Equilibrium-Based Finite Element Formulation for Timoshenko Curved Tapered Beams

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Abstract: *Curved tapered beams have been widely used in many engineering applications. Their complex geometries pose challenges to the development of robust approaches for the numerical modelling of their mechanical behaviour. The aim of the present contribution is to introduce a novel, simple and effective, finite element formulation for the quasi-static analysis of Timoshenko curved tapered beams. This formulation relies on a complementary variational approach based on a set of approximations that satisfy in strong form all equilibrium conditions of the boundary-value problem, resulting thus in a formulation that is free from both shear and membrane locking phenomena. The effectiveness of the formulation is numerically demonstrated through its application to different beam problems and the obtained results are analysed and discussed.*

1 INTRODUCTION

Due to their excellent mechanical performance and structural efficiency, curved beams have been widely used in many engineering applications, such as: bridge structures, piping systems, biomedical devices, aerospace and aeronautical structures, *etc.* Their complex geometries pose challenges to the development of robust approaches for the modelling of their mechanical behaviour.

Among the various approaches available in the literature for their analysis, those that are based on the finite element method have been the most successful. The simplest finite element modeling strategy for curved beams is an assembly of relatively short straight beam elements [1]. However, such an approach generally requires a large number of elements to obtain converged solutions. Furthermore, when applied to Timoshenko based structural models, some of these finite element approaches, in particular those that rely on the approximation of the displament fields based on lower-order shape functions, are prone to shear locking when the beam elements become slender and to membrane locking when their curvature increases [2]. Hybrid-mixed formulations, in which both displacement and force/bending moment fields are approximated, with the goal of avoiding the locking phenomena, were also explored [3]. However, most of the formulations that have been developed for the analysis of curved beams are limited to uniform cross-section cases. There are a few exceptions in which finite element formulations for curved beams with non-uniform cross-sections can be found in the literature, see *e.g.* [4, 5].

The aim of the present contribution is to introduce a novel, simple and effective, finite element formulation for the quasi-static analysis of Timoshenko curved tapered beams. Following the methodology adopted in [6, 7, 8], the proposed formulation relies on a complementary variational approach, only requiring the approximation of the internal force/bending moment fields. Such approximations are selected such that they satisfy in strong form all equilibrium conditions of the boundary-value problem. The formulation is naturally free from both shear and membrane locking phenomena. The effectiveness of the formulation is numerically demon-

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strated through its application to different beam problems and the obtained results are analysed and discussed.

2 BOUNDARY-VALUE PROBLEM

Consider a two-dimensional curved beam whose geometry is described by its centroidal axis denoted by $\mathcal{C}$. The centroidal axis $\mathcal{C}$ is parameterized by $s \in [0, L]$, with L denoting the length of the beam in its reference configuration. $\mathcal{C}$ is decomposed into an internal part, represented by $\Omega =]0, L[$, and a boundary part, identified by $\Gamma = \Gamma_N \cup \Gamma_D = \{0, L\}$, where Γ_N and Γ_D correspond to the Neumann and Dirichlet boundaries, respectively, such that $\Gamma_N \cap \Gamma_D = \emptyset$.

Let the beam be subjected to: distributed loads defined per unit length, denoted by p and q, and bending moments, denoted by m, applied in Ω and assumed to depend on s, concentrated loads $\bar{N}$ and $\bar{V}$ and a concentrated moment $\bar{M}$ applied on Γ_N; prescribed displacements, $\bar{u}$ and $\bar{w}$, and a prescribed rotation $\bar{\phi}$ defined on Γ_D. While p, $\bar{N}$ and $\bar{u}$ represent axial quantities, q, $\bar{V}$ and $\bar{w}$ represent transverse quantities. m, $\bar{M}$ and $\bar{\phi}$ represent rotational quantities. The loads are assumed to act at the centroidal axis of the beam.

The kinematical differential equations of the beam model under consideration are given in Ω as

$$\varepsilon_{ss} = \frac{1}{1 + \frac{z}{R}}(\varepsilon + z\chi) \tag{1a}$$

$$\gamma_{sz} = \frac{1}{1 + \frac{z}{R}}\gamma \tag{1b}$$

in which

$$\varepsilon = u' - \frac{w}{R} \tag{2a}$$

$$\gamma = w' + \frac{u}{R} - \phi \tag{2b}$$

$$\chi = \phi' \tag{2c}$$

with ε being the axial deformation, γ the shear deformation and χ the bending curvature of the beam. R stands for the radius of curvature of the beam centroidal axis, which, in general, may depend on s. As the shear deformation γ is not disregarded, the adopted model is based on Timoshenko's beam theory.

The Dirichlet (kinematical) boundary conditions of the problem are given as follows

$$u - \bar{u} = 0, \text{ on } \Gamma_D \tag{3a}$$

$$w - \bar{w} = 0, \text{ on } \Gamma_D \tag{3b}$$

$$\phi - \bar{\phi} = 0, \text{ on } \Gamma_D \tag{3c}$$

where u and w are the axial and transverse displacements of the beam axis, respectively, and ϕ the rotation of the beam cross-section.

The equilibrium of an infinitesimal beam element can be expressed by the following set of differential equations in Ω

$$N' - \frac{V}{R(s)} + p(s) = 0 \tag{4a}$$

$$V' + \frac{N}{R(s)} + q(s) = 0 \tag{4b}$$

$$-M' + V + m(s) = 0 \tag{4c}$$

representing equilibrium of axial forces, shear forces and bending moments, respectively, where $(\cdot)'$ stands for the derivative of $(\cdot)$ with respect to s.

The Neumann (or static) boundary conditions of the problem are

$$nN - \bar{N} = 0, \text{ on } \Gamma_N \tag{5a}$$
$$nV - \bar{V} = 0, \text{ on } \Gamma_N \tag{5b}$$
$$nM + \bar{M} = 0, \text{ on } \Gamma_N \tag{5c}$$

with

$$n = \begin{cases} 1 & \text{if } x = L \\ -1 & \text{if } x = 0 \end{cases}$$

The constitutive equations are taken as the following relationships defined in Ω

$$\sigma_{ss} = E\varepsilon_{ss} \tag{6a}$$
$$\tau_{sz} = G\gamma_{sz} \tag{6b}$$

with E and G denoting Young's modulus and shear modulus, respectively, of the beam, such that

$$G = \frac{E}{2(1+\nu)} \tag{7}$$

with ν standing for Poisson's coefficient. A linear elastic material behavior is, thus, assumed in this study. The material properties E and ν are taken as constants in Ω.

The internal forces and bending moment fields correspond to the following stress resultants on a beam cross-section

$$N = \int_A \sigma_{ss} dA \tag{8a}$$
$$V = \int_A \tau_{sz} dA \tag{8b}$$
$$M = \int_A \sigma_{ss} z dA \tag{8c}$$

with dA an infinitesimal area element of the beam cross-section. Making use of these definitions, and upon substitution of (1) and (6), leads to

$$N = C_{11}\varepsilon + C_{12}\chi \tag{9a}$$
$$V = C_{33}\gamma \tag{9b}$$
$$M = C_{12}\varepsilon + C_{22}\chi \tag{9c}$$

with

$$C_{11} = \int_A \frac{E}{1+\frac{z}{R}} dA$$
$$C_{12} = \int_A \frac{Ez}{1+\frac{z}{R}} dA$$
$$C_{22} = \int_A \frac{Ez^2}{1+\frac{z}{R}} dA$$
$$C_{33} = \int_A \frac{f_s G}{1+\frac{z}{R}} dA$$

where f_s stands for the shear correction factor. It is worth noting the coupling between N and M. Notably, if $\frac{h}{R} \ll 1$, then this coupling disappears.

3 VARIATIONAL BASIS

The strain energy functional of a beam element assumes the following form

$$U = \frac{1}{2}\int_V \left(E\varepsilon_{ss}^2 + f_s G\gamma_{sz}^2\right)\ dV \tag{11}$$

with dV an infinitesimal volume element of the beam. Upon substitution of equations (2) and (6), the strain energy can be recast as

$$\begin{aligned} U &= \frac{1}{2}\int_V \left(\frac{E}{\left(1+\frac{z}{R}\right)^2}\left(\varepsilon + z\chi\right)^2 + \frac{f_s G}{\left(1+\frac{z}{R}\right)^2}\gamma^2\right)\ dV \\ &= \frac{1}{2}\int_V \left(\frac{E}{\left(1+\frac{z}{R}\right)^2}\left(\varepsilon^2 + 2z\chi\varepsilon + z^2\chi^2\right) + \frac{f_s G}{\left(1+\frac{z}{R}\right)^2}\gamma^2\right)\ dV \end{aligned}$$

Since $dV = \left(1+\frac{z}{R}\right)dA d\Omega$, with $1+\frac{z}{R}$ the Jacobian of the transformation, and since the deformations ε, χ and γ only depend on the curvilinear coordinate s, the strain energy can be rewritten as

$$U = \frac{1}{2}\int_\Omega \left(\left(C_{11}\varepsilon + C_{12}\chi\right)\varepsilon + \left(C_{12}\varepsilon + C_{22}\chi\right)\chi + \left(C_{33}\gamma\right)\gamma\right)\ d\Omega \tag{13}$$

or, making use of (9), as

$$U = \frac{1}{2}\int_\Omega \left(N\varepsilon + M\chi + V\gamma\right)\ d\Omega \tag{14}$$

The total potential energy of the boundary-value problem under study is, thus, the functional $\Pi_p : \mathcal{U}_k(\Omega) \to \mathcal{R}$ given by

$$\Pi_p(u,w,\phi) = U(\varepsilon(u,w), \chi(\phi), \gamma(u,w,\phi)) + F(u,w,\phi) \tag{15}$$

where U is the strain energy functional defined in (14) and F represents the external potential energy given by

$$F(u,w,\phi) = -\int_\Omega (pu + qw + m\phi)\ d\Omega - [\bar{N}u]_{\Gamma_N} - [\bar{V}w]_{\Gamma_N} - [\bar{M}\phi]_{\Gamma_N} \tag{16}$$

$\mathcal{U}_k$ is the kinematically admissible space defined as

$$\mathcal{U}_k = \{(u,w,\phi) \in \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega) |\ u = \bar{u}, w = \bar{w}, \phi = \bar{\phi} \text{ on } \Gamma_D\} \tag{17}$$

where $\mathcal{H}^1(\Omega)$ represents a standard Sobolev space.

Inverting relations (9) gives

$$\varepsilon = \frac{C_{22}N - C_{12}M}{C_{11}C_{22} - C_{12}^2} \tag{18a}$$

$$\gamma = \frac{V}{C_{33}} \tag{18b}$$

$$\chi = \frac{C_{11}M - C_{12}N}{C_{11}C_{22} - C_{12}^2} \tag{18c}$$

On insertion of (18) into the strain energy functional (14) leads to the following complementary strain energy functional

$$U_c = \frac{1}{2}\int_\Omega \left(\frac{C_{22}N^2 - 2C_{12}NM + C_{11}M^2}{C_{11}C_{22} - C_{12}^2} + \frac{V^2}{C_{33}}\right)\ d\Omega \tag{19}$$

which only involves the internal forces/bending moment fields.

The associated total complementary energy $\Pi_c : \mathcal{U}_s(\Omega) \to \mathcal{R}$ comes out as

$$\Pi_c(N, V, M) = -U_c(N, V, M) + \Pi_{c,ext}(N, V, M) \tag{20}$$

in which $\Pi_{c,ext}$ represents the external complementary energy given as follows

$$\Pi_{c,ext}(N, V, M) = [nN\bar{u}]_{\Gamma_D} + [nV\bar{w}]_{\Gamma_D} + [nM\bar{\phi}]_{\Gamma_D} \tag{21}$$

and $\mathcal{U}_s$ stands for the statically admissible space defined as

$$\begin{aligned}\mathcal{U}_s = \{(N, V, M) \in \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega)|\\ N' - \frac{V}{R(s)} + p(s) = 0,\ V' + \frac{N}{R(s)} + q(s) = 0,\ -M' + V + m(s) = 0 \text{ in } \Omega;\\ nN - \bar{N} = 0,\ nV - \bar{V} = 0,\ nM + \bar{M} = 0 \text{ on } \Gamma_N\}\end{aligned}$$

(N, V, M) is said to be a stationary point of Π_c if, and only if, the following condition holds

$$\delta\Pi_c = 0,\ \forall\ (\delta N, \delta V, \delta M) \in \mathcal{V}_s \tag{22}$$

where $\mathcal{V}_s$ represents the homogeneous statically admissible space defined as

$$\begin{aligned}\mathcal{V}_s = \{(\delta N, \delta V, \delta M) \in \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega)|\ \delta N' - \frac{\delta V}{R(s)} = 0,\ \delta V' + \frac{\delta N}{R(s)} = 0,\\ -\delta M' + \delta V = 0,\ \text{in } \Omega; n\delta N = 0,\ n\delta V = 0,\ n\delta M = 0,\ \text{on } \Gamma_N\}\end{aligned}$$

A novel finite element formulation for the quasi-static analysis of Timoshenko curved tapered beams will be developed in the following on the basis of the complementary variational approach introduced above.

4 FINITE ELEMENT FORMULATION

As a starting point, let us define $\mathcal{H}^0_h$ and $\mathcal{H}^1_h$ as families of closed finite-dimensional subspaces of $\mathcal{H}^0$ and $\mathcal{H}^1$, respectively. A finite element approximation of (22) consists of seeking $(N^h, M^h, V^h) \in \mathcal{U}^h_s$ such that (22) holds for all $(\delta N^h, \delta M^h, \delta V^h) \in \mathcal{V}^h_s$, where $\mathcal{U}^h_s \subset \mathcal{U}_s$ and $\mathcal{V}^h_s \subset \mathcal{V}_s$ represent the discretized statically admissible spaces.

Let us assume that the entire domain Ω is partitioned in subdomains $\Omega_e \subset \Omega$, such that $\Omega = \cup_{e=1}^{n_e}\Omega_e$ in which n_e represents the number of beam elements. If the inter-element equilibrium conditions and Neumann boundary conditions are relaxed within the framework of the complementary energy principle, then, the following augmented Lagrangian, or hybrid complementary energy, has to be considered

$$L_c = \sum_{e=1}^{n_e} \Pi_{c,e} + \sum_{i=1}^{n_{int}} \left(\lambda_i^N [\![N]\!]_{\Gamma_i} + \lambda_i^V [\![V]\!]_{\Gamma_i} + \lambda_i^M [\![M]\!]_{\Gamma_i}\right) \tag{23}$$

where n_{int} represents the number of inter-element boundaries and Γ_i stands for the inter-element boundary i. $[\![(\cdot)]\!]$ stands for the jump of $(\cdot)$ on Γ_i. λ_i^N, λ_i^V and λ_i^M are the Lagrange multipliers, defined on Γ_i, that are energy-conjugate of N, V and M, respectively.

Without loss of generality, and only for the sake of simplicity, let us consider the case of beams with zero distributed loads, *i.e.*, $p(s) = q(s) = 0$ and $m(s) = 0$. Then, the solutions to the equilibrium differential equations (4) are as follows

$$N(s) = c_2 \sin(k(s)) + c_1 \cos(k(s)) \tag{24a}$$
$$V(s) = c_2 \cos(k(s)) - c_1 \sin(k(s)) \tag{24b}$$
$$M(s) = c_3 + c_2 \int \cos(k(s))\ ds - c_1 \int \sin(k(s))\ ds \tag{24c}$$

where c_1, c_2 and c_3 are constants and $k(s)$ is defined as

$$k(s) = \int \frac{1}{R(s)}\ ds \tag{25}$$

It is worth noting that, if, additionally, the beam radius of curvature R is constant, then $k(s)$ results as

$$k(s) = \frac{s}{R} \tag{26}$$

and, therefore, the internal forces/bending moment functions (24) come out as

$$N(s) = c_2 \sin\left(\frac{s}{R}\right) + c_1 \cos\left(\frac{s}{R}\right) \tag{27a}$$
$$V(s) = c_2 \cos\left(\frac{s}{R}\right) - c_1 \sin\left(\frac{s}{R}\right) \tag{27b}$$
$$M(s) = c_3 + c_2 R \sin\left(\frac{s}{R}\right) + c_1 R \cos\left(\frac{s}{R}\right) \tag{27c}$$

These functions are taken as the trial finite element approximations and a Galerkin approach is adopted, *i.e.*, the problem is numerically approached assuming the same trial and test approximation function spaces within the framework of the augmented Lagrangian given by (23). The involved integrals are numerically computed using a 5-point Gaussian quadrature rule.

Differentiation of L_c^h with respect to all the unknown parameters gives rise to a governing system of linear equations that involve the element constants c_i and the Lagrange multipliers as fundamental unknowns.

5 NUMERICAL TESTS

5.1 Quarter-Circular Cantilver Uniform Beam Under Shear Force at its Free End - Shear and Membrane Locking Tests

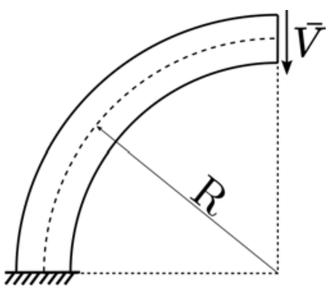

Figure 1: Quarter-Circular Cantilver Uniform Beam Under a Shear Force at its Free End

The classical problem of a quarter-circular uniform cantilever beam subjected to a shear force at its the free end as is illustrated in Figure 1 is analyzed first in order to test the capability of the proposed formulation to overcome the shear- and membrane-locking phenomena. The applied shear force, the cross-section width and the beam curvature radius were set to $\bar{V} = 1kN$,

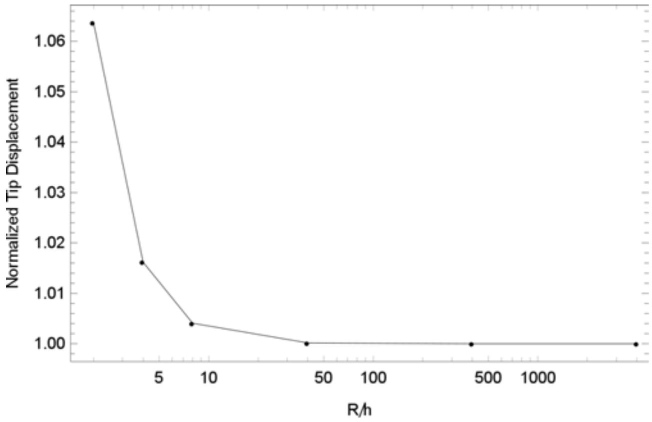

Figure 2: Quarter-Circular Cantilver Uniform Beam Under a Shear Force at its Free End - Shear and Membrane Locking Tests

$b = 0.4m$ and $R = 4m$. The analysis was carried out varying the slenderness ratio R/h of the beam. The beam was modeled using only one finite element. It is worth mentioning that finer finite element discretizations would exactly lead to the same results. The transverse displacements of the free end of the beam were computed and normalized with respect to their corresponding Euler-Bernoulli solutions, w_f^{EB}, for different values of the slenderness ratio, where

$$w_f^{EB} = \frac{\pi \bar{V} R^3}{4EI} + \frac{\pi \bar{V} R}{4EA} \tag{28}$$

The obtained results are shown in Figure 2. As it can be seen, as the slenderness ratio R/h increases, or, in other words, as the beam becomes thinner, the transverse tip displacements tend to the Euler-Bernoulli solutions. This shows that the proposed formulation does not suffer from either shear or membrane locking.

5.2 Clamped-Clamped Circular Beam with Tip Load

To validate and assess the accuracy and effectiveness of the proposed finite element formulation, a clamped-clamped circular beam with rectangular cross-section under tip loads as depicted in Figure 3 is herein analyzed. A uniform (constant h) beam is considered first and, afterwards, a tapered beam is studied. The following numerical parameters were considered for both situations: radius of curvature $R = 4m$, cross-section width $b = 0.4m$, shear correction factor $f_s = 5/6$ and opening angle $\theta_o = 2\pi/3$.

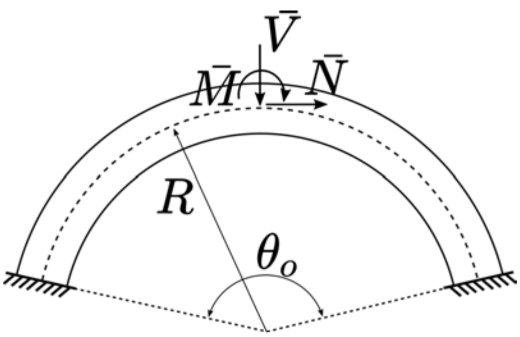

Figure 3: Clamped-Clamped Circular Beam

5.2.1 Clamped-Clamped Circular Uniform Beam with Tip Load - Accuracy test

In order to assess the accuracy of the proposed formulation, a uniform beam with a tip load is herein analysed. In this case, the cross-section height was set to $h = 0.6m$ and the Young's modulus and Poisson's ratio were set to $E = 30GPa$ and $\nu = 0.17$. The tip loads were assumed as $\bar{N} = \bar{V} = 1kN$ and $\bar{M} = 1kNm$. The beam was modeled using two finite elements. The accuracy of the proposed formulation is assessed by comparing the obtained results with the reference ones given in [4].

As it can be seen from the analysis of Table 1, the results produced by the proposed formulation are essentially the same as the reference ones.

Load Case	$\times 10^{-6}$	Ref. Sol. [4]	Present Study
$\bar{N} = 0,\ \bar{V} \neq 0,\ \bar{M} = 0$	$\frac{w}{R\theta_o}$	0.248781	0.248781
$\bar{N} \neq 0,\ \bar{V} = 0,\ \bar{M} = 0$	$\frac{u}{R\theta_o}$	0.12522	0.125221
	$\frac{\phi}{\theta_o}$	-0.379642	-0.379642
$\bar{N} = 0,\ \bar{V} = 0,\ \bar{M} \neq 0$	$\frac{u}{R\theta_o}$	-0.09491	-0.094910
	$\frac{\phi}{\theta_o}$	1.08224	1.082238

Table 1: Clamped-Clamped Circular Uniform Beam with Tip Load - Accuracy test

5.2.2 Clamped-Clamped Circular Tapered Beam with Tip Load

A tapered beam with a transverse tip load as illustrated in Figure 4 is now analysed. In this case, the initial cross-section height was set to $h_0 = h(0) = 0.6m$ and the Young's modulus and Poisson's ratio were set to $E = 70GPa$ and $\nu = 0.3$. The tip load was set to $\bar{V} = 1kN$. The beam was modeled firstly using two finite elements. The variation of the cross-section height was taken as $h(s) = h_0 \left(1 - \frac{s}{1.1L}\right)$, which corresponds to a beam with a cross-section height at $s = L$ given by $h(L) = \frac{h_0}{11}$.

The obtained diagrams of axial force, shear force and bending moment are shown in Figures 5, 6 and 7, respectively. As expected, neither the axial force nor the bending moment diagrams exhibit symmetry with respect to a vertical axis crossing the mid-span of the beam. Likewise, the shear force diagram is not anti-symmetric with respect to the mentioned axis. This is in opposition to what would be obtained if a uniform beam would have been considered. It is also interesting to note that the bending moment at $s = L$ is considerably lower than that at $s = 0$. This is clearly a consequence of the lower bending stiffness of the beam at $s = L$ when compared to that at $s = 0$.

Finally, a mesh convergence study was performed, in which the tip transverse displacement was computed using 2, 4, 8 and 16 finite elements. The obtained results are provided in Table 2, showing that the finite element formulation converges to a solution.

6 CONCLUSIONS

- A novel finite element formulation for the quasi-static analysis of Timoshenko curved tapered beams was proposed.
- The formulation relies on a hybrid complementary energy variational principle leading to statically admissible solutions.
- The formulation proved to be effective and naturally insensitive to the shear and membrane locking phenomena.

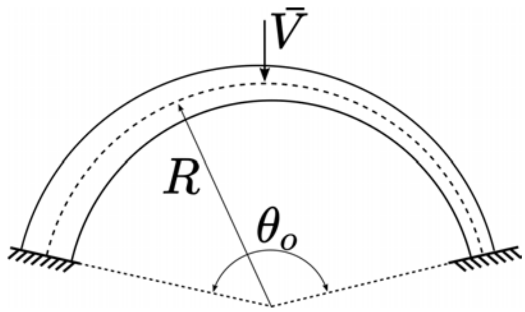

Figure 4: Clamped-Clamped Circular Tapered Beam with Tip Load

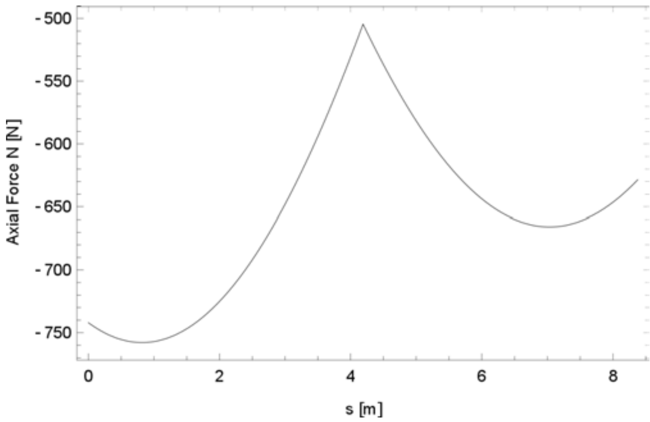

Figure 5: Clamped-Clamped Circular Tapered Beam with Tip Load - Axial Force Diagram

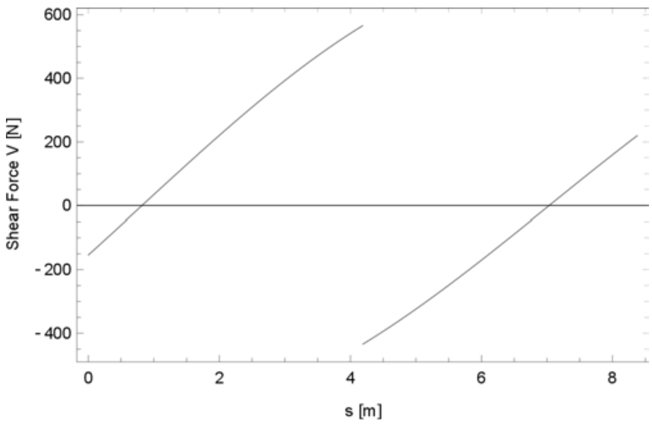

Figure 6: Clamped-Clamped Circular Tapered Beam with Tip Load - Shear Force Diagram

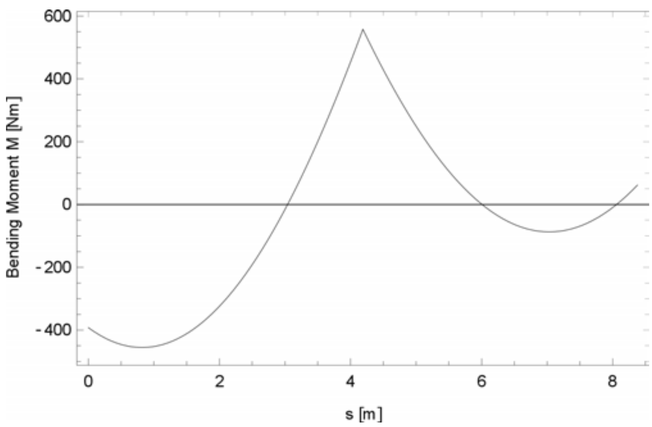

Figure 7: Clamped-Clamped Circular Tapered Beam with Tip Load - Bending Moment Diagram

n_e	$\frac{w\times10^{-6}}{R\theta_o}$
2	0.904064
4	0.938990
8	0.941257
16	0.941300

Table 2: Clamped-Clamped Circular Tapered Beam with Tip Load - Mesh Convergence Study

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