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$\alpha-\psi$-Contractive Type Mappings on Quasi-Metric Spaces

Salvador Romaguera[a], Pedro Tirado[a]

[a]*Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, 46022 Valencia, Spain*

Abstract. We introduce and discuss several types of $\alpha-\psi$-contractive mappings on quasi-metric spaces. We obtain some fixed point theorems in this setting and present suitable examples to show the validity of our approach and results. Finally, we give a characterization of doubly Hausdorff right K-sequentially complete quasi-metric spaces in terms of $\alpha-\psi$-contractive type mappings having fixed point.

1. Introduction and preliminaries

Samet, Vetro and Vetro obtained in [18] various general and important fixed point theorems by using $\alpha-\psi$-contractive mappings, where the so-called α-admissible functions play a crucial role. Since then, several authors have continued the research of this type of contractions from different approaches (see e.g. [1, 3, 4, 7, 10, 11, 19]). In particular, a slight variant of one of these theorems was recently extended to the setting of quasi-metric spaces in [17], where the authors also proved that this extension actually characterizes left K-sequential completeness for Hausdorff quasi-metric spaces and consequently that variant of Theorem 2.2 of [18] characterizes metric completeness. The fact that the α-admissible functions are not necessarily symmetric provides in the quasi-metric setting various appealing types of $\alpha-\psi$-contractive mappings that are useful to obtain some fixed point results not only for Hausdorff left K-sequentially complete quasi-metric spaces but also for Hausdorff right K-sequentially complete quasi-metric spaces, a relevant type of quasi-metric completeness to the study of completeness of hyperspaces and function spaces, and in characterizing the weak form of Ekeland's Variational Principle (see e.g. [12–14]). This approach will be discussed in Section 2 where we also provide several suitable examples to illustrate the obtained results. Finally, in Section 3 we characterize doubly Hausdorff right K-sequentially complete quasi-metric spaces in terms of $\alpha-\psi$- contractive mappings having fixed point, from which we will derive that Samet, Vetro and Vetro theorem [18, Theorem 2.2] characterizes the metric completeness.

Related to our research, the recent papers [9, 20–22], suggested by the referee, are of interest.

Throughout this paper by $\mathbb{R}$ and $\mathbb{N}$ we will denote the set of all real numbers and the set of all positive integer numbers, respectively. Our basic reference for general topology is [6] and for quasi-metric spaces it is [5].

In the rest of this section we recall some concepts and properties on quasi-metric spaces which will be useful along the paper. Our terminology is standard.

A quasi-metric on a set X is a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$:

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Email addresses: sromague@mat.upv.es (Salvador Romaguera), pedtipe@mat.upv.es (Pedro Tirado)

(i) $x = y \Leftrightarrow d(x, y) = d(y, x) = 0$, and

(ii) $d(x, z) \leq d(x, y) + d(y, z)$.

A quasi-metric space is a pair (X, d) such that X is a set and d is a quasi-metric on X.

Each quasi-metric d on X induces a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

It follows from the definition of the topology τ_d that a sequence $(x_n)_{n\in\mathbb{N}}$ converges to some $x \in X$ in τ_d if and only if $d(x, x_n) \to 0$ as $n \to \infty$.

If τ_d is a T_2 topology on X we say that (X, d) is a Hausdorff quasi-metric space.

If d is a quasi-metric on a set X, the function d^{-1} defined by $d^{-1}(x, y) = d(y, x)$ is also a quasi-metric on X, while the function d^s defined by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ is a metric on X.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a quasi-metric space (X, d). Then $(x_n)_{n\in\mathbb{N}}$ is called left K-Cauchy if for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ whenever $n_\varepsilon \leq n \leq m$; and it is called right K-Cauchy if is left K-Cauchy in (X, d^{-1}).

A quasi-metric space (X, d) is said to be left (resp. right) K-sequentially complete if every left (resp. right) K-Cauchy sequence converges in τ_d.

It is well known (see e.g. Examples 1, 4 and 5 in Section 2) that the notions of left K-sequential completeness and right K-sequential completeness are independent of each other.

2. $\alpha - \psi$-contractive mappings on quasi-metric spaces, fixed point theorems and examples

Let X be a (non-empty) set and $\alpha : X \times X \to [0, \infty)$ be a function. By α^{-1} we denote the function defined on $X \times X$ by

$\alpha^{-1}(x, y) = \alpha(y, x)$, for all $x, y \in X$.

Definition 1 [18, Definition 2.2]. Let X be a (non-empty) set. A self map T of X is called α-admissible if there is a function $\alpha : X \times X \to [0, \infty)$ such that, for any $x, y \in X$, $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.

Several interesting examples of α-admissible self maps may be found in [18].

Remark 1. Observe that a self map $T$of X is α-admissible if and only if it is α^{-1}-admissible.

Remark 2. If T is a self map of a set X with fixed point $x_0 \in X$, then T is α-admissible for the symmetric function $\alpha : X \times X \to [0, \infty)$ defined as $\alpha(x_0, x_0) = 1$ and $\alpha(x, y) = 0$ otherwise.

As usual (see e.g. [18, page 2154]) we shall denote by Ψ the set of all nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=0}^{\infty} \psi^n(t) < \infty$ for each $t \geq 0$. Recall that if $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$.

Definition 2. Let T be a self map of a quasi-metric space (X, d), $\alpha : X \times X \to [0, \infty)$ a function and $\psi \in \Psi$. Then

(1) T called $\alpha - \psi$-contractive if

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)),$$

for all $x, y \in X$.

(2) T is called $\alpha^{-1} - \psi$-contractive if

$$\alpha^{-1}(x, y)d(Tx, Ty) \leq \psi(d(x, y)),$$

for all $x, y \in X$.

Remark 3. Observe that, by the symmetry of d, a self map T of a metric space (X, d) is an $\alpha - \psi$-contractive mapping if and only if it is an $\alpha^{-1} - \psi$-contractive mapping.

In constrat to Remark 3 there exist $\alpha - \psi$-contractive mappings on quasi-metric spaces that are not $\alpha^{-1} - \psi$-contractive and there exist $\alpha^{-1} - \psi$-contractive mappings that are not $\alpha - \psi$-contractive mappings (see e.g. Examples 3, 4, 5 and 6 below).

As we indicated above, the theorem of reference in our context is the following.

Theorem A [18, Theorem 2.2]. *Let (X, d) be a complete metric space and $T : X \to X$ be an $\alpha - \psi$-contractive mapping satisfying the following conditions:*

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(ii) *T is α-admissible;*

(iii) *for any sequence $(x_n)_{n\in\mathbb{N}}$ in X satisfying $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and such that $d(x, x_n) \to 0$ for some $x \in X$ it follows that $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

A metric space (X, d) satisfying condition (iii) in Theorem A is said to be regular with respect to α.

A first question arises in a natural way: Is it possible to extend Theorem A to the realm of quasi-metric spaces?

Regarding to this question, next we present examples which show that Theorem A does not hold if we replace the condition that (X, d) be a complete metric space with the condition that (X, d) be a Hausdorff right K-sequentially complete or a Hausdorff left K-sequentially complete quasi-metric space. However, we shall show (Theorem 1) that, despite this, it is still possible to obtain a quasi-metric extension of Theorem A by using right K-sequential completeness and $\alpha^{-1} - \psi$-contractivity. At this point it seems interesting to recall that by replacing "$\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N}$" with "$\alpha(x, x_n) \geq 1$ *for all* $n \in \mathbb{N}$" in condition (iii) of Theorem A, a fixed point theorem for Hausdorff left K-sequentially complete quasi-metric spaces was obtained in [17].

Example 1. Let d be the quasi-metric on $\mathbb{N}$ given by $d(n, n) = 0$ for all $n \in \mathbb{N}$, $d(n, m) = 1/n$ if $n < m$, and $d(n, m) = 1$ if $n > m$. Since the right K-Cauchy sequences are those that are eventually constant, $(\mathbb{N}, d)$ is right K-sequentially complete. Moreover, it is Hausdorff because τ_d is the discrete topology on $\mathbb{N}$.

Now let T be the self map of $\mathbb{N}$ defined as $Tn = 2n$ for all $n \in \mathbb{N}$.

Define $\alpha : \mathbb{N} \times \mathbb{N} \to [0, \infty)$ by $\alpha(n, 2n) = 1$ for all $n \in \mathbb{N}$, and $\alpha(n, m) = 0$ otherwise.

We have that $\alpha(1, T1) = 1$.

Moreover T is α-admissible because if $\alpha(n, m) \geq 1$, then $m = 2n$, and thus $\alpha(Tn, Tm) = \alpha(2n, 4n) = 1$.

We also have that T is an $\alpha - \psi$-contractive mapping for $\psi \in \Psi$ given by $\psi(t) = t/2$ for all $t \geq 0$. Indeed, we get

$$\alpha(n, 2n)d(Tn, T2n) = d(2n, 4n) = \frac{1}{2n} = \frac{1}{2}d(n, 2n),$$

for all $n \in \mathbb{N}$.

Finally, if $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{N}$ such that $\alpha(x_n, x_{n+1}) \geq 1$, we deduce that $(x_n)_{n\in\mathbb{N}}$ does not converge in τ_d, so condition (iii) in Theorem A is obviously satisfied.

We have shown that all conditions of Theorem A are satisfied, but T has no fixed points. Notice that $(\mathbb{N}, d)$ is not left K-sequentially complete.

Example 2. Let d be the quasi-metric on the set $\mathbb{Z}$ of all integer numbers given by

$d(x, x) = 0$ for all $x \in \mathbb{Z}$,

$d(0, x) = 2^{-|x|}$ for all $x \in \mathbb{Z}\backslash\{0\}$,

$d(x, y) = 2^{-x}$ if $x, y \in \mathbb{N}$ with $x < y$,

$d(x, -y) = 2^{-x}$ if $x, y \in \mathbb{N}$ with $x \leq y$,

$d(x, -y) = 2^{-y}$ if $x, y \in \mathbb{N}$ with $x > y$, and

$d(x, y) = 1$ otherwise.

Clearly (X, d) is a Hausdorff quasi-metric space (note that 0 is the unique non-isolated point in τ_d). Moreover (X, d) is left K-sequentially complete (note that the sequence $(n)_{n\in\mathbb{N}}$ is left K-Cauchy and converges to 0 in τ_d).

Now let T be the self map of $\mathbb{Z}$ defined as $Tx = x + 1$ if $x \in \mathbb{N}$, and $Tx = x - 1$ otherwise.

Define $\alpha : \mathbb{N} \times \mathbb{N} \to [0, \infty)$ by

$\alpha(x, x + 1) = \alpha(x, 0) = 1$ if $x \in \mathbb{N}$,

$\alpha(x, -y) = 1$ if $x, y \in \mathbb{N}$ with $x > y$, and

$\alpha(x, y) = 0$ otherwise.

We have that $\alpha(1, T1) = \alpha(1, 2) = 1$.

Moreover T is α-admissible. Indeed, let $a, b \in \mathbb{Z}$ such that $\alpha(a, b) \geq 1$.

If $a = x$ and $b = x + 1$ for some $x \in \mathbb{N}$ we deduce that $\alpha(Ta, Tb) = \alpha(x + 1, x + 2) = 1$.

If $a = x$ for some $x \in \mathbb{N}$ and $b = 0$ we deduce that $\alpha(Ta, Tb) = \alpha(x + 1, -1) = 1$.

If $a = x$ and $b = -y$ for some $x, y \in \mathbb{N}$ with $x > y$ we deduce that $\alpha(Ta, Tb) = \alpha(x + 1, -(y + 1)) = 1$.

Now we prove that T is an $\alpha - \psi$-contractive mapping for $\psi \in \Psi$ given by $\psi(t) = t/2$ for all $t \geq 0$. Indeed, for any $x \in \mathbb{N}$ we get

$$\alpha(x, x + 1)d(Tx, T(x + 1)) = d(x + 1, x + 2) = 2^{-(x+1)} = \frac{1}{2}d(x, x + 1),$$

and

$$\alpha(x, 0)d(Tx, T0) = d(x + 1, -1) = \frac{1}{2} = \frac{1}{2}d(x, 0),$$

and for any $x, y \in \mathbb{N}$, with $x > y$,

$$\begin{aligned}\alpha(x, -y)d(Tx, T(-y)) &= d(x + 1, -(y + 1)) \\ &= 2^{-(y+1)} = \frac{1}{2}d(x, -y).\end{aligned}$$

We have shown that T is an $\alpha - \psi$-contractive mapping.

Finally, if $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Z}$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, and there is $x \in X$ for which $d(x, x_n) \to 0$ as $n \to \infty$, we deduce that $x = 0$ and $(x_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers, so $\alpha(x_n, x) = 1$ for all $n \in \mathbb{N}$.

Consequently, all conditions of Theorem A are satisfied, but T has no fixed points.

Lemma 1. *Let (X, d) be a quasi-metric space and $T : X \to X$ be an $\alpha^{-1} - \psi$-contractive mapping satisfying the following conditions:*

(i) *there exists $x_0 \in X$ for which $\alpha(x_0, Tx_0) \geq 1$;*

(ii) *T is α-admissible.*

Then, the sequence $(T^n x_0)_{n\in\mathbb{N}}$ is right K-Cauchy in (X, d).

Proof. The proof is an adaptation of the first part of the proof of Theorem A as given in [18, Theorem 2.2]. For each $n \in \mathbb{N}$ put $x_n := T^n x_0$. Then, by conditions (i) and (ii), $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$. Then

$$d(x_{n+1}, x_n) \leq \alpha^{-1}(x_{n+1}, x_n)d(x_{n+1}, x_n) \leq \psi(d(x_n, x_{n-1}),$$

for all $n \in \mathbb{N}$. So, by induction, $d(x_{n+1}, x_n) \leq \psi^n(d(x_1, x_0))$. Since $\psi \in \Psi$ we get (see [18, page 2156]) that $(x_n)_{n\in\mathbb{N}}$ is a right K-Cauchy sequence in (X, d).

Theorem 1. *Let (X, d) be a Hausdorff right K-sequentially complete quasi-metric space and $T : X \to X$ be an $\alpha^{-1} - \psi$-contractive mapping satisfying the following conditions:*

(i) *there exists $x_0 \in X$ for which $\alpha(x_0, Tx_0) \geq 1$;*

(ii) *T is α-admissible;*

(iii) *for any sequence $(x_n)_{n\in\mathbb{N}}$ in X satisfying $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and such that $d(x, x_n) \to 0$ for some $x \in X$ it follows that $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Proof. By Lemma 1, the sequence $(T^n x_0)_{n\in\mathbb{N}}$ is right K-Cauchy in (X, d). Since (X, d) is Haudorff and right K-sequentially complete, there is a unique $x \in X$ such that $d(x, T^n x_0) \to 0$ as $n \to \infty$, where we assume that $x \neq T^n x_0$ for all $n \in \mathbb{N} \cup \{0\}$.

We show that x is a fixed point of T. Indeed, from condition (iii) it follows that $\alpha(T^n x_0, x) \geq 1$ for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned} d(Tx, T^{n+1}x_0) &\leq \alpha^{-1}(x, T^n x_0) d(Tx, T^{n+1}x_0) \\ &\leq \psi(d(x, T^n x_0)) < d(x, T^n x_0), \end{aligned}$$

for all $n \in \mathbb{N}$. Hence $d(Tx, T^n x_0) \to 0$ as $n \to \infty$. Since (X, d) is Hausdorff we conclude that $x = Tx$. This finishes the proof.

Remark 4. By Remark 3 and the fact that every complete metric space is a right K-sequentially complete quasi-metric space, we deduce that Theorem 1 is, indeed, a quasi-metric generalization of Theorem A.

In the light of the conditions of regularity with respect to α involved in [17, Theorem 1] and in Theorems A and 1 above, we propose the following notions.

Definition 3. Let (X, d) be a quasi-metric space and $\alpha : X \times X \to [0, \infty)$ be a function.

(1) We say that (X, d) is (A_1)-regular with respect to α if for any sequence $(x_n)_{n\in\mathbb{N}}$ in X satisfying $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and such that $d(x, x_n) \to 0$ for some $x \in X$ it follows that $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

(2) We say that (X, d) is (A_2)-regular with respect to α if for any sequence $(x_n)_{n\in\mathbb{N}}$ in X satisfying $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and such that $d(x, x_n) \to 0$ for some $x \in X$ it follows that $\alpha(x, x_n) \geq 1$ for all $n \in \mathbb{N}$.

(3) We say that (X, d) is (A_1^{-1})-regular with respect to α if for any sequence $(x_n)_{n\in\mathbb{N}}$ in X satisfying $\alpha^{-1}(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and such that $d(x, x_n) \to 0$ for some $x \in X$ it follows that $\alpha^{-1}(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

(4) We say that (X, d) is (A_2^{-1})-regular with respect to α if for any sequence $(x_n)_{n\in\mathbb{N}}$ in X satisfying $\alpha^{-1}(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and such that $d(x, x_n) \to 0$ for some $x \in X$ it follows that $\alpha^{-1}(x, x_n) \geq 1$ for all $n \in \mathbb{N}$.

In Theorem 1 we have obtained a fixed point theorem for Hausdorff right K-sequentially complete quasi-metric space by using the notion of (A_1)-regularity. In the rest of this section we shall give fixed point theorems for Hausdorff right K- and left K-sequentially complete quasi-metric spaces with the help of the rest of notions of regularity given in the preceding definition. To this end, the following easy observation is appropriate.

Remark 5. Let (X, d) be a quasi-metric space and $\alpha : X \times X \to [0, \infty)$ be a function. Then:

(X, d) is (A_i^{-1})-regular with respect to α if and only if it is (A_i)-regular with respect to α^{-1}, $i = 1, 2$.

Theorem 2. *Let (X, d) be a Hausdorff right K-sequentially complete quasi-metric space and $T : X \to X$ be an $\alpha - \psi$-contractive mapping satisfying the following conditions:*

(i) *there exists* $x_0 \in X$ *for which* $\alpha^{-1}(x_0, Tx_0) \geq 1$;
(ii) T *is* α*-admissible;*
(iii) (X, d) *is* (A_1^{-1})*-regular with respect to* α.
Then T *has a fixed point.*

Proof. Put $\beta := \alpha^{-1}$. Then T is a $\beta^{-1} - \psi$-contractive mapping. Moreover $\beta(x_0, Tx_0) \geq 1$, and, by Remark 1, T is β-admissible. Finally, by Remark 5, (X, d) is (A_1)-regular with respect to β. We deduce from Theorem 1 that T has a fixed point.

Theorem 3 [17, Theorem 1]. *Let* (X, d) *be a Hausdorff left K-sequentially complete quasi-metric space and* $T : X \to X$ *be an* $\alpha - \psi$*-contractive mapping satisfying the following conditions:*
(i) *there exists* $x_0 \in X$ *for which* $\alpha(x_0, Tx_0) \geq 1$;
(ii) T *is* α*-admissible;*
(iii) (X, d) *is* (A_2)*-regular with respect to* α.
Then T *has a fixed point.*

Theorem 4. *Let* (X, d) *be a Hausdorff left K-sequentially complete quasi-metric space and* $T : X \to X$ *be an* $\alpha^{-1} - \psi$*-contractive mapping satisfying the following conditions:*
(i) *there exists* $x_0 \in X$ *for which* $\alpha^{-1}(x_0, Tx_0) \geq 1$;
(ii) T *is* α*-admissible;*
(iii) (X, d) *is* (A_2^{-1})*-regular with respect to* α.
Then T *has a fixed point.*

Proof. As in the proof of Theorem 2 put $\beta := \alpha^{-1}$. Then T is a $\beta - \psi$-contractive mapping. Moreover $\beta(x_0, Tx_0) \geq 1$, and, by Remark 1, T is β-admissible. Finally, by Remark 5, (X, d) is (A_2)-regular with respect to β. We deduce from Theorem 3 that T has a fixed point.

Next we give an example where we can apply Theorem 1 but not Theorems 2, 3 and 4, directly, because the $\alpha^{-1} - \psi$-contractive mapping involved is not $\alpha - \psi$-contractive, and, in addition, we get $\alpha^{-1}(x_0, Tx_0) < 1$ for all element x_0 of the quasi-metric space.

Example 3. Let Σ be an alphabet, i.e., a non-empty set, such that $|\Sigma| > 1$. The elements of Σ are usually called symbols or letters. Denote by Σ^F the set of all finite sequences (strings) of symbols over Σ, and by Σ^ω the set of all infinite sequences. We assume that the empty sequence $\emptyset$ is an element of Σ^F. Put $\Sigma^\infty = \Sigma^F \cup \Sigma^\omega$. As usual we denote by $\sqsubseteq$ the prefix order on Σ^∞, i.e., $x \sqsubseteq y \Leftrightarrow x$ is a prefix of y. If x is a prefix of y with $x \neq y$, we write $x \sqsubset y$. For each $x \in \Sigma^\infty$ we denote by $\ell(x)$ is length. Thus $\ell(\emptyset) = 0$, $\ell(x) < \infty$ if $x \in \Sigma^F$ and $\ell(x) = \infty$ if $x \in \Sigma^\omega$. (See e.g. [2, Chapter 1]).

Let d be the quasi-metric on Σ^∞ given by $d(x, y) = 2^{-\ell(y)} - 2^{\ell(x)}$ if $y \sqsubseteq x$, and $d(x, y) = 1$ otherwise. Then (Σ^∞, d) is a Hausdorff right K-sequentially complete quasi-metric space [15, Theorem 2 and Remark 4(a)].

Let T be any self map of Σ^∞ satisfying the following two conditions: (i) $\ell(Tx) = \ell(x) + 1$, and (ii) $x \sqsubseteq y \Rightarrow Tx \sqsubseteq Ty$.

Of course, if $x \in \Sigma^\omega$, $\ell(Tx) = \infty$, so $Tx \in \Sigma^\infty$. Notice also that condition (ii) implies that $Tx \sqsubset Ty$ whenever $x \sqsubset y$.

In fact, conditions (i) and (ii) above are not strange or very restrictive. Indeed, this type of self maps are key when discussing the existence and uniqueness of solution to recurrence equations associated to probabilistic divide and conquer algorithm by means of fixed point techniques (see e.g. [15, page 629], [16, Example 4.2], [8, Example 4.2]).

We first observe that T is not a Banach contraction on (Σ^∞, d) because for $x, y \in \Sigma^\infty$ such that y is not prefix of x, it follows that Ty is not a prefix of Tx, and thus $d(Tx, Ty) = 1 = d(x, y)$.

Let $\alpha : X \times X \to [0, \infty)$ given by $\alpha(x, y) = 1$ if $x \sqsubset y$, and $\alpha(x, y) = 0$ otherwise.

We will check that all conditions of Theorem 1 are satisfied for $\psi \in \Psi$ given by $\psi(t) = t/2$. Indeed:
We have $\alpha(\emptyset, T\emptyset) = 1$.
Moreover T is clearly α-admissible because if $\alpha(x, y) \geq 1$, then $x \sqsubset y$, so $Tx \sqsubset Ty$, and thus $\alpha(Tx, Ty) = 1$.
Next we show that T is an $\alpha^{-1} - \psi$-contractive mapping.
Let $x, y \in \Sigma^\infty$ such that $\alpha^{-1}(x, y) = 1$. Then $y \sqsubset x$. Hence $Ty \sqsubset Tx$, so

$$\begin{aligned}\alpha^{-1}(x,y)d(Tx,Ty) &= 2^{-(\ell(y)+1)} - 2^{-(\ell(x)+1)} \\ &= \frac{1}{2}(2^{-\ell(y)} - 2^{-\ell(x)}) = \frac{1}{2}d(x,y).\end{aligned}$$

It remains to show that (Σ^∞, d) is (A_1)-regular with respect to α. To this end, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in Σ^∞ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Then $x_n \sqsubset x_{n+1}$ for all $n \in \mathbb{N}$. Let x be the unique element of Σ^ω such that $x_n \sqsubset x$ for all $n \in \mathbb{N}$. Then $d(x, x_n) \to 0$ as $n \to \infty$, and $\alpha(x_n, x) = 1$ for all $n \in \mathbb{N}$. We conclude that (Σ^∞, d) is (A_1)-regular with respect to α.

However, we cannot apply Theorems 2 and 3 for this α because T is not an $\alpha - \psi$-contractive mapping for any $\psi \in \Psi$. Indeed, take $x = \emptyset$ and $y = T\emptyset$. Then, we get

$$\alpha(x, y)d(Tx, Ty) = d(Tx, Ty) = 1 > \psi(1) = \psi(d(x, y)).$$

Finally, we cannot apply Theorem 4 because $\alpha^{-1}(x_0, Tx_0) = 0$ for all $x_0 \in \Sigma^\infty$.

The following is an example where we can apply Theorem 2 but we cannot apply Theorems 3 and 4, nor can Theorem 1 directly. In fact, the involved quasi-metric space is not left K-sequentially complete and the $\alpha - \psi$-contractive mapping is not $\alpha^{-1} - \psi$-contractive.

Example 4. Let $(\mathbb{R}, d_S)$ the famous Sorgenfrey quasi-metric line (see e.g. [5, Example 1.1.6]), where d_S is the Hausdorff quasi-metric on $\mathbb{R}$ given by $d_S(x, y) = y - x$ if $x \leq y$, and $d_S(x, y) = 1$ if $x > y$.

It is well known that d_S induces the Sorgenfrey topology on $\mathbb{R}$ and that (X, d_S) is a right K-sequentially complete quasi-metric space

Let T be the self map on X given by $Tx = 1$ if $x > 1$, $Tx = (x + 1)/2$ if $x \in [0, 1]$, and $Tx = 1/2$ if $x < 0$.

We first observe that T is not a Banach contraction on $(\mathbb{R}, d_S)$ because $d_S(T2, T0) = d_S(1, 1/2) = 1 = d_S(2, 0)$.

Let $\alpha : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ given by $\alpha(x, y) = 1$ if $x \leq y$, and $\alpha(x, y) = 0$ if $x > y$.

We will check that all conditions of Theorem 2 are satisfied for $\psi \in \Psi$ given by $\psi(t) = t/2$. Indeed:

We have $\alpha^{-1}(2, T2) = \alpha(1, 2) = 1$.
Moreover T is clearly α-admissible because if $\alpha(x, y) \geq 1$, then $x \leq y$, so $Tx \leq Ty$, and thus $\alpha(Tx, Ty) = 1$.
Now let $x, y \in \mathbb{R}$ such that $\alpha(x, y) = 1$. If $x, y > 1$ or $x, y < 0$, we have $d_S(Tx, Ty) = 0$. If $x \leq 1 < y$, we get

$$\begin{aligned}\alpha(x,y)d_S(Tx,Ty) &= d_S(\frac{x+1}{2}, 1) = \frac{1-x}{2} \\ &< \frac{y-x}{2} = \frac{1}{2}d_S(x,y).\end{aligned}$$

If $x < 0$ and $y > 1$, we get

$$\alpha(x, y)d_S(Tx, Ty) = d_S(\frac{1}{2}, 1) = \frac{1}{2} < \frac{1}{2}(y - x) = \frac{1}{2}d_S(x, y).$$

If $x < 0$ and $y \in [0, 1]$, we get

$$\alpha(x, y)d_S(Tx, Ty) = d_S(\frac{1}{2}, \frac{y+1}{2}) = \frac{y}{2} < \frac{1}{2}(y - x) = \frac{1}{2}d_S(x, y),$$

and if $x, y \in [0, 1]$ we get

$$\alpha(x, y)d_S(Tx, Ty) = d_S(\frac{x+1}{2}, \frac{y+1}{2}) = \frac{y-x}{2} = \frac{1}{2}d_S(x, y).$$

We have shown that T is an $\alpha - \psi$-contractive mapping.

It remains to show that $(\mathbb{R}, d_S)$ is (A_1^{-1})-regular with respect to α. To this end, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $\alpha^{-1}(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Then $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$. Suppose there exists $x \in \mathbb{R}$ such that $d_S(x, x_n) \to 0$ as $n \to \infty$. Then $x = \inf_{n\in\mathbb{N}} x_n$. Consequently $\alpha^{-1}(x_n, x) = 1$ for all $n \in \mathbb{N}$. We conclude that $(\mathbb{R}, d_S)$ is (A_1^{-1})-regular with respect to α.

We have verified that all conditions of Theorem 2 are satisfied (in fact T has a (unique) fixed point $z = 1$). However, we can not apply to this example Theorems 3 and 4 because (X, d_S) is not left K-sequentially complete. Furthermore, we cannot apply Theorem 1 for this α because T is not $\alpha^{-1} - \psi$-contractive for any $\psi \in \Psi$. Indeed, let $\psi \in \Psi$. Then, for $x, y \in [0, 1]$ with $x > y$ we have

$$\alpha^{-1}(x, y)d_S(Tx, Ty) = 1 > \psi(1) = \psi(d_S(x, y)).$$

Next we present an example where we can apply Theorem 3 but not Theorems 1, 2 and 4.

Example 5. The one-point compactification of $\mathbb{N}$ consists of the set $\omega := \mathbb{N} \cup \{\infty\}$ endowed with the topology τ_ω whose open sets are all subsets of $\mathbb{N}$, and the sets of the form $\{\infty\} \cup (\mathbb{N}\backslash F)$ where F is a finite subset of $\mathbb{N}$ (see e.g. [6, page 222]).

Obviously τ_ω is a metrizable topology, but in our context we can construct an appealing and interesting quasi-metric on ω, denoted by d_ω and whose induced topology agrees with τ_ω, as follows:

$d_\omega(x, x) = 0$ for all $x \in \omega$;

$d_\omega(\infty, n) = 2^{-n}$ for all $n \in \mathbb{N}$;

$d_\omega(n, m) = 2^{-n}$ if $n, m \in \mathbb{N}$ with $n < m$, and

$d_\omega(x, y) = 1$ otherwise.

Then (ω, d_ω) is left K- and right K-sequentially complete (note, in particular, that the right K-Cauchy sequences are those that are eventually constant). Moreover $\tau_{d_\omega} = \tau_\omega$, $\tau_{(d_\omega)^{-1}}$ is the discrete topology on ω and $(d_\omega)^s$ is the discrete metric on ω.

Now let $A = \{1/n : n \in \mathbb{N}\backslash\{1\}\}$ and $X := \omega \cup A$.

Denote by q the quasi-metric on X given by

$q(x, y) = d_\omega(x, y)$ if $x, y \in \omega$,

$q(x, y) = |x - y|$ if $x, y \in A$, and

$q(x, y) = 1$ otherwise.

It is almost obvious that (X, q) is a Hausdorff left K-sequentially complete quasi-metric space. However, it is not right K-sequentially complete because $(1/n)_{n\in\mathbb{N}}$ is a right K-Cauchy sequence in (X, q) that does not converge in τ_q.

Let S be a strictly increasing self map of ω such that $n < Sn$ for all $n \in \mathbb{N}$. (Note that this implies $S\infty = \infty$ for all $n \in \mathbb{N}$).

Denote by T the self map of X defined as $Tx = Sx$ for all $x \in \omega$, and $Tx = x/2$ for all $x \in A$.

We first observe that T is not a Banach contraction on (X, q) because for $n, m \in \mathbb{N}$ with $n > m$ we have $Tn > Tm$, and thus $q(Tn, Tm) = 1 = q(n, m)$.

Let $\alpha : X \times X \to [0, \infty)$ given by $\alpha(n, m) = 1$ if $n, m \in \mathbb{N}$ with $n < m$, $\alpha(\infty, x) = 1$ for all $x \in \omega$, and $\alpha(x, y) = 0$ otherwise, and let $\psi \in \Psi$ given by $\psi(t) = t/2$.

We will check that all conditions of Theorem 3 are satisfied. Indeed:

We have $\alpha(1, T1) = 1$ because $1 < T1$.

Now let $\alpha(x, y) \geq 1$. If $x, y \in \mathbb{N}$ then $x < y$, so $Tx < Ty$, and thus $\alpha(Tx, Ty) = 1$. If $x = \infty$ and $y \in \omega$ we deduce that $\alpha(Tx, Ty) = \alpha(\infty, Ty) = 1$. Hence T is α-admissible.

Next we show that T is $\alpha - \psi$-contractive.

Let $x, y \in \omega$ such that $\alpha(x, y) \geq 1$.

If $x = y = \infty$, we have $q(Tx, Ty) = 0$.

If $x, y \in \mathbb{N}$, we have $x < y$, so $Tx < Ty$ and thus

$$\alpha(x,y)q(Tx,Ty) = 2^{-Tx} \leq 2^{-(x+1)} = \frac{1}{2}q(x,y).$$

If $x = \infty$ and $y \in \mathbb{N}$ we deduce

$$\alpha(x,y)q(Tx,Ty) = q(\infty,Ty) = 2^{-Ty} \leq 2^{-(y+1)} = \frac{1}{2}q(x,y).$$

It remains to check that (X,d) is (A_2)-regular with respect to α. To this end, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. We suppose, without loss of generality, that $x_n \in \mathbb{N}$ for all $n \in \mathbb{N}$. Then $x_{n+1} > x_n$ for all $n \in \mathbb{N}$. Therefore $q(\infty, x_n) \to 0$ as $n \to \infty$. Since $\alpha(\infty, x_n) = 1$ for all $n \in \mathbb{N}$, we conclude that (X,q) is (A_2)-regular with respect to α.

Thus, all conditions of Theorem 3 are satisfied. (In fact, ∞ is the unique fixed point of T).

However, we cannot apply Theorems 1 and 2 because (X,q) is not right K-sequentially complete. Moreover, we cannot apply Theorem 4 because T is not $\alpha^{-1} - \psi$-contractive for any $\psi \in \Psi$. Indeed, let $\psi \in \Psi$. Then, for $n, m \in \mathbb{N}$ with $n > m$ we have

$$\alpha^{-1}(n,m)q(Tn,Tm) = 1 > \psi(1) = \psi(q(n,m)).$$

We conclude this section with an example where we can apply Theorem 4 but not Theorems 1, 2 and 3.

Example 6. Let (X,q) and T the quasi-metric space and the self map of X of Example 5, respectively. We know that (X,q) is left K-sequentially complete.

Put $\beta := \alpha^{-1}$. Then $\beta^{-1} = \alpha$, so T is a $\beta^{-1} - \psi$-contractive mapping. Moreover $\beta^{-1}(1, T1) = 1$, and, by Remark 1, T is β-admissible. Furthermore (X,q) is (A_2^{-1})-regular with respect to β, by Remark 5. Thus, all conditions of Theorem 4 are satisfied.

However, we can not apply Theorems 1 and 2 because (X,q) is not right K-sequentially complete. Moreover, we cannot apply Theorem 3 for β because T is not $\beta - \psi$-contractive for any $\psi \in \Psi$.

3. A characterization of doubly Hausdorff right K-sequentially complete quasi-metric spaces

In [17, Theorem 3] its was proved that Theorem 3 above characterizes Hausdorff left K-sequentially complete quasi-metric spaces. More exactly it was stated the following.

Theorem 5 [17, Theorrem 3]. *A Hausdorff quasi-metric space (X,d) is left K-sequentially complete if and only if every $\alpha - \psi$-contractive mapping T* on (X,d) *satisfying the following conditions has a fixed point:*
(i) *there exists $x_0 \in X$ for which $\alpha(x_0, Tx_0) \geq 1$;*
(ii) *T is α-admissible;*
(iii) (X,d) *is (A_2)-regular with respect to α.*

As an immediate consequence it was deduced that a slight modification of [18, Theorem 2.2] provides a characterization of the metric completeness in the following way.

Corollary 1. *A metric space (X,d) is complete if and only if every $\alpha - \psi$-contractive mapping T* on (X,d) *satisfying the following conditions has a fixed point:*
(i) *there exists $x_0 \in X$ for which $\alpha(x_0, Tx_0) \geq 1$;*
(ii) *T is α-admissible;*
(iii) (X,d) *is (A_2)-regular with respect to α.*

In this section we show a characterization of doubly Hausdorff right K-sequentially complete quasi-metric spaces from which we will deduce (see Corollary 2 below) that the original fixed point theorem of Samet, Vetro and Vetro (Theorem A) also allows to characterize the metric completeness.

We say that a quasi-metric space (X,d) is doubly Hausdorff if both (X,d) and (X,d^{-1}) are Hausdorff quasi-metric spaces.

There exist doubly Hausdorff quasi-metric spaces in abundance. In fact, it is straightforward to check that all quasi-metric spaces given in our six examples above are doubly Hausdorff. In particular, the Sorgenfrey quasi-metric line provides a distinguished example of a doubly Hausdorff right K-sequentially complete quasi-metric space.

Theorem 6. *A doubly Hausdorff quasi-metric space* (X,d) *is right K-sequentially complete if and only if every* $\alpha^{-1}-\psi$*-contractive mapping* T on (X,d) *satisfying the following conditions has a fixed point:*
(i) *there exists* $x_0 \in X$ *for which* $\alpha(x_0, Tx_0) \geq 1$;
(ii) T *is* α*-admissible;*
(iii) (X,d) *is* (A_1)*-regular with respect to* α.

Proof. We only prove the "if" part because the "only if" part follows from Theorem 1.

To this end, we suppose that there is a right K-Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (X,d) of distinct points that does not converge in τ_d.

Put $F := \{x_n : n \in \mathbb{N}\}$. Then, the key step consists in observing that, without loss of gnerality, we can assume that $d(F\backslash\{x_n\}, x_n) > 0$ for all $n \in \mathbb{N}$.

Indeed, if the sequence $(x_n)_{n\in\mathbb{N}}$ does not converge in $\tau_{d^{-1}}$ or converges in $\tau_{d^{-1}}$ to some $x \in X\backslash F$, the inequality $d(F\backslash\{x_n\}, x_n) > 0$ is obviously satisfied for all $n \in \mathbb{N}$. In case that the sequence converges in $\tau_{d^{-1}}$ to some $x_{n_0} \in F$ we consider the sequence $(y_k)_{k\in\mathbb{N}}$ where $y_k := x_{n_0+k}$ for all $k \in \mathbb{N}$. Then $(y_k)_{k\in\mathbb{N}}$ is a right K-Cauchy sequence in (X,d) that does not converge in τ_d. Furthermore $d(F_0\backslash\{y_k\}, y_k) > 0$ for all $k \in \mathbb{N}$, where $F_0 := \{y_k : k \in \mathbb{N}\}$, because by the Hausdorffness of (X,d^{-1}), x_{n_0} would be the unique limit point of $(x_n)_{n\in\mathbb{N}}$ and, hence, of $(y_k)_{k\in\mathbb{N}}$ in $\tau_{d^{-1}}$.

Hence, since $(x_n)_{n\in\mathbb{N}}$ is right K-Cauchy and $d(F\backslash\{x_n\}, x_n) > 0$ for all $n \in \mathbb{N}$, we can find a subsequence $(s(n))_{n\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ such that $s(n+1) > s(n) > n$ and

$$d(x_r, x_q) < \frac{1}{2}d(F\backslash\{x_n\}, x_n),$$

for all $q, r \in \mathbb{N}$ such that $s(n) \leq q \leq r$.

Let $T : X \to X$ be the mapping given by

$Tx = x_1$ for all $x \in X\backslash F$, and $Tx_n = x_{s(n)}$ for all $n \in \mathbb{N}$.

Notice that T has no fixed points because $s(n) > n$ for all $n \in \mathbb{N}$.

Now let $\alpha : X \times X \to [0,\infty)$ given by

$\alpha(x,y) = 1$ if $x = x_n$ and $y = x_m$ for $n, m \in \mathbb{N}$ with $n < m$, and $\alpha(x,y) = 0$ otherwise.

We will prove that T is an $\alpha^{-1}-\psi$-contractive mapping for $\psi \in \Psi$ given by $\psi(t) = t/2$, that satisfies condition (i), (ii) and (iii) of the statement of Theorem 6.

We first note that $\alpha(x_1, Tx_1) = 1$ because $1 < s(1)$.

Now we are going to check that T is α-admissible. Suppose that $\alpha(x,y) \geq 1$. Then $x = x_n$ and $y = x_m$ with $n < m$. Therefore $\alpha(Tx, Ty) = \alpha(x_{s(n)}, x_{s(m)}) = 1$ because $s(n) < s(m)$.

Next we show that T is $\alpha^{-1}-\psi$-contractive. To this end, it suffices to take $x = x_n$ and $y = x_m$ with $n > m$. Thus, we have $s(n) > s(m)$ and we deduce

$$\begin{aligned}\alpha^{-1}(x,y)d(Tx,Ty) &= \alpha(x_m, x_n)d(x_{s(n)}, x_{s(m)}) < \frac{1}{2}d(F\backslash\{x_m\}, x_m) \\ &\leq \frac{1}{2}d(x_n, x_m) = \frac{1}{2}d(x,y) = \psi(d(x,y)).\end{aligned}$$

Finally, it is clear that (X,d) is (A_1)-regular with respect to α because the only τ_d-convergent sequences in F are those that are eventually constant and for any $x \in F$ we have $\alpha(x,x) = 0$.

Since T has no fixed points we have reached a contradiction. This concludes the proof.

Corollary 2. *A metric space (X, d) is complete if and only if every $\alpha - \psi$-contractive mapping T* on (X, d) *satisfying the following conditions has a fixed point:*
(i) *there exists $x_0 \in X$ for which $\alpha(x_0, Tx_0) \geq 1$;*
(ii) *T is α-admissible;*
(iii) (X, d) *is (A_1)-regular with respect to α.*

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