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C^*-algebra valued quasi metric spaces and fixed point results with an application

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Abstract

In this paper, we introduce the notion of C^*-algebra valued quasi metric space to generalize the notion of C^*-algebra valued metric space and investigate the topological properties besides proving some core fixed point results. Finally, we employ our one of the main results to examine the existence and uniqueness of the solution for a system of Fredholm integral equations.

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1. Introduction

The classical Banach contraction principle [8] continues to be one of the most motivating fixed point results, which has inspired several generations of mathematicians working in this domain. This principle is not merely an existence and uniqueness result but also offers a very effective computational procedure to

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compute the fixed point of the underlying contraction map. Several researchers attempted to improve this principle by enlarging the class of spaces. To accomplish this, the authors introduced various classes of metric spaces namely: (see [1–7, 9–11, 14, 15].

In 1931, W. A. Wilson [16] introduced the notion of quasi metric space. A quasi metric d on a non-empty set A is a function $d : A \times A \to \mathbb{R}_+$ which satisfies $d(a,b) \leq d(a,c) + d(c,b)$ and $d(a,b) = d(b,a) = 0$ if and only if $a = b$, for all $a, b \in A$. A quasi metric satisfies all the conditions of metric with the possible exception of symmetry (*i.e.*, the distance of a point 'a' to a point 'b' may not equal to the distance of a point 'b' to a point 'a').

On the other hand, Ma et al. [12] set up the class of C^*-algebra valued metric spaces (in short C^*-avMS) by interchanging $\mathbb{R}$ (the range set) with a unital C^*-algebra in 2014, which is a more broad class than the class of metric spaces, and used the equivalent to make some fixed point results in such spaces. After one year, Ma et al. [13] again presented the idea of C^*-algebra valued b-metric spaces as a generalization of C^*-algebra valued metric space and proved some fixed point results likewise utilized the of their work for an integral type operator as an application.

Enlivened by prior perceptions, we expand the class of C^*-algebra valued metric space by presenting the class of C^*-algebra valued quasi metric space and using the equivalent to make a fixed point result. Additionally, we concentrate on some topological properties of the C^*-algebra valued quasi metric space. Besides, we furnish some examples which show the utility of our main result.

2. Preliminaries

Recall some definitions, examples and useful results which are needed in our subsequent discussions. Now, we give the following definition of C^*-algebra valued metric space which is introduced by Ma et al. [12] in 2014.

Definition 2.1. Let A be a non-empty set. A mapping $d : A \times A \to \mathcal{A}$ is called a C^*-algebra valued metric on A, if it satisfies the following (for all $a, b, c \in A$):

(i) $d(a,b) \succcurlyeq 0_{\mathcal{A}}$ and $d(a,b) = 0_{\mathcal{A}}$ iff $a = b$;
(ii) $d(a,b) = d(b,a)$;
(iii) $d(a,b) \preccurlyeq d(a,c) + d(c,b)$.

The triplet $(A, \mathcal{A}, d)$ is called a C^*-algebra valued metric space.

Now, we introduce yet different type of generalized C^*-algebra valued metric space and quasi metric space, which we refer as C^*-algebra valued quasi metric space.

Definition 2.2. Let A be a non-empty set. A mapping $d : A \times A \to \mathcal{A}$ is called a C^*-algebra valued quasi metric on A, if it satisfies the following (for all $a, b, c \in A$):

(i) $d(a,b) \succcurlyeq 0_{\mathcal{A}}$ and $d(a,b) = d(b,a) = 0_{\mathcal{A}}$ iff $a = b$;
(ii) $d(a,b) \preccurlyeq d(a,c) + d(c,b)$.

The triplet $(A, \mathcal{A}, d)$ is called a C^*-algebra valued quasi metric space.

Example 2.3. Let $A = \mathbb{R}$ and $\mathcal{A} = M_2(\mathbb{C})$, the class of bounded and linear operators on a Hilbert space $\mathbb{C}^2$. Define $d : A \times A \to \mathcal{A}$ by (for all $a, b \in A$):

$$d(a,b) = \begin{cases} \begin{bmatrix} a-b & 0 \\ 0 & a-b \end{bmatrix} & \text{if } a \geq b \\ I_{2\times 2} \quad \text{if } a < b \end{cases}$$

where $I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $(A, \mathcal{A}, d)$ be a C^*-algebra valued quasi metric space.

It is clear that $d(a,b) = 0_{\mathcal{A}} \;\Leftrightarrow\; a = b$. Now, we consider the following two cases: **Case I:** For $a \geq b$, we have $d(a,b) = (a-b)I_{2\times 2}$.

- If $c < b$, then $d(a,c) = (a-c)I_{2\times 2}$ and $d(c,b) = I_{2\times 2}$.
- If $b \leq c < a$, then $d(a,c) = (a-c)I_{2\times 2}$ and $d(c,b) = (c-b)I_{2\times 2}$.
- If $a \leq c$, then $d(a,c) = I_{2\times 2}$ and $d(c,b) = (c-b)I_{2\times 2}$. Therefore, we have

$$d(a,b) \preccurlyeq d(a,c) + d(c,b).$$

Case 2: For $a < b$, we have $d(a,b) = I_{2\times 2}$.

- If $c < a$, then $d(a,c) = (a-c)I_{2\times 2}$ and $d(c,b) = I_{2\times 2}$.
- If $a \leq c < b$, then $d(a,c) = I_{2\times 2}$ and $d(c,b) = I_{2\times 2}$.
- If $b \leq c$, then $d(a,c) = I_{2\times 2}$ and $d(c,b) = (c-b)I_{2\times 2}$. Therefore, we have

$$d(a,b) \preccurlyeq d(a,c) + d(c,b).$$

By the above calculations, we can say that $(A, \mathcal{A}, d)$ is a C^*-algebra valued quasi metric space.

Example 2.4. Let $A = \mathbb{R}$ and $\mathcal{A} = M_2(\mathbb{C})$. Define $d : A \times A \to \mathcal{A}$ by (for all $a, b \in A$):

$$d(a,b) = \begin{cases} \begin{bmatrix} b-a & 0 \\ 0 & k(b-a) \end{bmatrix} & \text{if } a \leq b \\ \alpha \begin{bmatrix} a-b & 0 \\ 0 & k(a-b) \end{bmatrix} & \text{if } a > b \end{cases}$$

where $k, \alpha > 0$. Then $(A, \mathcal{A}, d)$ is a C^*-algebra valued quasi metric space.

Example 2.5. Let $A = [1, \infty)$ and $\mathcal{A} = M_2(\mathbb{C})$. Define $d : A \times A \to \mathcal{A}$ by (for all $a, b \in A$):

$$d(a,b) = \begin{cases} \begin{bmatrix} \ln b - \ln a & 0 \\ 0 & \ln b - \ln a \end{bmatrix} & \text{if } a \leq b \\ \frac{1}{3} \begin{bmatrix} \ln a - \ln b & 0 \\ 0 & \ln a - \ln b \end{bmatrix} & \text{if } a > b \end{cases}$$

where '$\ln'$ is natural logarithmic function. Then $(A, \mathcal{A}, d)$ is a C^*-algebra valued quasi metric space.

Now, we give some definitions of convergent, left-Cauchy, right-Cauchy and completeness of the quasi metric space as follows.

Definition 2.6. Let $(A, \mathcal{A}, d)$ be a C^*-algebra valued quasi metric space and $\{a_n\}$ a sequence in A. We say that

(i) The sequence $\{a_n\}$ is called convergent to $a \in A$, written $\lim_{n\to\infty} a_n = a$, if

$$\lim_{n\to\infty} d(a_n, a) = \lim_{n\to\infty} d(a, a_n) = 0_{\mathcal{A}}.$$

(ii) The sequence $\{a_n\}$ is called left-Cauchy if for each $\epsilon \succ 0_{\mathcal{A}}$ there exists a positive integer N such that

$$d(a_n, a_m) \prec \epsilon \text{ for all } n \geq m \geq N.$$

(iii) The sequence $\{a_n\}$ is called right-Cauchy if for each $\epsilon \succ 0_{\mathcal{A}}$ there exists a positive integer N such that

$$d(a_n, a_m) \prec \epsilon \text{ for all } m \geq n \geq N.$$

(iv) The sequence $\{a_n\}$ is called Cauchy if for each $\epsilon \succ 0_{\mathcal{A}}$ there exists a positive integer N such that

$$d(a_n, a_m) \prec \epsilon \text{ for all } m, n \geq N, \text{ i.e., } \lim_{n,m\to\infty} d(a_n, a_m) = 0_{\mathcal{A}}.$$

(v) The triplet $(A, \mathcal{A}, d)$ is left-complete if every left-Cauchy sequence in $(A, \mathcal{A}, d)$ is convergent.

(vi) The triplet $(A, \mathcal{A}, d)$ is right-complete if every right-Cauchy sequence in $(A, \mathcal{A}, d)$ is convergent.

(vii) The triplet $(A, \mathcal{A}, d)$ is complete if every Cauchy sequence in $(A, \mathcal{A}, d)$ is convergent.

Remark 2.7.

(1) Every C^*-algebra valued metric space is C^*-algebra valued quasi metric space but the converse is not true in general.

(2) In a C^*-algebra valued quasi metric space a sequence $\{a_n\}$ is Cauchy iff it is left-Cauchy and right-Cauchy.

Definition 2.8. Let $(A, \mathcal{A}, d)$ C^*-algebra valued quasi metric space. The conjugate (or dual) C^*-algebra valued quasi metric space is denoted by d^c and define by as follows:

$$d^c(a, b) = d(b, a), \text{ for all } a, b \in A.$$

A C^*-algebra valued quasi metric space is a C^*-algebra valued metric space iff it coincides with its conjugate d^c, for all $a, b \in A$.

Let w_c be a C^*-algebra positive valued function on A. The quadruplet $(A, \mathcal{A}, d, w_c)$ is called a C^*-algebra valued weighted quasi metric space, if for all $a, b \in A$

$$d(a, b) + w_c(a) = d(b, a) + w_c(b).$$

Then $(A, \mathcal{A}, d, w_c)$ is said to be C^*-algebra valued generalized weighted quasi metric space.

Proposition 2.9. *Let $(A, \mathcal{A}, d)$ C^*-algebra valued quasi metric space. The associated C^*-algebra valued metric d^s is define by:*

$$d^s(a,b) = \frac{1}{2}[d(a,b) + d(b,a)].$$

The associated C^*-algebra valued metric space d^s is the smallest C^*-algebra valued metric space majorising d.

Proof. To verify condition (i), for each $a, b \in A$, we have $d(a,b) \succcurlyeq 0_{\mathbb{A}}$. Also

$$\begin{array}{rcl} d^s(a,b) & = & 0_{\mathcal{A}} \\ & \Leftrightarrow & \frac{1}{2}[d(a,b) + d(b,a)] = 0_{\mathcal{A}} \\ & \Leftrightarrow & d(a,b) + d(b,a) = 0_{\mathcal{A}} \\ & \Leftrightarrow & d(a,b) = d(b,a) = 0_{\mathcal{A}} \\ & \Leftrightarrow & a = b. \end{array}$$

Now, for condition (ii), for each $a, b \in A$, we have

$$\begin{array}{rcl} d^s(a,b) & = & \frac{1}{2}[d(a,b) + d(b,a)] \\ & = & \frac{1}{2}[d(b,a) + d(a,b)] \\ & = & d^s(b,a). \end{array}$$

Finally, we show that condition (iii), for each $a, b, c \in A$, we have

$$\begin{array}{rcl} d^s(a,b) & = & \frac{1}{2}[d(a,b) + d(b,a)] \\ & = & \frac{1}{2}[d(a,c) + d(c,b) + d(b,c) + d(c,a)] \\ & = & \frac{1}{2}[d(a,c) + d(c,a)] + \frac{1}{2}[d(b,c) + d(c,b)] \\ & = & d^s(a,b) + d^s(b,a). \end{array}$$

Thus, $(A, \mathcal{A}, d^s)$ is C^*-algebra valued metric space. □

Definition 2.10. Let $(A, \mathcal{A}, d)$ be a C^*-algebra valued quasi metric space, $a \in A$, $N, M \subseteq A$ and $0_{\mathcal{A}} \prec \epsilon \in \mathcal{A}$. Denoted by:

- The diameter of set N

$$diam(N) = \sup\{d(a,b) : a, b \in N\}.$$

- The left-open ball of radius ϵ centered at a

$$B^L_\epsilon(a) = \{b \in A : d(a,b) \prec \epsilon\}.$$

- The right-open ball of radius ϵ centered at a

$$B^R_\epsilon(a) = \{b \in A : d(b,a) \prec \epsilon\}.$$

- The associated C^*-algebra valued quasi metric space open ball of radius ϵ centered at a

$$B_\epsilon = \{b \in A : d^s(a,b) \prec \epsilon\}.$$

- The left-distance from a to N

$$dist_d(a, N) = \inf\{d(a, b) : b \in N\}.$$

- The right-distance from a to N

$$dist_d(N, a) = \inf\{d(b, a) : b \in N\}.$$

- The left-ϵ-neighbourhood of N

$$\mathcal{N}^L_\epsilon = \inf\{a \in A : dist_d(N, a) \prec \epsilon\}.$$

- The right-ϵ-neighbourhood of N

$$\mathcal{N}^R_\epsilon = \inf\{a \in A : dist_d(a, N) \prec \epsilon\}.$$

- The associated metric ϵ-neighbourhood of N

$$\mathcal{N}^L_\epsilon = \inf\{a \in A : dist^s_d(N, a) \prec \epsilon\}.$$

- The distance between N and M

$$d(N, M) = \inf\{d(a, b) : a \in N, b \in M\}.$$

Proposition 2.11. *Let $(A, \mathcal{A}, d)$ be a C^*-algebra valued quasi metric space. Then the collection of all open left-balls $B^L_\epsilon(a)$ (right-balls $B^R_\epsilon(a)$) on A,*

$$\mathcal{U}^L_{\mathcal{A}} = \{B^L_\epsilon(a) : a \in A, \epsilon \succ 0_{\mathcal{A}}\}$$

forms a left-basis (right-basis) on A.

Proof. Take $a, b \in A$ and $\epsilon_1, \epsilon_2 \succ 0_{\mathcal{A}}$ such that $B^L_{\epsilon_1}(a) \cap B^L_{\epsilon_2}(b) \neq \varnothing$. Now, choose $c \in B^L_{\epsilon_1}(a) \cap B^L_{\epsilon_2}(b)$ and set $\epsilon_3 = \min\{\epsilon_1 - d(a, c), \epsilon_2 - d(b, c)\}$. Observe that $B^L_{\epsilon_3}(c) \subseteq B^L_{\epsilon_1}(a) \cap B^L_{\epsilon_2}(b)$. Therefore, $\mathcal{U}^{\mathcal{L}}_{\mathcal{A}}$ forms a left-basis on A.

Similarly, the collection of all open right-balls $B^L_\epsilon(a)$,

$$\mathcal{U}^{\mathcal{R}}_{\mathcal{A}} = \{B^R_\epsilon(a) : a \in A, \epsilon \succ 0_{\mathcal{A}}\}$$

forms a right-basis on A. □

Every C^*-algebra valued quasi metric space d naturally induces a T_0 topology $\mathcal{T}^L_{\mathcal{A}}$, where a set N is open if for each $a \in N$ there exists $\epsilon \succ 0_{\mathcal{A}}$ such that $B^L_\epsilon(a) \subseteq N$. Similarly, the topology $\mathcal{T}^R_{\mathcal{A}}$ can be define by using the right-balls (that is, $B^R_\epsilon(a)$) as its base and hence a C^*-algebra valued quasi metric space $(A, \mathcal{A}, d)$ can be naturally associated with a bi-topological space $(A, \mathcal{A}, \mathcal{T}^L_{\mathcal{A}}, \mathcal{T}^R_{\mathcal{A}})$. Moreover, if the map d satisfies $d(a, b) = 0 \Leftrightarrow a = b$ instead of condition (i) in Definition (2.2) then d induces a T_1 topology.

Proposition 2.12. *Let $(A, \mathcal{A}, d)$ be a C^*-algebra valued quasi metric space and associated $(A, \mathcal{A}, d^s)$ a C^*-algebra valued metric space. Then*

(1) A sequence $\{a_n\}$ is convergent to a in $(A, \mathcal{A}, d)$ if and only if $\{a_n\}$ is convergent to a in $(A, \mathcal{A}, d^s)$.
(2) A sequence $\{a_n\}$ is Cauchy in $(A, \mathcal{A}, d)$ if and only if $\{a_n\}$ is Cauchy in $(A, \mathcal{A}, d^s)$.
(3) The C^*-algebra valued quasi metric space $(A, \mathcal{A}, d)$ is complete if and only if C^*-algebra valued metric space $(A, \mathcal{A}, d^s)$ is complete.

Proof. (1) Suppose that $\{a_n\}$ is convergent to a in $(A, \mathcal{A}, d)$, that is,

$$\lim_{n\to\infty} d(a_n, a) = \lim_{n\to\infty} d(a, a_n) = 0_{\mathcal{A}}.$$

Which is equivalent to

$$\lim_{n\to\infty} d^s(a_n, a) = \frac{1}{2}[\lim_{n\to\infty} d(a_n, a) + \lim_{n\to\infty} d(a, a_n)] = 0_{\mathcal{A}}.$$

Hence, the sequence $\{a_n\}$ is convergent to a in $(A, \mathcal{A}, d^s)$.

(2) Suppose that $\{a_n\}$ is Cauchy in $(A, \mathcal{A}, d)$, that is,

$$\lim_{n\to\infty} d(a_n, a_m) = \lim_{n\to\infty} d(a_m, a_n) = 0_{\mathcal{A}}.$$

Which is equivalent to

$$\lim_{n\to\infty} d^s(a_n, a_m) = \frac{1}{2}[\lim_{n\to\infty} d(a_n, a_m) + \lim_{n\to\infty} d(a_m, a_n)] = 0_{\mathcal{A}}.$$

Therefore, the sequence $\{a_n\}$ is Cauchy in $(A, \mathcal{A}, d^s)$.

(3) It is a direct consequence of (1) and (2). □

Proposition 2.13. *Let $(A, \mathcal{A}, d_A)$ and $(B, \mathcal{A}, d_B)$ be two C^*-algebra valued quasi metric spaces. Then*

(1) $d(a, b) = d_A(a_1, a_2) + d_B(b_1, b_2)$, for all $a = (a_1, b_1)$, $b = (a_2, b_2) \in A \times B$ is a C^*-algebra valued quasi metric on $A \times B$.

(2) $\lim_{n\to\infty}(a_n, b_n) = (a, b)$ in $(A \times B, \mathcal{A}, d)$ if and only if $\lim_{n\to\infty} a_n = a$ in $(A, \mathcal{A}, d_A)$ and $\lim_{n\to\infty} b_n = b$ in $(B, \mathcal{A}, d_B)$. Particularly, the topology induced by d coincides the product topology on $A \times B$.

(3) $\{(a_n, b_n)\}$ is a Cauchy sequence in $(A \times B, \mathcal{A}, d)$ if and only if $\{a_n\}$ is a Cauchy sequence in $(A, \mathcal{A}, d_A)$ and $\{b_n\}$ is a Cauchy sequence in $(B, \mathcal{A}, d_B)$.

(4) $(A \times B, \mathcal{A}, d)$ is complete if and only if $(A, \mathcal{A}, d_A)$ and $(B, \mathcal{A}, d_B)$ are complete.

Proof. (1) Assume that $a = (a_1, b_1)$, $b = (a_2, b_2)$, $c = (c_1, c_2) \in A \times B$. Then, we have $d(a, b) = 0_{\mathcal{A}}$ if and only if $d_A(a_1, b_1) + d_B(a_2, b_2) = 0_{\mathcal{A}}$, that is, $d_A(a_1, b_1) = d_B(a_2, b_2) = 0_{\mathcal{A}}$, which implies that $a_1 = b_1$ and $a_2 = b_2$, that is, $a = b$. Now, to show the triangular inequality, we have

$$\begin{aligned} d(a, b) &= d_A(a_1, a_2) + d_B(b_1, b_2) \\ &\preccurlyeq d_A(a_1, c_1) + d_A(c_1, a_2) + d_B(b_1, c_2) + d_B(c_2, b_2) \\ &= d_A(a_1, c_1) + d_B(b_1, c_2) + d_A(c_1, a_2) + d_B(c_2, b_2) \\ &= d(a, c) + d(c, b). \end{aligned}$$

Therefore, $(A \times B, \mathcal{A}, d)$ is a C^*-algebra valued quasi metric space.

(2) Let $\lim_{n\to\infty}(a_n, b_n) = (a, b)$ in $(A \times B, \mathcal{A}, d)$ if and only if

$$\lim_{n\to\infty} d\big((a_n, b_n), (a, b)\big) = \lim_{n\to\infty} \big[d_A(a_n, a) + d_B(b_n, b)\big] = 0_{\mathcal{A}}$$

and

$$\lim_{n\to\infty} d\big((a, b), (a_n, b_n)\big) = \lim_{n\to\infty} \big[d_A(a, a_n) + d_B(b, b_n)\big] = 0_{\mathcal{A}}$$

which is equivalent to

$$\lim_{n\to\infty} d_A(a_n, a) = \lim_{n\to\infty} d_B(b_n, b) = \lim_{n\to\infty} d_A(a, a_n) = \lim_{n\to\infty} d_B(b, b_n) = 0_{\mathcal{A}}.$$

Therefore, $\lim_{n\to\infty} a_n = a$ in $(A, \mathcal{A}, d_A)$ and $\lim_{n\to\infty} b_n = b$ in $(B, \mathcal{A}, d_B)$. Therefore,

(3) Suppose the sequence $\{(a_n, b_n)\}$ is a Cauchy in $(A \times B, \mathcal{A}, d)$ if and only if

$$\lim_{n,m\to\infty} d\big((a_n, b_n), (a_m, b_m)\big) = \lim_{n,m\to\infty} \big[d_A(a_n, a_m) + d_B(b_n, b_m)\big] = 0_{\mathcal{A}}$$

which is equivalent to

$$\lim_{n,m\to\infty} d_A(a_n, a_m) = \lim_{n,m\to\infty} d_B(b_n, b_m) = 0_{\mathcal{A}}.$$

Therefore, $\{a_n\}$ is a Cauchy sequence in $(A, \mathcal{A}, d_A)$ and $\{b_n\}$ is a Cauchy sequence in $(B, \mathcal{A}, d_B)$.

(4) It is a direct consequence of (2) and (3). $\square$

3. Fixed Point Results

Now, we present our main result as follows:

Theorem 3.1. *Let $(A, \mathcal{A}, d)$ be complete C^*-algebra valued quasi metric space and $f : X \to X$ a mapping satisfies the following (for $\lambda \in \mathcal{A}$ with $\|\lambda\| < 1$):*

$$d(fa, fb) \preccurlyeq \lambda^* d(a, b)\lambda, \quad \forall\ a, b \in A. \tag{3.1}$$

Then f has a unique fixed point.

Proof. Firstly, select $a_0 \in A$ and extract an iterative sequence $\{a_n\}$ as:

$$a_n = fa_{n-1} = f^n a_0, \quad \forall \quad n \in \mathbb{N}.$$

Now, we want to show that $\lim_{n,m\to\infty} d(a_{n+1}, a_n) = 0_{\mathcal{A}}$. By choosing $a = a_{n+1}$ and $b = a_n$ in 3.1, we wet

$$\begin{aligned} d(a_{n+1}, a_n) &= d(fa_n, fa_{n-1}) = \lambda^* d(a_n, a_{n-1})\lambda \\ &\preccurlyeq (\lambda^*)^2 d(a_{n-1}, a_{n-2})\lambda^2 \\ &\preccurlyeq \ldots \\ &\preccurlyeq (\lambda^*)^n d(a_1, a_0)\lambda^n. \end{aligned}$$

Similarly, we can have

$$d(a_n, a_{n+1}) \preccurlyeq (\lambda^*)^n d(a_0, a_1)\lambda^n.$$

Now, we assert that $\{a_n\}$ is Cauchy sequence. For any $n, m \in \mathbb{N}$ such that $n+1 > m$, we have

$$\begin{aligned}
d(a_{n+1}, a_m) &\preccurlyeq d(a_{n+1}, a_n) + d(a_n, a_{n-1}) + \cdots + d(a_{m+1}, a_m) \\
&\preccurlyeq (\lambda^*)^n d(a_1, a_0)\lambda^n + \cdots + (\lambda^*)^m d(a_1, a_0)\lambda^m \\
&= \sum_{i=m}^{n} (\lambda^*)^i d(a_1, a_0)\lambda^i \\
&= \sum_{i=m}^{n} (\lambda^*)^i (d(a_1, a_0))^{\frac{1}{2}} (d(a_1, a_0))^{\frac{1}{2}} \lambda^i \\
&= \sum_{i=m}^{n} \left((d(a_1, a_0))^{\frac{1}{2}} \lambda^i\right)^* \left(d(a_1, a_0)^{\frac{1}{2}} \lambda^i\right) \\
&= \sum_{i=m}^{n} \left|(d(a_1, a_0))^{\frac{1}{2}} \lambda^i\right|^2 \\
&\preccurlyeq \left\| \sum_{i=m}^{n} \left|(d(a_1, a_0))^{\frac{1}{2}} \lambda^i\right|^2 \right\| I \\
&\preccurlyeq \sum_{i=m}^{n} \left\|(d(a_1, a_0))^{\frac{1}{2}}\right\|^2 \left\|\lambda^i\right\|^2 I \\
&\preccurlyeq \|d(a_1, a_0)\| \sum_{i=m}^{n} \|\lambda\|^{2i} I \\
&\preccurlyeq \|d(a_1, a_0)\| \frac{\|\lambda\|^{2m}}{1 - \|\lambda\|} I \to 0_{\mathbb{A}} \quad (\text{as } m \to \infty).
\end{aligned}$$

Thus, $\{a_n\}$ is left-Cauchy sequence, that is

$$\lim_{m\to\infty} d(a_n, a_m) = 0_{\mathbb{A}} \ \forall\, n \geq m \geq N.$$

Similarly, we can show that $\{a_n\}$ is right-Cauchy sequence, that is (for $m > n$)

$$\lim_{n\to\infty} d(a_m, a_n) = 0_{\mathbb{A}} \ \forall\, m \geq n \geq N.$$

Therefore, $\{a_n\}$ is Cauchy sequence. Since, $(A, \mathcal{A}, d)$ is complete C^*-algebra valued quasi metric space, then there exists a point a in A such that $\lim\limits_{n\to\infty} a_n = a$, that is, $\lim\limits_{n\to\infty} d(a_n, a) = \lim\limits_{n\to\infty} d(a, a_n) = 0_{\mathcal{A}}$. Then, we get

$$\begin{aligned}
d(fa, a) &\preccurlyeq d(fa, a_{n+1}) + d(a_{n+1}, a) \\
&\preccurlyeq d(fa, fa_n) + d(a_{n+1}, a) \\
&\preccurlyeq \lambda^* d(a, a_n)\lambda + d(a_{n+1}, a) \\
&\preccurlyeq \|\lambda^2\|\|d(a, a_n)\| I + d(a_{n+1}, a).
\end{aligned}$$

On taking limit as $n \to \infty$, we get $fa = a$. Hence, a is fixed point of f.

Now, to show that the fixed point is unique, we assume that there are two fixed points, say $a_1, a_2 \in A$ such that $fa_1 = a_1$ and $fa_2 = a_2$. Then by using

3.1, we have

$$\begin{array}{rcl}\|d(a_1,a_2)\| & = & \|d(fa_1,fa_2)\| \\ & \leq & \|\lambda^* d(a_1,a_2)\lambda\| \\ & \leq & \|\lambda^*\|\|d(a_1,a_2)\|\|\lambda\| \\ & = & \|\lambda\|^2\|(a_1,a_2)\|\end{array}$$

deals a contradiction. Hence, $a_1 = a_2$, that is, a_1 is a unique fixed point of f. This completes the proof. □

Example 3.2. In the Example 2.4, we define a self-mapping $f : A \to A$ by:

$$fa = \frac{a}{5}, \ \forall\ a \in A.$$

Notice that, $d(fa, fb) \preccurlyeq \lambda^* d(a,b)\lambda$, (for each $a, b \in A$) satisfies and

$$\lambda = \begin{bmatrix} \frac{\sqrt{5}}{5} & 0 \\ 0 & \frac{\sqrt{5}}{5} \end{bmatrix} \in A \text{ and } \|\lambda\| = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}} < 1.$$

Hence, all the assumptions of Theorem 3.1 are fulfilled and f unique fixed point, namely $a = 0_{\mathcal{A}}$.

Before presenting the next theorem we recall the following lemma which is needed is the sequel.

Lemma 3.3. *Let $\mathcal{A}$ be a unital C^*-algebra with a unit I. We have*

(1) If $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $I - a$ is invertible and $\|a(I-a)^{-1}\| < 1$;
(2) if $a, b \in \mathbb{A}_+$ with $ab = ba$, then $ab \in \mathcal{A}_+$;
(3) we denote $\mathbb{A}^{'} = \{a \in \mathcal{A} :\ ab = ba,\ \forall b \in \mathcal{A}\}$. Let $a \in \mathbb{A}^{'}$, if $b, c \in \mathcal{A}$ with $b \succcurlyeq c \succcurlyeq 0_{\mathcal{A}}$ and $I - a \in \mathbb{A}^{'}_+$ is an invertible operator, then $(I-a)^{-1}b \succcurlyeq (I-a)^{-1}c$.

Theorem 3.4. *Let $(A, \mathcal{A}, d)$ be complete C^*-algebra valued quasi metric space and $f : X \to X$ a continuous mapping satisfies that the following (for $\lambda \in \mathcal{A}$ with $\|\lambda\| < \frac{1}{2}$):*

$$d(fa, fb) \preccurlyeq \lambda[d(fa, b) + d(a, fb)], \quad \forall\ a, b \in A. \tag{3.2}$$

Then f has a unique fixed point.

Proof. Firstly, select $a_0 \in A$ and extract an iterative sequence $\{a_n\}$ as:

$$a_n = fa_{n-1} = f^n a_0, \ \ \forall\ \ n \in \mathbb{N}.$$

Now, we want to show that $\lim\limits_{n,m\to\infty} d(a_{n+1}, a_n) = 0_{\mathcal{A}}$. By choosing $a = a_{n+1}$ and $b = a_n$ in 3.2, we wet

$$\begin{array}{rcl} d(a_{n+1}, a_n) & = & d(fa_n, fa_{n-1}) \\ & = & \lambda[d(fa_n, a_{n-1}) + d(a_n, fa_{n-1})] \\ & = & \lambda[d(a_{n+1}, a_{n-1}) + d(a_n, a_n)] \\ & \preccurlyeq & \lambda[d(a_{n+1}, a_n) + d(a_n, a_{n-1})] \\ & = & \lambda d(a_{n+1}, a_n) + \lambda d(a_n, a_{n-1}). \end{array}$$

Thus,

$$(I-\lambda)d(a_{n+1},a_n) \preccurlyeq \lambda d(a_n,a_{n-1}).$$

Since, $\lambda \in \mathcal{A}$ with $\|\lambda\| < \frac{1}{2}$, then we have $(I-\lambda)^{-1} \in \mathcal{A}$ and also $\lambda(I-\lambda)^{-1} \in \mathcal{A}$ with $\|\lambda(I-\lambda)^{-1}\| < 1$ (by Lemma 3.3). Then, by assuming $u = \lambda(I-\lambda)^{-1}$, we obtain

$$d(a_{n+1},a_n) \preccurlyeq \lambda(I-\lambda)^{-1}d(a_n,a_{n-1}) = ud(a_n,a_{n-1}).$$

Similarly, we can have

$$d(a_n,a_{n+1}) \preccurlyeq ud(a_{n-1},a_n).$$

Now, we show that the sequence $\{a_n\}$ is Cauchy. Suppose $n+1 > m, \ \ \forall\, n,m \in \mathbb{N}$, so we have

$$\begin{array}{rcl} d(a_{n+1},a_m) & \preccurlyeq & d(a_{n+1},a_n) + d(a_n,a_{n-1}) + \cdots + d(a_{m+1},a_m) \\ & \preccurlyeq & (u^n + u^{n-1} + \cdots + u^m)d(a_1,a_0) \\ & = & \displaystyle\sum_{i=m}^{n} u^{\frac{i}{2}}u^{\frac{i}{2}}(d(a_1,a_0))^{\frac{1}{2}}(d(a_1,a_0))^{\frac{1}{2}} \\ & = & \displaystyle\sum_{i=m}^{n} (d(a_1,a_0))^{\frac{1}{2}}u^{\frac{i}{2}}u^{\frac{i}{2}}(d(a_1,a_0))^{\frac{1}{2}}\lambda^i \\ & = & \displaystyle\sum_{i=m}^{n} \left(u^{\frac{i}{2}}(d(a_1,a_0))^{\frac{1}{2}}\right)^* \left(u^{\frac{i}{2}}d(a_1,a_0)^{\frac{1}{2}}\right) \\ & = & \displaystyle\sum_{i=m}^{n} \left|u^{\frac{i}{2}}(d(a_1,a_0))^{\frac{1}{2}}\right|^2 \\ & \preccurlyeq & \displaystyle\left\|\sum_{i=m}^{n} \left|u^{\frac{i}{2}}(d(a_1,a_0))^{\frac{1}{2}}\right|^2\right\| I \\ & \preccurlyeq & \displaystyle\sum_{i=m}^{n} \left\|(d(a_1,a_0))^{\frac{1}{2}}\right\|^2 \left\|u^{\frac{i}{2}}\right\|^2 I \\ & \preccurlyeq & \displaystyle\|d(a_1,a_0)\| \sum_{i=m}^{n} \left\|u^{\frac{i}{2}}\right\|^i I \\ & \preccurlyeq & \displaystyle\|d(a_1,a_0)\| \frac{\|u\|^m}{1-\|u\|} I \to 0_{\mathbb{A}} \quad \text{(as } m \to \infty). \end{array}$$

Thus, $\{a_n\}$ is left-Cauchy sequence, that is

$$\lim_{n,m\to\infty} d(a_n,a_m) = 0_{\mathbb{A}}.$$

Similarly, we can have $\{a_n\}$ is right-Cauchy sequence, that is (for $m > n$)

$$\lim_{n,m\to\infty} d(a_m,a_n) = 0_{\mathbb{A}}.$$

Therefore, the sequence $\{a_n\}$ is Cauchy. Since, $(A,\mathcal{A},d)$ is complete C^*-algebra valued quasi metric space, then there exists a point a in A such that $\lim\limits_{n\to\infty} a_n =$

a, that is, $\lim_{n\to\infty} d(a_n, a) = \lim_{n\to\infty} d(a, a_n) = 0_{\mathcal{A}}$. Now, by using the continuity of f, we have

$$\begin{array}{rcl} d(fa, a) & \preccurlyeq & d(fa, a_{n+1}) + d(a_{n+1}, a) \\ & \preccurlyeq & d(fa, fa_n) + d(a_{n+1}, a) \end{array}$$

On taking limit as $n \to \infty$, we get $fa = a$. Hence, a is fixed point of f.

Now, to show that the fixed point is unique, we assume that there are two fixed points, say $a_1, a_2 \in A$ such that $fa_1 = a_1$ and $fa_2 = a_2$. Then by employing 3.4, we have

$$\begin{array}{rcl} \|d(a_1, a_2)\| & = & \|d(fa_1, fa_2)\| \\ & \leq & \|\lambda[d(fa_1, a_2) + d(a_1, fa_2)]\| \\ & \leq & \|\lambda\|\|d(a_1, a_2) + d(a_1, a_2)\| \\ & \leq & \|\lambda\|[\|d(a_1, a_2)\| + \|d(a_1, a_2)\|] \\ & = & 2\|\lambda\|\|(a_1, a_2)\| \end{array}$$

a contradiction (since $2\|\lambda\| < 1$). Hence, $a_1 = a_2$, that is, a_1 is a unique fixed point of f. This completes the proof. □

Now, we obtain following corollaries:

Remark 3.5. By taking $d(a, b) = d(b, a)$, for all $a, b \in A$ in Theorem 3.1, we obtain Theorem 2.1 of Z. Ma et al. [12].

Remark 3.6. By taking $d(a, b) = d(b, a)$, for all $a, b \in A$ in Theorem 3.4, we obtain Theorem 2.3 of Z. Ma et al. [12].

4. Application

To find the existence and uniqueness results of a contractive mapping on complete C^*-algebra valued metric space for the integral type equation is carried out by Z. Ma et al. [12] in 2014 whose lines are as under:

Example 4.1 ([12]). Consider the integral equation

$$a(\xi) = \int_{\Delta} G(\xi, \omega, a(\omega))d\omega + h(\xi), \ \forall \ \xi, \omega \in \Delta, \tag{4.1}$$

where Δ is a Lebesgue measurable set.

Suppose that

(1) h is an essentially bounded measurable function defined on Δ and $G : \Delta^2 \times \mathbb{R} \to \mathbb{R}$,
(2) there exists a continuous function $\eta : \Delta \times \Delta \to \mathbb{R}$ and $\lambda \in (0, 1)$ such that

$$\mid G(\xi, \omega, a(\omega)) - G(\xi, \omega, b(\omega)) \mid \leq \lambda \mid \eta(\xi, \omega) \mid (\mid a(\omega) - b(\omega) \mid),$$

for all $\xi, \omega \in \Delta$ and $a, b \in \mathbb{R}$.
(3) $\sup_{\xi \in \Delta} \int_{\Delta} \mid \eta(\xi, \omega) \mid d\omega \leq 1$.

Then the integral equation has a unique solution in A, where A stands for the space of essentially bounded measurable functions defined on Δ.

Now, we will utilize Theorem 3.1 to find the solution of following integral equation:

$$a(\xi) = \int_{\Delta} G(\xi, \omega, a(\omega))d\omega + h(\xi), \ \forall \ \xi, \omega \in \Delta, \tag{4.2}$$

where, Δ is a Lebesgue measurable set with $m(\Delta) < \infty$, $G : \Delta \times \Delta \times \mathbb{R} \to \mathbb{R}$ and $h \in A$. Define $d : A \times A \to \mathcal{A}$ by (for all $a, b \in A$),:

$$d(a,b) = \begin{cases} \pi_{|a-b|+|a|} & \text{if } a \neq b \\ 0_{\mathcal{A}} & \text{if } a = b. \end{cases}$$

where $L(H) = \mathcal{A}$, H stand for the set of square integrable functions defined on Δ, and $\pi_a : H \to H$ is the multiplicative operator defined by:

$$\pi_a(\theta) = a.\theta, \quad \text{for all } \theta \in H.$$

Now, we present our following theorem:

Theorem 4.2. *Assume that (for all $a, b \in A$)*

(1) $\exists$ a continuous function $\eta : \Delta \times \Delta \to \mathbb{R}$ and $\lambda \in (0,1)$ such that

$$\mid G(\xi, \omega, a(\omega)) - G(\xi, \omega, b(\omega)) \mid \leq \lambda \mid \eta(\xi, \omega) \mid (\mid a(\omega) - b(\omega) \mid + \mid a(\omega) \mid),$$

$for \ all \ \xi, \omega \in \Delta.$

(2) $\sup_{\xi \in \Delta} \int_{\Delta} \mid \eta(\xi, \omega) \mid d\omega \leq 1.$

Then the integral equation (4.2) has a unique solution in A.

Proof. Define $f : A \to A$ by:

$$fa(\xi) = \int_{\Delta} G(\xi, \omega, a(\omega))d\omega + h(\xi), \ \forall \ \xi, \omega \in \Delta.$$

Set $k = \lambda I$, then $k \in \mathcal{A}$ and $\|k\| = \lambda < 1$. For any point u in H, we have

$$\begin{aligned}
\|d(fa, fb)\| &= \sup_{\|u\|=1} (\pi_{|fa-fb|+|fa|}(u), u) \\
&= \sup_{\|u\|=1} \int_\Delta \left[\left| \int_\Delta G(\xi, \omega, a(\omega)) - G(\xi, \omega, b(\omega)) d\omega \right| \right] u(\xi) u(\bar{\xi}) d\xi \\
&\quad + \sup_{\|u\|=1} \int_\Delta \left(\int_\Delta G(\xi, \omega, a(\omega)) \right) u(\xi) u(\bar{\xi}) d\xi \\
&\leq \sup_{\|u\|=1} \int_\Delta \left[\int_\Delta \big| G(\xi, \omega, a(\omega)) - G(\xi, \omega, b(\omega)) \big| d\omega \right] |u(\xi)|^2 d\xi \\
&\quad + \sup_{\|u\|=1} \int_\Delta \left| \int_\Delta G(\xi, \omega, a(\omega)) \right| |u(\xi)|^2 d\xi \\
&\leq \sup_{\|u\|=1} \int_\Delta \left[\int_\Delta \big| \lambda \eta(\xi, \omega)(a(\omega) - b(\omega) + \mid a(\omega) \mid) \big| d\omega \right] |u(\xi)|^2 d\xi \\
&\leq \sup_{\|u\|=1} \int_\Delta \left[\int_\Delta \big| \lambda \mid \eta(\xi, \omega) \mid (\mid a(\omega) - b(\omega) \mid + \mid a(\omega) \mid) \big| d\omega \right] |u(\xi)|^2 d\xi \\
&\leq \lambda \sup_{\|u\|=1} \int_\Delta \left[\int_\Delta |\eta(\xi, \omega)| d\omega \right] |u(\xi)|^2 d\xi \|a - b\|_\infty \\
&\leq \lambda \sup_{\xi \in E} \int_\Delta |\eta(\xi, \omega)| d\omega \sup_{\|u\|=1} \int_\Delta |u(\xi)|^2 d\xi \|a - b\|_\infty \\
&\leq \lambda \|a - b\|_\infty \\
&= \|k\| \, \|d(a, b)\|.
\end{aligned}$$

Since, $\|k\| < 1$, so one can easily seen that the mapping f satisfies all the assumptions of Theorem 3.1. Hence, (4.2) has a unique solution, means that f has a unique fixed point. ☐

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