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On finite soluble groups in which Sylow permutability is a transitive relation*

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Abstract

A characterisation of finite soluble groups in which Sylow permutability is a transitive relation by means of subgroup embedding properties enjoyed by all the subgroups is proved in the paper. The key point is an extension of a subnormality criterion due to Wielandt.

1 Introduction and statements of results

One of the principal objectives of this paper is to give characterisations of finite soluble groups in which Sylow permutability is a transitive relation by means of two subgroup embedding properties, weak S-permutability and S-subpermutiser condition, which will be defined below.

Our approach involves an analysis of the relation between the above properties and Sylow permutability. In this context, a nice extension of a well-known subnormality criterion due to Wielandt turns out to be crucial.

Recall that a subgroup H of a finite group G is said to be *S-permutable* in G if H permutes with all Sylow subgroups of G. According to a theorem of Kegel [10], every S-permutable subgroup is subnormal. A group G is said to be a *PST-group* if every subnormal subgroup of G is S-permutable in G. Subclasses of PST-groups are the class of *PT-groups* or groups in which permutability is transitive and the class of *T-groups* or groups in which normality is transitive.

There are several characterisations of finite soluble T-groups, PT-groups and PST-groups in terms of normal structure and Sylow structure ([1, 2, 3, 4, 5, 7, 9, 12]).

Theorem 3 of [4] explains clearly the parallelism between these characterisations. Roughly speaking, one can get a T-characterisation (respectively, a

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PT-characterisation) from a PST-characterisation just by adding 'Dedekind' (respectively, 'modular') to the Sylow subgroups and substituting 'S-permutable' by 'normal' (respectively, 'permutable').

Recently, Bianchi, Gillio Berta Mauri, Herzog and Verardi [6] present a new characterisation of soluble T-groups using the following embedding property:

> A subgroup H of G is said to be an *$\mathcal{H}$-subgroup* of G if for all $g \in G$, $N_G(H) \cap H^g \leq H$.

They prove:

Theorem 1 ([6, Theorem 10]). *A group G is a soluble T-group if and only if every subgroup of G is an $\mathcal{H}$-subgroup.*

The above embedding property is closely related to the *weak normality*, studied by the authors in [3]:

> A subgroup H of G is called *weakly normal* in G if $H^g \leq N_G(H)$ implies that $g \in N_G(H)$.

If H is weakly normal in G and H is normal in a subgroup K of G, then $N_G(K)$ is contained in $N_G(H)$. This fact is crucial in the proof of [6, Theorem 10] and is a subgroup embedding property also studied in [3]:

> A subgroup H of G is said to satisfy the *subnormaliser condition* if for every subgroup K of G such that $H \trianglelefteq K$, it follows that $N_G(K) \leq N_G(H)$.

Although neither a weakly normal subgroup is an $\mathcal{H}$-subgroup nor a subgroup satisfying the subnormaliser condition is weakly normal ([3, Example 2]), we have:

Theorem 2 ([3]). *The following statements are equivalent:*

1. *G is a soluble T-group.*
2. *Every subgroup of G is weakly normal in G.*
3. *Every p-subgroup of G is weakly normal in G for all primes p.*
4. *Every subgroup of G satisfies the subnormaliser condition in G.*
5. *Every p-subgroup of G satisfies the subnormaliser condition in G for all primes p.*

In view of the parallelism between the characterisations of finite soluble T-, PT- and PST-groups in terms of the normal structure and Sylow structure, it is of interest to investigate the following situation:

> Is it possible to define PT- and PST-versions of the above embedding properties to get the PT- and PST-versions of Theorems 1 and 2?

This paper tries to give the complete answer to this question.

Let us begin with the following elementary equivalences:

- A subgroup H of a group G is weakly normal in G if and only if H satisfies the following property: if $g \in G$ and H is normal in $\langle H, H^g \rangle$, then H is normal in $\langle H, g \rangle$.
- A subgroup H of a group G satisfies the subnormaliser condition in G if and only if for every subgroup K of G such that H is normal in K and for every element $x \in G$ such that K is normal in $\langle K, x \rangle$, we have that H is normal in $\langle H, x \rangle$.

Therefore it seems natural to consider the following embedding properties, which can be regarded as the PST-versions of the abovementioned ones:

Definition 1. We say that a subgroup H of a group G is *weakly S-permutable* in G when the following condition holds:

> If $g \in G$ and H is S-permutable in $\langle H, H^g \rangle$, then H is S-permutable in $\langle H, g \rangle$.

Definition 2. We say that a subgroup H of a group G satisfies the *S-subpermutiser condition* in G when the following condition holds:

> If H is S-permutable in K and x is an element of G such that K is S-permutable in $\langle K, x \rangle$, then H is S-permutable in $\langle H, x \rangle$.

Note that there exist subgroups H such that H is S-permutable in $\langle H, H^g \rangle$ for all $g \in G$, but H is not S-permutable in G, as Example 1 shows.

Example 1. Consider the group $G = \Sigma_4$, the symmetric group of degree 4, and $H = \langle (1,2)(3,4) \rangle$. For every $g \in G$, $\langle H, H^g \rangle \leq \mathrm{Soc}(G)$. In fact, if $g \notin N_G(H)$, $\langle H, H^g \rangle = \mathrm{Soc}(G) \trianglelefteq G$, hence H is S-permutable in $\langle H, H^g \rangle$, but H is not S-permutable in $\langle H, g \rangle$ for some $g \in G$, e g, $g = (1,2,3)$ (notice that $\langle H, g \rangle = A_4$). In particular, H is not S-permutable in G.

Clearly S-permutable subgroups are weakly S-permutable. Maximal subgroups, Sylow subgroups and self-normalising subgroups are weakly S-permutable, too.

The following proposition shows the relation between the above properties and the corresponding T-versions.

Proposition 1. *Let H be a subgroup of a group G. Then:*

1. *If H is weakly normal in G, then H is weakly S-permutable in G.*
2. *If H satisfies the subnormaliser condition in G, then H satisfies the S-subpermutiser condition in G.*

Obviously the next step will be to analyse the relation between weak S-permutability and S-subpermutiser condition. There exist subgroups satisfying the S-subpermutiser condition which are not weakly S-permutable (see Example 2 below). However, we prove in the following that weak S-permutability implies the S-subpermutiser condition. The strategy used is the following:

It is clear that a subgroup H of a group G is normal (respectively, permutable) in G if and only if H is normal in $\langle H, g \rangle$ for every $g \in G$. Less trivial is the following result of Wielandt:

Theorem 3. *For a subgroup H of a group G, the following statements are equivalent:*

1. *H is subnormal in G.*
2. *H is subnormal in $\langle H, H^g \rangle$ for all $g \in G$.*
3. *H is subnormal in $\langle H, g \rangle$ for all $g \in G$.*

Example 1 shows that the equivalence between 1 and 2 does not hold neither for normality, nor permutability nor S-permutability. Nevertheless, the equivalence between 1 and 3, already noted above for normality and permutability, also holds for S-permutability, and it is a key result which helps to relate weak S-permutability and S-subpermutiser condition to S-permutability.

Theorem A. *A subgroup H of a group G is S-permutable in G if and only if H is S-permutable in $\langle H, g \rangle$ for every $g \in G$.*

Applying Theorem A we have:

Corollary 1. *If H satisfies the S-subpermutiser condition in a group G and H is a subnormal subgroup of a subgroup K of G, then H is S-permutable in K.*

Corollary 2. *If H is weakly S-permutable in G, then H satisfies the S-subpermutiser condition in G.*

Next we deal with certain localisations of PST-, PT- and T-groups.

Fix a prime p. Robinson [11] introduced the class $\mathcal{C}_p$ of all groups G such that each subgroup of every Sylow p-subgroup P of G is normal in $N_G(P)$. He proves that a group G is a soluble T-group if and only if it belongs to the class $\mathcal{C}_p$ for all primes p. The PT-version of the class $\mathcal{C}_p$ is the class $\mathcal{X}_p$ introduced by Beidleman, Brewster and Robinson in [5]: a group G belongs to $\mathcal{X}_p$ if and only if each subgroup of every Sylow p-subgroup P of G is permutable in $N_G(P)$. A group G is a soluble PT-group if and only if G belongs to the class $\mathcal{X}_p$ for all primes p ([5, Theorem A]). The PST-version of the above classes is the class $\mathcal{Y}_p$ introduced by the authors in [4]: a group G belongs to $\mathcal{Y}_p$ if and only if when H and K are p-subgroups of G such that $H \leq K$, then H is S-permutable in $N_G(K)$. A group G is a soluble PST-group if and only if G belongs to the class $\mathcal{Y}_p$ for all primes p ([4, Theorem 4]).

Bryce and Cossey [7] characterise in the soluble universe the groups in the class $\mathcal{C}_p$ as the groups G in which every p'-perfect subnormal subgroup of G is normal in G. We also prove in [3] that a soluble group G belongs to the class $\mathcal{C}_p$ if and only if every p'-perfect subgroup is weakly normal in G.

It is natural then to ask for the relation between the class $\mathcal{Y}_p$ and weak S-permutability and S-subpermutiser condition. First of all, note that there exist groups in the class $\mathcal{Y}_p$ with p'-perfect subnormal subgroups which are neither weakly S-permutable nor satisfy the S-subpermutiser condition (see Section 3). The best result we get is:

Theorem B. *Let G be a group. The following statements are equivalent:*

1. *G is a $\mathcal{Y}_p$-group.*
2. *Every p-subgroup of G satisfies the S-subpermutiser condition in G.*

With the above results at hand, we are able to prove the following characterisations of soluble PST-groups.

Theorem C. *Let G be a group. The following statements are equivalent:*

1. *G is a soluble PST-group.*
2. *Every subgroup of G is weakly S-permutable in G.*
3. *For every prime number p, every p-subgroup of G is weakly S-permutable in G.*
4. *Every subgroup of G satisfies the S-subpermutiser condition in G.*
5. *For every prime number p, every p-subgroup of G satisfies the S-subpermutiser condition in G.*

2 Proofs

Proof of Proposition 1. 1. Suppose that H is a weakly normal subgroup of G. Let g be an element of G such that H is S-permutable in $\langle H, H^g\rangle$. By Kegel's Theorem [10] we know that H is subnormal in $\langle H, H^g\rangle$. Now applying [3, Lemma 1] we have that H is normal in $\langle H, H^g\rangle$. The weak normality of H in G implies that H is normal in $\langle H, g\rangle$ and, in particular, H is S-permutable in $\langle H, g\rangle$. Consequently, H is weakly S-permutable in G.

With the same arguments to those used in the proof of statement 1 and applying Kegel's theorem and [3, Lemma 1], we have that each subgroup satisfying the subnormaliser condition in G also satisfies the S-subpermutiser condition in G. □

Proof of Theorem A. Suppose that G is a group of minimal order with a subgroup H such that H is S-permutable in $\langle H, g\rangle$ for every $g \in G$, but H is not S-permutable in G. Since H is a subnormal subgroup of $\langle H, g\rangle$ for every $g \in G$, from Theorem 3 it follows that H is a subnormal subgroup of G. Let M be a maximal normal subgroup of G containing H. Since H is not S-permutable in G, there exists a prime p and a Sylow p-subgroup P of G such that P does not permute with H.

Suppose that there exists a maximal subgroup M_1 of G such that $H \leq M_1$ and M is not contained in M_1. Then $MM_1 = G$. From the minimality of G, it follows that H is S-permutable in M and M_1. Moreover, there exists a Sylow p-subgroup Q of M and a Sylow p-subgroup Q_1 of M_1 such that their product $QQ_1 = P_0$ is a Sylow p-subgroup of G. Then H permutes with both Q and Q_1, hence H permutes with P_0. Consider a minimal normal subgroup N of G contained in M. By minimality of G, HN/N permutes with PN/N, hence HN permutes with P and $P(HN)$ is a subgroup of G. If $P(HN)$ is a proper subgroup of G, then H permutes with P, a contradiction. Consequently we have that $P(HN) = G$. There exists an element $x \in G$ such that $P_0 = P^x$, and x can be expressed as $x = x_1x_2$, with $x_1 \in P$ and $x_2 \in HN$. Therefore $P_0 = P^x = P^{x_2}$. Hence H permutes with P^{x_2}, or, equivalently, $H^{x_2^{-1}}$ permutes with P. Since H is a subnormal subgroup of G, by a theorem of Wielandt [8, A,14.3] we have that $\mathrm{Soc}(G)$ normalises each subnormal subgroup of G. In particular, H is a normal subgroup of HN, and since $x_2 \in HN$, we have that $H = H^{x_2^{-1}}$. This implies that H permutes with P, a contradiction. Consequently, if M_1 is a maximal subgroup of G containing H, then $M \leq M_1$. Since $P(HN) = G$ and $HN \leq M$, it follows that $|G : M|$ is a power of p. Hence all maximal subgroups of G/M are normal. Thus M is actually a maximal subgroup, and it is the unique maximal subgroup of G containing H. Therefore if $x \in G \setminus M$, we have that $\langle H, x \rangle = G$: otherwise there would exist another maximal subgroup of G containing H. From the hypothesis, H is S-permutable in $\langle H, x \rangle = G$, the final contradiction.

The converse is clear. □

Note by Theorem A that a subgroup H of a group G satisfies the S-subpermutiser condition in G if and only if H satisfies the following property:

> If H is S-permutable in K and K is S-permutable in L, then H is S-permutable in L.

Proof of Corollary 1. Suppose that H satisfies the S-subpermutiser condition in G and that H is subnormal in a subgroup K of G. Arguing by induction we can suppose, without loss of generality, that H is S-permutable in a proper normal subgroup L of K. Consider $g \in K$. Since H is S-permutable in L and L is S-permutable in $\langle L, g \rangle$, from the S-subpermutiser condition we have that H is S-permutable in $\langle H, g \rangle$. Since this happens for every $g \in K$, from Theorem A we obtain that H is an S-permutable subgroup of K. □

Proof of Corollary 2. Assume that H is a weakly S-permutable subgroup of G. Let K be a subgroup of G such that H is S-permutable in K. Suppose in addition that x is an element of G such that K is S-permutable in $\langle K, x \rangle$. By Kegel's theorem, we have that H is subnormal in $\langle K, x \rangle$. By Corollary 1 we obtain that H is S-permutable in $\langle K, x \rangle$, as desired. □

Proof of Theorem B. Suppose that every p-subgroup of G satisfies the S-subpermutiser condition in G. Suppose that $H \leq L \leq P$, where P is a Sylow p-subgroup of G. Since H is a subnormal subgroup of $N_G(L)$ and H satisfies the S-subpermutiser condition in G, we have that H is S-permutable in $N_G(L)$ by Corollary 1. Therefore G is in the class $\mathcal{Y}_p$.

Now suppose that G is in the class $\mathcal{Y}_p$. Assume that H is an S-permutable p-subgroup of K, and K is an S-permutable subgroup of L. Arguing by induction, we can suppose that $H \leq K \trianglelefteq L$ and that H is S-permutable in K. Since G belongs to the class $\mathcal{Y}_p$, H is S-permutable in $N_G(K)$, which contains L. In particular, H is S-permutable in L. □

Proof of Theorem C. Let us see that 1 implies 2. Suppose that G is a soluble PST-group. Applying the results of [1], $G = AB$, where A is the nilpotent residual of G, A is abelian of odd order, $|A|$ and $|B|$ are coprime and every subgroup of A normal in G. Let $g \in G$ and $H \leq G$ such that H is S-permutable in $\langle H, H^g \rangle$. We can suppose that G is not nilpotent, and so $A \neq 1$. Let N be a minimal normal subgroup of G such that $N \leq A$. By minimality of G, HN/N is weakly S-permutable in G/N. Hence HN/N is S-permutable in $\langle H, g \rangle N/N$. Consequently HN is S-permutable in $\langle H, g \rangle N$. If $\langle H, g \rangle$ is a proper subgroup of G, then H is S-permutable in $\langle H, g \rangle$. Therefore $G = \langle H, g \rangle$ and HN is S-permutable in G. This implies that HN is a subnormal subgroup of G.

Assume that H is not weakly S-permutable and let p be a prime number dividing $|G|$ and P a Sylow p-subgroup of G such that H does not permute with P. If $(HN)P$ is a proper subgroup of G, then H permutes with P by induction. Consequently, $G = (HN)P$. Suppose that p divides $|A|$, then $P \leq A$ and P is a normal subgroup of G. Hence H permutes with P, a contradiction. Therefore $|P|$ and $|A|$ are coprime. Moreover, $\mathrm{Core}_G(H) = 1$. Thus $H \cap A = 1$ and $|H|$ and $|A|$ are coprime. As a consequence, if π is the set of primes dividing $|A|$ and n_π is the π-part of the number n, then

$$|G|_\pi = \frac{|HN|_\pi |P|_\pi}{|HN \cap P|_\pi} = |HN|_\pi = |N|_\pi$$

and hence $A = N$.

Let us denote $T = \langle H, H^g \rangle$ and let q be the prime dividing $|N|$. If $|T|_q \neq 1$, then $N \cap T$ is a nontrivial normal subgroup of G. Hence $N \leq T$. Since H is S-permutable in T, we have that H is a subnormal subgroup of T and so H is subnormal in HN. Therefore H is a subnormal subgroup of G. Since G is a PST-group, we have that H is S-permutable in G, a contradiction. Therefore $|T|_q = 1$. We can suppose that $T \leq B$. The element g can be expressed as $g = bn$, with $b \in B$ and $n \in N = \langle x \rangle$, with $o(x) = p$ (notice that G is supersoluble). If $n = 1$, then H is S-permutable in $\langle H, b \rangle = \langle H, g \rangle$, because B is nilpotent. Hence $n \neq 1$ and $N = \langle n \rangle$ and $H^g \leq B^g = B^n$, therefore $H^g \leq B \cap B^n = C_B(n)$ (see [8, A,16.3]). Consequently $H^g \leq C_G(n)$, whence $H^b \leq \left(C_G(n)\right)^{n^{-1}} = C_G(n)$. This implies that $H^b \leq C_G(N)$ and so $H^b N$ is a nilpotent group. But in this case H^b is a subnormal subgroup of G, because $H^b N$ is a subnormal subgroup of G. Therefore H is a subnormal subgroup of G. Since G is a PST-group, we have that H is S-permutable in G, the final contradiction.

It is obvious that 2 implies 3 and that 4 implies 5. From Proposition 1, it follows that 2 implies 4 and that 3 implies 5. From Theorem B and [4, Theorem 5], it follows that 5 implies 1. This completes the proof. □

3 An example

Example 2. Consider $P = \langle x, y \mid x^2 = y^8 = 1, y^x = y^5 \rangle$, a modular group of order 16. P has an irreducible and faithful module over the field of 17 elements, $V = \langle w_1, w_2 \rangle$, such that the action of P is described by $w_1^x = w_2$, $w_2^x = w_1$, $w_1^y = w_1^9$, $w_2^y = w_2^8$. We construct the semidirect product $G = [V]P$. We observe that x centralises the element $w_1 w_2$. Let $g = w_1 w_2 y$. Let $H = \langle x \rangle$. We have that $H^g = \langle x^y \rangle = \langle x y^4 \rangle \leq P$. Consequently the subgroup $H = \langle x \rangle$ is S-permutable in $\langle H, H^g \rangle$. But H is not S-permutable in $\langle H, g \rangle = G$: it suffices to see that H does not permute with, e g, P^{w_1}.

It is clear that G is a 2-nilpotent group, and so G belongs to the class $\mathcal{Y}_2$ by [4, Theorem 5]. Applying Theorem B, all 2-subgroups of G, in particular H, satisfy the S-subpermutiser condition in G (the reader is invited to prove directly that H satisfies the S-subpermutiser condition in G).

Consider the subgroup $L = \langle x, w_1 w_2^{-1} \rangle$. Then L is a $2'$-perfect subnormal subgroup of G which is not permutable with P. However, L is S-permutable in $M = \langle x, y^2, w_1, w_2 \rangle \trianglelefteq G$ and M is S-permutable in $G = \langle M, g \rangle$, but L is not S-permutable in $G = \langle L, g \rangle$. It follows that L does not satisfy the S-subpermutiser condition in G.

4 Postscript: An extension to PT-groups

In this section we introduce two new embedding properties useful to give characterisations of PT-groups.

Definition 3. We say that a subgroup H of a group G is *weakly permutable* when the following condition holds:

> If H is permutable in $\langle H, H^g \rangle$, then H is permutable in $\langle H, g \rangle$.

Definition 4. We say that a subgroup H of a group G satisfies the *subpermutiser condition* in G when the following condition holds:

> If H is permutable in K and x is an element of G such that K is permutable in $\langle K, x \rangle$, then H is permutable in $\langle H, x \rangle$.

Weak permutability and the subpermutiser condition extend weak normality and the subnormaliser condition, respectively, to permutability. The following results hold:

Theorem 4.

1. *If H is a weakly normal subgroup of G, then H is a weakly permutable subgroup of G.*
2. *If H is a weakly permutable subgroup of G, then H is a weakly S-permutable subgroup of G.*
3. *If H satisfies the subnormaliser condition in G, then H satisfies the subpermutiser condition in G.*
4. *If H satisfies the subpermutiser condition in G, then H satisfies the S-subpermutiser condition in G.*
5. *If H is a weakly permutable subgroup of G, then H satisfies the subpermutiser condition in G.*
6. *If H is weakly permutable in G and H is a subnormal subgroup of a subgroup K of G, then H is permutable in K.*
7. *If H satisfies the subpermutiser condition in G and H is a subnormal subgroup of a subgroup K of G, then H is permutable in K.*

We can give now PT-versions of Theorem B and Theorem C.

Theorem D. *Let G be a group. The following statements are equivalent:*

1. *G belongs to $\mathcal{X}_p$.*
2. *Every p-subgroup of G satisfies the subpermutiser condition.*

Theorem E. *Let G be a group. The following statements are equivalent:*

1. *G is a soluble PT-group.*

2. *Every subgroup of G is weakly permutable in G.*

3. *For every prime number p, every p-subgroup of G is weakly permutable in G.*

4. *Every subgroup of G satisfies the subpermutiser condition in G.*

5. *For every prime number p, every p-subgroup of G satisfies the subpermutiser condition in G.*

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