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A NOTE ON FINITE $\mathcal{PST}$-GROUPS

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Abstract. A finite group G is said to be a $\mathcal{PST}$-group if, for subgroups H and K of G with H Sylow-permutable in K and K Sylow-permutable in G, it is always the case that H is Sylow-permutable in G. A group G is a $\mathcal{T}^*$-group if, for subgroups H and K of G with H normal in K and K normal in G, it is always the case that H is Sylow-permutable in G. In this paper, we show that finite $\mathcal{PST}$-groups and finite $\mathcal{T}^*$-groups are one and the same. A new characterisation of soluble $\mathcal{PST}$-groups is also presented.

1. Introduction and statement of results

Throughout this paper, all groups considered are finite. A subgroup H of a group G is called *Sylow-permutable* in G, or *S-permutable*, if $HS = SH$ for every Sylow subgroup S of G. Kegel [9] has shown that S-permutable subgroups are subnormal. However there exist subnormal subgroups which are not S-permutable. Robinson [10] called $\mathcal{PST}$-groups the groups in which every subnormal subgroup is S-permutable. From Kegel's result, a group G is a $\mathcal{PST}$-group if and only if S-permutability is a transitive relation in G.

Many papers have studied $\mathcal{PST}$-groups in detail. Agrawal initiated the study in [1] where he characterised the soluble $\mathcal{PST}$-groups as follows:

Theorem 1. *A group G is a soluble $\mathcal{PST}$-group if and only the nilpotent residual D of G is an abelian Hall subgroup of odd order such that G induces power automorphisms in D.*

Robinson, in [10], gave the following characterisation of $\mathcal{PST}$-groups:

Theorem 2 ([10])**.** *A group G is a $\mathcal{PST}$-group if and only if it has a perfect normal subgroup D such that:*

1. *G/D is a soluble $\mathcal{PST}$-group;*
2. *$D/\operatorname{Z}(D) = U_1/\operatorname{Z}(D) \times \cdots \times U_k/\operatorname{Z}(D)$ where $U_i/\operatorname{Z}(D)$ is simple and $U_i \trianglelefteq G$;*

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(3) *if* $\{i_1, i_2, \dots, i_r\} \subseteq \{1, 2, \dots, k\}$, *where* $1 \le r < k$, *then the factor group* $G/U'_{i_1}U'_{i_2}\cdots U'_{i_r}$ *satisfies* N_p *for all* $p \in \pi\bigl(\mathrm{Z}(D)\bigr)$.

Here, a group G satisfies N_p if, for all soluble normal subgroups N, the p'-elements of G induce power automorphisms in $\mathrm{O}_p(G/N)$.

On the other hand, Asaad and Csörgő defined in [4] $\mathcal{T}^*$*-groups* as the groups G such that if H is a normal subgroup of K and K is a normal subgroup of G, then H is S-permutable in G. In other words, a group G is a $\mathcal{T}^*$-group whenever every subnormal subgroup of G of defect at most 2 is S-permutable in G. The proofs of most results of this paper seem to use the requirement that all subnormal subgroups of a $\mathcal{T}^*$-group are S-permutable, as in $\mathcal{PST}$-groups, without explicitly stating the equivalence between the concepts of $\mathcal{T}^*$-group and $\mathcal{PST}$-group. Therefore, in order to check the validity of the proofs of [4], it is necessary to show whether $\mathcal{PST}$-groups can be characterised as the groups in which every subnormal subgroup of defect at most 2 is S-permutable. Our first main result shows that this question has an affirmative answer.

Robinson established in [10] that $\mathcal{PST}$-groups are $\mathcal{SC}$-groups, that is, groups whose chief factors are all simple. With little alteration of Robinson's proof of that result, one can arrive at the same conclusion for $\mathcal{T}^*$-groups.

Lemma 3. *A* $\mathcal{T}^*$*-group is an* $\mathcal{SC}$*-group.*

$\mathcal{SC}$-groups are also characterised by Robinson in [10].

Theorem 4. *A group* G *is an* $\mathcal{SC}$*-group if and only if there is a perfect normal subgroup* D *such that* G/D *is supersoluble,* $D/\mathrm{Z}(D)$ *is a direct product of* G*-invariant simple groups, and* $\mathrm{Z}(D)$ *is supersolubly embedded in* G *(i.e., there is a* G*-admissible series of* $\mathrm{Z}(D)$ *with cyclic factors).*

A $\mathcal{U}_p^*$-group is defined in [2] to be a p-supersoluble group G in which all p-chief factors are G-isomorphic when regarded as modules over G. In [2, Corollary 3], the following characterisation of soluble $\mathcal{PST}$-groups is given.

Theorem 5. G *is a soluble* $\mathcal{PST}$*-group if and only if* G *satisfies* $\mathcal{U}_p^*$ *for all primes* p.

Our first main result shows that $\mathcal{PST} = \mathcal{T}^*$:

Theorem A. G *is a* $\mathcal{T}^*$*-group if and only if* G *is a* $\mathcal{PST}$*-group.*

In [3, Theorem 3.1], Asaad proved that a group G is a soluble $\mathcal{T}$-group if and only if for all primes p dividing the order of $\mathrm{F}^*(G)$, the generalised Fitting subgroup of G, every p-subgroup of G is pronormal in G. As a consequence, he proved that a group G is a soluble $\mathcal{T}$-group if and only if for all primes p dividing the order of $\mathrm{F}^*(G)$, G satisfies property $\mathcal{C}_p$, that is, every subgroup of a Sylow p-subgroup P of G is normal in $\mathrm{N}_G(P)$ ([3, Corollary 3.2]). He extended this result to permutability by showing that a group G is a soluble $\mathcal{PT}$-group if and only if G satisfies $\mathcal{X}_p$ for all primes p dividing the order of $\mathrm{F}^*(G)$. Here a group G satisfies $\mathcal{X}_p$ when every subgroup of a Sylow p-subgroup P of G is permutable in $\mathrm{N}_G(P)$. This property was introduced and studied in [7].

The $\mathcal{PST}$-version of the properties $\mathcal{C}_p$ and $\mathcal{X}_p$ is the property $\mathcal{Y}_p$ introduced in [5]. Recall that a group G satisfies $\mathcal{Y}_p$ if whenever H and K are p-subgroups of G such that $H \leq K$, then H is S-permutable in $\mathrm{N}_G(K)$. In [5, Theorem 4], it is proved that a group G is a soluble $\mathcal{PST}$-group if and only if G satisfies $\mathcal{Y}_p$ for all primes p. Asaad's results admit the following generalisation to $\mathcal{PST}$-groups:

Theorem B. *A group G is a soluble $\mathcal{PST}$-group if and only if G satisfies $\mathcal{Y}_p$ for all primes p dividing the order of $\mathrm{F}^*(G)$.*

Unlike previous characterisations of soluble $\mathcal{PST}$-groups, this one does not follow quickly from the classification of minimal non-$\mathcal{PST}$-groups given by Robinson in [11].

2. Proofs

Proof of Theorem A. Only the necessity of the condition is in doubt. We assume that it does not hold and derive a contradiction. Let G be a group of minimal order such that G is a $\mathcal{T}^*$-group but G is not a $\mathcal{PST}$-group. An argument similar to the one used in [1] to show that quotients of $\mathcal{PST}$-groups are $\mathcal{PST}$-groups shows that all quotient groups of G are $\mathcal{T}^*$-groups. Therefore, by minimality of G, we have that every proper quotient group of G is a $\mathcal{PST}$-group. Applying Lemma 3, G is an $\mathcal{SC}$-group. Thus, from Theorem 4, we have that G has a normal perfect subgroup D such that $D/\mathrm{Z}(D) = U_1/\mathrm{Z}(D) \times \cdots \times U_k/\mathrm{Z}(D)$, with all $U_i/\mathrm{Z}(D)$ simple, and $\mathrm{Z}(D)$ is supersolubly embedded in G.

Assume that $D \neq 1$, i.e., G is not soluble. Then G/D is a soluble $\mathcal{PST}$-group. Since $U_i/\mathrm{Z}(D)$ is simple for all i, we have that $U_i' \neq 1$ for all i. Therefore if $\{i_1, i_2, \dots, i_r\} \subseteq \{1, 2, \dots, k\}$, with $r < k$, we have that G/U_{i_j}' is a $\mathcal{PST}$-group and so $G/U_{i_1}'U_{i_2}' \dots U_{i_r}'$ satisfies N_p for all primes p. Theorem 2 implies that G is a $\mathcal{PST}$-group, contrary to assumption. Therefore $D = 1$ and G is soluble. Since all chief factors of G are simple, we have that G is supersoluble. Let p be the largest prime dividing the order of G. Then G has a normal Sylow p-subgroup, P say. Moreover, G/P is a $\mathcal{PST}$-group by the choice of G. Hence G/P satisfies $\mathcal{U}_q^*$ for all primes $q \neq p$ by Theorem 5. This implies that G satisfies $\mathcal{U}_q^*$ for for all primes $q \neq p$. Since G is not a $\mathcal{PST}$-group, it follows that G does not satisfy $\mathcal{U}_p^*$.

Suppose that $\mathrm{O}_{p'}(G) \neq 1$. Then $G/\mathrm{O}_{p'}(G)$ is a soluble $\mathcal{PST}$-group. Therefore $G/\mathrm{O}_{p'}(G)$ satisfies $\mathcal{U}_p^*$ by Theorem 5, and so G satisfies $\mathcal{U}_p^*$. This is a contradiction. Consequently $\mathrm{O}_{p'}(G) = 1$. Assume that G has two different minimal normal subgroups N_1 and N_2. Both of them have order p, and G/N_1 and G/N_2 satisfy $\mathcal{U}_p^*$. If N_1N_2 is a proper subgroup of P, then by considering all chief factors of G between N_1 and N_1N_2, between N_2 and N_1N_2, and between N_1N_2 and P, we obtain that G satisfies $\mathcal{U}_p^*$. This contradiction shows that $P = N_1N_2$. Note that $P = N_1 \times N_2$ is abelian. If D is a subgroup of P, then D is normal in P and D is S-permutable in G; hence D is normalised by all p'-elements of G and D is normal in G. Thus elements of G induce power automorphisms in P, from which it follows that G satisfies $\mathcal{U}_p^*$, contrary to the choice of G.

Hence G has a unique minimal normal subgroup N, which is contained in P, and G/N is a $\mathcal{PST}$-group. Moreover, $P = \mathrm{O}_p(G) = \mathrm{F}(G)$. If N is not contained in the Frattini subgroup $\Phi(G)$ of G, then G is a primitive group and so $N = \mathrm{F}(G)$ has order p. In particular, G satisfies $\mathcal{U}_p^*$. This contradiction yields $N \leq \Phi(G)$. If G/N is p-nilpotent, then G is p-nilpotent and so G satisfies $\mathcal{U}_p^*$. This is not possible. Consequently, G/N is not nilpotent. Since G/N is a $\mathcal{PST}$-group, the nilpotent residual R/N of G/N is an abelian Hall subgroup of G/N and all elements of G induce power automorphisms on R/N. Moreover, N is the unique minimal normal subgroup of G. In particular, R/N is a p-group and so $P/N = R/N$. In particular, p'-elements of G/N induce power automorphisms on P/N. Let S be a subgroup of P. Then SN is normal in G because P/N is abelian and $\mathrm{O}^p(G/N)$ normalises SN/N. In addition, since either $SN = S$ or S is a maximal subgroup of the p-group SN, we have that S is a normal subgroup of SN. Since G is a $\mathcal{T}^*$-group, S is S-permutable in G. Then all p'-elements of G normalise S and so induce power automorphisms in P. Hence G satisfies $\mathcal{U}_p^*$. This is the final contradiction. □

The following lemma is needed in the proof of Theorem B.

Lemma 6. *Let p be a prime and let M be a normal p'-subgroup of a group G. Then G satisfies $\mathcal{Y}_p$ if and only if G/M satisfies $\mathcal{Y}_p$.*

Proof. Let p be a prime and let M be a normal p'-subgroup of a group G. By [5, Lemma 2], we have that if G satisfies $\mathcal{Y}_p$, then G/M satisfies $\mathcal{Y}_p$. Conversely, assume that G/M satisfies $\mathcal{Y}_p$. By [5, Theorem 5], we have that either G/M is p-nilpotent, or G/M has abelian Sylow p-subgroups and G/M satisfies $\mathcal{C}_p$. In the first case, we have that G is p-nilpotent and so G satisfies $\mathcal{Y}_p$ by [5, Theorem 5]. Assume that G/M has abelian Sylow p-subgroups and satisfies $\mathcal{C}_p$. Let P be a Sylow p-subgroup of G. Consider a subgroup H of P, and $g \in \mathrm{N}_G(P)$. We have that $H^gM = HM$ because HM/M is normalised by $gM \in G/M$. Therefore $H^g = H^gM \cap P = HM \cap P = H$. This implies that G satisfies $\mathcal{C}_p$ and so G satisfies $\mathcal{Y}_p$ by [5, Theorem 5]. □

Proof of Theorem B. If G is a soluble $\mathcal{PST}$-group, we can apply [5, Theorem 4] to conclude that G satisfies $\mathcal{Y}_p$ for all primes p. Let G be a group satisfying $\mathcal{Y}_p$ for all primes p dividing the order of $\mathrm{F}^*(G)$. We shall prove that G is a soluble $\mathcal{PST}$-group by induction on $|G|$. By [5, Theorem 4], we can suppose that $\mathrm{F}^*(G)$ is a proper subgroup of G. Note that the class $\mathcal{Y}_p$ is subgroup-closed for all primes p. Hence $\mathrm{F}^*(G)$ satisfies $\mathcal{Y}_p$ for all primes p. Applying [5, Theorem 4], we have that $\mathrm{F}^*(G)$ is soluble. Therefore $1 \neq \mathrm{F}^*(G) = \mathrm{F}(G)$ by [8, X, 13].

Suppose that there exists a prime p dividing $|\mathrm{F}^*(G)|$ such that a Sylow p-subgroup P of G is not abelian. In this case, G is p-nilpotent by [5, Theorem 5]. Moreover, since $\mathrm{F}^*\big(\mathrm{O}_{p'}(G)\big)$ is contained in $\mathrm{F}^*(G)$, we have that $\mathrm{O}_{p'}(G)$ is a soluble $\mathcal{PST}$-group by induction. This implies that G is soluble. Let N be a minimal normal subgroup of G contained in $\mathrm{O}_p(G)$. Since $1 \neq N \cap \mathrm{Z}(P)$ is contained in the centre of G, we have that $N \cap \mathrm{Z}(P) = N$. Thus $\mathrm{F}(G/N) = \mathrm{F}(G)/N$. Consequently G/N satisfies $\mathcal{Y}_p$ for all primes p dividing $|\mathrm{F}^*(G/N)|$. Hence G/N is a soluble $\mathcal{PST}$-group by induction and so G/N satisfies $\mathcal{Y}_q$ for all primes q dividing $|G/N|$ by [5, Theorem 4]. By Lemma 6, G satisfies $\mathcal{Y}_q$ for all primes $q \neq p$. Since G satisfies $\mathcal{Y}_p$ by hypothesis, it follows that G satisfies $\mathcal{Y}_p$ for all primes p and so G is a soluble $\mathcal{PST}$-group by [5, Theorem 4].

Therefore we can assume, by [5, Theorem 5], that for every prime p dividing $|\mathrm{F}^*(G)|$, G has an abelian Sylow p-subgroup P and G satisfies $\mathcal{C}_p$. In this case, every cyclic subgroup of p-power order of $\mathrm{F}(G)$ is normal in G, because G satisfies $\mathcal{C}_p$, and so centralised by G'. Hence G' is contained in $\mathrm{C}_G\big(\mathrm{F}(G)\big)$, which is contained in $\mathrm{F}(G)$ by [8, X, 13]. Thus G' is abelian and so G is soluble.

Let q be a prime. If q divides $|G'|$, then q divides $|\mathrm{F}(G)|$ and so G satisfies $\mathcal{Y}_q$ by hypothesis. Suppose that q does not divide $|G'|$. Consider a q-subgroup H of G. We have that HG' is a normal subgroup of G and so every Sylow subgroup of HG' is pronormal in G. Hence H is pronormal in G. According to [6, Lemma 2], G satisfies $\mathcal{C}_q$ and so G satisfies $\mathcal{Y}_q$ by [5, Theorem 3]. Consequently, G is a soluble $\mathcal{PST}$-group. □

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