], there would exist a subspace $F \subset H^0_v$ isomorphic to c_0 such that $W_{\psi,\varphi}|_F\colon F \to F$ would be an isomorphism. However, since $W_{\psi,\varphi}|_F$ is weakly compact, by Lemma 5.1.12, c_0 would be reflexive, which is a contradiction (see [37, Corollary 7.10]).

On the other hand, consider now the diagram

$$\begin{array}{ccc} H^0_v(\mathbb{D},E) & \xrightarrow{W_{\psi,\varphi}} & H^0_w(\mathbb{D},E) \\ P\big\uparrow & & \big\downarrow Q \\ E & \dashrightarrow & E \end{array}$$

where operators P and Q are the same as in Proposition 5.1.11. Since $Q \circ W_{\psi,\varphi} \circ P = I_E$, I_E is weakly compact because it is the composition of weakly compact and continuous operators. Then, by Lemma 5.1.12, E is reflexive.

$\boxed{iii) \Leftrightarrow iv)}$ The same way as in Proposition 5.1.11, this follows from Propositions 2.0.1, 2.0.2, 2.0.3, 2.0.4 and 3.2.3, since $\psi \in H^0_w$.

$\boxed{iii) \Rightarrow i)}$ It is done in Lemma 5.1.13. □

5.2 Inductive limits of weighted Banach spaces of vector-valued functions

Let $V = (v_n)_n$ be a sequence of strictly positive, radial, typical, continuous, decreasing weights on $\mathbb{D}$ such that $v_n(z) \geq v_{n+1}(z)$ for each $n \in \mathbb{N}$ and $z \in \mathbb{D}$. We assume sometimes later that for every $n \in \mathbb{N}$ there exists $m > n$ such that

$$\lim_{r \to 1^-} \frac{v_m(r)}{v_n(r)} = 0. \tag{V}$$

This is condition (V) as described in [10, Section 0.4].

Let E be a Banach space. The space $VH(\mathbb{D}, E)$ is defined as the inductive limit of the Banach spaces $H^\infty_{v_n}(\mathbb{D}, E)$. That is,

$$VH(\mathbb{D}, E) := \operatorname*{ind}_n H^\infty_{v_n}(\mathbb{D}, E).$$

This space is a DF-space (see [25, Proposition 8.3.16] and [36, (5) p. 403]). In the same way the space $V_0H(\mathbb{D}, E)$ is defined:

$$V_0H(\mathbb{D}, E) := \operatorname*{ind}_n H^0_{v_n}(\mathbb{D}, E).$$

Observe that, in the scalar case, Korenblum type LB-spaces $A^{-\infty}$ and $A^{-\alpha}_{-}$ with $\alpha > 0$ are of this type. The space $VH(\mathbb{D}, E)$ is an LB-space because each of the step spaces $H^\infty_{v_n}(\mathbb{D}, E)$ is a Banach space.

Definition 5.2.1. A locally convex inductive limit $F = \text{ind}_n F_n$ is called:

- *regular* if every bounded subset of F is contained and bounded in a step F_n;
- *compactly regular* if every compact subset of F is contained and compact in a step F_n;
- *boundedly retractive* if every bounded subset B of F is contained in a step F_n and the topologies of F and F_n coincide on B;
- *strongly boundedly retractive* if F is regular and, for each $n \in \mathbb{N}$ there is $m > n$ such that F and F_m induce the same topology on the bounded subsets of F_n;
- *sequentially retractive* if every convergent sequence in F is contained in a step F_n and converges there.

We say that a Hausdorff locally convex space X satisfies the *countable neighbourhood property (c.n.p.)* if for every sequence $(U_n)_n$ of 0−neighbourhoods in X there are $c_n > 0$ such that $\bigcap_n c_n U_n$ is a 0−neighbourhood in X. For more information about spaces satisfying the countable neighbourhood property, see [15].

In Proposition 5.2.2, we show that $VH(\mathbb{D}, E)$ is a regular LB-space. Further, if condition (V) is satisfied, $VH(\mathbb{D}, E)$ is strongly boundedly retractive (Proposition 5.2.4). In particular, every relatively (weakly) compact subset of $VH(\mathbb{D}, E)$ is contained and satisfies the same properties in a step. See Proposition 5.2.7.

Observe that, if given $n \in \mathbb{N}$, m is selected as in condition (V), then $H^\infty_{v_n}(\mathbb{D}, E) \subseteq H^0_{v_m}(\mathbb{D}, E)$ with continuous inclusion. Therefore, condition (V) implies that $VH(\mathbb{D}, E) = V_0H(\mathbb{D}, E)$ (with the same locally convex topology).

Proposition 5.2.2. *If E is a Banach space, then $VH(\mathbb{D}, E)$ is a regular LB-space.*

Proof. Consider the following inductive limit of spaces of continuous functions

$$VC(\mathbb{D}, E) := \operatorname*{ind}_n C_{v_n}(\mathbb{D}, E)$$

where $C_{v_n}(\mathbb{D}, E) := \{F \in C(\mathbb{D}, E) : \sup_{z \in \mathbb{D}} v_n(z)\|F(z)\| < \infty\}$ and $C(\mathbb{D}, E)$ denotes the space of all continuous functions from $\mathbb{D}$ into E. For every $n \in \mathbb{N}$, the space $C_{v_n}(\mathbb{D}, E)$ is a Banach space with the norm $\|F\|_{v_n} = \sup_{z \in \mathbb{D}} v_n(z)\|F(z)\|$.

The space E satisfies the c.n.p.. Indeed, let $(U_n)_n$ be a sequence of 0-neighbourhoods. Then, there exists a sequence of balls centered in 0, $(B_n)_n$, with radii r_n, such that $B_n \subseteq U_n$ for all $n \in \mathbb{N}$. Take $(c_n)_n := (1/r_n)_n$, thus, $\bigcap_n c_n U_n \supseteq \bigcap_n c_n B_n = B(0, 1)$. Then, $\bigcap_n c_n U_n$ is a 0-neighbourhood.

By [8, Corollary 2.5], $VC(\mathbb{D}, E)$ is a regular inductive limit. Moreover, since $H^\infty_{v_n}(\mathbb{D}, E) \subseteq C_{v_n}(\mathbb{D}, E)$ with continuous inclusion for all $n \in \mathbb{N}$, then $VH(\mathbb{D}, E) \subseteq VC(\mathbb{D}, E)$ with continuous inclusion by Lemma 1.2.2.

Let $B \subset VH(\mathbb{D}, E)$ be bounded, then $B \subset VC(\mathbb{D}, E)$ is bounded because the inclusion $VH(\mathbb{D}, E) \hookrightarrow VC(\mathbb{D}, E)$ is continuous. There is $n \in \mathbb{N}$ such that $B \subset C_{v_n}(\mathbb{D}, E)$ and it is bounded. Thus, $B \subset H^\infty_{v_n}(\mathbb{D}, E)$ because all elements of B are analytic, and B is bounded there because both spaces $H^\infty_{v_n}(\mathbb{D}, E)$ and $C_{v_n}(\mathbb{D}, E)$ have the same norm $\| \, \|_{v_n}$. □

The following theorem gives some equivalences of the concepts seen before, for an inductive limit of Banach spaces $E = \text{ind}_n E_n$. This theorem is [44, Theorem 6.4], where it is written for countable inductive limits of Fréchet spaces.

Theorem 5.2.3. *Let $E = ind_n E_n$ be an LB-space. The following conditions are equivalent:*

(i) E is sequentially retractive,

(ii) E is compactly regular,

(iii) E is boundedly retractive,

(iv) E is strongly boundedly retractive,

(v) E is regular and for each $n \in \mathbb{N}$ there is $m > n$ such that for all $k \geq m$ the spaces E_m and E_k induce the same topology on the bounded subsets of E_n.

Moreover, these conditions imply that E is complete.

Proposition 5.2.4. *If $V = (v_n)_n$ satisfies condition (V), then $VH(\mathbb{D}, E) = V_0 H(\mathbb{D}, E)$ is strongly boundedly retractive.*

Proof. This is stated explicitly in the consequences of condition (V) in [10, p. 114]. It is consequence (b). □

Moreover, by consequence (a) in [10, p. 114], if $m > n$ is selected for n according to (V), the spaces $H^\infty_{v_m}(\mathbb{D}, E), VH(\mathbb{D}, E)$ and the compact open topology all induce the same topology on the bounded subsets of $H^\infty_{v_n}(\mathbb{D}, E)$.

Proposition 5.2.5. *If $V = (v_n)_n$ satisfies condition (V), then every weakly (relatively) compact subset of $VH(\mathbb{D}, E)$ is contained and weakly (relatively) compact in some step $H^\infty_{v_n}(\mathbb{D}, E)$.*

This is a consequence of Proposition 5.2.7, which is a more general result. In its proof, we use the following definition.

Definition 5.2.6 ([14], Definition 10, p. 105). A countable locally convex inductive limit $E = \text{ind}_n E_n$ of metrizable locally convex spaces satisfies

- *condition (M)* if there exists an increasing sequence of absolutely convex $0-$neighbourhoods $U_n \subseteq E_n$ such that, for each $n \in \mathbb{N}$, there is $m \geq n$ with the property that, for every $k \geq m$, the topologies of E_k and E_m induce the same topology on U_n;

- *condition (M_0)* if there exists an increasing sequence of absolutely convex $0-$neighbourhoods $U_n \subseteq E_n$ such that, for each $n \in \mathbb{N}$, there is $m \geq n$ with the property that, for every $k \geq m$, for every $f \in E'_m$ and for every $\varepsilon > 0$, there exists $g \in E'_k$ with
$$|f(x) - g(x)| < \varepsilon \quad \text{for all } x \in U_n.$$

In [44] condition (M) of $\text{ind}_n E_n$ is called *acyclic*. [44, Theorem 6.4] implies that all conditions in Theorem 5.2.3 are equivalent to condition (M) for LB-spaces.

Proposition 5.2.7. *Let $E = \text{ind}_n E_n$ be a strongly boundedly retractive LB-space. Then every weakly compact subset of E is contained and weakly compact in some E_n.*

Proof. If E is strongly boundedly retractive then E has condition (M), hence condition (M_0) (see [14, p. 105]).

By condition (M_0) there is an increasing sequence $(U_n)_n$ such that U_n is an absolutely convex neighbourhood in E_n such that for every $n \in \mathbb{N}$ there exists $m > n$ with $\sigma(E_m, E'_m)|_{U_n} = \sigma(E, E')|_{U_n}$.

Take $B \subset E$ weakly compact ($\sigma(E, E')$-compact). Since E is regular, then there is $n \in \mathbb{N}$ such that $B \subset E_n$ and is bounded in E_n. Find $\lambda > 0$ with $B \subset \lambda U_n$. Since $\sigma(E, E')|_{\lambda U_n} = \sigma(E_m, E'_m)|_{\lambda U_n}$ and B is $\sigma(E, E')$-compact, it follows that B is $\sigma(E_m, E'_m)$-compact. □

Next we characterize when linear operators $T: F \to G$ between LB-spaces $F = \text{ind}_n F_n$ and $G = \text{ind}_m G_m$ are bounded, Montel, reflexive, compact or weakly compact.

Lemma 5.2.8. *Let E be a locally convex metrizable space, and let $(A_n)_n \subset E$ be a sequence of precompact subsets. Then, there exists $(\varepsilon_n)_n$ with $\varepsilon_n > 0$ for all $n \in \mathbb{N}$ such that $\bigcup_n \varepsilon_n A_n$ is precompact.*

Proof. Let $(U_n)_n$ be a basis of absolutely convex 0-neighbourhoods in E with $U_{n+1} \subseteq U_n$ for all $n \in \mathbb{N}$. Since each A_n is bounded, for each $n \in \mathbb{N}$ there exists $\varepsilon_n > 0$ with $\varepsilon_n A_n \subseteq U_n$.

Now, fix $m \in \mathbb{N}$. If $n > m$ then $\varepsilon_n A_n \subseteq U_n \subseteq U_m$ and so, $\bigcup_{n=m+1}^{\infty} \varepsilon_n A_n \subseteq U_m$. On the other hand, $\bigcup_{n=1}^{m} \varepsilon_n A_n$ is precompact. Thus, there exist $x_1, \ldots, x_s \in E$ such that $\bigcup_{n=1}^{m} \varepsilon_n A_n \subseteq \bigcup_{i=1}^{s} (x_i + U_m)$. Therefore,

$$\bigcup_{n=1}^{\infty} \varepsilon_n A_n \subseteq \bigcup_{n=1}^{m} \varepsilon_n A_n \cup \bigcup_{n=m+1}^{\infty} \varepsilon_n A_n \subseteq \bigcup_{i=1}^{s} (x_i + U_m) \cup U_m.$$

Applying Remark 5.1.2, $\bigcup_{n=1}^{\infty} \varepsilon_n A_n$ is precompact. □

Lemma 5.2.9. *Let (B_n) be a sequence of weakly relatively compact subsets in a complete metrizable locally convex space H. Then, there is a sequence $(\varepsilon_n)_n$ with $\varepsilon_n > 0$ for all $n \in \mathbb{N}$ such that $\bigcup_n \varepsilon_n B_n$ is weakly relatively compact in H.*

Proof. Let $(U_n)_n$ be a basis of absolutely convex 0-neighbourhoods in H with $U_{n+1} \subseteq U_n$ for all $n \in \mathbb{N}$. Since, for each $n \in \mathbb{N}$, B_n is bounded, there exists $\varepsilon_n > 0$ such that $\varepsilon_n B_n \subseteq U_n$.

Now, fix a 0-neighbourhood $V \subseteq H$. Thus, there exists $m \in \mathbb{N}$ with $U_k \subseteq V$ for all $k \geq m$. Moreover, $\bigcup_{i=1}^{m-1} \varepsilon_i B_i$ is weakly relatively compact (since it is a finite union of weakly relatively compact sets). By [17, Lemma 2.2(a)], there is a weakly compact set $C_V \subseteq H$ such that $\bigcup_{i=1}^{m-1} \varepsilon_i B_i \subseteq V + C_V$. Since $C_V \cup \{0\}$ is also weakly compact and $V \subseteq V + (C_V \cup \{0\})$, we have that

$$\bigcup_n \varepsilon_n B_n \subseteq V + (C_V \cup \{0\}).$$

Applying again [17, Lemma 2.2(a)] we obtain the result. □

Proposition 5.2.10. *Let $T\colon F \to G$ be a continuous linear operator between two LB-spaces $F = ind_n F_n$ and $G = ind_m G_m$.*

a) Assume that G is regular. Then T is bounded if, and only if, there is m such that $T\colon F_n \to G_m$ is continuous for all $n \in \mathbb{N}$.

b) Assume that F is regular and that G is strongly boundedly retractive.

(i) T is Montel if, and only if, for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $T(F_n) \subseteq G_m$ and $T\colon F_n \to G_m$ is compact.

(ii) T is reflexive if, and only if, for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $T(F_n) \subseteq G_m$ and $T\colon F_n \to G_m$ is weakly compact.

(iii) T is compact if, and only if, there exists $m \in \mathbb{N}$ such that $T(F_n) \subseteq G_m$ and $T\colon F_n \to G_m$ is compact for all $n \in \mathbb{N}$.

(iv) T is weakly compact if, and only if, there exists $m \in \mathbb{N}$ such that $T(F_n) \subseteq G_m$ and $T\colon F_n \to G_m$ is weakly compact for all $n \in \mathbb{N}$.

Proof. (a) If T is bounded, then there exists a 0-neighbourhood $U \subseteq F$ such that $T(U)$ is bounded in G. Since G is regular there is some $m \in \mathbb{N}$ such that $T(U) \subseteq G_m$ and is bounded there. Since U can be taken absolutely convex, it is absorbent, that is, $F = \bigcup_{k\in\mathbb{N}} kU$. Then, for each $f \in F$ there is some $k \in \mathbb{N}$ with $T(f) = kT(u) \in kG_m = G_m$, which implies that $T(F) \subseteq G_m$ and so, $T(F_n) \subseteq G_m$ for all $n \in \mathbb{N}$. Moreover, since, for each $n \in \mathbb{N}$, $U \cap F_n$ is a neighbourhood of F_n and $T(U \cap F_n)$ is bounded in G_m (because $T(U)$ is), then $T\colon F_n \to G_m$ is well defined and bounded (or, equivalently, continuous, since we are in Banach spaces).

On the other hand, assume that there is some $m \in \mathbb{N}$ such that $T\colon F_n \to G_m$ is continuous for all $n \in \mathbb{N}$. Then, there exists $m \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ there is $\varepsilon_n > 0$ such that $T(\varepsilon_n \overline{B}_{F_n}) \subseteq \overline{B}_{G_m}$. If U denotes the absolutely convex hull of $\varepsilon_n \overline{B}_{F_n}$, then U is a 0-neighbourhood in F and $T(U) \subseteq \overline{B}_{G_m}$ is bounded in G_m, hence, in G.

(b.i) Assume T is Montel. Fix $n \in \mathbb{N}$. The closed unit ball $\overline{B}_{F_n}$ of F_n is bounded in F. By hypothesis, $T(\overline{B}_{F_n})$ si relatively compact in G. Since G is strongly boundedly retractive, by Theorem 5.2.3, G is compactly regular. Then, there is some $m \in \mathbb{N}$ such that $T(\overline{B}_{F_n})$ is relatively compact in G_m. This implies that $T(\overline{B}_{F_n}) \subseteq G_m$ and

$T\colon F_n \to G_m$ is compact. Moreover, $\overline{B}_{F_n}$ is absorbent since it is absolutely convex, then $T(F_n) \subseteq G_m$.

For the converse, let $B \subseteq F$ be bounded. Since F is regular, there is some $n \in \mathbb{N}$ such that $B \subseteq F_n$ and it is bounded there. By hypothesis, $T(B) \subseteq G_m$ and $T(B)$ relatively compact in G_m, hence in G.

(b.ii) Suppose T is reflexive. Fix $n \in \mathbb{N}$. Since $\overline{B}_{F_n}$ is bounded in F, by hypothesis $T(\overline{B}_{F_n})$ is weakly relatively compact in G. Since G is strongly boundedly retractive, by Proposition 5.2.7 there exists $m \in \mathbb{N}$ such that $T(\overline{B}_{F_n})$ is weakly relatively compact in G_m. This implies that $T(\overline{B}_{F_n}) \subseteq G_m$ and $T\colon F_n \to G_m$ is weakly compact.

On the other hand, let B be a bounded set of F. Since F is regular, there is $n \in \mathbb{N}$ such that $B \subseteq F_n$ and it is bounded there. By hypothesis there is some $m \in \mathbb{N}$ such that $T(B)$ is weakly relatively compact in G_m, hence in G.

(b.iii) First, suppose T is compact. That is, there exists a 0-neighbourhood $U \subseteq F$ such that $T(U)$ is relatively compact in G. Since G is strongly boundedly retractive, by Theorem 5.2.3, it is compactly regular and so, there is $m \in \mathbb{N}$ such that $T(U)$ is relatively compact in G_m. Now, since $T(F) \subseteq G_m$ (because $T(U) \subseteq G_m$), then $T(F_n) \subseteq G_m$ for all $n \in \mathbb{N}$. Moreover, for each $n \in \mathbb{N}$, $U \cap F_n$ is a neighbourhood in F_n and $T(U \cup F_n)$ is relatively compact in G_m. Therefore, $T\colon F_n \to G_m$ is compact.

For the converse, we have, by assumption, that there is $m \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ $T(\overline{B}_{F_n})$ is relatively compact in G_m. Since in Banach spaces relatively compact and precompact sets are the same, we use Lemma 5.2.8 and we get that there exists $(\varepsilon_n)_n$, with $\varepsilon_n > 0$ for all $n \in \mathbb{N}$, such that $\bigcup_n \varepsilon_n T(\overline{B}_{F_n})$ is relatively compact in G_m.

If A denotes the absolutely convex hull of $\bigcup_n \varepsilon_n T(\overline{B}_{F_n})$, by Krein's Theorem ([36, 24.5(4) p. 325]), A is also relatively compact in G_m. Now, if we call B the absolutely convex hull of $\bigcup_n \varepsilon_n \overline{B}_{F_n}$, B is a 0-neighbourhood in F and $T(B) \subseteq A$ is relatively compact.

(b.iv) First, assume T is weakly compact. Then, there is a 0-neighbourhood U in F such that $T(U)$ is relatively weakly compact. Since G is strongly boundedly retractive, we can apply Proposition 5.2.7 and obtain that there exists some $m \in \mathbb{N}$ such that $T(U)$ is relatively weakly compact in G_m. For each $n \in \mathbb{N}$, $U \cap F_n$ is a neighbourhood in F_n. Since $T(F) \subseteq G_m$ (because $T(U) \subseteq G_m$) thus $T(F_n) \subseteq G_m$. Then, $T(U \cap F_n)$ is relatively weakly compact in G_m.

On the other hand, by hypothesis there is $m \in \mathbb{N}$ such that $T(\overline{B}_{F_n}) \subseteq G_m$ is relatively weakly compact for all $n \in \mathbb{N}$. By Lemma 5.2.9 there exist $(\varepsilon_n)_n$ with $\varepsilon_n > 0$ for all $n \in \mathbb{N}$ such that $\bigcup_n \varepsilon_n T(\overline{B}_{F_n})$ is relatively weakly compact in G_m.

If A denotes the absolutely convex hull of $\bigcup_n \varepsilon_n T(\overline{B}_{F_n})$ then A is relatively weakly compact in the Banach space G_m (see [36, 24.5(4') p. 325]). If B denotes the absolutely convex hull of $\bigcup_n \varepsilon_n \overline{B}_{F_n}$ then B is a 0-neighbourhood in F and $T(B) \subseteq A$ is relatively weakly compact.

□

5.2.1 Weighted composition operators on inductive limits of vector-valued spaces

We now investigate the weighted composition operator

$$W_{\psi,\varphi}\colon VH(\mathbb{D},E)\to VH(\mathbb{D},E)$$

where $\varphi\colon \mathbb{D}\to\mathbb{D}$ is analytic, $\psi\in H(\mathbb{D})$ and E is a Banach space. We consider $V=(v_n)_n$ and $VH(\mathbb{D},E)$ as in Section 5.2.

Since $VH(\mathbb{D},E)$ is regular (see Proposition 5.2.2), from Propositions 2.0.1, 5.1.6 and Lemma 1.2.2(i) we obtain the next result.

Proposition 5.2.11. *Let E be a Banach space, $\varphi\colon \mathbb{D}\to\mathbb{D}$ analytic and $\psi\in H(\mathbb{D})$. The following are equivalent:*

(i) $W_{\psi,\varphi}\colon VH(\mathbb{D},E)\to VH(\mathbb{D},E)$ *is continuous,*

(ii) for all $n\in\mathbb{N}$ there is $m>n$ such that $W_{\psi,\varphi}\colon H^\infty_{v_n}(\mathbb{D},E)\to H^\infty_{v_m}(\mathbb{D},E)$ is continuous,

(iii) for all $n\in\mathbb{N}$ there is $m>n$ such that $\sup_{z\in\mathbb{D}}|\psi(z)|v_m(z)/\tilde{v}_n(\varphi(z))<+\infty$.

Proposition 5.2.12. *Let E be a Banach space, $\varphi\colon \mathbb{D}\to\mathbb{D}$ analytic and $\psi\in H(\mathbb{D})$. The following are equivalent:*

(i) $W_{\psi,\varphi}\colon VH(\mathbb{D},E)\to VH(\mathbb{D},E)$ *is bounded,*

(ii) there is $m\in\mathbb{N}$ such that $W_{\psi,\varphi}\colon H^\infty_{v_n}(\mathbb{D},E)\to H^\infty_{v_m}(\mathbb{D},E)$ is continuous for all $n\in\mathbb{N}$,

(iii) there is $m\in\mathbb{N}$ such that $\sup_{z\in\mathbb{D}}|\psi(z)|v_m(z)/\tilde{v}_n(\varphi(z))<+\infty$ for all $n\in\mathbb{N}$.

Proof. Since every bounded operator is continuous and $VH(\mathbb{D},E)$ is regular, we can apply Proposition 5.2.10(a) and obtain $(i)\Leftrightarrow(ii)$. Implications $(ii)\Leftrightarrow(iii)$ are a direct consequence of Propositions 2.0.1 and 5.1.6. □

Proposition 5.2.13. *Let E be a Banach space, $\varphi\colon \mathbb{D}\to\mathbb{D}$ analytic and $\psi\in VH(\mathbb{D},\mathbb{C})$. Suppose that $V=(v_n)_n$ has condition (V) and $W_{\psi,\varphi}\colon VH(\mathbb{D},E)\to VH(\mathbb{D},E)$ is continuous. The following are equivalent:*

(i) $W_{\psi,\varphi}\colon VH(\mathbb{D},E)\to VH(\mathbb{D},E)$ *is Montel,*

(ii) for all $n\in\mathbb{N}$ there is $m>n$ such that $W_{\psi,\varphi}\colon H^\infty_{v_n}(\mathbb{D},E)\to H^\infty_{v_m}(\mathbb{D},E)$ is compact,

(iii) for all $n\in\mathbb{N}$ there is $m>n$ such that $W_{\psi,\varphi}\colon H^\infty_{v_n}\to H^\infty_{v_m}$ is compact and E has finite dimension,

(iv) E has finite dimension and for all $n\in\mathbb{N}$ there is $m>n$ such that

$$\lim_{r\longrightarrow 1^-}\sup_{|\varphi(z)|>r}|\psi(z)|\frac{v_m(z)}{\tilde{v}_n(\varphi(z))}=0.$$

Proof. Since $VH(\mathbb{D},E)$ is strongly boundedly retractive (see Proposition 5.2.4), we can apply Proposition 5.2.10(b.i) and obtain $(i)\Leftrightarrow(ii)$. Implications $(ii)\Leftrightarrow(iii)$ and $(iii)\Leftrightarrow(iv)$ are direct consequences of Propositions 5.1.11, 3.2.3, 2.0.3 and 2.0.4. □

Proposition 5.2.14. *Let E be a Banach space, $\varphi\colon \mathbb{D} \to \mathbb{D}$ analytic and $\psi \in VH(\mathbb{D}, \mathbb{C})$. Suppose that $V = (v_n)_n$ has condition (V) and $W_{\psi,\varphi}\colon VH(\mathbb{D}, E) \to VH(\mathbb{D}, E)$ is continuous. The following are equivalent:*

(i) $W_{\psi,\varphi}\colon VH(\mathbb{D}, E) \to VH(\mathbb{D}, E)$ *is reflexive,*

(ii) *for all* $n \in \mathbb{N}$ *there is* $m > n$ *such that* $W_{\psi,\varphi}\colon H^\infty_{v_n}(\mathbb{D}, E) \to H^\infty_{v_m}(\mathbb{D}, E)$ *is weakly compact,*

(iii) *for all* $n \in \mathbb{N}$ *there is* $m > n$ *such that* $W_{\psi,\varphi}\colon H^\infty_{v_n} \to H^\infty_{v_m}$ *is compact and* E *is reflexive,*

(iv) E *is reflexive and for all* $n \in \mathbb{N}$ *there is* $m > n$ *such that*

$$\lim_{r \longrightarrow 1^-} \sup_{|\varphi(z)|>r} |\psi(z)| \frac{v_m(z)}{\tilde{v}_n(\varphi(z))} = 0.$$

Proof. Since $VH(\mathbb{D}, E)$ is strongly boundedly retractive (see Proposition 5.2.4), we can apply Proposition 5.2.10(b.ii) and obtain $(i) \Leftrightarrow (ii)$. Implications $(ii) \Leftrightarrow (iii)$ and $(iii) \Leftrightarrow (iv)$ are direct consequences of Propositions 5.1.14, 3.2.3, 2.0.3 and 2.0.4. □

Proposition 5.2.15. *Let E be a Banach space, $\varphi\colon \mathbb{D} \to \mathbb{D}$ analytic and $\psi \in VH(\mathbb{D}, \mathbb{C})$. Suppose that $V = (v_n)_n$ has condition (V) and $W_{\psi,\varphi}\colon VH(\mathbb{D}, E) \to VH(\mathbb{D}, E)$ is continuous. The following are equivalent:*

(i) $W_{\psi,\varphi}\colon VH(\mathbb{D}, E) \to VH(\mathbb{D}, E)$ *is compact,*

(ii) *there is* $m \in \mathbb{N}$ *such that* $W_{\psi,\varphi}\colon H^\infty_{v_n}(\mathbb{D}, E) \to H^\infty_{v_m}(\mathbb{D}, E)$ *is compact for all* $n \in \mathbb{N}$,

(iii) *there is* $m \in \mathbb{N}$ *such that* $W_{\psi,\varphi}\colon H^\infty_{v_n} \to H^\infty_{v_m}$ *is compact for all* $n \in \mathbb{N}$ *and* E *has finite dimension,*

(iv) E *has finite dimension and there is* $m \in \mathbb{N}$ *such that*

$$\lim_{r \longrightarrow 1^-} \sup_{|\varphi(z)|>r} |\psi(z)| \frac{v_m(z)}{\tilde{v}_n(\varphi(z))} = 0 \quad \textit{for all } n \in \mathbb{N}.$$

Proof. Since $VH(\mathbb{D}, E)$ is strongly boundedly retractive (see Proposition 5.2.4), we can apply Proposition 5.2.10(b.iii) and obtain $(i) \Leftrightarrow (ii)$. Implications $(ii) \Leftrightarrow (iii)$ and $(iii) \Leftrightarrow (iv)$ are direct consequences of Propositions 5.1.11, 3.2.3, 2.0.3 and 2.0.4. □

Proposition 5.2.16. *Let E be a Banach space, $\varphi\colon \mathbb{D} \to \mathbb{D}$ analytic and $\psi \in VH(\mathbb{D}, \mathbb{C})$. Suppose that $V = (v_n)_n$ has condition (V) and $W_{\psi,\varphi}\colon VH(\mathbb{D}, E) \to VH(\mathbb{D}, E)$ is continuous. The following are equivalent:*

(i) $W_{\psi,\varphi}\colon VH(\mathbb{D}, E) \to VH(\mathbb{D}, E)$ *is weakly compact,*

(ii) *there is* $m \in \mathbb{N}$ *such that* $W_{\psi,\varphi}\colon H^\infty_{v_n}(\mathbb{D}, E) \to H^\infty_{v_m}(\mathbb{D}, E)$ *is weakly compact for all* $n \in \mathbb{N}$,

(iii) *there is* $m \in \mathbb{N}$ *such that* $W_{\psi,\varphi}\colon H^\infty_{v_n} \to H^\infty_{v_m}$ *is compact for all* $n \in \mathbb{N}$ *and* E *is reflexive,*

(iv) E *is reflexive and there is* $m \in \mathbb{N}$ *such that*

$$\lim_{r \longrightarrow 1^-} \sup_{|\varphi(z)|>r} |\psi(z)| \frac{v_m(z)}{\tilde{v}_n(\varphi(z))} = 0 \quad \textit{for all } n \in \mathbb{N}.$$

Proof. Since $VH(\mathbb{D}, E)$ is strongly boundedly retractive (see Proposition 5.2.4), we can apply Proposition 5.2.10(b.iv) and obtain $(i) \Leftrightarrow (ii)$. Implications $(ii) \Leftrightarrow (iii)$ and $(iii) \Leftrightarrow (iv)$ are direct consequences of Propositions 5.1.14, 3.2.3, 2.0.3 and 2.0.4. □

5.2.2 Weighted composition operators on Korenblum type spaces of vector-valued functions

Following the notation of Korenblum type spaces in section 1.5, we denote by $A^{-\infty}(E)$ the space

$$A^{-\infty}(E) := \bigcup_{n\in\mathbb{N}} H_n^\infty(\mathbb{D}, E) \ .$$

Here, $H_n^\infty(\mathbb{D}, E)$ denotes the weighted Banach space of vector-valued functions for the weight $v_n(z) = (1-|z|)^n$. Such weights are typical and essential (See Section 1.4.1). The space $A^{-\infty}(E)$ is endowed with the inductive limit topology: $A^{-\infty}(E) = \operatorname{ind}_n H_n^\infty(\mathbb{D}, E)$. Observe that $A^{-\infty}(E) = VH(\mathbb{D}, E)$ when $V = (v_n)_n = ((1-|z|)^n)_n$ is such decreasing sequence of weights.

Also, we denote by $A_-^{-\alpha}(E)$ the space

$$A_-^{-\alpha}(E) := \bigcup_{n\in\mathbb{N}} H_{\alpha_n}^\infty(\mathbb{D}, E) \ ,$$

where $H_{\alpha_n}^\infty(\mathbb{D}, E)$ denotes the weighted Banach space of vector-valued functions with the weight $v_{\alpha_n}(z) = (1-|z|)^{\alpha-\frac{1}{n}}$, where $n \geq n_0$ such that $\alpha - \frac{1}{n_0} > 0$. These weights are also typical and essential. The space $A_-^{-\alpha}(E)$ is endowed with the inductive limit topology: $A_-^{-\alpha}(E) = \operatorname{ind}_n H_{\alpha_n}^\infty(\mathbb{D}, E)$. Observe that $A_-^{-\alpha}(E) = VH(\mathbb{D}, E)$ when $V = (v_{\alpha_n})_n = (1-|z|)^{\alpha-\frac{1}{n}}$ is such decreasing sequence of weights.

Both spaces, $A^{-\infty}(E)$ and $A_-^{-\alpha}(E)$, are regular (see Proposition 5.2.2). Notice that both sequences of weights, $(v_n)_n$ and $(v_{\alpha_n})_n$, satisfy condition (V). Thus, $A^{-\infty}(E)$ and $A_-^{-\alpha}(E)$ are strongly boundedly retractive (see Proposition 5.2.4).

Corollary 5.2.17. *Let E be a Banach space, $\varphi\colon \mathbb{D} \to \mathbb{D}$ analytic and $\psi \in H(\mathbb{D})$. The following are equivalent:*

(i) $W_{\psi,\varphi}\colon A^{-\infty}(E) \to A^{-\infty}(E)$ is continuous,

(ii) for all $n \in \mathbb{N}$ there is $m > n$ such that $W_{\psi,\varphi}\colon H_n^\infty(\mathbb{D}, E) \to H_m^\infty(\mathbb{D}, E)$ is continuous,

(iii) for all $n \in \mathbb{N}$ there is $m > n$ such that

$$\sup_{z\in\mathbb{D}} |\psi(z)| \frac{(1-|z|)^m}{(1-|\varphi(z)|)^n} < +\infty.$$

Corollary 5.2.18. *Let E be a Banach space, $\varphi\colon \mathbb{D} \to \mathbb{D}$ analytic and $\psi \in H(\mathbb{D})$. The following are equivalent:*

(i) $W_{\psi,\varphi}\colon A_-^{-\alpha}(E) \to A_-^{-\alpha}(E)$ is continuous,

(ii) for all $n \in \mathbb{N}$ there is $m > n$ such that $W_{\psi,\varphi}\colon H_{\alpha_n}^\infty(\mathbb{D}, E) \to H_{\alpha_m}^\infty(\mathbb{D}, E)$ is continuous,

(iii) for all $n \in \mathbb{N}$ there is $m > n$ such that

$$\sup_{z\in\mathbb{D}} |\psi(z)| \frac{(1-|z|)^{\alpha-\frac{1}{m}}}{(1-|\varphi(z)|)^{\alpha-\frac{1}{n}}} < +\infty.$$

Corollaries 5.2.17 and 5.2.18 are consequences of Proposition 5.2.11.

Corollary 5.2.19. *Let E be a Banach space, $\varphi\colon \mathbb{D} \to \mathbb{D}$ analytic and $\psi \in H(\mathbb{D})$. The following are equivalent:*

(i) $W_{\psi,\varphi}\colon A^{-\infty}(E) \to A^{-\infty}(E)$ is bounded,

(ii) there is $m \in \mathbb{N}$ such that $W_{\psi,\varphi}\colon H_n^\infty(\mathbb{D},E) \to H_m^\infty(\mathbb{D},E)$ is continuous for all $n \in \mathbb{N}$,

(iii) there is $m \in \mathbb{N}$ such that

$$\sup_{z\in\mathbb{D}} |\psi(z)| \frac{(1-|z|)^m}{(1-|\varphi(z)|)^n} < +\infty \quad \textit{for all } n \in \mathbb{N}.$$

Corollary 5.2.20. *Let E be a Banach space, $\varphi\colon \mathbb{D} \to \mathbb{D}$ analytic and $\psi \in H(\mathbb{D})$. The following are equivalent:*

(i) $W_{\psi,\varphi}\colon A_-^{-\alpha}(E) \to A_-^{-\alpha}(E)$ is bounded,

(ii) there is $m \in \mathbb{N}$ such that $W_{\psi,\varphi}\colon H_{\alpha_n}^\infty(\mathbb{D},E) \to H_{\alpha_m}^\infty(\mathbb{D},E)$ is continuous for all $n \in \mathbb{N}$,

(iii) there is $m \in \mathbb{N}$ such that

$$\sup_{z\in\mathbb{D}} |\psi(z)| \frac{(1-|z|)^{\alpha-\frac{1}{m}}}{(1-|\varphi(z)|)^{\alpha-\frac{1}{n}}} < +\infty \quad \textit{for all } n \in \mathbb{N}.$$

Corollaries 5.2.19 and 5.2.20 are consequences of Proposition 5.2.12.

Corollary 5.2.21. *Let E be a Banach space, $\varphi\colon \mathbb{D} \to \mathbb{D}$ analytic and $\psi \in A^{-\infty}$. Suppose that $W_{\psi,\varphi}\colon A^{-\infty}(E) \to A^{-\infty}(E)$ is continuous. The following are equivalent:*

(i) $W_{\psi,\varphi}\colon A^{-\infty}(E) \to A^{-\infty}(E)$ is Montel,

(ii) for all $n \in \mathbb{N}$ there is $m > n$ such that $W_{\psi,\varphi}\colon H_n^\infty(\mathbb{D},E) \to H_m^\infty(\mathbb{D},E)$ is compact,

(iii) for all $n \in \mathbb{N}$ there is $m > n$ such that $W_{\psi,\varphi}\colon H_n^\infty \to H_m^\infty$ is compact and E has finite dimension,

(iv) E has finite dimension and for all $n \in \mathbb{N}$ there is $m > n$ such that

$$\lim_{r\longrightarrow 1^-} \sup_{|\varphi(z)|>r} |\psi(z)| \frac{(1-|z|)^m}{(1-|\varphi(z)|)^n} = 0.$$

Corollary 5.2.22. *Let E be a Banach space, $\varphi\colon \mathbb{D} \to \mathbb{D}$ analytic and $\psi \in A_-^{-\alpha}$. Suppose that $W_{\psi,\varphi}\colon A_-^{-\alpha}(E) \to A_-^{-\alpha}(E)$ is continuous. The following are equivalent:*

(i) $W_{\psi,\varphi}\colon A_-^{-\alpha}(E) \to A_-^{-\alpha}(E)$ is Montel,

(ii) *for all* $n \in \mathbb{N}$ *there is* $m > n$ *such that* $W_{\psi,\varphi}\colon H^\infty_{\alpha_n}(\mathbb{D}, E) \to H^\infty_{\alpha_m}(\mathbb{D}, E)$ *is compact,*

(iii) *for all* $n \in \mathbb{N}$ *there is* $m > n$ *such that* $W_{\psi,\varphi}\colon H^\infty_{\alpha_n} \to H^\infty_{\alpha_m}$ *is compact and* E *has finite dimension,*

(iv) E *has finite dimension and for all* $n \in \mathbb{N}$ *there is* $m > n$ *such that*

$$\lim_{r \longrightarrow 1^-} \sup_{|\varphi(z)|>r} |\psi(z)| \frac{(1-|z|)^{\alpha-\frac{1}{m}}}{(1-|\varphi(z)|)^{\alpha-\frac{1}{n}}} = 0.$$

Corollaries 5.2.21 and 5.2.22 are consequences of Proposition 5.2.13.

Corollary 5.2.23. *Let* E *be a Banach space,* $\varphi\colon \mathbb{D} \to \mathbb{D}$ *analytic and* $\psi \in A^{-\infty}$. *Suppose that* $W_{\psi,\varphi}\colon A^{-\infty}(E) \to A^{-\infty}(E)$ *is continuous. The following are equivalent:*

(i) $W_{\psi,\varphi}\colon A^{-\infty}(E) \to A^{-\infty}(E)$ *is reflexive,*

(ii) *for all* $n \in \mathbb{N}$ *there is* $m > n$ *such that* $W_{\psi,\varphi}\colon H^\infty_n(\mathbb{D}, E) \to H^\infty_m(\mathbb{D}, E)$ *is weakly compact,*

(iii) *for all* $n \in \mathbb{N}$ *there is* $m > n$ *such that* $W_{\psi,\varphi}\colon H^\infty_n \to H^\infty_m$ *is compact and* E *is reflexive,*

(iv) E *is reflexive and for all* $n \in \mathbb{N}$ *there is* $m > n$ *such that*

$$\lim_{r \longrightarrow 1^-} \sup_{|\varphi(z)|>r} |\psi(z)| \frac{(1-|z|)^{m}}{(1-|\varphi(z)|)^{n}} = 0.$$

Corollary 5.2.24. *Let* E *be a Banach space,* $\varphi\colon \mathbb{D} \to \mathbb{D}$ *analytic and* $\psi \in A^{-\alpha}_{-}$. *Suppose that* $W_{\psi,\varphi}\colon A^{-\alpha}_{-}(E) \to A^{-\alpha}_{-}(E)$ *is continuous. The following are equivalent:*

(i) $W_{\psi,\varphi}\colon A^{-\alpha}_{-}(E) \to A^{-\alpha}_{-}(E)$ *is reflexive,*

(ii) *for all* $n \in \mathbb{N}$ *there is* $m > n$ *such that* $W_{\psi,\varphi}\colon H^\infty_{\alpha_n}(\mathbb{D}, E) \to H^\infty_{\alpha_m}(\mathbb{D}, E)$ *is weakly compact,*

(iii) *for all* $n \in \mathbb{N}$ *there is* $m > n$ *such that* $W_{\psi,\varphi}\colon H^\infty_{\alpha_n} \to H^\infty_{\alpha_m}$ *is compact and* E *is reflexive,*

(iv) E *is reflexive and for all* $n \in \mathbb{N}$ *there is* $m > n$ *such that*

$$\lim_{r \longrightarrow 1^-} \sup_{|\varphi(z)|>r} |\psi(z)| \frac{(1-|z|)^{\alpha-\frac{1}{m}}}{(1-|\varphi(z)|)^{\alpha-\frac{1}{n}}} = 0.$$

Corollaries 5.2.23 and 5.2.24 are consequences of Proposition 5.2.14.

Corollary 5.2.25. *Let* E *be a Banach space,* $\varphi\colon \mathbb{D} \to \mathbb{D}$ *analytic and* $\psi \in A^{-\infty}$. *Suppose that* $W_{\psi,\varphi}\colon A^{-\infty}(E) \to A^{-\infty}(E)$ *is continuous. The following are equivalent:*

(i) $W_{\psi,\varphi}\colon A^{-\infty}(E) \to A^{-\infty}(E)$ *is compact,*

(ii) *there is* $m \in \mathbb{N}$ *such that* $W_{\psi,\varphi}\colon H^\infty_n(\mathbb{D}, E) \to H^\infty_m(\mathbb{D}, E)$ *is compact for all* $n \in \mathbb{N}$,

(iii) *there is* $m \in \mathbb{N}$ *such that* $W_{\psi,\varphi}\colon H^\infty_n \to H^\infty_m$ *is compact for all* $n \in \mathbb{N}$ *and* E *has finite dimension,*

(iv) E has finite dimension and there is $m \in \mathbb{N}$ such that

$$\lim_{r \longrightarrow 1^-} \sup_{|\varphi(z)|>r} |\psi(z)| \frac{(1-|z|)^m}{(1-|\varphi(z)|)^n} = 0 \quad \text{for all } n \in \mathbb{N}.$$

Corollary 5.2.26. *Let E be a Banach space, $\varphi\colon \mathbb{D} \to \mathbb{D}$ analytic and $\psi \in A_-^{-\alpha}$. Suppose that $W_{\psi,\varphi}\colon A_-^{-\alpha}(E) \to A_-^{-\alpha}(E)$ is continuous. The following are equivalent:*

(i) $W_{\psi,\varphi}\colon A_-^{-\alpha}(E) \to A_-^{-\alpha}(E)$ is compact,

(ii) there is $m \in \mathbb{N}$ such that $W_{\psi,\varphi}\colon H^\infty_{\alpha_n}(\mathbb{D},E) \to H^\infty_{\alpha_m}(\mathbb{D},E)$ is compact for all $n \in \mathbb{N}$,

(iii) there is $m \in \mathbb{N}$ such that $W_{\psi,\varphi}\colon H^\infty_{\alpha_n} \to H^\infty_{\alpha_m}$ is compact for all $n \in \mathbb{N}$ and E has finite dimension,

(iv) E has finite dimension and there is $m \in \mathbb{N}$ such that

$$\lim_{r \longrightarrow 1^-} \sup_{|\varphi(z)|>r} |\psi(z)| \frac{(1-|z|)^{\alpha-\frac{1}{m}}}{(1-|\varphi(z)|)^{\alpha-\frac{1}{n}}} = 0 \quad \text{for all } n \in \mathbb{N}.$$

Corollaries 5.2.25 and 5.2.26 are consequences of Proposition 5.2.15.

Corollary 5.2.27. *Let E be a Banach space, $\varphi\colon \mathbb{D} \to \mathbb{D}$ analytic and $\psi \in A^{-\infty}$. Suppose that $W_{\psi,\varphi}\colon A^{-\infty}(E) \to A^{-\infty}(E)$ is continuous. The following are equivalent:*

(i) $W_{\psi,\varphi}\colon A^{-\infty}(E) \to A^{-\infty}(E)$ is weakly compact,

(ii) there is $m \in \mathbb{N}$ such that $W_{\psi,\varphi}\colon H^\infty_n(\mathbb{D},E) \to H^\infty_m(\mathbb{D},E)$ is weakly compact for all $n \in \mathbb{N}$,

(iii) there is $m \in \mathbb{N}$ such that $W_{\psi,\varphi}\colon H^\infty_n \to H^\infty_m$ is compact for all $n \in \mathbb{N}$ and E is reflexive,

(iv) E is reflexive and there is $m \in \mathbb{N}$ such that

$$\lim_{r \longrightarrow 1^-} \sup_{|\varphi(z)|>r} |\psi(z)| \frac{(1-|z|)^m}{(1-|\varphi(z)|)^n} = 0 \quad \text{for all } n \in \mathbb{N}.$$

Corollary 5.2.28. *Let E be a Banach space, $\varphi\colon \mathbb{D} \to \mathbb{D}$ analytic and $\psi \in A_-^{-\alpha}$. Suppose that $W_{\psi,\varphi}\colon A_-^{-\alpha}(E) \to A_-^{-\alpha}(E)$ is continuous. The following are equivalent:*

(i) $W_{\psi,\varphi}\colon A_-^{-\alpha}(E) \to A_-^{-\alpha}(E)$ is weakly compact,

(ii) there is $m \in \mathbb{N}$ such that $W_{\psi,\varphi}\colon H^\infty_{\alpha_n}(\mathbb{D},E) \to H^\infty_{\alpha_m}(\mathbb{D},E)$ is weakly compact for all $n \in \mathbb{N}$,

(iii) there is $m \in \mathbb{N}$ such that $W_{\psi,\varphi}\colon H^\infty_{\alpha_n} \to H^\infty_{\alpha_m}$ is compact for all $n \in \mathbb{N}$ and E is reflexive,

(iv) E is reflexive and there is $m \in \mathbb{N}$ such that

$$\lim_{r \longrightarrow 1^-} \sup_{|\varphi(z)|>r} |\psi(z)| \frac{(1-|z|)^{\alpha-\frac{1}{m}}}{(1-|\varphi(z)|)^{\alpha-\frac{1}{n}}} = 0 \quad \text{for all } n \in \mathbb{N}.$$

Corollaries 5.2.27 and 5.2.28 are consequences of Proposition 5.2.16.

5.3 Projective limits of weighted Banach spaces of vector-valued functions

Let $W = (w_n)_n$ be a sequence of strictly positive, radial, typical, continuous, decreasing weights on $\mathbb{D}$ such that $w_n(z) \leq w_{n+1}(z)$ for each $n \in \mathbb{N}$ and $z \in \mathbb{D}$. We say that W satisfies condition (W) if for all $n \in \mathbb{N}$ there exists $m > n$ such that

$$\lim_{r \longrightarrow 1^-} \frac{w_n(r)}{w_m(r)} = 0. \tag{W}$$

Let E be a Banach space. The space $HW(\mathbb{D}, E)$ is defined as the projective limit of the Banach spaces $H^\infty_{w_n}(\mathbb{D}, E)$. That is,

$$HW(\mathbb{D}, E) = \bigcap_{n \in \mathbb{N}} H^\infty_{w_n}(\mathbb{D}, E) = \operatorname*{proj}_n H^\infty_{w_n}(\mathbb{D}, E).$$

This space is a Fréchet space endowed with the norms

$$\|F\|_{w_n} := \sup_{z \in \mathbb{D}} w_n(z) \|F(z)\|, \quad n \in \mathbb{N}.$$

If we assume that (W) holds and $m > n$ is selected as in (W), then $H^\infty_{w_m}(\mathbb{D}, E) \subseteq H^0_{w_n}(\mathbb{D}, E)$ and $HW(\mathbb{D}, E) = HW_0(\mathbb{D}, E) = \operatorname{proj}_n H^0_{w_n}(\mathbb{D}, E)$.

If we denote by $\overline{B}_n$ the closed unit ball of $H_{w_n}(\mathbb{D}, E)$, a basis of absolutely convex neighbourhoods of $HW(\mathbb{D}, E)$ is given by $(\frac{1}{n}\overline{B}_n \cap HW(\mathbb{D}, E)))_n$.

We associate with $W = (w_n)_n$ the following family of weights on $\mathbb{D}$:

$$V(W) := \{v \colon \mathbb{D} \to]0, +\infty[\ ;\ v \text{ is continuous, radial and } w_n v \text{ is bounded in } \mathbb{D} \text{ for each } n\}.$$

Definition 5.3.1 ([11], Definitions 1.1)**.** Let P be a set of real valued functions defined on an index set I. The set P is said to be a *Köthe set* on I if the following three properties are satisfied:

- $\alpha(i) \geq 0$ for each $i \in I$ and each $\alpha \in P$;
- for every pair $(\alpha, \beta) \in P \times P$ there exists $\gamma \in P$ such that $\max(\alpha(i), \beta(i)) \leq \gamma(i)$ for all $i \in I$;
- for each $i \in I$ there exists $\alpha \in P$ with $\alpha(i) > 0$.

Corresponding to each index set I and Köthe set P we associate the *echelon space*

$$\lambda_\infty(I, P) := \{x \colon I \to \mathbb{R} \text{ such that } \sup_{i \in I} \alpha(i)|x(i)| < \infty \text{ for all } \alpha \in P\}.$$

Lemma 5.3.2. *A subset $B \subseteq HW(\mathbb{D}, E)$ is bounded if, and only if, there is $v \in V(W)$ such that*

$$B \subseteq B_v := \{F \in HW(\mathbb{D}, E) \ :\ \|F(z)\| \leq v(z) \text{ for all } z \in \mathbb{D}\}.$$

Proof. Since for each $F \in B_v$

$$\sup_{z \in \mathbb{D}} w_n(z) \|F(z)\| \leq \sup_{z \in \mathbb{D}} w_n(z) v(z) < +\infty,$$

we have that B_v is bounded in $HW(\mathbb{D}, E)$.

Given a bounded set $B \subseteq HW(\mathbb{D}, E)$, for each $n \in \mathbb{N}$ there is M_n such that

$$\sup_{F \in B} \sup_{z \in \mathbb{D}} w_n(z) \|F(z)\| \leq M_n.$$

Then, the set $\tilde{B} := \{\|F(\cdot)\| : \mathbb{D} \to \mathbb{R} \ ; F \in B\}$ is bounded in the Köthe echelon space $\lambda_\infty(\mathbb{D}, W)$. By the characterization of bounded subsets in the Köthe echelon space of infinite order, [11, Proposition 2.5], there exists $\overline{w} \colon \mathbb{D} \to]0, +\infty[$ such that $w_n \overline{w}$ is bounded for each $n \in \mathbb{N}$ and

$$\sup_{z \in \mathbb{D}} \frac{\|F(z)\|}{\overline{w}(z)} \leq 1 \quad \text{for each } F \in B.$$

We can apply [10, Proposition in p. 112] to show that $\overline{w}$ is dominated by $v \in V(W)$. This implies that $\|F(z)\| \leq v(z)$ for all $z \in \mathbb{D}$, and $B \subseteq B_v$. □

Let X be a locally convex space, and A an absolutely convex subset of X. The space

$$X_A := \bigcup_{\mu > 0} \mu A$$

is a normed space with the norm

$$p_A(x) := \inf\{\lambda > 0 : x \in \lambda A\}, \quad x \in X_A.$$

The norm p_A is called *Minkowski functional*. For more information about this functional, we refer the reader to [37, p. 47].

Remark 5.3.3. If $(X, \|\cdot\|)$ is a Banach space and $\overline{B}_X$ its closed unit ball, then

$$\|x\| = \inf\{\lambda > 0 : x \in \lambda \overline{B}_X\}.$$

In fact, for each $x \in X$, $x/\|x\| \in \overline{B}_X$ if, and only if $x \in \|x\|\overline{B}_X$. Suppose there exists $0 < \lambda < \|x\|$ with $x \in \lambda \overline{B}_X$, that is, $x/\lambda \in \overline{B}_X$. This implies that $\|x\|/\lambda \leq 1$ and $\|x\| \leq 1$, which is a contradiction. Thus, the norm coincides with the infimum.

Lemma 5.3.4. *If $v \in V(W)$, then the Banach space $HW(\mathbb{D}, E)_{B_v}$ is isometrically isomorphic to the Banach space $H^\infty_{\frac{1}{v}}(\mathbb{D}, E)$.*

Proof. Observe that $F \in HW(\mathbb{D}, E)_{B_v}$ if, and only if, there exists $\mu > 0$ with $F \in \mu B_v$, that is, $F/\mu \in B_v$. This occurs if, and only if, $\|F(z)/\mu\| \leq v(z)$ for all $z \in \mathbb{D}$ or, equivalently, $\sup_{z \in \mathbb{D}} \|F(z)\|/v(z) \leq \mu$. That is, $F \in H^\infty_{\frac{1}{v}}(\mathbb{D}, E)$.

Moreover, since B_v is the closed unit ball of $H^\infty_{\frac{1}{v}}(\mathbb{D}, E)$, by Remark 5.3.3 we have that the norm $\|\cdot\|_{\frac{1}{v}}$ coincides with p_{B_v}. □

Given any holomorphic function $F \in H(\mathbb{D}, E)$ there is $(x_k)_k \subseteq E$ such that

$$F(z) = \sum_{k=0}^{\infty} x_k z^k, \quad z \in \mathbb{D},$$

and the series converges uniformly on the compact subsets of $\mathbb{D}$. The k-th Taylor polynomial of F is denoted by P_k:

$$P_k(z) := \sum_{j=0}^{k} x_j z^j.$$

Clearly, if w is a strictly positive radial decreasing weight on $\mathbb{D}$ with $\lim_{r\longrightarrow 1^-} w(r)=0$, then every (vector-valued) polynomial $P(z)=\sum_{j=0}^{k} x_j z^j$ belongs to $H_w^\infty(\mathbb{D},E)$.

Given $G\in H(\mathbb{D},E)$, $G(z)=\sum_{k=0}^{\infty} x_k z^k$, $z\in\mathbb{D}$, we denote by $C_t(G)$ the *Cesàro sums* of the Taylor polynomial of G. That is,

$$C_t(G)(z) := \frac{1}{t+1}\sum_{j=0}^{t}\left(\sum_{k=0}^{j} x_k z^k\right).$$

Clearly, $C_t(G)$ tends to G uniformly on compact sets in $\mathbb{D}$ whenever $t\longrightarrow +\infty$.

Lemma 5.3.5. *Let w be a strictly positive, continuous, radial weight on $\mathbb{D}$ with $\lim_{r\longrightarrow 1^-} w(r)=0$. Then, for each $G\in H_w^\infty(\mathbb{D},E)$ we have*

$$\sup_{z\in\mathbb{D}} w(z)\|C_n(G)(z)\| \le \sup_{z\in\mathbb{D}} w(z)\|G(z)\|, \quad \textit{for all } n\in\mathbb{N}.$$

Proof. For every $u'\in E'$ and $G\in H_w^\infty(\mathbb{D},E)$, $u'\circ G\in H_w^\infty$. By [12, Proposition 1.2],

$$\sup_{z\in\mathbb{D}} w(z)|C_n(u'\circ G)(z)| \le \sup_{z\in\mathbb{D}} w(z)|(u'\circ G)(z)|, \quad \text{for all } n\in\mathbb{N}.$$

Moreover, applying this inequality and the Hahn-Banach Theorem, that is, $\|x\| = \sup\{|u'(x)|, u'\in E', \|u'\|\le 1\}$, we obtain that, for each $n\in\mathbb{N}$,

$$\begin{aligned}
\sup_{z\in\mathbb{D}} w(z)\|C_n(G)(z)\| &= \sup_{z\in\mathbb{D}} w(z) \sup_{\|u'\|\le 1} |u'\circ(C_n(G))(z)| = \sup_{z\in\mathbb{D}} w(z) \sup_{\|u'\|\le 1} |C_n(u'\circ G)(z)| \\
&= \sup_{z\in\mathbb{D}} \sup_{\|u'\|\le 1} w(z)|C_n(u'\circ G)(z)| = \sup_{\|u'\|\le 1} \sup_{z\in\mathbb{D}} w(z)|C_n(u'\circ G)(z)| \\
&\le \sup_{\|u'\|\le 1} \sup_{z\in\mathbb{D}} w(z)|(u'\circ G)(z)| = \sup_{z\in\mathbb{D}} \sup_{\|u'\|\le 1} w(z)|(u'\circ G)(z)| \\
&= \sup_{z\in\mathbb{D}} w(z) \sup_{\|u'\|\le 1} |(u'\circ G)(z)| = \sup_{z\in\mathbb{D}} w(z)\|G(z)\|.
\end{aligned}$$

□

Corollary 5.3.6. *For each $F\in H_{w_n}^\infty(\mathbb{D},E)$ there is a sequence $(G_k)_k\subset HW(\mathbb{D},E)$ such that*

$$\sup_{z\in\mathbb{D}} w_n(z)\|G_k(z)\| \le \sup_{z\in\mathbb{D}} w_n(z)\|F(z)\|$$

and $G_k\longrightarrow F$ uniformly on the compact subsets of $\mathbb{D}$, as $k\longrightarrow\infty$.

Proof. It is enough to take $G_k := C_k(F)$ for all $k\in\mathbb{N}$. □

Proposition 5.3.7. *Let w be a strictly positive, continuous, radial weight on $\mathbb{D}$ with $\lim_{r\longrightarrow 1^-} w(r)=0$. For each $F\in H_w^0(\mathbb{D},E)$ the sequence $(C_k(F))_k$ converges to F in $H_w^0(\mathbb{D},E)$.*

Proof. Since $F\in H_w^0(\mathbb{D},E)$, for each $\varepsilon>0$ there is $0<r<1$ with $w(z)\|F(z)\|<\varepsilon/2$ for all $|z|>r$. Take $M>0$ such that $\max_{|z|\le r} w(z)\le M$, and choose $k_0\in\mathbb{N}$ with

$$\max_{|z|\le r}\|F(z)-C_k(F)(z)\| < \frac{\varepsilon}{2M} \quad \text{for all } k\ge k_0.$$

Then, applying [12, Lemma 1.1] and the Hahn-Banach Theorem to the vector case, we obtain that for all $k \geq k_0$,

$$\begin{aligned}
\sup_{z\in\mathbb{D}} w(z)\|F(z)-C_k(F)\| &\leq \max\left\{\max_{|z|\leq r} w(z)\|F(z)-C_k(F)(z)\|, \sup_{|z|>r} w(z)\|F(z)-C_k(F)(z)\|\right\}\\
&< \max\left\{M\frac{\varepsilon}{2M}, \frac{\varepsilon}{2} + \sup_{|z|>r} w(z)\|C_k(F)(z)\|\right\}\\
&\leq \max\left\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2} + \sup_{|z|>r} w(z)\max_{|\lambda|=1}\|F(\lambda z)\|\right\}\\
&\leq \max\left\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2} + \sup_{|z|>r} w(z)\|F(z)\|\right\} \leq \max\left\{\frac{\varepsilon}{2}, 2\frac{\varepsilon}{2}\right\} = \varepsilon.
\end{aligned}$$

That is, $\|F - C_k(F)\|_w$ tends to 0 as $k \longrightarrow \infty$. □

Corollary 5.3.8. *If the sequence $W = (w_n)_n$ satisfies condition (W), then $HW(\mathbb{D}, E)$ is dense in $H^0_{w_n}(\mathbb{D}, E)$ for each $n \in \mathbb{N}$.*

5.3.1 Weighted composition operators on projective limits of vector-valued spaces

Let H be a Hausdorff locally convex space. For each $n \in \mathbb{N}$, E_n and F_n are Banach spaces with closed unit balls C_{E_n} and C_{F_n} with norms $\|\cdot\|_{E_n}$ and $\|\cdot\|_{F_n}$ respectively. We assume that $E_{n+1} \subseteq E_n \subseteq E_1 \subseteq H$ with continuous inclusion, $C_{E_{n+1}} \subseteq C_{E_n}$, $F_{n+1} \subseteq F_n \subseteq F_1 \subseteq H$ with continuous inclusion and $C_{F_{n+1}} \subseteq C_{F_n}$ for all $n \in \mathbb{N}$.

We define the Fréchet spaces

$$E := \bigcap_n E_n, \quad F := \bigcap_n F_n,$$

both endowed with the projective limit topology.

We also assume the following two conditions:

(C1) For all $n \in \mathbb{N}$, $x \in E_n$ there is $(y_k)_k \subseteq E$ such that $y_k \longrightarrow x$ in H and $\|y_k\|_{E_n} \leq \|x\|_{E_n}$ for each k.

(C2) Each C_{F_n} is closed in H.

Proposition 5.3.9 ([17], Proposition 4.2)**.** *Let H be a Hausdorff locally convex space and $T\colon H \to H$ continuous. Let $E, F \subseteq H$ be two projective limits of Banach spaces, $E = proj_n E_n$, $F = proj_n F_n$, where $E_{n+1} \subseteq E_n \subseteq E_1 \subseteq H$ and $F_{n+1} \subseteq F_n \subseteq F_1 \subseteq H$ with continuous inclusion for all $n \in \mathbb{N}$. Also, assume conditions (C1) and (C2) are satisfied.*

a) The following conditions are equivalent:

- *(i) $T(E) \subseteq F$,*
- *(ii) $T \in \mathcal{L}(E, F)$,*
- *(iii) for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ with $T(E_m) \subseteq F_n$,*
- *(iv) for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $T\colon E_m \to F_n$ is well-defined and continuous.*

b) The following conditions are equivalent:

(i) $T\colon E \to F$ *is bounded,*

(ii) there exists $m \in \mathbb{N}$ *such that* $T(E_m) \subseteq F_n$ *for all* $n \in \mathbb{N}$*,*

(iii) there exists $m \in \mathbb{N}$ *such that* $T\colon E_m \to F_n$ *is well-defined and continuous for all* $n \in \mathbb{N}$*.*

c) The following conditions are equivalent:

(i) $T\colon E \to F$ *is Montel (respectively reflexive),*

(ii) for all absolutely convex closed bounded subset $B \subseteq E$*,* $T\colon E_B \to F_n$ *is compact (resp. weakly compact) for all* $n \in \mathbb{N}$*.*

d) The following conditions are equivalent:

(i) $T\colon E \to F$ *is compact (respectively weakly compact),*

(ii) there exists $m \in \mathbb{N}$ *such that* $T\colon E_m \to F_n$ *is compact (resp. weakly compact) for all* $n \in \mathbb{N}$*.*

Proof. Part a). The equivalences (i)⇔(ii) and (iii)⇔(iv) follow from the Closed Graph Theorem. The definition of projective topology yields (iv)⇒(ii). We show (ii)⇒(iii). Fix $m \in \mathbb{N}$. Since $T \in \mathcal{L}(E,F)$, we find $n \in \mathbb{N}$, $\lambda > 0$ such that $T(E \cap C_{E_n}) \subseteq \lambda C_{F_m}$. We show that $T(E_n) \subseteq F_m$. Given $x \in E_n$, there exists $\mu > 0$ such that $x/\mu \in C_{E_n}$. By condition (C1), there is a sequence $(y_k)_k \subseteq E \cap C_{E_n}$ such that y_k tends to x/μ in H. This implies that $T(y_k) \longrightarrow T(x)/\mu$ in H. Since $T(y_k) \in \lambda C_{F_m}$ for all $k \in \mathbb{N}$ and C_{F_m} is closed in H by condition (C2), then $T(x)/\mu \in \lambda C_{F_m}$ and so, $T(x) \in \mu\lambda C_{F_m} \subset F_m$. The other parts of the proof are exactly like in [17, Proposition 4.2].

□

In the applications we have in mind, H is $H(\mathbb{D}, E)$ with the compact-open topology, $E_n = F_n = H^\infty_{w_n}(\mathbb{D}, E)$ for each $n \in \mathbb{N}$, and

$$C_{E_n} = C_{F_n} = \{f \in H(\mathbb{D}, E) : \sup_{z\in\mathbb{D}} w_n(z)\|F(z)\| \le 1\}.$$

We denote this closed unit balls by C_n in the rest of this chapter.

Remark 5.3.10. Observe that condition (C1) for $HW(\mathbb{D}, E)$ follows from Corollary 5.3.6.

We show that condition (C2) is also satisfied. We need to see that C_n is closed in $H(\mathbb{D}, E)$ for the τ_{co} topology. Fix $n \in \mathbb{N}$. Let $(F_k)_k \subseteq C_n$ be a sequence that converges to some $F \in H(\mathbb{D}, E)$ with the τ_{co} topology. Since for every $z \in \mathbb{D}$ the set $\{z\}$ is a compact set, we have that for a fixed $z \in \mathbb{D}$, $F_k(z) \longrightarrow F(z)$. Then, for every $\varepsilon > 0$ there is k_0 such that $w_n(z)\|F_k(z) - F(z)\| < \varepsilon$ for all $k \ge k_0$. Since $F_k \in C_n$ for all $k \in \mathbb{N}$, we have that $w_n(z)\|F_k(z)\| < 1$ for each $k \in \mathbb{N}$. Therefore,

$$w_n(z)\|F(z)\| \le w_n(z)\|F_k(z) - F(z)\| + w_n(z)\|F_k(z)\| < 1 + \varepsilon, \quad \text{for all } k \ge k_0.$$

Now, since z is arbitrary, we have that $\sup_{z\in\mathbb{D}} w_n(z)\|F(z)\| \le 1$. That is, $F \in C_n$.

In the next proposition we give some characterizations that should be compared with [17, Theorem 4.3] and the comments below it.

Proposition 5.3.11. *Let $W = (w_n)_n$ be an increasing sequence of strictly positive, radial, continuous weights on $\mathbb{D}$ with $\lim_{r\longrightarrow 1^-} w_n(r) = 0$ for each $n \in \mathbb{N}$. Let $\varphi, \psi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Let E be a Banach space.*

a) The following conditions are equivalent:

(i) $W_{\psi,\varphi}\colon HW(\mathbb{D}, E) \to HW(\mathbb{D}, E)$ is continuous,

(ii) for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $W_{\psi,\varphi}\colon H^\infty_{w_m}(\mathbb{D}, E) \to H^\infty_{w_n}(\mathbb{D}, E)$ is continuous,

(iii) for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $W_{\psi,\varphi}\colon H^\infty_{w_m} \to H^\infty_{w_n}$ is continuous,

(iv) for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that

$$\sup_{z\in\mathbb{D}} \frac{w_n(z)}{\tilde{w}_m(\varphi(z))}|\psi(z)| < \infty.$$

b) The following conditions are equivalent:

(i) $W_{\psi,\varphi}\colon HW(\mathbb{D}, E) \to HW(\mathbb{D}, E)$ is bounded,

(ii) there exists $m \in \mathbb{N}$ such that $W_{\psi,\varphi}\colon H^\infty_{w_m} \to H^\infty_{w_n}$ is continuous for all $n \in \mathbb{N}$,

(iii) there exists $m \in \mathbb{N}$ such that

$$\sup_{z\in\mathbb{D}} \frac{w_n(z)}{\tilde{w}_m(\varphi(z))}|\psi(z)| < \infty \quad \text{for all } n \in \mathbb{N}.$$

c) The following conditions are equivalent:

(i) $W_{\psi,\varphi}\colon HW(\mathbb{D}, E) \to HW(\mathbb{D}, E)$ is Montel (reflexive),

(ii) for all $v \in V(W)$ there is $n \in \mathbb{N}$ such that $\psi \in H^0_{w_n}$ and $W_{\psi,\varphi}\colon H^\infty_{\frac{1}{v}}(\mathbb{D}, E) \to H^\infty_{w_n}(\mathbb{D}, E)$ is compact (resp. weakly compact),

(iii) for all $v \in V(W)$ there is $n \in \mathbb{N}$ such that $\psi \in H^0_{w_n}$, $W_{\psi,\varphi}\colon H^\infty_{\frac{1}{v}} \to H^\infty_{w_n}$ is compact and E has finite dimension (resp. E is reflexive),

(iv) E has finite dimension (resp. E is reflexive) and for all $v \in V(W)$ there is $n \in \mathbb{N}$ such that $\psi \in H^0_{w_n}$ and

$$\limsup_{|z|\longrightarrow 1^-} \frac{w_n(z)}{\widetilde{(1/v)}(\varphi(z))}|\psi(z)| = 0.$$

d) The following conditions are equivalent:

(i) $W_{\psi,\varphi}\colon HW(\mathbb{D}, E) \to HW(\mathbb{D}, E)$ is (weakly) compact,

(ii) there exists $m \in \mathbb{N}$ such that $W_{\psi,\varphi}\colon H^\infty_{w_m}(\mathbb{D}, E) \to H^\infty_{w_n}(\mathbb{D}, E)$ is compact (resp. weakly compact) and $\psi \in H^0_{w_n}$ for all $n \in \mathbb{N}$,

(iii) E has finite dimension (resp. E is reflexive) and there exists $m \in \mathbb{N}$ such that $\psi \in H^0_{w_n}$ and

$$\limsup_{|z|\longrightarrow 1^-} \frac{w_n(z)}{\tilde{w}_m(\varphi(z))}|\psi(z)| = 0 \quad \text{for all } n \in \mathbb{N}.$$

Proof. Equivalences of (a) and (b) can be directly deduced from Propositions 5.3.9, 5.1.6 and 2.0.1. Applying Propositions 5.3.9, 5.1.11, 5.1.14, 2.0.4 and Lemma 5.3.4 we obtain (c). For part (d) we use Propositions 5.3.9, 5.1.11, 5.1.14 and 2.0.4. □

5.3.2 Weighted composition operators on projective limits of Korenblum type of vector-valued functions

Following the notation of Korenblum type spaces in section 1.5, for a fixed $\alpha \geq 0$, we denote by $A_+^{-\alpha}(E)$ the space

$$A_+^{-\alpha}(E) := \bigcap_{n\in\mathbb{N}} H_{\alpha_n}^\infty(\mathbb{D}, E) \ .$$

Here, $H_{\alpha_n}^\infty(\mathbb{D}, E)$ denotes the weighted Banach space of vector-valued functions where the weights are $v_{\alpha_n}(z) = (1 - |z|)^{\alpha+\frac{1}{n}}$. It is endowed with the projective limit topology: $A_+^{-\alpha}(E) = \operatorname{proj}_n H_{\alpha_n}^\infty(\mathbb{D}, E)$. Observe that $A_+^{-\alpha}(E) = HW(\mathbb{D}, E)$ when $W = (v_{\alpha_n})_n$ is the increasing sequence of weights, which are essential (see Section 1.4.1).

Corollary 5.3.12. *Let E be a Banach space, $\alpha \geq 0$ and $\varphi, \psi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$.*

a) The following conditions are equivalent:

(i) $W_{\psi,\varphi}\colon A_+^{-\alpha}(E) \to A_+^{-\alpha}(E)$ is continuous,

(ii) for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $W_{\psi,\varphi}\colon H_{\alpha_m}^\infty(\mathbb{D}, E) \to H_{\alpha_n}^\infty(\mathbb{D}, E)$ is continuous,

(iii) for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $W_{\psi,\varphi}\colon H_{\alpha_m}^\infty \to H_{\alpha_n}^\infty$ is continuous,

(iv) for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|)^{\alpha+\frac{1}{n}}}{(1-|\varphi(z)|)^{\alpha+\frac{1}{m}}} |\psi(z)| < \infty.$$

b) The following conditions are equivalent:

(i) $W_{\psi,\varphi}\colon A_+^{-\alpha}(E) \to A_+^{-\alpha}(E)$ is bounded,

(ii) there exists $m \in \mathbb{N}$ such that $W_{\psi,\varphi}\colon H_{\alpha_m}^\infty \to H_{\alpha_n}^\infty$ is continuous for all $n \in \mathbb{N}$,

(iii) there exists $m \in \mathbb{N}$ such that

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|)^{\alpha+\frac{1}{n}}}{(1-|\varphi(z)|)^{\alpha+\frac{1}{m}}} |\psi(z)| < \infty \quad \text{for all } n \in \mathbb{N}.$$

c) The following conditions are equivalent:

(i) $W_{\psi,\varphi}\colon A_+^{-\alpha}(E) \to A_+^{-\alpha}(E)$ is Montel (reflexive),

(ii) for all $v \in V((v_{\alpha_n})_n)$ there is $n \in \mathbb{N}$ such that $\psi \in H_{\alpha_n}^0$ and $W_{\psi,\varphi}\colon H_{\frac{1}{v}}^\infty(\mathbb{D}, E) \to H_{\alpha_n}^\infty(\mathbb{D}, E)$ is compact (resp. weakly compact),

(iii) for all $v \in V((v_{\alpha_n})_n)$ there is $n \in \mathbb{N}$ such that $\psi \in H_{\alpha_n}^0$, $W_{\psi,\varphi}\colon H_{\frac{1}{v}}^\infty \to H_{\alpha_n}^\infty$ is compact and E has finite dimension (resp. E is reflexive),

(iv) E has finite dimension (resp. E is reflexive) and for all $v \in V((v_{\alpha_n})_n)$ there is $n \in \mathbb{N}$ such that $\psi \in H_{\alpha_n}^0$ and

$$\limsup_{|z|\longrightarrow 1^-} \frac{(1-|z|)^{\alpha+\frac{1}{n}}}{\widetilde{(1/v)}(\varphi(z))} |\psi(z)| = 0.$$

d) The following conditions are equivalent:

(i) $W_{\psi,\varphi}\colon A_+^{-\alpha}(E) \to A_+^{-\alpha}(E)$ *is (weakly) compact,*

(ii) there exists $m \in \mathbb{N}$ *such that* $W_{\psi,\varphi}\colon H_{\alpha_m}^\infty(\mathbb{D}, E) \to H_{\alpha_n}^\infty(\mathbb{D}, E)$ *is compact (resp. weakly compact) and* $\psi \in H_{\alpha_n}^0$ *for all* $n \in \mathbb{N}$*,*

(iii) E *has finite dimension (resp.* E *is reflexive) and there exists* $m \in \mathbb{N}$ *such that* $\psi \in H_{\alpha_n}^0$ *and*

$$\limsup_{|z|\longrightarrow 1^-} \frac{(1-|z|)^{\alpha+\frac{1}{n}}}{(1-|\varphi(z)|)^{\alpha+\frac{1}{m}}} |\psi(z)| = 0 \quad \textit{for all } n \in \mathbb{N}.$$

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