A Complex Fractional Mathematical Modeling for the Love Story of Layla and Majnun

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Abstract

In this article, we provide numerical simulations to show the importance and the effects of fractional order derivatives in psychological studies. As it is well-known, complex variables are more realistic for defining structures in different cases. In this paper, we evidence that such dynamics become more realistic when we use fractional derivatives. We study a non-integer order, non-linear mathematical model for defining a love story of Layla and Majnun (a couple in a romantic relationship). We exemplify all necessary practical calculations to study this serious psychological phenomena. The existence of the unique solution for the given model is exhibited. We use a very recent and strong modified Predictor-Corrector algorithm to evaluate the model structure. Stability of the proposed method is also given. We exemplify that the given complex fractional model is more realistic and represents reality more closely. The proposed model is very basic, significant, and efficient at introducing distinct natures by only replacing one control parameter. In this study, we found that in some of the cases there are stable limit cycles, in some cases periodic behaviours and sometimes transiently chaotic solutions exist which cannot be observed for integer order models at same parameter values. The principal contribution of this article is to exhibit the importance of non-integer order derivatives for analysing complex dynamics. The use of complex variables makes this study more effective because they have both magnitude and phase to better explore the love and can describe different emotions such as coexisting love and hate.

Keywords: Psychological modelling, Complex variables, Fractional mathematical model, Numerical algorithm, New generalised Caputo type fractional derivative
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1. Introduction

According to the American Psychological Association, Psychology is the scientific study of the mind. This theory is multifaceted and contains many sub-areas of study such areas

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as human development, health, social nature, clinical and cognitive processes. Generally, what do we know about love? Love is dinkum, maddening and sometimes scary. Our lives succumb on it and it frequently seems like our planet would stay to spinning if love did not exist. We study the biological reason to explain these reactions; we analyse cultural effects on how to deal with it and try to understand psychological phenomena that influence us to fall in love. Moreover, this feeling is not only present in human being but also in all animal kingdom.

Figure 1: Monogamous Relationships Exist Throughout The Animal Kingdom

There exist many famous love stories in humans. One of the most famous is the one about Layla and Majnun who was a poet. After seeing Layla, he fell blusterous in love with her and asked her to marry him. More details about this story can be read in [12]. In Figure 2, we have pasted some famous lines of Majnun's poem.

Since last few decades, fractional derivatives have been applied in different scientific areas such as biology, chemistry, physics, and engineering among others. One of the main advantages that fractional derivatives offer is the capture of memory effect on dynamical systems and they are widely used to model the outcome of short term memory. Fractional models have been considered for describing the dynamics of epidemiological systems [5, 6, 21, 22, 24, 8], biological processes [11, 27] or physical modelling [7, 34, 2, 10, 14]. As an example, in [16], the authors provided an ecological study by using new generalised Caputo type non-integer order derivative. In [35] the authors considered the same fractional derivative for modelling pine wilt disease. Caputo fractional derivatives were also considered in [9] for analysing COVID-19 transmission. Some very recent applications of fractional derivatives can be observed from [1, 15, 18, 19, 17, 20, 25].

On the other hand, there exist a wide research on mathematical modelling in psychology [4, 32]. More concretely, there are a great number of references that contain mathematical models which describe romantic relationships, see [31, 28, 29, 33] and the references therein. One of the main difficulties for modelling the evolution of sentimental relationships is defining and quantifying feelings such as love in a mathematical way. Another strain arises because love stories are affected by the turbulence of the surrounding social environment, which usually leads to these models exhibiting chaos [12, 30]. Moreover, the parameters defining the interaction between the couple in a romantic relationship vary over time, and are suceptible to learning and accomodation [30].

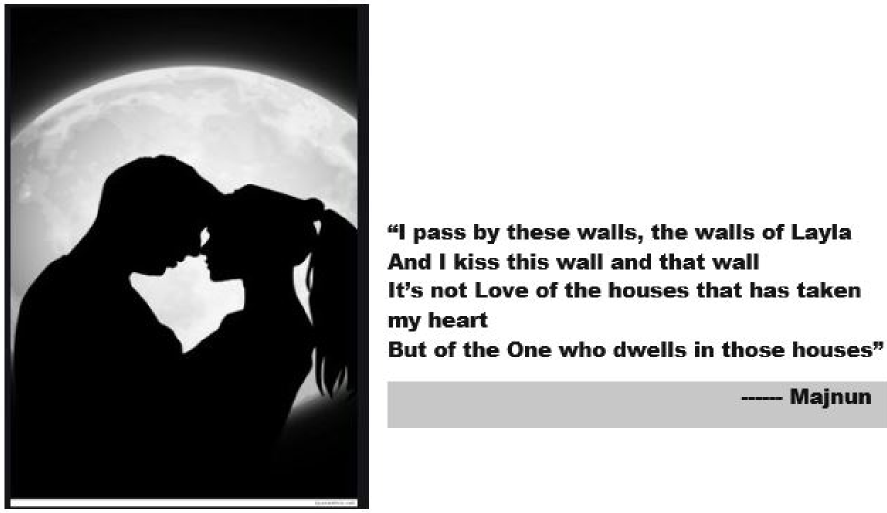


Figure 2: Some lines of Majnun's famous poems

Our aim in this paper is to take advantage of fractional equations for describing anomalous dynamics in complex systems, see [3] where fractional-order dynamical models of love where introduced for the first time. More concretely, we generalize the model of love proposed in [12] allowing the state dynamics of the system to assume non-integer orders. The reality that romantic relationships are effected by memory makes non-integer derivatives suitable for modeling such kind of dynamical models. The proposed model considered in [12] and motivated by the relationship between Layla and Majnun, is a complex dynamical system which involves two differential equations defining the changes over time of the feelings exhibited by them in their romantic relationship. Due to the fact that complex variables have both magnitude and phase, they are more effective for describing a wider range of emotions. Our main contribution in this paper is the generalization of the Layla and Majnun love model in terms of the recent modified version of Caputo type non-integer order derivative given by Odibat et al. [26] and to explore a different nature of the suggested non-linear model. We observed that such non-integer order models can exhibit stable limit cycles, periodic nature and chaos in the company of non-linearity by using appropriate parameters.
Our paper is framed as follows: we first review some basic definitions that will be needed in the paper. In section 3 we define the fractional version of the complex non-linear dynamical model given by Jafari et al. [12] in which we base our study. Section 4 is devoted to the existence of the unique solution of the model, implementation of the recent modified form of Predictor-Corrector scheme and the analysis of stability of such algorithm. In section 5 we include the graphical solutions of the resulting system by using different sets of parameter values to better demonstrate the outputs. Section 6 contains the conclusions of our study.

2. Preliminaries

Here, we recall some important concepts and results used in this paper.

Definition 1. *[26] The recent version of Caputo-type fractional derivative (called new generalised Caputo), $D_{d_+}^{\Theta,\rho}$, of order $\Theta > 0$ is given by:*

$$(D_{d_+}^{\Theta,\rho}\Psi)(\xi) = \frac{\rho^{\Theta-n+1}}{\Gamma(n-\Theta)}\int_d^{\xi} s^{\rho-1}(\xi^{\rho}-s^{\rho})^{n-\Theta-1}\left(s^{1-\rho}\frac{d}{ds}\right)^n \Psi(s)ds, \quad \xi > d, \tag{1}$$

where $\rho > 0, d \geq 0,$ and $n-1 < \Theta \leq n$.

Lemma 1. *[23] If $0 < b < 1$ and ϱ is an nonnegative integer, then there exist constants $\mathcal{C}_{b,1} > 0$ and $\mathcal{C}_{b,2} > 0$ which only depend on b, such that*

$$(\varrho+1)^b - \varrho^b \leq \mathcal{C}_{b,1}(\varrho+1)^{b-1},$$

and

$$(\varrho+2)^{b+1} - 2(\varrho+1)^{b+1} + \varrho^{b+1} \leq \mathcal{C}_{b,2}(\varrho+1)^{b-1}.$$

Lemma 2. *[23] Assume that $d_{q,s} = (s-q)^{b-1}(q = 1,2,...,s-1)$ and $d_{q,s} = 0$ for $q \geq s, b, M, h, T > 0, rh \leq T$ and r is a positive integer. Let $\sum_{q=r}^{q=s} d_{q,s}|e_q| = 0$ for $\varrho > s \geq 1$. If*

$$|e_s| \leq Mh^b \sum_{q=1}^{s-1} d_{q,s}|e_q| + |\beta_0|, \quad s = 1,2,...,r,$$

then

$$|e_r| \leq \mathcal{C}|\beta_0|, \quad r = 1,2,...$$

where $\mathcal{C} > 0$ is a constant which does not depend on r and h.

3. Model description

As we have already discussed above there exist a great number of mathematical models which have been introduced in psychology to describe the dynamics of love feelings. In this paper, we are considering an ordinary complex non-linear dynamical model given by *Jafari et al.* [12]. Firstly, they proposed a non-linear model containing two complex variables and four parameters for describing the dynamics:

$$\begin{cases} \dfrac{dM}{dt} = \beta_a + L^2 + \beta_c M, \\ \dfrac{dL}{dt} = \beta_b + M^2 + \beta_d L, \end{cases} \tag{2}$$

where $\beta_a > 0,\ \beta_b < 0,\ \beta_c < 0,\ \beta_d < 0$. In the above model, the complex variables M and L represent the sequent emotions that Majnun and Layla felt for each other. Here, β_a and β_b are constant parameters defining the impact of the environment on their feelings. They fixed $\beta_a > 0$ on the basis that Majnun was a jolly man and everybody had compassion for him. Consequently, the total environmental impact on him was hopeful. However, $\beta_b < 0$ settled because Layla was incriminated by her own family and society since she was married. The square terms L^2 and M^2 are used to define that they were madly in love, and any gesture of attention from the other stimulated them extensively. The reason to fix $\beta_c < 0,\ \beta_d < 0$ was to indicate that their love was a kind of true love, answering flatly to the spirits of the other but blank of seduction and self-hood which represents a kind of secure love. Significance of

such parameters is also given in [30].
After giving the above model database, authors extended the model for the complex plane by setting $M = M_r + iM_i$ and $L = L_r + iL_i$, and defined the model as follows:

$$\begin{cases} \dfrac{dM_r}{dt} = \beta_a + L_r^2 - L_i^2 + \beta_c M_r, \\ \dfrac{dM_i}{dt} = 2L_r L_i + \beta_c M_i, \\ \dfrac{dL_r}{dt} = \beta_b + M_r^2 - M_i^2 + \beta_d L_r, \\ \dfrac{dL_i}{dt} = 2M_r M_i + \beta_d L_i. \end{cases} \tag{3}$$

Here, $M_r,\ M_i$ are the feelings of Majnun defined for real and imaginary part. Similarly $L_r,\ L_i$ are the feelings of Layla defined for real and imaginary part. They also calculated the equilibrium points of the given model by setting the parameters $\beta_a = 1, \beta_b = -1, \beta_c = 0$, which are the following:

$$M_r = \pm\frac{\beta_d}{2}\frac{1}{\sqrt{\dfrac{-1+\sqrt{1+\beta_d{}^2}}{2}}},\ \ M_i = \pm\sqrt{\frac{-1+\sqrt{1+\beta_d{}^2}}{2}},\ \ L_r = 0,\ \ L_i = \pm 1.$$

Our task is to study the given model 3 via fractional derivatives. So now we formulate the above model according to a recent modified version of Caputo type fractional derivative. *Odibat et al.* [26] proposed a modified form of new generalised Caputo type non-integer order derivative which is a specific form of *Katugmpola* derivative [13]. Now the generalisation of the above model in fractional sense is given as follows:

$$\begin{cases} {}^C D_t^{\Theta,\rho} M_r(t) = \beta_a + L_r^2 - L_i^2 + \beta_c M_r, \\ {}^C D_t^{\Theta,\rho} M_i(t) = 2L_r L_i + \beta_c M_i, \\ {}^C D_t^{\Theta,\rho} L_r(t) = \beta_b + M_r^2 - M_i^2 + \beta_d L_r, \\ {}^C D_t^{\Theta,\rho} L_i(t) = 2M_r M_i + \beta_d L_i. \end{cases} \tag{4}$$

where ${}^C D_t^{\Theta,\rho}$ is the fractional order derivative operator of new generalised Caputo type of order Θ.
For the simplification of the model 4, we write its compact and equivalent form in the sense of singular type kernels as follows:

$$\begin{cases} {}^C D_t^{\Theta,\rho} M_r(t) = \zeta_1(t, M_r), \\ {}^C D_t^{\Theta,\rho} M_i(t) = \zeta_2(t, M_i), \\ {}^C D_t^{\Theta,\rho} L_r(t) = \zeta_3(t, L_r), \\ {}^C D_t^{\Theta,\rho} L_i(t) = \zeta_4(t, L_i). \end{cases} \tag{5}$$

4. Numerical Solution

4.1. Existence and uniqueness analysis

Firstly, we discuss the existence of unique solution for the proposed model 4. We establish the calculations for $M_r(t)$. The analysis for the remainder equations of 5 follow similarly.

Let us adopt the initial value problem (IVP)

$$ {}^{C}D_t^{\Theta,\rho} M_r(t) = \zeta_1(t, M_r), \tag{6a} $$
$$ M_r(0) = M_{r_0}. \tag{6b} $$

The Volterra type integral equation of above IVP [5] is given by

$$ M_r(t) = M_r(0) + \frac{\rho^{1-\Theta}}{\Gamma(\Theta)} \int_0^t \mu^{\rho-1}(t^\rho - \mu^\rho)^{\Theta-1} \zeta_1(\mu, M_r) d\mu. \tag{7} $$

Theorem 1. *[5, 13] (Existence). Let $0 < \Theta \leq 1$, $M_{r_0} \in \mathbb{R}$, $K > 0$ and $T^* > 0$. Settle $P := \{(t, M_r) : \ t \in [0, T^*], \ |M_r - M_{r_0}| \leq K\}$ and let the function $\zeta_1 : P \to \mathbb{R}$ be continuous. Further, settle $M_x := \sup_{(t,M_r)\in P} |\zeta_1(t, M_r)|$ and*

$$ T = \begin{cases} T^*, & if M_x = 0, \\ min\{T^*, \left(\dfrac{K\Gamma(\Theta+1)\rho^\Theta}{M_x}\right)^{\dfrac{1}{\Theta}}\} & otherwise. \end{cases} \tag{8} $$

Then, there is a function $M_r \in C[0, T]$ which is a solution of the IVP (6a) *and* (6b).

Lemma 3. *[5, 13] If the hypothesis of Theorem 1 holds then the function $M_r \in C[0, T]$ satisfies the IVP* (6a) *and* (6b) *if and only if, it solves the non-linear Volterra type integral equation* (7).

Theorem 2. *[5, 13] (Uniqueness). Let $M_r(0) \in \mathbb{R}$, $K > 0$ and $T^* > 0$. Also assume $0 < \Theta \leq 1$ and $m = \lceil \Theta \rceil$. Consider the set P as in Theorem 1 and let the mapping $\zeta_1 : P \to \mathbb{R}$ be continuous and Lipschitz relating to the second variable, i.e.*

$$ |\zeta_1(t, M_{r_1}) - \zeta_1(t, M_{r_2})| \leq L|M_{r_1} - M_{r_2}|, $$

for some constant $L > 0$ independent of t, M_{r_1}, and M_{r_2}. Then, a unique solution $M_r \in C[0, T]$ exists for the IVP (6a) *and* (6b).

4.2. Methodology

There is a large family of computational methods in classical sense and in fractional sense for solving the various types of differential equations. Many of them have great efficiency to solve the models with minimum error. In fractional calculus, a big amount of numerical methods are proposed to study non-linear dynamics in different scientific fields. In our methodology, we are considering a recent modified version of Predictor-Corrector method specified in [26], by making some favourable changes. In this regard, consider the Volterra integral equation given below

$$ M_r(t) = M_r(0) + \frac{\rho^{1-\Theta}}{\Gamma(\Theta)} \int_0^t \mu^{\rho-1}(t^\rho - \mu^\rho)^{\Theta-1} \zeta_1(\mu, M_r) d\mu. \tag{9} $$

We have given the proof of existence of solution for the proposed non-linear system on the interval $[0, T]$. Here we split $[0, T]$ into N disparate subintervals $\{[t_k, t_{k+1}], \ k = 0, 1, ..., N-1\}$ taking the mesh points

$$ \begin{cases} t_0 = 0, \\ t_{i+1} = (t_i^\rho + h)^{1/\rho}, \quad i = 0, 1, ..., N-1, \end{cases} \tag{10} $$

where $h = \dfrac{T^\rho}{N}$. Now, to establish the approximation terms $M_{r_i},\ i = 0, 1, ..., N$, we denote $M_{r_k} \approx M_r(t_k)(k = 1, 2, ..., i)$, and now we compute the approximation $M_{r_{i+1}} \approx M_r(t_{i+1})$ in the sense of the integral equation

$$M_r(t_{i+1}) = M_r(0) + \frac{\rho^{1-\Theta}}{\Gamma(\Theta)} \int_0^{t_{i+1}} \mu^{\rho-1}(t_{i+1}^\rho - \mu^\rho)^{\Theta-1} \zeta_1(\mu, M_r) d\mu. \tag{11}$$

Let us assume $z = \mu^\rho$. Then we get

$$M_r(t_{i+1}) = M_r(0) + \frac{\rho^{-\Theta}}{\Gamma(\Theta)} \int_0^{t_{i+1}^\rho} (t_{i+1}^\rho - z)^{\Theta-1} \zeta_1(z^{1/\rho}, M_r(z^{1/\rho}))dz, \tag{12}$$

that is,

$$M_r(t_{i+1}) = M_r(0) + \frac{\rho^{-\Theta}}{\Gamma(\Theta)} \sum_{k=0}^{i} \int_{t_i^\rho}^{t_{i+1}^\rho} (t_{i+1}^\rho - z)^{\Theta-1} \zeta_1(z^{1/\rho}, M_r(z^{1/\rho}))dz. \tag{13}$$

To demonstrate the right-hand side of Equation (13), we apply the trapezoidal quadrature rule according to the weight function $(t_{i+1}^\rho - z)^{\Theta-1}$, by substituing the function $\zeta_1(z^{1/\rho}, M_r(z^{1/\rho}))$ by its piecewise linear interpolant selecting the nodes t_k^ρ $(k = 0, 1, ..., i+1)$. Then, we establish

$$\int_{t_k^\rho}^{t_{i+1}^\rho} (t_{i+1}^\rho - z)^{\Theta-1} \zeta_1(z^{1/\rho}, M_r(z^{1/\rho}))dz \approx \frac{h^\Theta}{\Theta(\Theta+1)} \Bigg[\Big((i-k)^{\Theta+1} - (i-k-\Theta)(i-k+1)^\Theta \Big)$$
$$\zeta_1(t_k, M_r(t_k)) + \Big((i-k+1)^{\Theta+1} - (i-k+\Theta+1)(i-k)^\Theta \Big) \zeta_1(t_{k+1}, M_r(t_{k+1})) \Bigg]. \tag{14}$$

Now using the above terms in Equation (13), we evaluate the corrector formula for $M_r(t_{i+1}),\ i = 0, 1, ..., N-1$,

$$M_r(t_{i+1}) \approx M_r(0) + \frac{\rho^{-\Theta} h^\Theta}{\Gamma(\Theta+2)} \sum_{k=0}^{i} a_{k,i+1} \zeta_1(t_k, M_r(t_k)) + \frac{\rho^{-\Theta} h^\Theta}{\Gamma(\Theta+2)} \zeta_1(t_{i+1}, M_r(t_{i+1})), \tag{15}$$

where

$$a_{k,i+1} = \begin{cases} i^{\Theta+1} - (i-\Theta)(i+1)^\Theta \quad if \quad k = 0, \\ (i-k+2)^{\Theta+1} + (i-k)^{\Theta+1} - 2(i-k+1)^{\Theta+1} \quad if \ 1 \le k \le i. \end{cases} \tag{16}$$

Now, we proceed to evaluate the predictor value $M_r^P(t_{i+1})$, using the one-step Adams-Bashforth scheme to the integral equation (12). For that reason, we shift the function $\zeta_1(z^{1/\rho}, M_r(z^{1/\rho}))$ by the weight $\zeta_1(t_k, M_r(t_k))$ at each integral in Eq. (13) and we get

$$\begin{aligned} M_r^P(t_{i+1}) &\approx M_r(0) + \frac{\rho^{-\Theta}}{\Gamma(\Theta)} \sum_{k=0}^{i} \int_{t_k^\rho}^{t_{k+1}^\rho} (t_{i+1}^\rho - z)^{\Theta-1} \zeta_1(t_k, M_r(t_k))dz, \\ &= M_r(0) + \frac{\rho^{-\Theta} h^\Theta}{\Gamma(\Theta+1)} \sum_{k=0}^{i} [(i+1-k)^\Theta - (i-k)^\Theta] \zeta_1(t_k, M_r(t_k)). \end{aligned} \tag{17}$$

Now replacing $M_r(t_{i+1})$ in right side of (15) by $M_r^P(t_{i+1})$, our P-C algorithm, for defining the approximations $M_{r_{i+1}} \approx M_r(t_{i+1})$, is fully evaluated by the formula

$$M_{r_{i+1}} \approx M_r(0) + \frac{\rho^{-\Theta}h^{\Theta}}{\Gamma(\Theta+2)}\sum_{k=0}^{i} a_{k,i+1}\zeta_1(t_k, M_{r_k}) + \frac{\rho^{-\Theta}h^{\Theta}}{\Gamma(\Theta+2)}\zeta_1(t_{i+1}, M_{r_{i+1}}^P), \tag{18}$$

where $M_{r_k} \approx M_r(t_k),\ k = 0, 1, ..., i$, and the predicted value $M_{r_{i+1}}^P \approx M_r^P(t_{i+1})$ is given in Eq. (17) with the terms $a_{k,i+1}$ provided in (16).
So Eqs. (17) and (18) are the Predictor and Corrector algorithms for the first equation of the proposed model respectively. Applying the same algorithm for the whole system we get:

$$\begin{aligned}
M_{ri+1} &\approx M_r(0) + \frac{\rho^{-\Theta}h^{\Theta}}{\Gamma(\Theta+2)}\sum_{k=0}^{i} a_{k,i+1}\zeta_1(t_k, M_{rk}) + \frac{\rho^{-\Theta}h^{\Theta}}{\Gamma(\Theta+2)}\zeta_1(t_{i+1}, M_{ri+1}^P),\\
M_{ii+1} &\approx M_i(0) + \frac{\rho^{-\Theta}h^{\Theta}}{\Gamma(\Theta+2)}\sum_{k=0}^{i} a_{k,i+1}\zeta_2(t_k, M_{ik}) + \frac{\rho^{-\Theta}h^{\Theta}}{\Gamma(\Theta+2)}\zeta_2(t_{i+1}, M_{ii+1}^P),\\
L_{ri+1} &\approx L_r(0) + \frac{\rho^{-\Theta}h^{\Theta}}{\Gamma(\Theta+2)}\sum_{k=0}^{i} a_{k,i+1}\zeta_3(t_k, L_{rk}) + \frac{\rho^{-\Theta}h^{\Theta}}{\Gamma(\Theta+2)}\zeta_3(t_{i+1}, L_{ri+1}^P),\\
L_{ii+1} &\approx L_i(0) + \frac{\rho^{-\Theta}h^{\Theta}}{\Gamma(\Theta+2)}\sum_{k=0}^{i} a_{k,i+1}\zeta_4(t_k, L_{ik}) + \frac{\rho^{-\Theta}h^{\Theta}}{\Gamma(\Theta+2)}\zeta_4(t_{i+1}, L_{ii+1}^P),
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
{M_r}^P(t_{i+1}) &\approx M_r(0) + \frac{\rho^{-\Theta}h^{\Theta}}{\Gamma(\Theta+1)}\sum_{k=0}^{i}[(i+1-k)^{\Theta} - (i-k)^{\Theta}]\zeta_1(t_k, M_r(t_k)),\\
{M_i}^P(t_{i+1}) &\approx M_i(0) + \frac{\rho^{-\Theta}h^{\Theta}}{\Gamma(\Theta+1)}\sum_{k=0}^{i}[(i+1-k)^{\Theta} - (i-k)^{\Theta}]\zeta_2(t_k, M_i(t_k)),\\
{L_r}^P(t_{i+1}) &\approx L_r(0) + \frac{\rho^{-\Theta}h^{\Theta}}{\Gamma(\Theta+1)}\sum_{k=0}^{i}[(i+1-k)^{\Theta} - (i-k)^{\Theta}]\zeta_3(t_k, L_r(t_k)),\\
{L_i}^P(t_{i+1}) &\approx L_i(0) + \frac{\rho^{-\Theta}h^{\Theta}}{\Gamma(\Theta+1)}\sum_{k=0}^{i}[(i+1-k)^{\Theta} - (i-k)^{\Theta}]\zeta_4(t_k, L_i(t_k)).
\end{aligned} \tag{20}$$

4.3. Stability of the above algorithm

Theorem 3. *Assume that $\zeta_1(t, M_r)$ satisfies the Lipschitz property and M_{r_j} ($j = 1, ..., r+1$) are the solutions of Predictor-Corrector algorithm* (19) *and* (20). *Then, the proposed numerical method is conditionally stable.*

Proof. Let $\tilde{M}_{r_0},\ \tilde{M}_{r_j}\ (j = 0, ..., r+1)$ and $\tilde{M}_{r_{r+1}}^P\ (r = 0, ..., N-1)$ be perturbations of M_{r_0}, M_{r_j} and $M_{r_{r+1}}^P$. Then, the approximation equations are stated by using Eqs. (19) and (20) as:

$$\tilde{M}_{r_{r+1}}^P = \tilde{M}_{r_0} + \frac{\rho^{-\Theta}h^{\Theta}}{\Gamma(\Theta+1)}\sum_{j=0}^{r} b_{j,r+1}(\zeta_1(t_j, M_{r_j} + \tilde{M}_{r_j}) - \zeta_1(t_j, M_{r_j})), \tag{21}$$

with $b_{j,r+1} = [(r+1-j)^{\Theta} - (r-j)^{\Theta}]$

$$\tilde{M_{r_{r+1}}} = \tilde{M}_{r_0} + \frac{\rho^{-\Theta}h^{\Theta}}{\Gamma(\Theta+2)}(\zeta_1(t_{r+1}, M^P_{r_{r+1}} + \tilde{M^P_{r_{r+1}}}) - \zeta_1(t_{r+1}, M^P_{r_{r+1}}))+ \\ \frac{\rho^{-\Theta}h^{\Theta}}{\Gamma(\Theta+2)}\sum_{j=0}^{r} a_{j,r+1}(\zeta_1(t_j, M_{r_j} + \tilde{M}_{r_j}) - \zeta_1(t_j, M_{r_j})). \tag{22}$$

Applying the Lipschitz definition, we get

$$|\tilde{M_{r_{r+1}}}| \leq \zeta_0 + \frac{\rho^{-\Theta}h^{\Theta}m_1}{\Gamma(\Theta+2)}\left(|\tilde{M^P_{r_{r+1}}}| + \sum_{j=1}^{r} a_{j,r+1}|\tilde{M}_{r_j}|\right), \tag{23}$$

where $\zeta_0 = \max_{0\leq k\leq N}\{|\tilde{M}_{r_0}| + \frac{\rho^{-\Theta}h^{\Theta}m_1 a_{r,0}}{\Gamma(\Theta+2)}|\tilde{M}_{r_0}|\}$. Also, from Eq.(3.18) in [23] we derive

$$|\tilde{M^P_{r_{r+1}}}| \leq \eta_0 + \frac{\rho^{-\Theta}h^{\Theta}m_1}{\Gamma(\Theta+1)}\sum_{j=1}^{r} b_{j,r+1}|\tilde{M}_{r_j}|, \tag{24}$$

where $\eta_0 = \max_{0\leq r\leq N}\{|\tilde{M}_{r_0}| + \frac{\rho^{-\Theta}h^{\Theta}m_1 b_{r,0}}{\Gamma(\Theta+1)}|\tilde{M}_{r_0}|\}$. Substituting $|\tilde{M^P_{r_{r+1}}}|$ from Eq. (24) into Eq. (23) results

$$|\tilde{M_{r_{r+1}}}| \leq \gamma_0 + \frac{\rho^{-\Theta}h^{\Theta}m_1}{\Gamma(\Theta+2)}\left(\frac{\rho^{-\Theta}h^{\Theta}m_1}{\Gamma(\Theta+1)}\sum_{j=1}^{r} b_{j,r+1}|\tilde{M}_{r_j}| + \sum_{j=1}^{r} a_{j,r+1}|\tilde{M}_{r_j}|\right), \tag{25}$$

$$\leq \gamma_0 + \frac{\rho^{-\Theta}h^{\Theta}m_1}{\Gamma(\Theta+2)}\sum_{j=1}^{r}\left(\frac{\rho^{-\Theta}h^{\Theta}m_1}{\Gamma(\Theta+1)}b_{j,r+1} + a_{j,r+1}\right)|\tilde{M}_{r_j}|, \tag{26}$$

$$\leq \gamma_0 + \frac{\rho^{-\Theta}h^{\Theta}m_1\mathcal{C}_{\Theta,2}}{\Gamma(\Theta+2)}\sum_{j=1}^{r}(r+1-j)^{\theta-1}|\tilde{M}_{r_j}|, \tag{27}$$

where $\gamma_0 = \max\{\zeta_0 + \frac{\rho^{-\Theta}h^{\Theta}m_1 a_{r+1,r+1}}{\Gamma(\Theta+2)}\eta_0\}$. $C_{\Theta,2} > 0$ is a constant only dependent on Θ (Lemma 1) and h is supposed to be small enough. Lemma 2 gives $|\tilde{M_{r_{r+1}}}| \leq \mathcal{C}\gamma_0$. which finishes the proof. □

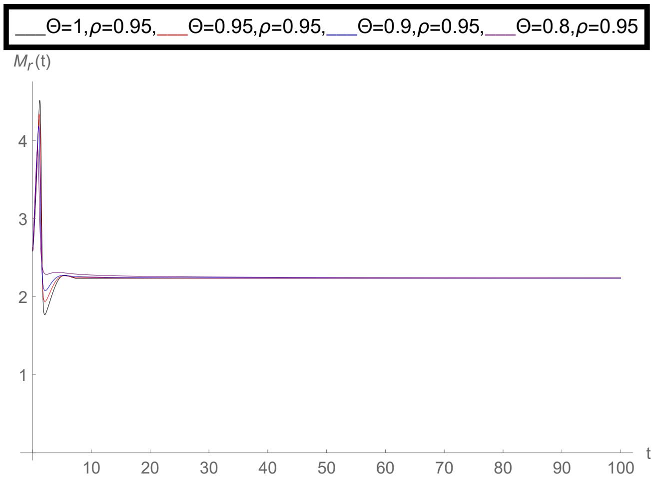
5. Graphical Simulations

Now, we perform some practical simulations by using *Mathematica* software. In [12], authors used number of sets to perform the graphs. In our simulations, we are also following the same sets of values to compare the effects of fractional mathematical model to ordinary model. The different sets of parameters values are given in Table 1.

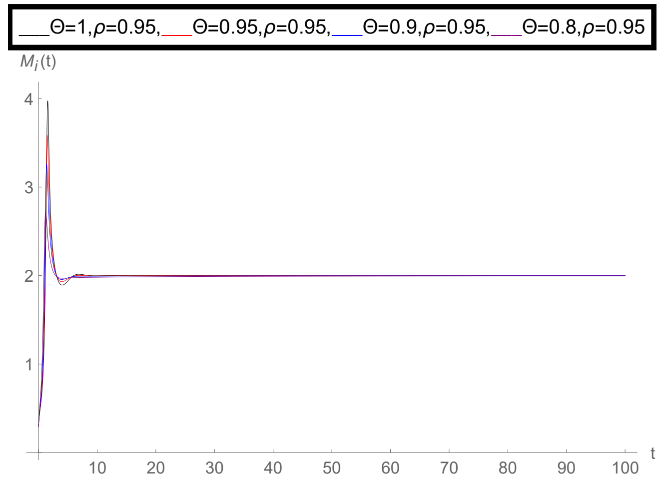
Table 1: Different sets of numerical values

Set	a	b	c	d	$M_r(0)$	$M_i(0)$	$L_r(0)$	$L_i(0)$
1	1	-1	0	$-4\sqrt{5}$	2.6291	0.29267	0.22786	1.6856
2	1	-1	0	$-\sqrt{3}$	0.78118	0.56896	-0.82171	-0.26561
3	1	-1	0	0	-0.13224	-0.33745	0.19911	0.5954
4	1	-1	0	0	0.49098	-0.27848	0.0030098	-0.31316
5	1	-1	-0.01	-0.01	-1.1878	-2.2023	0.98634	-0.5186

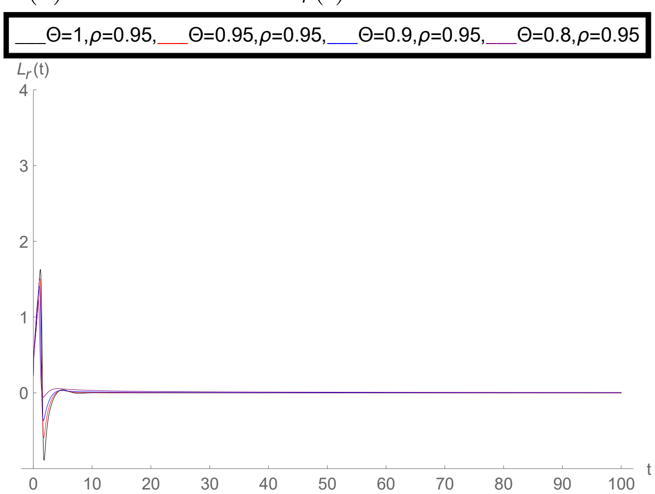
The graphs of the different classes of model 4 for the numerical values of set-1 are given in Figure 3. In this case, the dynamics linger to a fixed point and stay there for all time, so the system converges to a particular value for all the fractional order values Θ. Sub-figures 3e and 3f correspond to the 2D dynamics of M_r versus M_i and L_r versus L_i. For each fractional order, such relationship is convergent to a stable circumstance after a brief short-term which is reasonable for ever-after stories. In Figure 4, we performed the plots for set-2. In this case, all model classes converge to a periodic behaviour after a initial period of time. This condition is very usual in real life and many present models for romantic relationships show such nature. We find a stable limit cycle for all fractional orders but in this case, we can clearly see the role of different orders. At $\Theta = 1$ and $\Theta = 0.95$, the graphs are nearly same but when we shift to the lower order then the nature of the given classes goes to differ. After $\Theta = 0.85$ the periodic behaviour starts disappearing and classes show a stable solution which is similar to Figure 3. In sub-figure 4e and 4f, periodic behaviour exists for both relationships at every fractional values but at integer order case, periods exist more stable as compare to other fractional orders.



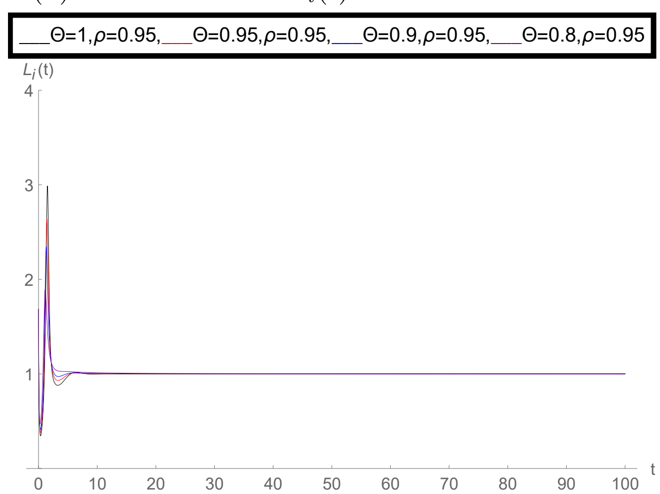
(a) Nature of class $M_r(t)$ versus time variable t



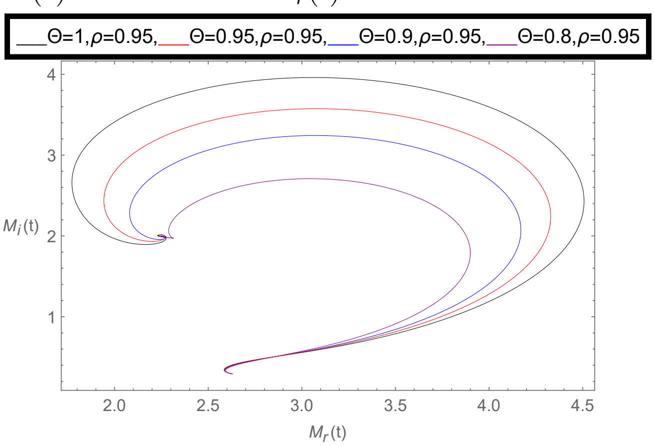
(b) Nature of class $M_i(t)$ versus time variable t



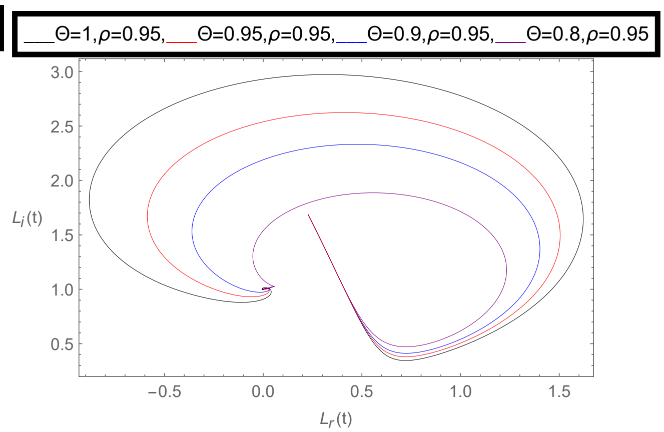
(c) Nature of class $L_r(t)$ versus time variable t



(d) Nature of class $L_i(t)$ versus time variable t

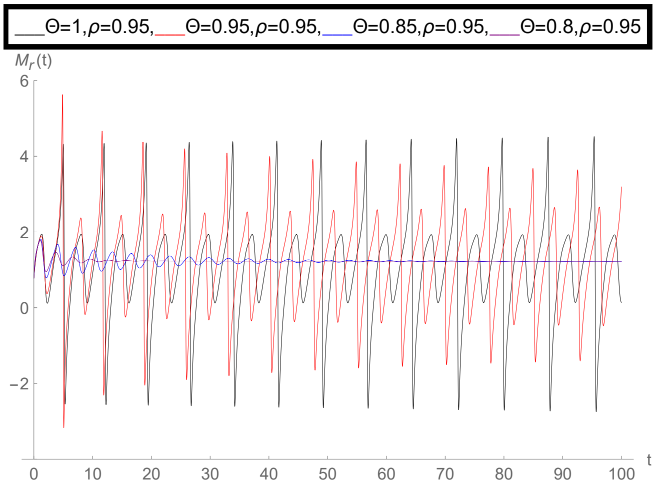


(e) 2D plot M_r versus M_i

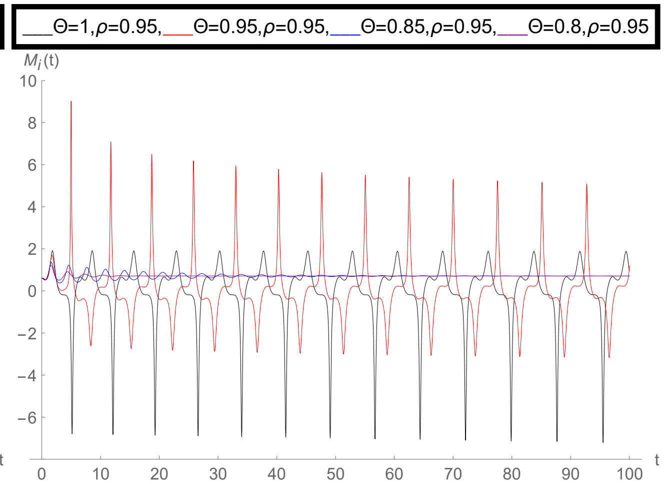


(f) 2D plot L_r versus L_i

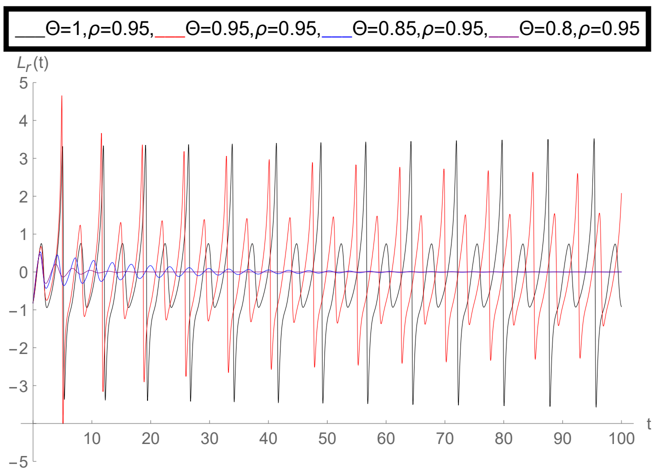
Figure 3: Behaviour of the model for $\beta_a = 1, \beta_b = -1, \beta_c = 0, \beta_d = -4\sqrt{5}$, and initial conditions $M(0) = 2.6291 + 0.29267i$ & $L(0) = 0.22786 + 1.6856i$



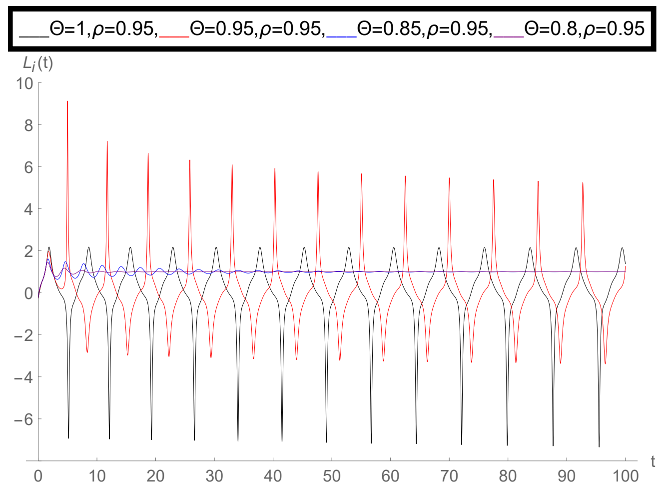
(a) Nature of class $M_r(t)$ versus time variable t



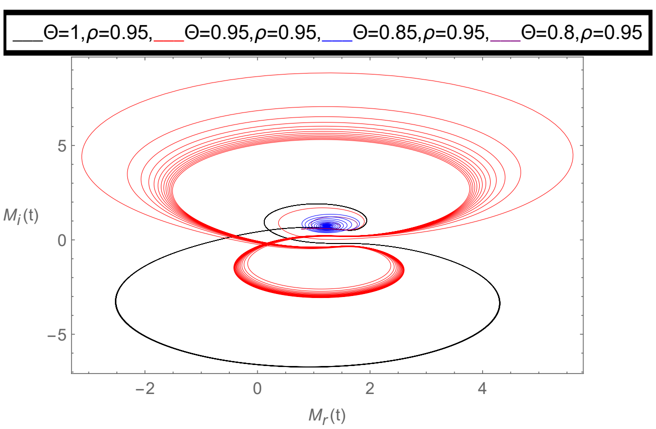
(b) Nature of class $M_i(t)$ versus time variable t



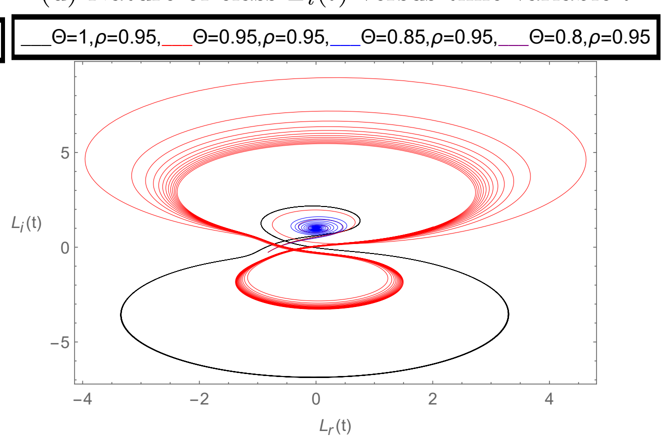
(c) Nature of class $L_r(t)$ versus time variable t



(d) Nature of class $L_i(t)$ versus time variable t



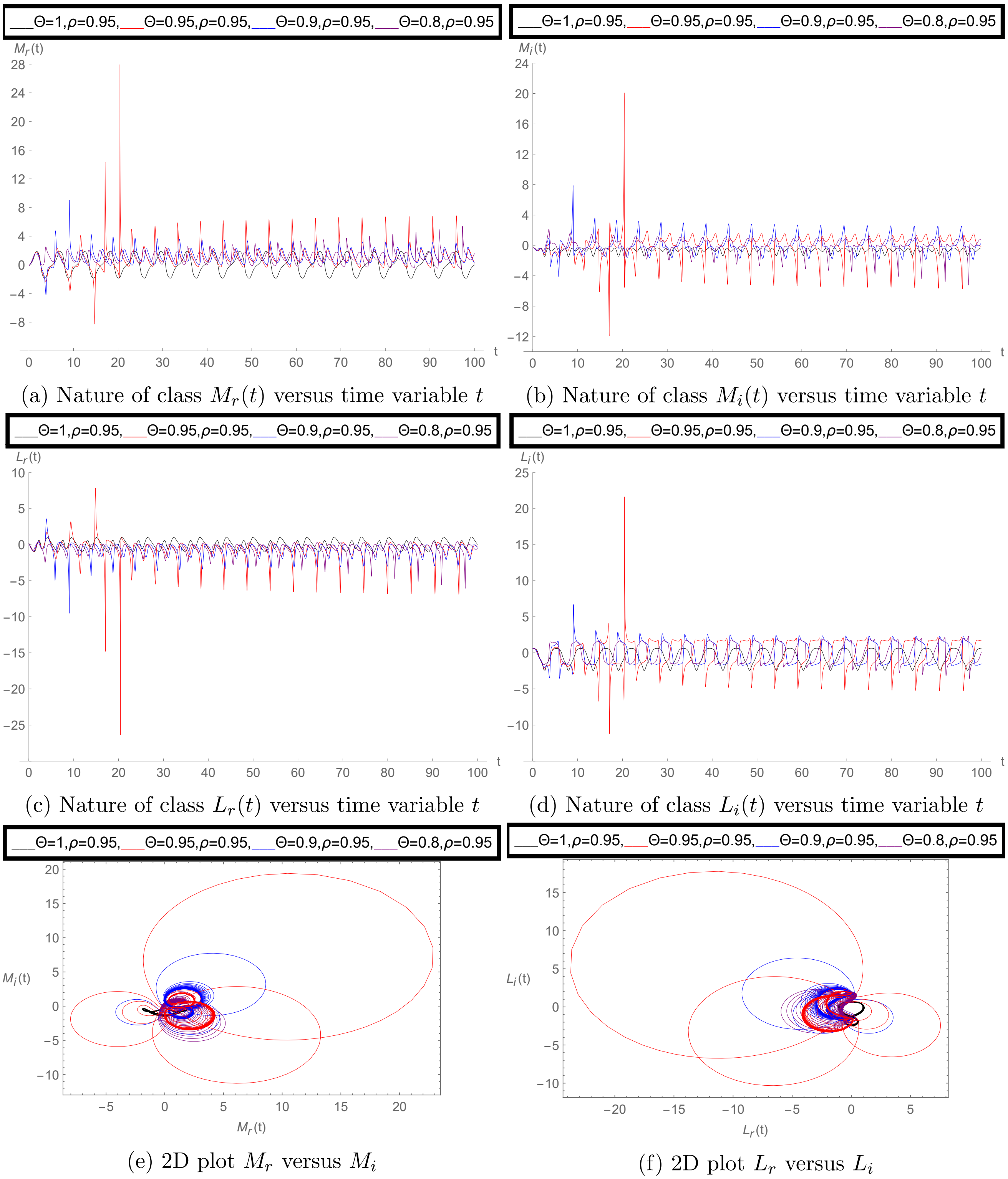
(e) 2D plot M_r versus M_i



(f) 2D plot L_r versus L_i

Figure 4: Behaviour of the model for $\beta_a = 1, \beta_b = -1, \beta_c = 0, \beta_d = -\sqrt{3}$, and initial conditions $M(0) = 0.78118 + 0.56896i$ & $L(0) = -0.82171 - 0.26561i$

It is well-known that life is tender and unpredictable, particularly when you are in a love relationship and irrelevant, unexpected, and fulminating episodes take place. So we use set-3, where $\beta_c = 0$ and $\beta_d = 0$. In that case, the time rows are transiently chaotic with sharp occasional spikes at every fractional order values. The effects of these orders can be seen from the family of Figure 5. In sub-figures 5e and 5f, the 2-D relationship between M_r & M_i and L_r & L_i is specified and it describes that this relationship is temporarily chaotic. In this case the trajectory is also like little periodic at some small fractional orders which is much more realistic than simple periodic behaviour.



(a) Nature of class $M_r(t)$ versus time variable t

(b) Nature of class $M_i(t)$ versus time variable t

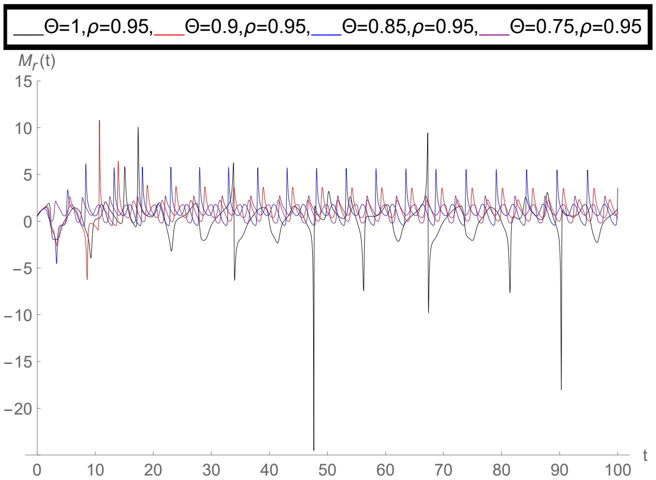
(c) Nature of class $L_r(t)$ versus time variable t

(d) Nature of class $L_i(t)$ versus time variable t

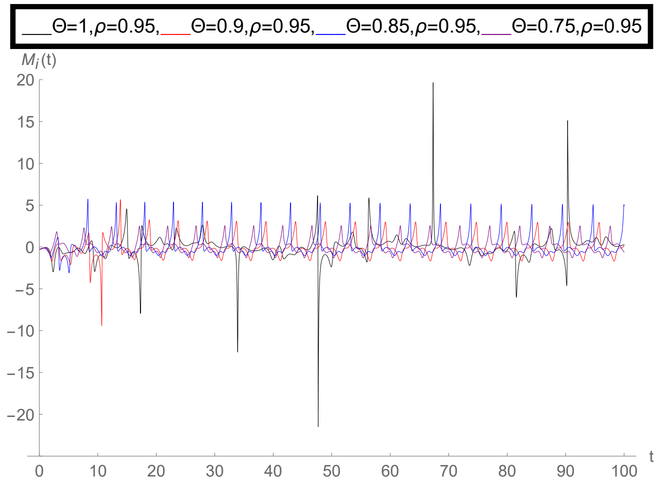
(e) 2D plot M_r versus M_i

(f) 2D plot L_r versus L_i

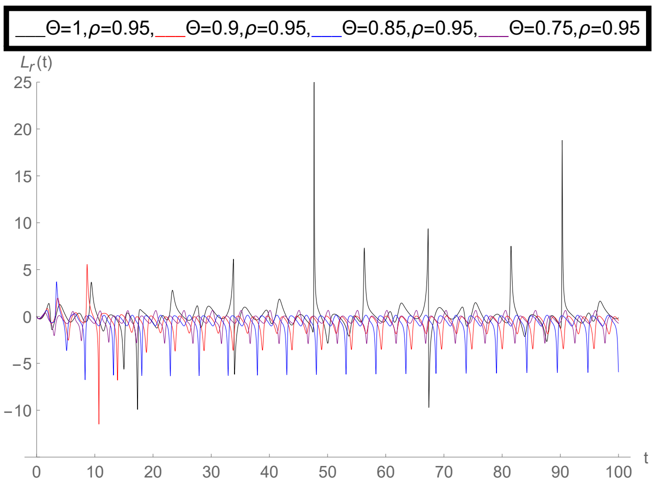
Figure 5: Behaviour of the model for $\beta_a = 1, \beta_b = -1, \beta_c = 0, \beta_d = 0$, and initial conditions $M(0) = -0.13224 - 0.33745i$ & $L(0) = 0.19911 + 0.5954i$



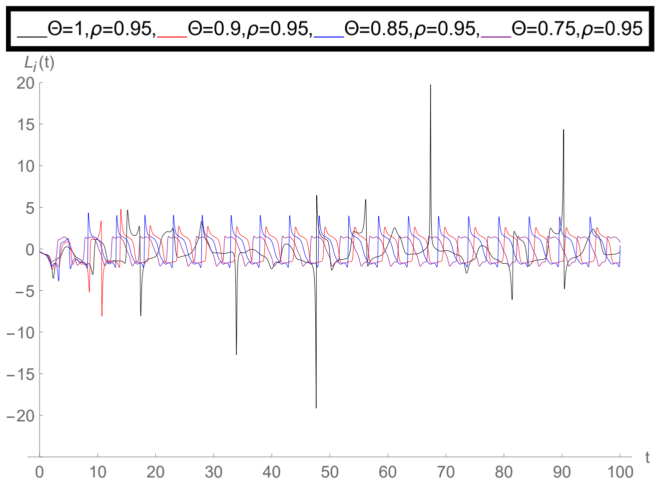
(a) Nature of class $M_r(t)$ versus time variable t



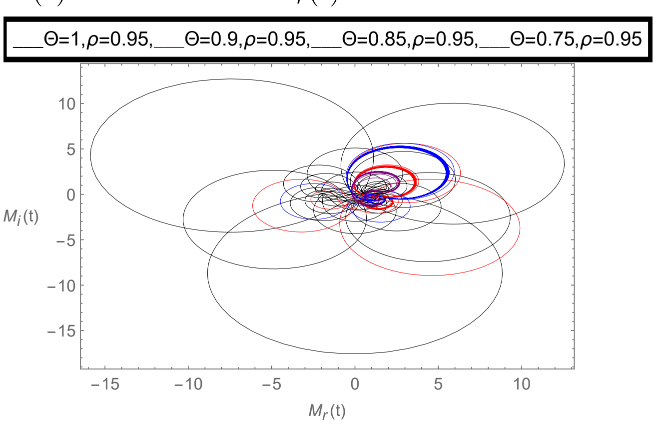
(b) Nature of class $M_i(t)$ versus time variable t



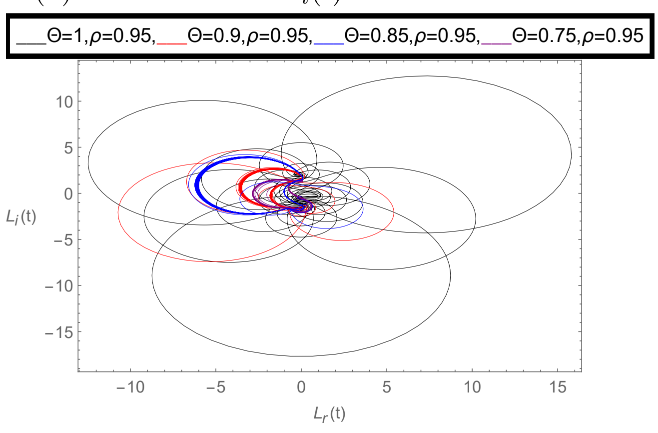
(c) Nature of class $L_r(t)$ versus time variable t



(d) Nature of class $L_i(t)$ versus time variable t



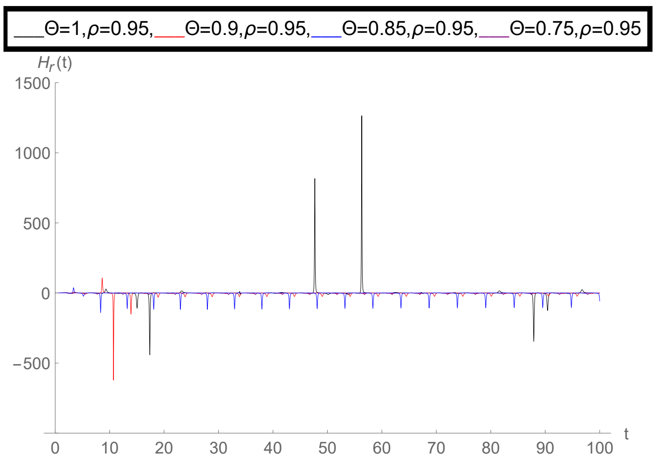
(e) 2D plot M_r versus M_i



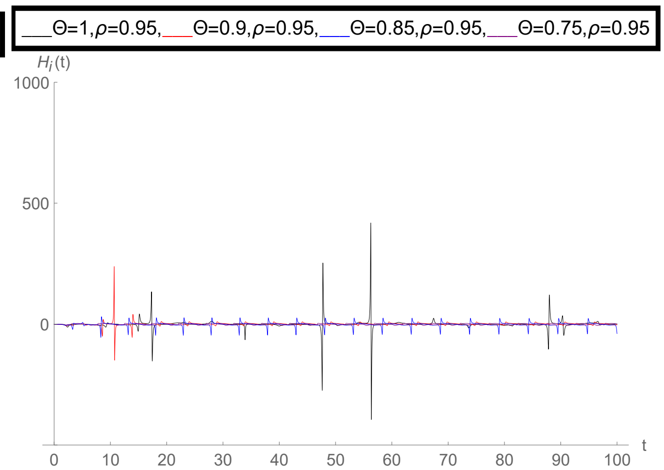
(f) 2D plot L_r versus L_i

Figure 6: Behaviour of the model for $\beta_a = 1, \beta_b = -1, \beta_c = 0, \beta_d = 0$, and initial conditions $M(0) = 0.49098 - 0.27848i$ & $L(0) = 0.0030098 - 0.31316i$

In [12], authors noticed that when they continued simulating the graphs for a longer time for set-3, then there would be an unbounded explosion after a later time period. Then they identified that case as a special one due to the fact it is a conservative Hamiltonian system with a Hamiltonian defined by $H = L + \dfrac{L^3}{3} + M - \dfrac{M^3}{3}$. So, it has no attractor, and all initial condition leads to a unique behaviour. To demonstrate the dynamics of the given Hamiltonian at fractional order values, we specified the plots in Figure 7. Sub-figures 7a and 7b show the calculated H_r and H_i for the given set-4. We can see that the value of H stays almost constant. In Figure 6, we performed the separate graphs of all classes along with the 2D trajectory of Majnun and Layla's love in the complex plane where we see that the dynamics are transiently chaotic without those accidental eruptions.



(a) Nature of $H_r(t)$ versus time variable t



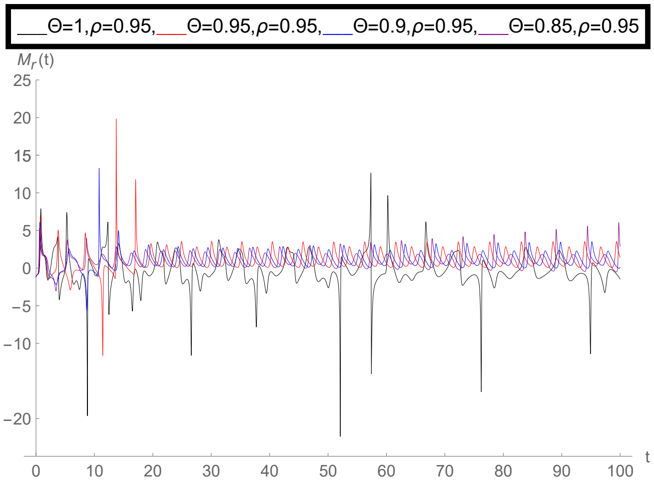
(b) Nature of $H_i(t)$ versus time variable t

Figure 7: Behaviour of the Hamiltonian function for $\beta_a = 1, \beta_b = -1, \beta_c = 0, \beta_d = 0$, and initial conditions $M(0) = 0.49098 - 0.27848i$ & $L(0) = 0.0030098 - 0.31316i$

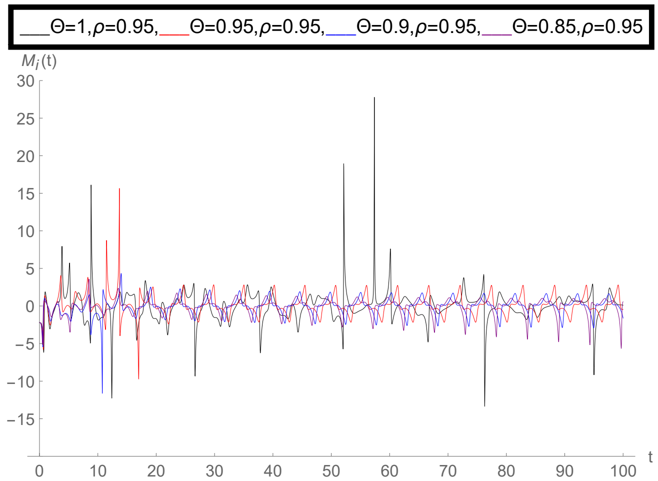
We noticed that at every fractional order values, the sensitive dependence on initial values which characterizes chaos behaviour is smoothly observed. In Figure 8, we established the plots for the numerical values of the set-5. From sub-figures 8e and 8f, some big changes in the behaviour of 2-D trajectories can be seen when the fractional order varies.

Table 2: CPU time in seconds for sets 1-5.

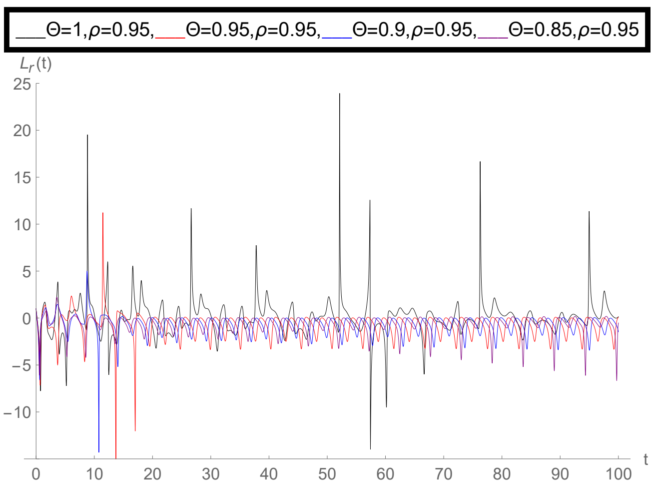
t	set1	set2	set3	set4	set5
1	0.12	0.11	0.12	0.61	0.14
10	2.13	2.11	2.11	2.12	2.07
50	10.67	10.55	10.55	10.50	10.37
100	46.36	54.88	53.44	54.43	52.55



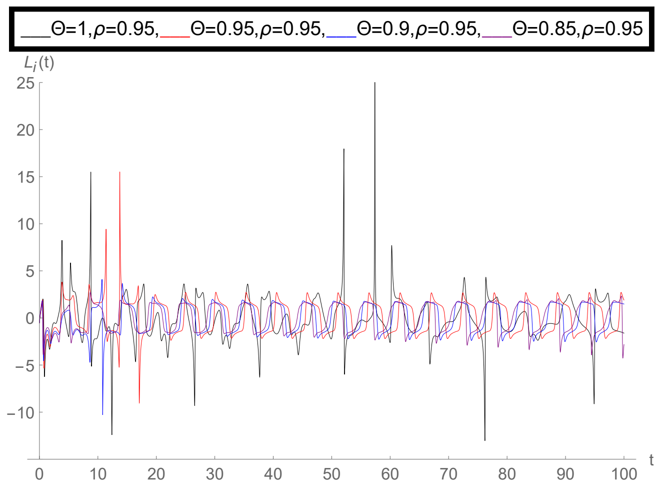
(a) Nature of class $M_r(t)$ versus time variable t



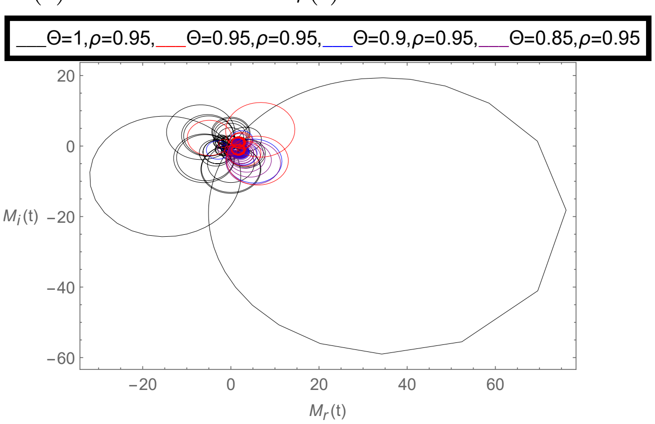
(b) Nature of class $M_i(t)$ versus time variable t



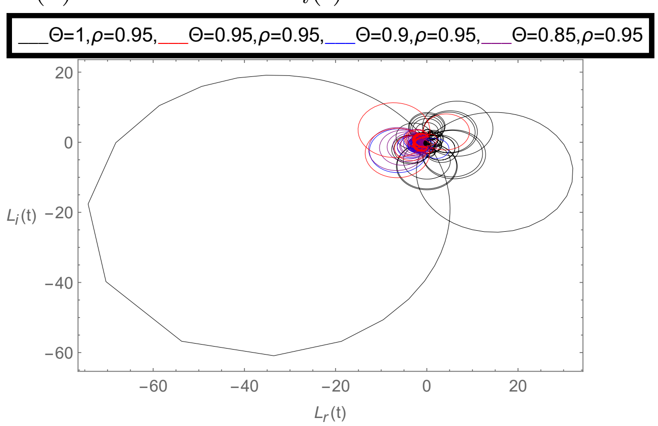
(c) Nature of class $L_r(t)$ versus time variable t



(d) Nature of class $L_i(t)$ versus time variable t



(e) 2D plot M_r versus M_i



(f) 2D plot L_r versus L_i

Figure 8: Behaviour of the model for $\beta_a = 1, \beta_b = -1, \beta_c = -0.01, \beta_d = -0.01$, and initial conditions $M(0) = -1.1878 - 2.2023i$ & $L(0) = 0.98634 - 0.51864i$

As it has been noticed, there are various types of nature shown by the proposed model which depend on the parameters β_c and β_d. In some cases there are stable limit cycles, periodic behaviours and temporarily chaotic solutions. In the last cases, chaos appears clearly and expands to long time interval (may be infinity), but the trajectory usually recoils sharply. This kind of explosive dynamics can take place in some relationships. Actually, among the three possible dealings, transient chaos is the one which best describes Layla and Majnun's relationship. We used *Mathematica* software for performing the above graphs. We also provided the CPU output time in Table 2.

6. Conclusion

Since complex variables are more realistic for describing certain dynamics we have considered a complex dynamical system endowed with fractional derivatives. In this paper, we have established some novel simulations to specify the beauty and the needs of non-integer order derivatives in psychological studies. We have provided a non-linear fractional order mathematical model for defining a love story. We have exemplified the all necessary practical simulations to study this serious psychological phenomena. Outcomes about the existence of the unique solution for the model have been established. We have used a very recent and strong modified Predictor-Corrector method to evaluate the model structure. Stability of the proposed algorithm is also stated. We have concluded that the given complex fractional model is more realistic and represents reality more closely. The proposed model is very basic, significant, and efficient at introducing a wide range of different natures by only changing the control parameter β_d. For upcoming works, this model can be studied considering other fractional derivatives likes conformal, Caputo-Fabrizio and Atangana-Baleanu which have different kernel properties. Some other fractional numerical algorithms can be applied on the given model to check its dynamics.

Availability of data and materials

All the data is included in the paper.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Pushpendra Kumar: Investigation, Conceptualization, Formal analysis, Methodology, Visualization, Resources, Writing - original draft.
Vedat Suat Erturk: Conceptualization, Investigation, Supervision, Software, Visualization, Writing- review & editing.
Marina Murillo-Arcila: Conceptualization, Formal analysis, Writing- review & editing.

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Bibliography

[1] Abboubakar, H., Kumar, P., Erturk, V. S., and Kumar, A. (2021). A mathematical study of a tuberculosis model with fractional derivatives. *International Journal of Modeling, Simulation, and Scientific Computing*.

[2] Acay, B. and Inc, M. (2021). Fractional modeling of temperature dynamics of a building with singular kernels. *Chaos, Solitons & Fractals*, 142:110482.

[3] Ahmad, W. M. and El-Khazali, R. (2007). Fractional-order dynamical models of love. *Chaos, Solitons & Fractals*, 33(4):1367–1375.

[4] Baghdadi, G., Jafari, S., Sprott, J., Towhidkhah, F., and Golpayegani, M. H. (2015). A chaotic model of sustaining attention problem in attention deficit disorder. *Communications in Nonlinear Science and Numerical Simulation*, 20(1):174–185.

[5] Erturk, V. S. and Kumar, P. (2020). Solution of a covid-19 model via new generalized caputo-type fractional derivatives. *Chaos, Solitons & Fractals*, page 110280.

[6] Gao, W., Veeresha, P., Baskonus, H. M., Prakasha, D., and Kumar, P. (2020). A new study of unreported cases of 2019-ncov epidemic outbreaks. *Chaos, Solitons & Fractals*, page 109929.

[7] Goldfain, E. (2008). Fractional dynamics and the standard model for particle physics. *Communications in Nonlinear Science and Numerical Simulation*, 13(7):1397–1404.

[8] Hussain, G., Khan, T., Khan, A., Inc, M., Zaman, G., Nisar, K. S., and Akgl, A. (2021). Modeling the dynamics of novel coronavirus (covid-19) via stochastic epidemic model. *Alexandria Engineering Journal*, 60(4):4121–4130.

[9] Inc, M., Acay, B., Berhe, H. W., Yusuf, A., Khan, A., and Yao, S.-W. (2021). Analysis of novel fractional covid-19 model with real-life data application. *Results in Physics*, 23:103968.

[10] Inc, M., Akinlar, M., Tchier, F., Bal, C., Bousbahi, F., Tawfiq, F., and Weber, G. (2020). Solutions of fractional-stochastic baos system. *Alexandria Engineering Journal*, 59(6):4997–5006.

[11] Iomin, A. (2006). Toy model of fractional transport of cancer cells due to self-entrapping. *Physical Review E*, 73(6):061918.

[12] Jafari, S., Sprott, J. C., and Golpayegani, S. M. R. H. (2016). Layla and majnun: a complex love story. *Nonlinear Dynamics*, 83(1-2):615–622.

[13] Katugampola, U. N. (2014). Existence and uniqueness results for a class of generalized fractional differential equations. *arXiv preprint arXiv:1411.5229*.

[14] Korpinar, Z., Tchier, F., Inc, M., Bousbahi, F., Tawfiq, F. M., and Akinlar, M. A. (2020). Applicability of time conformable derivative to wick-fractional-stochastic pdes. *Alexandria Engineering Journal*, 59(3):1485–1493. A Special Section on: Computational Methods in Engineering and Artificial Intelligence.

[15] Kumar, P. and Erturk, V. S. (2021a). A case study of covid-19 epidemic in india via new generalised caputo type fractional derivatives. *Mathematical Methods in the Applied Sciences*, pages 1–14.

[16] Kumar, P. and Erturk, V. S. (2021b). Environmental persistence influences infection dynamics for a butterfly pathogen via new generalised caputo type fractional derivative. *Chaos, Solitons & Fractals*, 144:110672.

[17] Kumar, P., Erturk, V. S., Abboubakar, H., and Nisar, K. S. (2021a). Prediction studies of the epidemic peak of coronavirus disease in brazil via new generalised caputo type fractional derivatives. *Alexandria Engineering Journal*.

[18] Kumar, P., Erturk, V. S., and Almusawa, H. (2021b). Mathematical structure of mosaic disease using microbial biostimulants via caputo and atangana-baleanu derivatives. *Results in Physics*, page 104186.

[19] Kumar, P., Erturk, V. S., and Murillo-Arcila, M. (2021c). A new fractional mathematical modelling of covid-19 with the availability of vaccine. *Results in Physics*, page 104213.

[20] Kumar, P., Erturk, V. S., Yusuf, A., Nisar, K. S., and Abdelwahab, S. F. (2021d). A study on canine distemper virus (cdv) and rabies epidemics in the red fox population via fractional derivatives. *Results in Physics*, page 104281.

[21] Kumar, P., Rangaig, N. A., Abboubakar, H., and Kumar, S. (2020). A malaria model with caputo-fabrizio and atangana-baleanu derivatives. *International Journal of Modeling, Simulation, and Scientific Computing*.

[22] Kumar, P. and Suat Erturk, V. The analysis of a time delay fractional covid-19 model via caputo type fractional derivative. *Mathematical Methods in the Applied Sciences*.

[23] Li, C. and Zeng, F. (2013). The finite difference methods for fractional ordinary differential equations. *Numerical Functional Analysis and Optimization*, 34(2):149–179.

[24] Nabi, K. N., Abboubakar, H., and Kumar, P. (2020). Forecasting of covid-19 pandemic: From integer derivatives to fractional derivatives. *Chaos, Solitons & Fractals*, page 110283.

[25] Nabi, K. N., Kumar, P., and Erturk, V. S. (2021). Projections and fractional dynamics of covid-19 with optimal control strategies. *Chaos, Solitons & Fractals*, page 110689.

[26] Odibat, Z. and Baleanu, D. (2020). Numerical simulation of initial value problems with generalized caputo-type fractional derivatives. *Applied Numerical Mathematics*.

[27] Petráš, I. and Magin, R. L. (2011). Simulation of drug uptake in a two compartmental fractional model for a biological system. *Communications in Nonlinear Science and Numerical Simulation*, 16(12):4588–4595.

[28] Rinaldi, S., Landi, P., and Rossa, F. D. (2013). Small discoveries can have great consequences in love affairs: the case of beauty and the beast. *International Journal of Bifurcation and Chaos*, 23(11):1330038.

[29] Rinaldi, S., Rossa, F. D., and Dercole, F. (2010). Love and appeal in standard couples. *International Journal of Bifurcation and Chaos*, 20(08):2443–2451.

[30] Sprott, J. (2004). Dynamical models of love. *Nonlinear dynamics, psychology, and life sciences*, 8(3):303–314.

[31] Sprott, J. (2005). Dynamical models of happiness. *Nonlinear Dynamics, Psychology, and Life Sciences*, 9(1):23–36.

[32] Szolnoki, A., Xie, N.-G., Ye, Y., and Perc, M. (2013). Evolution of emotions on networks leads to the evolution of cooperation in social dilemmas. *Physical Review E*, 87(4):042805.

[33] Tabatabaei, S. S., Yazdanpanah, M. J., Jafari, S., and Sprott, J. C. (2014). Extensions in dynamic models of happiness: effect of memory. *International Journal of Happiness and Development*, 1(4):344–356.

[34] Tarasov, V. E. (2015). Fractional-order difference equations for physical lattices and some applications. *Journal of Mathematical Physics*, 56(10):103506.

[35] Yusuf, A., Acay, B., Mustapha, U. T., Inc, M., and Baleanu, D. (2021). Mathematical modeling of pine wilt disease with caputo fractional operator. *Chaos, Solitons & Fractal*, 143:110569.