EIGENMEASURES AND STOCHASTIC DIAGONALIZATION OF BILINEAR MAPS

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Abstract. A new stochastic approach is presented to understand general spectral type problems for (not necessarily linear) functions between topological spaces. In order to show its potential applications, we construct the theory for the case of bilinear forms acting in couples of a Banach space and its dual. Our method consists of using integral representations of bilinear maps that satisfy particular domination properties, which is shown to be equivalent to having a certain spectral structure. Thus, we develop a measure-based technique for the characterization of bilinear operators having a spectral representation, introducing the notion of eigenmeasure, which will become the central tool of our formalism. Specific applications are provided for operators between finite dimensional linear spaces.

1. Introduction

Diagonalization of linear maps is one of the main tools in vector and tensor analysis. In view of their universal applications, some of the main results of the operator theory are related to the computation of eigenvectors and eigenvalues of linear maps. All areas of physics, statistics, data analysis, and all sciences using mathematical techniques use also spectral analysis. Since there is a large history on applications to these topics, it is difficult to get an updated picture of the state of the art, also because there are contributions from different areas of mathematics. However, let us try to show a panorama of different approaches that our results intend to unify.

Certainly, the spectrum contains some important properties of the operator in the linear case, which has led to the attempt to extend this notion to non-linear operators. In fact, this topic can already be considered a classic; there are books published since the middle of the 20th century on the subject (see [1, 7]). It is clear that to define the notion of diagonalizable operator is more complicated in the non-linear context, so the first difficulty appeared in how the definition of spectrum should be in this case. In the linear case, the spectral analysis is related to the solvability of the equation $\lambda x - Tx = 0$ for the map T and the identification of the set of all its solutions (eigenvalues and eigenvectors). In this respect, a reasonable definition of spectrum for non-linear operators is expected to have some restrictions: it must be reduced to the usual spectrum for linear operators, it must cover the notion of operator's eigenvalue, and it must also

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find non-trivial applications (which may not be obtained with known tools) (see [p.2][1]). The simplest nonlinear analogue of the spectral equation is given by the expression $\lambda B_0 x - Bx = 0$, where B_0 and B are nonlinear Banach-space-valued maps and B_0 is in some sense a canonical map. To obtain non-trivial situations it is necessary to assume that the non-linear map B_0 acts as the identity, and this requires setting in advance the specific non-linear operator B_0 (see [p.2][7]).

On the other hand, some probabilistic ideas for the real calculation of eigenvalues and eigenvectors coming from computational mathematics and physics appeared in the last decades of the past century. As a consequence of the historical increase in computing power, a significant number of probabilistic techniques requiring a large number of calculations began to be introduced, allowing good results to be obtained that were not possible in the past. Thus, in the nineties an approach called "stochastical diagonalization" was developed. As an specific tool in quantum mechanics, some computational methods agroupated by this label appeared in the nineteens (see for example the Introduction in [2]), based on the joint application of some exact calculus together with Monte Carlo methods for the computation of eigenvectors of relevant physical operators—the Hamiltonian—. For example, the method proposed in [2] is carried out with the aim of obtaining a good estimate of the lowest eigenvalue, associated with the so-called "ground state" in applied quantum mechanics, mainly in quantum chemistry (see [2, §3]). It is presented as a variant of the Jacobi method—which consists of the successive application of plane rotations—, and, in practical terms, is based on the construction of a set of suitable eigenvectors, choosing one of them (a trial state) by means of a Monte Carlo procedure (see [2, §5.2]). In fact, the final algorithm is based on the calculation of good approximations to one of the eigenvectors by means of a probabilistic method, in which some information about the probability distribution helps to make a correct choice.

The notion of "probabilistic diagonalization" for Hamiltonians can also be found in Section 4 (p.255) of [12]. In other related paper ([11]), the author claims that: "An exact analytic calculation, i.e. an exact diagonalization of (the operator) V, is not practical for a large number N of sites. However, the very fact that N is a large number may be turned to our profit to obtain exact results in the probabilistic sense, i.e. to obtain ratios for the coefficients. This is the basis of the method which we shall call 'probabilistic diagonalization'." Mixing quantum states is sometimes complicated, and some probabilistic ideas help to make approximate calculations, even if the probability of a state to occur is unknown (see [2, p.109]; see also [9, 15]). Sometimes the problem is produced by the truncation error associated with any subspace diagonalization calculation. In this case, Monte Carlo sampling allows to obtain good numerical results in the computation of the contribution of the remaining basis vectors not included in the initial diagonalization, combining in this way both diagonalization and Monte Carlo techniques [10]. Useful tools where obtained, and related methods were successfully checked in different contexts; see for example [8] for the comparison of the ground state of the Hubbard model by both exact and stochastic methods. All these contributions—together with a lot of more works not cited here—allowed to develop an open source project called HANDE (Highly Accurate N-Determinant) to provide computational tools for stochastic diagonalization in quantum chemistry (see [14]).

Other kind of problems of spectral analysis of operators involving stochastic components are the ones in which the operators depends on some random variables, and hence the problem is inherently stochastic ([5]). Although we are interested in introducing probabilistic elements for the spectral analysis of deterministic operators—and so the problem is different—, there are some similarities with this approach. Thus, some of the main tools of our methodological construction can also be found in the classical approach for the analysis of stochastic differential equations—for example, metric probability spaces, conditional expectations associated to martingales [13]—. Finally, let us say that some typical numerical techniques for the effective computation of eigenvalues and eigenvectors use also probabilistic arguments, even if the theoretical framework has no relation with stochastic analysis (see for example [18]).

Motivated by the introduction of this type of tools in other important scientific areas, in this work we intend to develop a theoretical framework built in analytical terms to justify the abstract foundations of the arguments underlying these tools. Based on probabilistic ideas, we face the problem of extending the notion of spectral analysis to general functions acting on topological measure spaces, without a priori assuming any algebraic structure, which will be introduced in later steps. Although the problem can be formulated in a completely general way and the fundamental results can be obtained that way, to ensure clarity of presentation we will focus our attention on the case of bilinear forms. Thus, the main aim of the present paper is to present a direct stochastic approach to the diagonalization of linear operators by means of the study of the associated bilinear forms.

We will use standard measure theory and Banach space theory concepts and notations. For vector measures and vector valued integration, we refer to the books [4, 16]. For integral domination inequalities concerning summability in Banach spaces, general separation arguments and Ky Fan's Lemma, we refer to [3]; for these topics applied in a non-linear setting, see [6]. Finally, for bilinear maps, tensor products and vector valued integration, see [17].

2. Probabilistic diagonalization: the main notion

Let us introduce first some specific concepts. If Λ is a set, we say that a real valued function $\varphi : \Lambda \to \mathbb{R}$ is *an evaluation of* Λ.

Definition 2.1. *Consider a measure space (Ω, Σ, μ), and let Λ be a set. Let Λ^* be a set of evaluations of Λ, and assume that $(\Lambda^*, \Sigma^*, \nu)$ is also a measure space. Take two functions $f, g : \Omega \to \Lambda$. We say that $x \in \Omega$ is an eigenelement of f with respect to g, Λ^* and ν if*

$$\int_N \varphi(f(x))\, d\nu(\varphi) = \lambda(x) \int_N \varphi(g(x))\, d\nu(\varphi), \quad N \in \Sigma^*,$$

where $\lambda(x)$ is a real number not depending on N.

All the elements appearing in this definition show the aim of this new abstract framework we are introducing. First, we reduce the notion of eigenelement for a function to a set of evaluations of the functions involved. Also, the fact that x belongs to a measure space allows the pointwise defined notion of eigenelement to be automatically translated into stochastic terms. Indeed, taking as a reference the transformation of x into its Dirac's measure δ_x, we can generalize the above introduced notion as follows.

We say that a measure μ defined on the space (Ω, Σ) is an **eigenmeasure** of f with respect to g, Λ^* and ν if there is a measurable function $\lambda : \Omega \to \mathbb{R}$ such that

$$\int_M \Big(\int_N \varphi(f(x))\, d\nu(\varphi) \Big)\, d\mu(x) = \int_M \lambda(x) \Big(\int_N \varphi(g(x))\, d\nu(\varphi) \Big)\, d\mu(x)$$

is well-defined —that is, all the integrals appearing act over integrable functions— and the equality holds for every $M \in \Sigma$ and $N \in \Sigma^*$.

This new definition opens the door to the use of typically stochastic ideas. For example, an element N can be fixed in the integral with respect to ν, obtaining a function λ_N that depends on N in the equation above, with the meaning of "average action" of the elements of the "dual space" Λ^*. Also, we can fix a set $M \in \Sigma$ in this equation, and the integral provides a formula for an "average eigenvalues equation" by considering all the elements of M together.

Remark 2.2. *As explained in the Introduction, our concern is to provide a generalization of the usual concepts of eigenvector and eigenvalue to a non-linear setting, introducing also stochastic ideas in the definition. Let us present now the simplest reference case. Take a diagonalizable linear continuous operator $T : E \to E$ on a Banach space E. Suppose that $x \in S_E$ is an eigenvector. Take $\Lambda^* = S_{E^*}$, the sphere of the dual space of E, and suppose that λ is the eigenvalue associated to x. Then, for every $x^* \in E^*$, $T(x) = \lambda\, x$, which holds if and only if*

$$\langle T(x), x^* \rangle = \lambda \langle x, x^* \rangle \quad \text{for all } x^* \in \Lambda^*.$$

Suppose that S_{E^*} is separable and take a sequence of elements $S_0 := \{x_n^* \in S_{E^*} : n \in \mathbb{N}\}$ that is dense in S_{E^*}. Consider the Borel regular measure $\nu_0 := \sum_{n=1}^{\infty} 2^{-n} \delta_{x_n^*}$ acting in the Borel sigma-algebra $\mathcal{B}(S_{E^*})$, where $\delta_{x_n^*}$ are the Dirac's deltas of all the elements of the set S_0.

Fix a norm one eigenvector $x_0 \in S_E$ with eigenvalue λ_0. Take also the Borel measurable space associated to S_E and the Dirac's delta δ_{x_0} in it as the measure μ. Then the following equation holds for every $M \in \mathcal{B}(S_E)$ containing x_0, and $N \in \mathcal{B}(S_{E^*})$;

$$\int_N \langle T(x_0), x^* \rangle\, d\nu_0(x^*)$$
$$= \int_M \Big(\int_N \langle T(x), x^* \rangle\, d\nu_0(x^*) \Big)\, d\delta_{x_0}(x) = \int_M \lambda_0 \Big(\int_N \langle x, x^* \rangle\, d\nu_0(x^*) \Big)\, d\delta_{x_0}(x)$$
$$= \lambda_0 \int_M \Big(\int_N \langle x, x^* \rangle\, d\nu_0(x^*) \Big)\, d\delta_{x_0}(x) = \lambda_0 \int_N \langle x, x^* \rangle\, d\nu_0(x^*).$$

(If M does not contain x_0, obviously the equalities hold also.) Using this integral equation for all the (Borel measurable) singletons $N = \{x_n^*\}$, we get that these integral equalities are equivalent to the equation $T(x) = \lambda_0 x_0$.

The theory corresponding to the generalization of this case is the main contribution of this work, and will be developed in the following sections.

3. Eigenmeasures for bilinear forms

A linear form is determined by its kernel, in the sense that, if the kernel of a linear form is a subset of the kernel of other linear form, then both forms coincide up to a constant. In particular, if for $x_1^*, x_2^* \in E^*$ we have

$$|\langle x, x_1^* \rangle| \leq K |\langle x, x_2^* \rangle| \quad \text{for all } x \in E$$

and for a certain constant K, we have that there is a real number λ such that $x_1^* = \lambda\, x_2^*$, and the converse is trivially true. Of course, this argument can be also used for the dual situation, for elements $x \in E$ considered as vectors in E^{**}. Moreover, this is true if the set of x's that are considered are dense in the sphere of E. This is the starting point of our characterization of diagonalizable bilinear forms, based on probabilistic estimates of the inequality written above. The idea of eigenmeasure explained in the previous section will be central.It provides the appropriate tool for the development of a formalism for the idea of approximate spectrum of a linear operator, which depends on the set of vectors that are considered and the set of linear forms that are used to distinguish them.

Suppose now that we are considering a linear space E of dimension equal to n, and a linear operator $T : E \to E$ that is diagonalizable. Take a set of pairwise linearly independent eigenvectors $x_1, ..., x_n$ with associated eigenvalues $\lambda_1, ..., \lambda_n$. Write $K := \max_{i=1,...,n} |\lambda_i|$. Then the following inequality holds for any finite subset $N \in \{1, ..., n\}$, and any finite set $\{x_i^* \in E^* : i \in N\}$,

$$\sum_{n\in N} |\langle T(x_i), x_i^*\rangle| \leq K\, n \max_{i\in N} |\langle x_i, x_i^*\rangle|.$$

Actually, it is immediate to see that this condition is equivalent to the fact that T is diagonalizable and $x_1, ..., x_n$ are eigenvectors associated to certain (not known) eigenvalues $\lambda_1, ..., \lambda_n$.

More can be said about. If we have the inequalities only for a concrete subset of n linear forms, they still characterize that T is diagonalizable and its eigenvectors are $x_1, ..., x_n$. This is a consequence of the following easy fact. Given a basis $\mathcal{B} = \{x_1, ..., x_n\} \subset S_E$, we define as usual the dual basis as $\mathcal{B}^* = \{x_1^*, ..., x_n^*\} \subset S_{E^*}$ to be the (unique) set of elements of the dual satisfying $\langle x_i, x_j^*\rangle = \delta_{i,j}$, $i, j \in \{1, ..., n\}$.

Remark 3.1. *Let E be an n-dimensional space. The following statemens are equivalent for a linear operator $T : E \to E$.*

(i) *T is diagonalizable with a basis of eigenvectors $\mathcal{B} = \{x_1, ..., x_n \in S_E\}$.*
(ii) *There is a constant $K > 0$ such that for every $N \subseteq \{1, ..., n\}$ and every subset of $m = |N|$ elements of the dual basis $x_{j(1)}^*, ..., y_{j(m)}^*$,*

$$\sum_{i\in N} |\langle T(x_i), x_{j(i)}^*\rangle| \leq K\, n \max_{i\in N} |\langle x_i, x_{j(i)}^*\rangle|.$$

This is just a consequence of the arguments above; if we have the inequality in (ii) for each such selection, we have that a vector $T(x_i)$ satisfies that $|\langle T(x_i), x_j^*\rangle| \leq K|\langle x_i, x_j^*\rangle|$ for all elements x_j^* of the dual basis, what implies that it is equal to 0 for all $j \neq i$. Then we have that necessarily $T(x_i) = \lambda x_i$ for some λ. The converse is a consequence of the definition of eigenvectors, just taking finite additions and for K the maximum of all the absolute values of the eigenvalues.

This suggests the definition of canonical eigenmeasure associated to a basis of eigenvectors and its dual basis. This idea will be developed throughout the paper; let us now provide a standard example for the case of finite dimensional spaces.

Example 3.2. *Consider a finite dimensional Banach space E, a diagonalizable operator $T : E \to E$ and a basis of eigenvectors $\mathcal{B} = \{x_1, ..., x_n\}$ with dual basis $\mathcal{B}^* = \{x_1^*, ..., x_n^*\}$. Consider the (maybe repeated) eigenvalues $\lambda_1, ..., \lambda_n$. Let us show how to define a natural measure structure associated to them. First, we*

can define a measure by means of the elements of $\mathcal{B}$ and other measure by the vectors in $\mathcal{B}^*$. Take the sigma-algebras of all the subsets of S_E and S_{E^*}, and the probability measures given by the formulas

$$\mu_0(M) := \frac{1}{n}\sum_{x\in\mathcal{B}} \delta_x(M), \quad M \subset S_{E^*}$$

and

$$\nu_0(N) := \frac{1}{n}\sum_{x^*\in\mathcal{B}^*} \delta_{x^*}(N), \quad N \subset S_{E^*}.$$

Then we get the following representation: for the operator T and the sets $M \subset S_E$ and $N \subset S_{E^*}$, we have that

$$\int_M \Big(\int_N \langle T(x), x^*\rangle d\nu_0(x^*)\Big) d\mu_0(x)$$

$$= \frac{1}{n}\sum_{x\in\mathcal{B}\cap M} \Big(\frac{1}{n}\sum_{x^*\in\mathcal{B}^*\cap N} \langle T(x), x^*\rangle\Big) = \frac{1}{n^2}\sum_J \lambda_j,$$

where $\{\lambda_j : j \in J\}$, are the eigenvalues associated to the eigenvectors $x_j \in M$ that satisfy that the corresponding dual elements x_j^* are also in N. We will say that the measures μ_0 and ν_0 defined above are canonical eigenmeasures for T.

We are interested in finding a characterization of when a measure is in some sense an eigenmeasure, based on integral inequalities. Our idea is to use an integral formalism with an easy interpretation in terms of stochastic notions. In order to prove it, we need first some previous results. As we will see, there is a weaker property than the existence of an eigenmeasure with an associated eigenvalues function λ that still allows some easy representation of a bilinear operator in terms of integral averages. We will call this notion a **weak eigenmeasure**, and the analysis of such a notion will be the content of the next section. Let us introduce it below.

Definition 3.3. *Let E be a reflexive Banach space. Fix a Borel regular measure ν on B_{E^*} and let μ be a Borel regular measure on B_E. We say that μ is a weak eigenmeasure for a bilinear map $B : E \times E^* \to \mathbb{R}$ if for each $N \in \mathcal{B}(B_{E^*})$ there is a μ-integrable function such that*

$$\int_N B(x, x^*)\, d\nu = \lambda_N(x) \int_N \langle x, x^*\rangle\, d\nu, \quad \mu - a.e. \quad x \in B_E,$$

and the functions $\lambda_N(x)$ are uniformly bounded in N and $x \in B_E$.

Note that the existence of such a measure for a bilinear map B opens the door for the design of an approximation procedure involving integral representations of B in terms of the integrals $(x, A) \mapsto \alpha(x, N) := \int_A \langle x, x^*\rangle d\nu$. We will precisely define and use this function α in the next section. Using α to get a uniform approximation to B is the main technical idea of our construction: it aims to recall a "weak stochastic version" of the exact eigenvector equation: for $x \in B_E$,

$$B(x, x^*) = \lambda\langle x, x^*\rangle, \qquad x^* \in B_{E^*}.$$

3.1. **Weak eigenmeasures and approximation of weak diagonalizable bilinear operators.** In this section we present our main characterization of weak eigenmeasures for bilinear maps. We start with a required technical result. Since for a given $x \in E$, the function $x \to \langle x, \cdot \rangle : E^* \to \mathbb{R}$ is w^*-continuous, it is clear by the Riesz Theorem that it is integrable for every regular Borel measure ν over B_{E^*}.

Lemma 3.4. *Let ν be a regular probability Borel measure on (B_{E^*}, w^*), and let N be a Borel measurable subset of $\mathcal{B}(B_{E^*})$. Then the functions*

$$x \mapsto \int_N \langle x, y \rangle d\nu(y), \quad x \mapsto \int_N |\langle x, y \rangle| d\nu(y),$$

are $w-$continuous.

Note that the proof below can be adapted for every bilinear form $B(x, y)$, since it is only using one side $w-$continuity and every norm continuous linear map is weak-to-weak continuous.

Proof. We prove the lemma for the second function; for the first one the proof is the same, since the first inequality below can be written as well. Fix a net $(x_\tau) \subset B_E$ that converges to $x_0 \in B_E$ in the weak topology. For every τ, we have that

$$\Big| \int_N |\langle x_\tau, y \rangle| d\nu(y) - \int_N |\langle x_0, y \rangle| d\nu(y) \Big| \leq \int_N |\langle x_\tau - x_0, y \rangle| d\nu(y),$$

and, by definition, for every $y \in E^*$ we have that $\lim_\tau \langle x_\tau - x_0, y \rangle = 0$. Notice also that all these functions are pointwise dominated by the constant function 2, and that the measure involved is finite. So, by Lebesgue Dominated Convergence Theorem, we have that

$$\lim_\tau \int_N |\langle x_\tau - x_0, y \rangle| d\nu(y) = 0.$$

Therefore, the function is $w-$continuous. □

However, the most interesting result for the linear case is given when a measure is fixed in the second variable. In a sense, this represents the situation when the actions of the elements of the dual are given by some sort of average, where the probability measure used with this aim opens the door of an approximation method for the understanding of the notion of eigenvector. We propose an approach that allows to introduce topological concepts instead of algebraic arguments for supporting the idea of eigenvector.

Given a probability regular Borel measure $\nu \in \mathcal{M}(B_{E^*}$, let us define the *$\nu-$average duality function* $\alpha_\nu : B_E \times \mathcal{B}(B_{E^*}) \to \mathbb{R}$ by

$$\alpha_\nu(x, N) := \int_N \langle x, x^* \rangle \, d\nu(x^*), \quad x \in B_E, \quad N \in \mathcal{B}(B_{E^*}).$$

Theorem 3.5. *Let E be a reflexive Banach space and $B : E \times E^* \to \mathbb{R}$ a bilinear map. Fix probability measures $\mu \in \mathcal{M}(B_E)$ and $\nu \in \mathcal{M}(B_{E^*})$. The following statements are equivalent.*

(i) *There is a constant $K > 0$ such that for every finite set of Borel measurable sets $M_1, ..., M_n$ and $N_1, ..., N_m$,*

$$\sum_{i=1}^n \int_{M_i} \Big| \int_{N_i} B(x, y) d\nu(y) \Big| \, d\mu(x) \leq K \sup_{x \in B_E} \left(\sum_{i=1}^n \chi_{M_i}(x) \Big| \int_{N_i} \langle x, y \rangle d\nu(y) \Big| \right).$$

(ii) *There is a constant $K > 0$ such that there is regular measure η over the Borel measure space of (B_E, w^*) such that*

$$\int_M \Big| \int_N B(x,y) d\nu(y) \Big| \, d\mu(x) \leq K \int_M \Big| \int_N \langle x, y \rangle d\nu(y) \Big| \, d\eta(x)$$

for all Borel sets $M \in \mathcal{B}(B_E)$ and $N \in \mathcal{B}(B_{E^*})$.

(iii) *There is a regular Borel measure η on $\mathcal{B}(B_E)$ such that $\mu \ll \eta$ and for every $N \in \mathcal{B}(B_{E^*})$ there is a bounded measurable function $x \mapsto \lambda_N(x)$,*

$$\int_N B(x,y) \, d\nu(y) = \lambda_N(x) \int_N \langle x, y \rangle \, d\nu(y) \qquad \eta - a.e. \quad x \in E,$$

and the set of all the functions λ_N are uniformly bounded by a constant K $\eta-$a.e. In other words, μ is a weak eigenmeasure.

Moreover, when these statements are satified, the following equality holds for every $x \in B_E$ η-a.e. and for all pairs of disjoint measurable sets $N, N' \in \mathcal{B}(B_{E^*})$,

$$\alpha(x, N \cup N') \cdot \lambda_{N \cup N'}(x) = \alpha(x, N) \cdot \lambda_N(x) + \alpha(x, N') \cdot \lambda_{N'}(x). \tag{1}$$

We will need two more lemmata for proving this result; we write and prove them separately below.

Lemma 3.6. *Let E be a reflexive Banach space and fix a regular probability Borel measure ν on (B_{E^*}, w^*). Let μ, η be regular probability Borel measures on B_E. Fix a measurable subset $N \subseteq B_{E^*}$ and a w-closed subset $M \subseteq B_E$, and suppose that there is a constant $K > 0$ such that the inequality*

$$\int_M \Big| \int_N B(x,y) d\nu(y) \Big| d\mu(x) \leq K \int_{B_E} \varphi(x) \Big| \int_N \langle x, y \rangle d\nu(y) \Big| \, d\eta(x)$$

holds for all continuous extensions φ of χ_M in $C(B_E)$. Then

$$\int_M \Big| \int_N B(x,y) d\nu(y) \Big| d\mu(x) \leq K \int_M \Big| \int_N \langle x, y \rangle d\nu(y) \Big| \, d\eta(x).$$

Proof. The set $M \subset B_E$ is $w-$closed by assumption, and so $A = B_E \setminus M$ is $w-$open, and in particular it is Borel measurable. We have also that E is reflexive, what means that B_E is $w-$compact. Since it is also Hausdorff, we have that it is a normal topological space. On the other hand, η is regular, and so $\eta(A) = \sup_{Q \subset A} \eta(Q)$, where the supremum is computed over all compact subsets of A. Therefore, for each ϵ we can find a compact set $Q_\varepsilon \subset A$ such that $\eta(A \setminus Q_\varepsilon) = \eta(A) - \eta(Q_\varepsilon) < \epsilon$.

By Urysohn's Lemma, we can find a continuous function φ_{Q_ε} that is equal to one in M and to 0 in K. Therefore, by using the continuous function $\min\{|\varphi_{Q_\varepsilon}|, 1\}$ instead of φ_{Q_ε} if necessary, we get

$$\int_M \Big| \int_N B(x,y) d\nu(y) \Big| d\mu(x) \leq K \int_{B_E} \varphi_{Q_\varepsilon}(x) \Big| \int_N \langle x, y \rangle d\nu(y) \Big| \, d\eta(x)$$

$$= K \int_M \varphi_{Q_\varepsilon}(x) \Big| \int_N \langle x, y \rangle d\nu(y) \Big| \, d\eta(x) + K \int_{Q_\varepsilon} \varphi_{Q_\varepsilon}(x) \Big| \int_N \langle x, y \rangle d\nu(y) \Big| \, d\eta(x)$$

$$+ K \int_{A \setminus Q_\varepsilon} \varphi_{Q_\varepsilon}(x) \Big| \int_N \langle x, y \rangle d\nu(y) \Big| \, d\eta(x)$$

$$\leq K \int_M \Big| \int_N \langle x, y \rangle d\nu(y) \Big| \, d\eta(x) + K\eta(A \setminus Q_\varepsilon) \cdot \sup_{x \in B_E} \Big| \int_N \langle x, y \rangle d\nu(y) \Big|$$

$$\leq K \int_M \Big| \int_N \langle x, y\rangle d\nu(y)\Big| \, d\eta(x) + K\,\varepsilon.$$

Since this can be done for every $\varepsilon > 0$, we get the result.

□

Lemma 3.7. *In the setting of Lemma 3.6, if the inequality there holds for a fixed measurable set N and for all $w-$closed M, we have that there is an integrable function $\lambda_N(x) \in L^\infty(\eta)$ of ∞-norm less or equal to K such that*

$$\int_N B(x,y)d\nu(y) = \lambda_N(x) \int_N \langle x, y\rangle \, d\nu(y) \quad \textit{for all } x \;\; \eta - a.e.$$

Proof. It is just an application of the Radon-Nikodym Theorem for measures. By Lemma 3.4, for every x the function $x \mapsto \int_N |B(x,y)|d\nu$ is continuous, and so integrable with respect to any Borel regular measure over B_E. Thus, the map

$$\mathcal{B}(B_E) \ni M \mapsto \Delta_N(M) = \int_M \Big| \int_N B(x,y)d\nu\Big| d\mu$$

is a countably additive measure. The same can be said about the map

$$\mathcal{B}(B_E) \ni M \mapsto \Gamma_N(M) = \int_M \Big| \int_N \langle x, y\rangle d\nu\Big| d\eta.$$

Since both measures μ and η are regular and all the functions involved are uniformly bounded, the inequality provided by Lemma 3.6 can be extended to all Borel measurable sets: indeed, we approximate the integral using $w-$closed sets, that are compact in the space, and so the measure on every measurable set—an so all the integrals—can be written as a supremum over these compact sets, what allows to preserve the inequality.

The domination gives that the (finite) measure Δ_N is absolutely continuous with respect to the (positive finite) measure Γ_N. An application of Riesz's Theorem gives the existence of a function λ_N^0, such that

$$\Big| \int_N B(x,y)d\nu(y)\Big| = \lambda_N^0(x)\Big| \int_N \langle x, y\rangle \, d\nu(y)\Big| \quad \text{for all } x \;\; \eta - a.e.$$

and is essentially bounded due to the inequality. Writing $s(x)$ as the product of the signs of both integrals for every element x and setting $\lambda_N(x) = s(x) \cdot \lambda_N(x)$, we get the result. □

Proof. (Proof of Theorem 3.5)

(i)⇒(ii) Fix $n \in \mathbb{N}$. Fix a pair of Borel sets M and N in $\mathcal{B}(B_E)$ and $\mathcal{B}(B_{E^*})$, respectively, and consider any extension of the function $\chi_M(x)$ to a continuous non-negative function φ_M bounded above by 1. By the inequality in (i), we have that for every finite set of Borel measurable sets $M_1, ..., M_n$ and $N_1, ..., N_m$, and every set of extensions φ_{M_i},

$$\sum_{i=1}^n \int_{M_i} \Big| \int_{N_i} B(x,y)d\nu(y)\Big| \, d\mu(x) \leq K \sup_{x\in B_E} \left\{ \sum_{i=1}^n \varphi_{M_i}(x)\Big| \int_{N_i} \langle x, y\rangle d\nu(y)\Big| \right\}.$$

Note that the properties of the L^1-norm and this inequality allow to write the same expression for convex combinations, as

$$\sum_{i=1}^n \alpha_i \int_{M_i} \Big| \int_{N_i} B(x,y)d\nu(y)\Big| \, d\mu(x) \leq K \sup_{x\in B_E} \left\{ \sum_{i=1}^n \alpha_i\varphi_{M_i}(x)\Big| \int_{N_i} \langle x, y\rangle d\nu(y)\Big| \right\},$$

$0 \leq \alpha_i \leq 1$, $\sum_{i=1}^n \alpha_i = 1$. Indeed, it is enough to repeat the same term all the times needed and divide by the adequate natural number to obtain the inequality for rational numbers. Then by approximating each real number by rational numbers we get the inequality (this is an observation due to Mendel and Schechtman, that can be found in the paper by Farmer and Johnson; see the introduction in [6]).

A standard separation argument gives a measure η such that

$$\int_M \Big| \int_N B(x,y)d\nu(y)\Big| d\mu(x) \leq K \int_{B_E} \varphi_M(x)\Big| \int_N \langle x,y\rangle d\nu(y)\Big| d\eta(x) \qquad (2)$$

for all Borel sets M and N and any non-negative extension φ_M of the characteristic function χ_M bounded by 1. Indeed, we can consider the convex family of w^*-continuous convex functions $\Delta : \mathcal{M}(B_E) \to \mathbb{R}$ defined as

$$\Delta(\eta) := \sum_{i=1}^n \alpha_i \int_{M_i} \Big| \int_{N_i} B(x,y)d\nu(y)\Big| d\mu(x)$$

$$-K \int_{B_E} \sum_{i=1}^n \alpha_i \varphi_{M_i}(x)\Big| \int_{N_i} \langle x,y\rangle d\nu(y)\Big| d\eta(x),$$

for each selection of $n \in \mathbb{N}$, α_i's, extensions φ_{M_i}'s, and sets N_i's, M_i's. The inequalities obtained above prove that for each such a function, there is a Dirac's delta at the point (x_0, y_0) where the supremum is attained, and so $\Delta(\delta_{(x_0,y_0)}) \leq 0$. Ky Fan's Lemma gives a measure μ_0 such that $\Delta(\mu_0) \leq 0$ for every function Δ of the family. In particular, we obtain (2) for each couple of Borel sets M and N. Lemma 3.6 gives (ii).

For (ii)⇒(iii) we apply Lemma 3.7, based on the Radon-Nikodym Theorem for absolutely continuous measures. Finally, for getting (iii)⇒(i) it is enough to consider a finite family of sets and compute the inequality by a direct calculation, taking into account that the functions λ_N are uniformly bounded by K.

Finally, for the last equality, consider $x \in B_E$ and $N, N' \in \mathcal{B}(B_{E^*})$. Then the bilinearity of B directly gives

$$\lambda_{N\cup N'}(x) \int_{N\cup N'} \langle x,y\rangle d\nu = \int_{N\cup N'} B(x,y)d\nu = \int_N B(x,y)d\nu + \int_{N'} B(x,y)d\nu$$
$$= \lambda_N(x) \int_N \langle x,y\rangle d\nu + \lambda_{N'}(x) \int_{N'} \langle x,y\rangle d\nu.$$

□

This result provides the desired characterization of a class of measures that partially satisfy the requirements that we are searching in order to provide an stochastic representation of diagonalizable bilinear maps. A domination as the one provided in (i) of Theorem 3.5 implies that for every fixed measurable set $N \subset B_{E^*}$, the average of the actions by the elements of N of the values of $B(x,\cdot)$ can be computed as the product of a function $\lambda_N(x)$ by the average action of $\langle x,\cdot\rangle$. This is indeed what the formula (1) means. The comments in the next remark will leave us to define what we call a weak eigenmeasure, as a measure that satisfy exactly the equivalent statements of Theorem 3.5.

Remark 3.8.

(1) *The equality appearing at the end of the statement of Theorem 3.5 is the key for getting that in fact, μ is an eigenmeasure. Indeed, for this to hold we need the function $\lambda_N(\cdot)$ to be independent of N, that is, to obtain a*

unique function that provides the desired "eigenvalues function". Indeed, if there is a unique function $\lambda(\cdot)$ defined $\eta-$a.e. such that $\lambda(\cdot) = \lambda_N(\cdot)$ $\eta-$a.e. for all measurable sets N, the equality in the theorem clearly holds, independently of how the function $\alpha(\cdot,\cdot)$ is. However, the converse does not hold. In order to see this, just consider any diagonal operator $T : \mathbb{R}^2 \to \mathbb{R}^2$, where $T(e_1) = \lambda_1 e_1$ and $T(e_2) = \lambda_2 e_2$, where $\{e_1, e_2\}$ is the canonical basis. The example is given by the bilinear form B associated to T by $B(x, x^*) = \langle T(x), x^*\rangle$, as usual. Take the dual basis $\{e_1^*, e_2^*\}$ and consider the (Borel regular) measures

$$\nu_0 = \frac{1}{2}\delta_{e_1^*} + \frac{1}{2}\delta_{e_2^*}, \qquad \mu_0 = \delta_{\frac{1}{\sqrt{2}}(e_1+e_2)}.$$

Note that the only non μ_0-null sets are the ones containing $1/\sqrt{2}(e_1+e_2)$, and the ones that are not null with respect to ν_0 have to contain e_1^* or e_2^*. We can choose the sets $N_1 = \{e_1^*\}$ and $N_2 = \{e_2^*\}$. Then we have that

$$\int_{N_1} \big\langle \frac{1}{\sqrt{2}}(e_1+e_2), x^*\big\rangle \, d\nu_0 = \frac{1}{2\sqrt{2}} = \int_{N_2} \big\langle \frac{1}{\sqrt{2}}\,(e_1+e_2), x^*\big\rangle \, d\nu_0,$$

and

$$\int_{N_1} \big\langle T\big(\frac{1}{\sqrt{2}}(e_1+e_2)\big), x^*\big\rangle \, d\nu_0 = \frac{\lambda_1}{2\sqrt{2}},$$
$$\int_{N_2} \big\langle T\big(\big(\frac{1}{\sqrt{2}}\,(e_1+e_2)\big), x^*\big\rangle \, d\nu_0 = \frac{\lambda_2}{2\sqrt{2}}.$$

Consequently, the function $\lambda_N(\cdot)$ is independent of N if and only if $\lambda_1 = \lambda_2$, what of course do not hold in general.

Note also that all the elements provided in this example define a situation that fits completely with the requirements of Theorem 3.5; in particular, the inequalities that are required in (i) are trivially satisfied. However, although of course the equation (1) holds, we cannot expect that the theorem provides a unique eigenvalues function λ independent of the set $N \in \mathcal{B}(B_{E^*})$, that is the condition appearing in Definition 3.3.

Thus, we see that the measures μ satisfying the equivalent statements of Theorem 3.5 are weak eigenmeasures. As we have shown, this definition include both cases in which the functions $\lambda_N(\cdot)$ do not depend on N, and cases as the one explained above.

(2) *Example 3.2 provides a clear case in which the functions $\lambda_N(\cdot)$ do not depend on N. Indeed, in this case the only non-ν_0-null sets are all the subsets of $\mathcal{B}^*$, and the non-μ_0-null sets are all the subsets of $\mathcal{B}$. It can be easily seen that the functions $\lambda_N(x_i) = \lambda_i$ for every $x_i \in \mathcal{B}$ and every measurable set $N \in \mathcal{B}(B_{E^*})$ give a suitable eigenvalues function for T associated to ν_0 and μ_0. As we indicated in the explanation of Example 3.2, this is in fact the canonical case. Clearly, this example satisfies also the inequalities in (i) of Theorem 3.5.*

Next result proves a characterization under certain positivity restrictions of when a weak eigenmeasure is in fact an eigenmeasure.

Corollary 3.9. *Let E be a reflexive Banach space and take a bilinear form $B : E \times E^* \to \mathbb{R}$. Fix ν a Borel regular measure over B_{E^*}, and suppose that μ is a weak eigenmeasure for B with respect to ν. Suppose also that $\alpha_\nu(x, N) \geq 0$ for μ-a.e. $x \in B_E$ and $N \in \mathcal{B}(B_{E^*})$. Then the following statements are equivalent.*

(i) *For every $N \in \mathcal{B}(B_{E^*})$, $\lambda_{B_E(x)} \geq \lambda_N(x)$ $\mu-$a.e.*

(ii) *The measure μ is an eigenmeasure for B with respect to ν.*

Proof. (ii)⇒(i), is obvious, since by definition we have that $\lambda_N \geq \lambda_{N'}(x)$ for every $N, N' \in \mathcal{B}(B_{E^*})$.

For (i)⇒(ii), consider the (only non trivial) case for which $\alpha_\nu(x, B_{E^*}) = \int_{B_{E^*}} \langle x, x^* \rangle d\nu > 0$ (otherwise, all the integrals have to be equal to zero, by the requirement $\alpha_\nu(\cdot, \cdot) \geq 0$). Fix N, and consider the equality coming from the definition of weak eigenmeasure,

$$\lambda_{B_{E^*}}(x)\, \alpha_\nu(x, B_{E^*}) = \lambda_N(x)\, \alpha_\nu(x, N) + \lambda_{N'}(x)\, \alpha_\nu(x, N').$$

Write $\beta := \alpha_\nu(x, N)/\alpha_\nu(x, B_{E^*})$, and assume that $0 < \beta < 1$ (otherwise, the result is obvious). Note that we obtain, using the inequality above,

$$\lambda_{B_{E^*}}(x) = \beta\, \lambda_N(x) + (1 - \beta)\, \lambda_{N'}(x) \leq \max\{\lambda_N(x), \lambda_{N'}(x)\} \leq \lambda_{B_{E^*}}(x),$$

and so we have that $\lambda_{B_{E^*}}(x) = \lambda_N$ or $\lambda_{B_{E^*}}(x) = \lambda_{N'}$, what clearly implies that both of them are equal to $\lambda_{B_{E^*}}(x)$. The result is proven. □

3.2. **Associated probabilistic factorizations for linear operators.** Let us explain now the abstract picture of our construction, which involves vector measures and vector valued integrable functions. Let us center our attention on linear operators $T : E \to E$ by considering the associated bilinear maps $B(x, x^*) = \langle T(x), x^* \rangle$. The measures μ and ν "eliminate" those vectors of the space E for which a diagonal representation—a formula like $\langle T(x), x^* \rangle = \lambda \langle x, x^* \rangle$—is not possible. Therefore, the associated integral formulas do not allow to represent the whole operator but only those elements for which this can be done. This means that we obtain a kind of "weak representation", with the advantage of allowing simple calculation formulas for those elements for which the formalism works.

Let us write some factorization diagrams with the aim of helping to understand these ideas visually. Suppose that ν is a measure and μ is an eigenmeasure. Suppose that $\lambda : B_E \to \mathbb{R}$ is an associated eigenvalues function that is defined for all $x \in B_E$, and not only μ-a.e. We are searching for a factorization related to the following one,

$$\begin{array}{ccc} B_E \times B_{E^*} & \xrightarrow{\langle T(\cdot), \cdot \rangle} & \mathbb{R}, \\ \big(\lambda(\cdot)\, \cdot, \cdot\big) \big\downarrow & & \big\downarrow Id \\ E \times E^* & \xrightarrow{\langle \cdot, \cdot \rangle} & \mathbb{R} \end{array}$$

which however cannot be considered to be commutative as it is written but only in probabilistic way. That is, the equality only happens when the integrals with respect to ν and μ are computed with the ranges of both the ways the factorization shows.

Therefore, this "exact" factorization is not adequate for our purposes. Alternatively, our construction allows to associate a representable vector measure to each linear operator in $\mathcal{L}(E, E)$. As a consequence of Lemma 3.4, for every subset $N \subseteq B_{E^*}$, the map $\mathcal{L}(E, E) \ni T \mapsto \int_M \langle T(x), \cdot \rangle\, d\mu(x) \in \mathcal{M}(B_E, L^1(B_{E^*} \cap N, \nu))$ assigns to each operator a vector measure of bounded variation having values in the space $L^1(B_{E^*}, \nu)$. It can be easily checked that the associated map is indeed a countably additive vector measure.

An integral representation is allowed for such a vector measure to get a map $\mathcal{L}(E,E) \ni T \mapsto \langle T(\cdot),\cdot\rangle \in L^1(B_E,\mu,L^1(\nu,N))$. This provides a (vector valued Bochner) integrable function to each operator, that satisfies that

$$\Big\| \big\| \langle T(x),\cdot\rangle \big\|_{L^1(N,\nu)} \Big\|_{L^1(B_E,\mu)} \leq \|T\|\,\|x\|, \quad x \in E.$$

The properties of the Bochner integrable functions imply that, for every $h \in L^\infty(B_E,\mu)$,

$$\Big\langle \int_{B_E} \Big(\int_{B_{E^*}} \langle T(x),x^*\rangle d\nu \Big)\, d\mu, h(x) \Big\rangle = \int_{B_E} \Big\langle \int_{B_{E^*}} \langle T(x),x^*\rangle d\nu, h(x)\Big\rangle\, d\mu.$$

Summing up all these facts, we can get the following "diagonal" characterization of linear maps with an associated eigenmeasure μ: such a map T can be factored as

$$\begin{array}{ccc} L^1\big(B_E,\mu,L^1(B_{E^*},\nu)\big) & \xrightarrow{T_{\mu,\nu}} & L^1\big(B_E,\mu,L^1(B_{E^*},\nu)\big), \\ \Big\downarrow {\scriptstyle M_\lambda} & \nearrow {\scriptstyle I_{\langle\cdot,\cdot\rangle}} & \\ L^1\big(B_E,\mu,L^1(B_{E^*},\nu)\big) & & \end{array}$$

where $T_{\mu,\nu}$ is the extension of the measure defined by T and μ,ν to all integrable functions, $I_{\langle\cdot,\cdot\rangle}$ and M_λ are the multiplication operators given by $M_\lambda\big(f\big)(x) := \lambda(x)f(x)$ and $I_{\langle\cdot,\cdot\rangle}(f)(x) := f(x)\cdot\langle x,\cdot\rangle$, $f \in L^1\big(B_E,\mu,L^1(B_{E^*},\nu)$, for a certain measurable function λ that belongs to $L^\infty(B_E,\mu)$. As we have seen in Theorem 3.5, diagonalizable operators are related to a certain domination map, which leads to an integral domination and representation by means of a function belonging to $L^\infty(\mu)$.

3.3. **Stochastic representations of diagonalizable operators and integral approximation of bilinear maps.** Let us explain in what follows our approximation formulas for getting integral averages of eigenvector equations for bilinear operators. In our formalism, the measures ν and μ are the tools that are used to choose which elements of the spaces E and E^* are allowed for approximating the values of $B(x,x^*)$ by means of diagonal formulas. We present here the direct results associated to countable families of relevant sets—in the sense we define below—; however, it can be easily seen that the same ideas can be used for getting a general martingale-type approximation method.

Let us start by defining the technical notion of sufficient family of Borel subsets of B_{E^*}. We will say that such a family $\mathcal{F} = \{N_k : k \in J\} \subseteq \{N \in \mathcal{B}(B_{E^*}) : \nu(N_k) > 0\}$ is *sufficient* if for every $x \in B_E$, the set of integrals

$$\Big\{ \int_{N_k} \langle x,x^*\rangle\, d\nu(x^*) : k \in J \Big\},$$

determines univocally the element x. That is, if there are elements $x,y \in B_E$ such that $\int_{N_k}\langle x,x^*\rangle\, d\nu = \int_{N_k}\langle y,x^*\rangle\, d\nu$, then $x = y$. For some of the applications, this definition could be also made for integral (vector valued) averages $\int_M x d\mu(x)$, on non-$\mu-$null subsets $M \in \mathcal{B}(B_{E^*})$, where x is understood to be a Bochner integrable function $B_E \ni x \mapsto x \in L^1(B_E,\mu,E)$.

We will also use the following concept: a sufficient family $\mathcal{F}$ is *complete* if it covers the unit ball of E^* $\nu-$a.e., that is, if $\cup_{N\in\mathcal{F}} N \in \mathcal{B}(B_{E^*})$ and $\nu(B_{E^*} \setminus \cup_{N\in\mathcal{F}} N = 0$.

The most basic example of a family of Borel subsets that is sufficient and complete is the one given by a measure as $\nu = \sum_{i=1}^{\infty} \frac{1}{2^i}\delta_{e_i}$ for a space E such that E^* has an unconditional basis $\{e_i : i \in \mathbb{N}\}$. The class of measurable subsets $\big\{\{e_i\} : i \in \mathbb{N}\big\}$ clearly gives a sufficient family. Since the only basic sets of non-null measure are the sets $\{e_i\}$, we have also that $\nu\big(B_{E^*} \setminus \cup_{i\in\mathbb{N}}\{e_i\}\big) = 0$. Note that it is also countable, what will be also a requirement in the result below.

Next result present the main consequences of the properties obtained for weak eigenmeasures concerning the computation of average values of the bilinear forms using "diagonal information". It gives a tool for describing the bilinear form B in terms of its diagonal values.

Corollary 3.10. *Let E be a reflexive Banach space and $B : E \times E^* \to \mathbb{R}$ a bilinear form. Let ν be a regular Borel measure on B_{E^*} having a sufficient and complete countable disjoint family of subsets $\mathcal{F}$, and let μ be a weak eigenmeasure. Then*

(i) *For every $N \in \mathcal{B}(B_{E^*})$,*

$$\int_N B(x,x^*)\,d\nu = \lim_n \sum_{k=1}^{n} \alpha(x, N\cap N_k)\,\lambda_{N\cap N_k}(x) \quad \textit{uniformly for all } x \in B_E \ \mu\textit{−a.e.}$$

(ii) *For every $N \in \mathcal{B}(B_{E^*})$,*

$$\lambda_N(x) = \lim_n \sum_{k=1}^{n} \frac{\alpha(x, N\cap N_k)}{\alpha(x,N)}\,\lambda_{N\cap N_k}(x) \quad \textit{uniformly for all } x \in B_E \ \mu\textit{−a.e.}$$

for couples of x and N such that $\alpha(x,N) \neq 0$.

(iii) *Moreover, if μ is also an eigenmeasure and , then $\alpha(x, B_{E^*}) \neq 0$,*

$$\lambda(x) = \lim_n \sum_{k=1}^{n} \frac{\alpha(x, N_k)}{\alpha(x,B_{E^*})}\,\lambda_{N_k}(x) \quad \textit{uniformly for all } x \in B_E \ \mu\textit{−a.e.}$$

Proof. The main tool for proving this result is Theorem 3.5, and Equation (1) given in it. (i) Consider a measurable set N. By the definition of ν and the fact that the N_k's define a sufficient complete set, we have that

$$\Big|\int_N B(x,x^*)\,d\nu - \int_{N\cap(\cup_k N_k)} B(x,x^*)\,d\nu\Big| = 0.$$

Fix $\varepsilon > 0$. Then, taking into account that ν is countably additive, we have that there is a natural number n_0 such that

$$\Big|\int_N B(x,x^*)\,d\nu - \sum_{k=1}^{n_0} \alpha(x, N\cap N_k)\,\lambda_{N\cap N_k}(x)\Big|$$

$$\Big|\int_N B(x,x^*)\,d\nu - \int_{N\cap(\cup_{k=1}^{n_0} N_k)} B(x,x^*)\,d\nu\Big|$$

$$= \Big|\int_{N\cap(\cup_k N_k)} B(x,x^*)\,d\nu - \int_{\cup_{k=1}^{n_0} N_k\cap N} B(x,x^*)\,d\nu\Big|$$

$$= \Big|\int_{N\cap(\cup_{k=n_0+1}^{\infty} N_k)} B(x,x^*)\,d\nu\Big| \leq \nu(\cup_{k=n_0+1}^{\infty} N_k) \sup_{x\in B_E,\, x^*\in B_{E^*}} \big|B(x,x^*)\big| \leq \varepsilon\,\|B\|.$$

Therefore, we have (i) uniformly (since ε does not depend on x), for all x μ−a.e.

(ii) This is just a consequence of (i) and the fact that for every $N \in \mathbb{B}(B_{E^*})$,

$$\lambda_N(x)\,\alpha(x,N) = \int_N B(x,x^*)\,d\nu$$

Finally, (iii) is given by (ii), taking into account the definition of eigenmeasure. □

4. Actual calculations for the diagonal representation of the operators

Basically, the formalism that we have developed in the previous section remarks the fact that domination of the linear forms associated to an operator T by $\langle\cdot, x^*\rangle$ for enough functionals $x^* \in B_{E^*}$ provides some sort of weak diagonal representation for T, that have to be written in terms of integrals. Far from being an abstract theoretical result, our aim is to show that this is a computational tool associated with probability sampling, in which Monte-Carlo type methods could find a natural context. Our concern in this section is to show a basic computational scheme, which could be followed to perform calculations for the diagonal representation of operators.

Consider a (linear and continuous) operator $T : E \to E$ between and Euclidean or a separable Hilbert space E. An algorithm to get some "diagonal" information of different types could follow the next steps.

(1) Find a (finite) sample $N_0 = \{x_1^*, ..., x_m^*\}$ of elements of the unit sphere of E^*. It can be computed following several rules, depending on the objective of the computation. Some of them are: use a probabilistic distribution to choose vectors centered in any vector of the sphere, use a uniform distribution in this sphere, or choose some elements of a normalized basis of E^*. These elements have to be used to define the measure ν of our formalism, using for example the formula $\nu := 1/m \sum_{i=1}^m \delta_{x_i^*}$.

(2) Fix a constant $K > 0$ as large as needed, depending on the size of $\|T\|$ and other parameters of the space and the operator, and use either a deterministic procedure or a sampling method to fix a set M_0 of elements of the unit sphere of E. The measure μ is then defined in the same way that it has been done in Step (1).

(3) Compute the set $T(M_0)$ and restrict the set M_0 if needed in order to get the domination properties required in Theorem 3.5: for every subsets $N \subset N_0$ and $M \subset M_0$,

$$\Big|\sum_{x^*\in N}\langle T(x), x^*\rangle\Big| \le K\,\Big|\sum_{x^*\in N}\langle x, x^*\rangle\Big| \quad \text{for all } x \in M.$$

The constant K can be increased as much as needed to allow the inequalities to hold. Note that, in case T is diagonalizable, a constant K satisfying $\max\{\,|\lambda_i| : \lambda_i \text{ is an igenvalue of } T\} \le K$ satisfies also these inequalities for adequate sets N_0 and M_0.

(4) In case the inequalities do not hold, the set M have to be reduced to get them, and increased again following any systematic random procedure. The set N can also to be changed. In the next section we will show some example of application following a deterministic method, that give concrete computation formulas for triangular matrices (Section 5.2).

(5) Item (iii) in Theorem 3.5 provides a weak diagonal representation of T; of course, the functions λ_N that give such a representation can be easily computed by

$$\lambda_N(x) := \frac{\sum_{x^* \in N} \langle T(x), x^* \rangle}{\sum_{x^* \in N} \langle x, x^* \rangle}, \quad x \in M_0.$$

Corollary 3.9 can also be applied to obtain a unique function λ not depending on N. The explanations in the previous sections show in which sense these results provide information about the diagonal representation of T.

Finally, let us remark that Equality (1) allows to relate the different functions λ_N appearing in our formalism: whenever the requirements above are satisfied, if $x \in M_0$, for all pairs of disjoint measurable sets $N, N' \in \mathcal{B}(B_{E^*})$,

$$\lambda_{N \cup N'}(x) \cdot \Big(\sum_{x^* \in N \cup N'} \langle x, x^* \rangle \Big) = \lambda_N(x) \cdot \Big(\sum_{x^* \in N} \langle x, x^* \rangle \Big) + \lambda_{N'}(x) \cdot \Big(\sum_{x^* \in N'} \langle x, x^* \rangle \Big).$$

5. Applications

Let us show two applications of the ideas presented in this paper. The first one concerns the computation of average eigenvalues of linear and bilinear operators with respect to a couple of measures on B_E and B_{E^*}. The second one uses the domination properties that inspire Theorem 3.5 for computing the set of eigenvectors of a triangular operator without explicitly using its eigenvalues.

5.1. **Eigenvalues averages and approximation.** Let E be a finite dimensional Banach space. Fix $M \in \mathcal{B}(B_E)$ and $N \in \mathcal{B}(B_{E^*})$. To obtain the average eigenvalue of a bilinear map $B : E \times E^* \to \mathbb{R}$ in M and N with respect to a couple of measures ν and μ we only have to compute the minimum of the equation

$$\begin{aligned} \varepsilon &= \int_M \Big(\int_N |B(x,y) - \lambda \langle x, y \rangle|^2 d\nu(y) \Big) d\mu(x) \\ &= \int_M \Big(\int_N |B^2(x,y) - 2\lambda B(x,y)\langle x, y \rangle + \lambda^2 \langle x, y \rangle^2 | d\nu(y) \Big) d\mu(x), \end{aligned}$$

that can be understood as an error formula representing how far the bilinear operator B is from being diagonal. Note that we do not need μ to be an eigenmeasure for the following computations.

Therefore, computing the right value of λ in

$$\frac{d\varepsilon}{d\lambda} = \int_M \Big(\int_N (-2B(x,y)\langle x, y \rangle + 2\lambda \langle x, y \rangle^2) d\nu(y) \Big) d\mu(x) = 0,$$

we get

$$\lambda = \frac{\int_M \Big(\int_N (B(x,y)\langle x, y \rangle) d\nu(y) \Big) d\mu(x)}{\int_M \Big(\int_N \langle x, y \rangle^2 d\nu(y) \Big) d\mu(x)}. \tag{3}$$

We can consider also the minimization of this expression when λ is assumed to be also a function of x. Let us show an easy example. Consider a diagonalizable operator $T : E \to E$, where the dimension of E is n, and the associated bilinear form $B(\cdot, \cdot) = \langle T(\cdot), \cdot \rangle$.

(1) Take a basis of eigenvectors $\mathcal{B} = \{e_1, ..., e_n\}$ with dual basis $\mathcal{B}^* = \{e_1^*, ..., e_n^*\}$, that is, $\langle e_i, e_j^* \rangle = \delta_{ij}$, and the associated canonical probability measures defined on B_E and B_{E^*} as $\mu := \sum_{i=1}^n \frac{1}{n}\delta_{e_i}$, and $\nu := \sum_{j=1}^n \frac{1}{n}\delta_{e_j^*}$. The "error formula" can in this case be written considering λ as a function of x,

$$\int_{B_E} \Big(\int_{B_{E^*}} |B(x,y) - \lambda(x)\langle x, y\rangle|^2 d\nu(y) \Big) d\mu(x).$$

Let us compute this integral. First notice that for measurable subsets A not containing any element of the basis $\{e_1, ..., e_n\}$, we have that $\mu(A) = 0$. So we can define $\lambda(x) = 0$ for every x not in the basis. Let us write $\lambda(e_i) = \lambda_i$ for every $i = 1, ..., n$. Thus,

$$\begin{aligned}
\varepsilon &= \int_{B_E} \Big(\int_{B_{E^*}} |B(x,y) - \lambda(x)\langle x, y\rangle|^2 d\nu(y) \Big) d\mu(x) \\
&= \sum_{i=1}^n \frac{1}{n} \Big(\sum_{j=1}^n \frac{1}{n} |B(e_i, e_j^*) - \lambda(e_i)\langle e_i, e_j^*\rangle|^2 \Big) \\
&= \frac{1}{n^2} \sum_{i=1}^n \Big(|B(e_i, e_i^*) - \lambda_i \langle e_i, e_i^*\rangle|^2 + \sum_{j=1, j\neq i}^n |B(e_i, e_j^*)|^2 \Big)
\end{aligned}$$

Thus, it is enough to compute the derivative with respect to every λ_i to get the minimum of the function. Indeed, we have that the equation

$$\frac{d\varepsilon}{d\lambda_i} = \frac{-2}{n^2} \big(B(e_i, e_i^*) - \lambda_i \langle e_i, e_i^*\rangle \big) = 0$$

gives the expected optimal solution $\lambda_i = \lambda(e_i) = B(e_i, e_i^*)$, which shows that this is given by the coincidence of $\lambda(x)$ with the set of eigenvalues λ_i $\mu-$a.e.

(2) Let us present now a completely different example. Consider the bilinear form $B : \mathbb{R}^2 \times (\mathbb{R}^2)^* \to \mathbb{R}$ and compute the average eigenvalue λ with respect to Lebesgue measures defined on the unit balls of $\mathbb{R}^2$ and $(\mathbb{R}^2)^* = \mathbb{R}^2$. We have

$$\begin{aligned}
&\int_{B_{\mathbb{R}^2}} \Big(\int_{B_{\mathbb{R}^{2*}}} |B(x,y) - \lambda\langle x, y\rangle|^2 dy \Big) dx \\
&= \int_{B_{\mathbb{R}^2}} \Big(\int_{B_{\mathbb{R}^2}} |B(x,y) - \lambda\langle x, y\rangle|^2 dy \Big) dx \\
&= \int_{B_{\mathbb{R}^2}} \Big(\int_{B_{\mathbb{R}^2}} \big(B^2(x,y) - 2\lambda B(x,y)\langle x, y\rangle + \lambda^2 \langle x, y\rangle^2 \big) dy \Big) dx.
\end{aligned}$$

Then, by Equation 3, we get

$$\lambda = \frac{\int_{B_{\mathbb{R}^2}} \Big(\int_{B_{\mathbb{R}^2}} (B(x,y)\langle x, y\rangle) dy \Big) dx}{\int_{B_{\mathbb{R}^2}} \Big(\int_{B_{\mathbb{R}^2}} \langle x, y\rangle^2 dy \Big) dx} \tag{4}$$

For the elements $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$, we get $\langle x, y\rangle = x_1y_1 + x_2y_2$ and by computing the double integral

$$\int_{B_{\mathbb{R}^2}} \Big(\int_{B_{\mathbb{R}^2}} (x_1y_1 + x_2y_2)^2 d(y_1, y_2) \Big) d(x_1, x_2)$$

appearing in the denominator of the above equation by using polar coordinates on l $B_{\mathbb{R}^2}$, we get

$$\lambda = \frac{9\int_{B_{\mathbb{R}^2}}\Big(\int_{B_{\mathbb{R}^2}}(B(x,y)\langle x,y\rangle)dy\Big)dx}{2\pi^2} \tag{5}$$

Let us give a concrete example. Let us specify the bilinear map $B : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as $B(x,y) = \langle P_1 x, y\rangle$, where $P_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is the first orthogonal projection defined by $P_1(x_1,x_2) = (x_1,0)$. Then, we get

$$\int_{B_{\mathbb{R}^2}}\Big(\int_{B_{\mathbb{R}^2}}(B(x,y)\langle x,y\rangle)dy\Big)dx$$

$$= \int_{B_{\mathbb{R}^2}}\Big(\int_{B_{\mathbb{R}^2}}(x_1^2y_1^2 + x_1y_1x_2y_2)d(y_1,y_2)\Big)d(x_1,x_2) = \frac{\pi^2}{9}.$$

Therefore, we obtain that the average eigenvalue for the bilinear map B defined in this way is

$$\lambda = \frac{9\int_{B_{\mathbb{R}^2}}\Big(\int_{B_{\mathbb{R}^2}}(B(x,y)\langle x,y\rangle)dy\Big)dx}{2\pi^2} = 1/2. \tag{6}$$

In a similar way, λ is found to be also $1/2$ when the bilinear map $B : \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}$ is considered to be $B(x,y) = \langle P_2 x, y\rangle$, where $P_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is the second orthogonal projection given by $P_2(x_1,x_2) = (0,x_2)$. Moreover, if $B(x,y) = \langle CP_1x, y\rangle$, where C is a constant and P_1 is the first orthogonal projection, we obtain

$$\begin{aligned}\lambda &= \frac{\int_{B_{\mathbb{R}^2}}\Big(\int_{B_{\mathbb{R}^2}}(B(x,y)\langle x,y\rangle)dy\Big)dx}{\int_{B_{\mathbb{R}^2}}\Big(\int_{B_{\mathbb{R}^2}}\langle x,y\rangle^2 dy\Big)dx}\\ &= \frac{\int_{B_{\mathbb{R}^2}}\Big(\int_{B_{\mathbb{R}^2}}C(x_1^2y_1^2 + x_1y_1x_2y_2)d(y_1,y_2)\Big)d(x_1,x_2)}{\int_{B_{\mathbb{R}^2}}\Big(\int_{B_{\mathbb{R}^2}}(x_1y_1+x_2y_2)^2 d(y_1,y_2)\Big)d(x_1,x_2)}\\ &= \frac{C\pi^2/9}{2\pi^2/9} = C/2.\end{aligned}$$

5.2. **An algorithm for the finite dimensional case: a recursion formula for triangular matrices.** Consider a diagonalizable linear map $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. Then it has a matrix representation $[A]_{(n+1)\times(n+1)}$. Let us write

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1(n+1)} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2(n+1)} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{(n+1)1} & a_{(n+1)2} & a_{(n+1)3} & \cdots & a_{(n+1)(n+1)} \end{bmatrix}.$$

In order to simplify the presentation, assume that this matrix is (lower) triangular. Suppose that the vector $\alpha^1 = (\alpha_0^1, \alpha_1^1, ..., \alpha_n^1)$ is the first possible eigenvector with $\alpha_0^1 = 1$; of course, it may happens that there is not such an eigenvector, then we jump to the next step. Then, we consider vectors $t = (t_0, ..., t_n)$ of

the dual space that are orthogonal to α^1; that is, $\langle \alpha^1, t\rangle = 0$. The inequalities $|\langle T(\alpha^1), t\rangle| \leq K|\langle \alpha^1, t\rangle|$ that are at the heart of our procedure give

$$\begin{bmatrix} t_0 & \cdots & t_n \end{bmatrix} \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 & 0 \\ : & : & : & \ddots & 0 & 0 \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & 0 \\ a_{(n+1)1} & a_{(n+1)2} & a_{(n+1)3} & \cdots & a_{(n+1)n} & a_{(n+1)(n+1)} \end{bmatrix} \begin{bmatrix} \alpha_0^1 \\ \alpha_1^1 \\ \alpha_2^1 \\ : \\ \alpha_n^1 \end{bmatrix} = 0. \tag{7}$$

Similarly, this equation can be written for the other possible eigenvectors $\alpha^i = (\alpha_0^i, \alpha_1^i, ..., \alpha_n^i)$ such that $\alpha_k^i = 0$ if $k < i-1$ and $\alpha_k^i = 1$ if $k = i-1$, where $i = 2, ..., n+1$.

An $(n+1)\times(n+1)-$ lower triangular matrix has a characteristic polynomial given by $P(\lambda) = \prod_{i=1}^{n+1}(a_{ii} - \lambda)$. If we use the standard procedure, to find the eigenvectors we need to compute the kernel of the operators $T - a_{ii}Id$. Let us show that we can use Equation 7 instead.

There are $n+1$ linearly independent eigenvectors $\{\alpha^1, ..., \alpha^{n+1}\}$ for this matrix. For all $m = 1, ..., n+1$, the eigenvector $\alpha^m = \{\alpha_k^m\}_{k=1}^m$ has the coordinates

$$\alpha_k^m = \begin{cases} 0 & k < m-1 \\ 1 & k = m-1 \\ \alpha_k^m & k \geq m \end{cases}.$$

Thus, for every m we want to solve Equation 7 for some specific vectors $t = (t_0, ..., t_n)$ such that $t \perp \alpha^m$. Let us consider $m = m_0 \in \{1, ..., n+1\}$. Then we can consider $n+1-m_0$ vectors that are orthogonal to α^{m_0} defined as

$$t_k = \begin{cases} \alpha_{m_0+i}^{m_0} & k = m_0 - 1 \\ -1 & k = m_0 + i \\ 0 & k \in \{0, ..., n\} \setminus \{m_0 - 1, m_0 + i\} \end{cases},$$

where $i = 0, ..., n - m_0$ and every such i gives a vector that is orthogonal to the vector α^{m_0}. Therefore, there are n such vectors for α^1, $n-1$ vectors for α^2,..., and 1 vector for α^n; since $\alpha^{n+1} = (0, ..., 0, 1)$, there is nothing to compute in this case. Summing up, we need to consider Equation 7 for $\dfrac{n(n+1)}{2}$ vectors. By solving the resulting system of linear equations, we get the coordinates of some eigenvectors with the following recursion formula:

$$\alpha_k^m = \frac{\sum_{j=m}^k \alpha_{j-1}^m a_{(k+1)j}}{(a_{mm} - a_{(k+1)(k+1)})}, \qquad k \geq m, \tag{8}$$

that can be applied under the restriction $a_{mm} \neq a_{(k+1)(k+1)}$. To make it more understandable, let us apply the procedure for an operator $T : \mathbb{R}^5 \to \mathbb{R}^5$ that has a lower triangular representation

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}.$$

We are searching for eigenvectors as

$$\begin{gathered}\alpha^1=(\alpha_0^1,\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1)=(1,\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1),\\ \alpha^2=(\alpha_0^2,\alpha_1^2,\alpha_2^2,\alpha_3^2,\alpha_4^2)=(0,1,\alpha_2^2,\alpha_3^2,\alpha_4^2),\\ \alpha^3=(\alpha_0^3,\alpha_1^3,\alpha_2^3,\alpha_3^3,\alpha_4^3)=(0,0,1,\alpha_3^3,\alpha_4^3),\\ \alpha^4=(\alpha_0^4,\alpha_1^4,\alpha_2^4,\alpha_3^4,\alpha_4^4)=(0,0,0,1,\alpha_4^4),\\ \alpha^5=(\alpha_0^5,\alpha_1^5,\alpha_2^5,\alpha_3^5,\alpha_4^5)=(0,0,0,0,1).\end{gathered}$$

The vectors t of the dual space that are defined as

$$\begin{gathered}(\alpha_1^1,-1,0,0,0),\\ (\alpha_2^1,0,-1,0,0),\\ (\alpha_3^1,0,0,-1,0),\\ (\alpha_4^1,0,0,0,-1),\end{gathered}$$

are orthogonal to α^1. By using the recursion formula written in Equation 8, we get that

$$\alpha_1^1=\frac{a_{21}}{a_{11}-a_{22}}$$

$$\alpha_2^1=\frac{a_{31}+\alpha_1^1a_{32}}{a_{11}-a_{33}}$$

$$\alpha_3^1=\frac{a_{41}+\alpha_1^1a_{42}+\alpha_2^1a_{43}}{a_{11}-a_{44}}$$

$$\alpha_4^1=\frac{a_{51}+\alpha_1^1a_{52}+\alpha_2^1a_{53}+\alpha_3^1a_{54}}{a_{11}-a_{55}}$$

In the same way, for the second eigenvector $\alpha^2=(\alpha_0^2,\alpha_1^2,\alpha_2^2,\alpha_3^2,\alpha_4^2)=(0,1,\alpha_2^2,\alpha_3^2,\alpha_4^2)$, we consider the orthogonal vectors

$$\begin{gathered}(0,\alpha_2^2,-1,0,0),\\ (0,\alpha_3^2,0,-1,0),\\ (0,\alpha_4^2,0,0,-1),\end{gathered}$$

and we find the coordinates

$$\alpha_2^2=\frac{a_{32}}{a_{22}-a_{33}},\quad \alpha_3^2=\frac{a_{42}+\alpha_2^2a_{43}}{a_{22}-a_{44}},\quad \alpha_4^2=\frac{a_{52}+\alpha_2^2a_{53}+\alpha_3^2a_{54}}{a_{22}-a_{55}}.$$

For the third eigenvector $\alpha^3=(\alpha_0^3,\alpha_1^3,\alpha_2^3,\alpha_3^3,\alpha_4^3)=(0,0,1,\alpha_3^3,\alpha_4^3)$ we use the orthogonal vectors $(0,0,\alpha_3^3,-1,0)$ and $(0,0,\alpha_4^3,0,-1)$, where

$$\alpha_3^3=\frac{a_{43}}{a_{33}-a_{44}},\quad \alpha_4^3=\frac{a_{53}+\alpha_3^3a_{54}}{a_{33}-a_{55}}.$$

Finally, the fourth eigenvector $\alpha^4=(\alpha_0^4,\alpha_1^4,\alpha_2^4,\alpha_3^4,\alpha_4^4)=(0,0,0,1,\alpha_4^4)$ is computed by using the orthogonal vector $(0,0,0,\alpha_4^4,-1)$ to get

$$\alpha_4^4=\frac{a_{54}}{a_{44}-a_{55}}.$$

The last eigenvector is obviously $(0,0,0,0,1)$. The normalization of the obtained eigenvectors give the result.

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