APPLICATION OF THE FINITE ELEMENT METHOD TO THE UNDERSTANDING OF A NON-STEADY STATE HEAT TRANSPORT PROBLEM

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Abstract

Nowadays, the search for new active and constructive learning methodologies is one of the pedagogical objectives in universities and other educational institutions. In the degrees taught at the Escola Tècnica Superior d'Enginyeria Agronòmica i del Medi Natural (ETSEAMN), the phenomena associated with heat transport and the underlying physics are studied. In the master classes, within the framework of food science and technology subjects, the phenomenon of this energy balance in a one-dimensional system is explained. In the present work, the analytical solution is developed and evaluated, and a numerical model achieving the same solution is shown. One of the advantages of using numerical tools is that more complex solutions can be obtained, even if the corresponding analytical solution does not exist or is not known, which is useful for engineering students. In this work, it is shown that changing the boundary conditions, geometry, or dimension in the system and the mathematical model, and solving it by applying a numerical solution method is easier and more comprehensive for students rather than facing the complexity of the analytical solutions. To demonstrate the applications and possibilities of this approach, a simple way of modifying the degrees of freedom of the same problem applied in the case of a vegetable subjected to a heat treatment is shown. Thus, the students can understand how these modifications affect the solution and comprehend that the dimension of some problems can be simplified in order to contextualize them. The development of PoliformaT, the e-learning platform implemented in the Universitat Politècnica de València facilitates the use of these new teaching models that combine traditional on-site laboratory tasks assignments with other learning assignments carried out autonomously online by students.

Keywords: finite elements method, active learning, meaningful learning, heat transfer, teaching methodologies.

1 INTRODUCTION

Nowadays, thermodynamics and heat transfer problems contextualized to different cases are used for students to implement them in their future professional careers. Heat transfer problems are applied in various fields of physics, both in engineering and science, including in industrial applications [1], water heating or food treatment [2], automotive [3], electronic systems [4], design of hot-cold systems [5], and so on.

Following the zero principle of thermodynamics [6], a heat transfer process occurs when two systems that do not have the same temperature come into contact. As long as more elements with different temperatures are added to the system, the energy transfer process will continue to occur until these surfaces reach thermal equilibrium. The mechanisms of heat transfer are radiation, convection, and conduction. In this work, a problem that contains the mechanism of conduction and convection is proposed. These mechanisms will be studied for a non-steady state case for which energy variations occur in time because the temperature changes, unlike in the steady state case. It is known that analytical solutions are complex for studying non-steady states, but the understanding of the physical laws that govern them is not. Therefore, it is easy for the student to understand the concept of heat transport, because it is simple, but solving problems can present difficulties for her/him.

In the first part of this work, the heat transfer phenomenon is presented showing the equations and physical laws that govern it. The analytical resolution of the mathematical expressions is also shown, offering, in addition, a numerical resolution [7]. This definition is important since it allows to demonstrate that by means of numerical tools, the solution to an analytical problem can be obtained. Considering and knowing that the complexity of the solution increases as we increase the degrees of freedom of the problem (such as the spatial dimensions), we propose a problem in which the student can visualize the different solutions by adding all those degrees of freedom that he/she considers. For example, the initial or boundary conditions of the problem can be changed in a simple way, even if there is no analytical solution to the problem. Using simulation tools by means of the finite element method, these solutions can be obtained and the evolution of the temperature over time in the system can be visualized.

Teaching through these numerical tools also allows to improve the student's computational skills. These tools can be used in other subjects in which they can apply the knowledge of numerical problem definition to solve other physical phenomena with different conditions and degrees of freedom. All this is designed for engineering students of the degrees offered by the «Escuela Técnica Superior del Medio Agronómico y Natural» in Universitat Politècnica de València. These students need to solve problems of thermodynamics, mechanics, electromagnetism, among others. In this work, we present a problem of heat transmission in a non-stationary state in a fruit or vegetable that must be treated by a thermal process for its preservation and consumption.

2 METHODOLOGY

The practical case of food preservation is an excellent example of a non-steady state heat transfer problem. Thus, we pose the following question: Consider that we want to produce slices of a vegetable for its use in salads. In this case, we will choose carrots to be able to assign a numerical value to the characteristic parameters of the material. The manufacturing procedure consists of immersing the carrot slices in a solution of different compounds at 25°C to prevent browning. In addition, the slices must be kept at a temperature of 1°C. For this reason, after the previous treatment, they are placed in another bath with other preservatives at a temperature of -1ºC. The slices will remain in this second bath until all their points reach a temperature less than or equal to the desired temperature (1ºC), so we will have to determine how long they should remain there. We will consider that the slices are cylindrical in shape with a diameter of 2.5 cm and a thickness (2L) of 1.5 mm. The thermal diffusivity (α) of the carrot is 0.002 cm^2/s [8], and we will consider that the convective heat transfer coefficient is very high, *i.e.,* the external resistance to convective heat transfer is negligible.

2.1 Heat conduction equations and analytical solution

This first part of the paper summarizes the physical laws governing heat conduction and its energy balance for the one-dimensional case. To simplify the case, the system is defined as an infinite sheet (Figure 1a). By means of the superposition principle, the cases for higher spatial dimensions could be solved. For the resolution, it is considered that the sum of the Input and the Generation is equal to the Output plus the Accumulation; the initial temperature is uniform throughout the system; the heat flow by conduction that reaches the surface of the system is exchanged by convection with the surrounding fluid (Figure 1b); there is a symmetrical temperature distribution with respect to the central axis; the temperature surrounding the system does not change with time, the material does not shrink, and the physical properties remain constant throughout the process.

Considering the previous statements, the general heat conduction equation is defined as follows:

$$q_{x+\delta x} - q_x + \frac{\partial\left(\rho C_p T dV\right)}{\partial t} = g dV \quad (1)$$

Applying the conditions of the proposed problem and making a change of variable, it can be reformulated as shown:

$$\psi(x,t) = \frac{T(x,t) - T_\infty}{T_o - T_\infty} \quad (2)$$

The boundary conditions and constraints of the problem result in the following expressions:

$$\alpha \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial \psi}{\partial t} \quad (3)$$

$$\psi(x,0) = 1 \quad (4)$$

$$-k\frac{\partial\psi(L,t)}{\partial x} = h\,\psi(L,t) \tag{5}$$

$$\frac{\partial\psi(0,t)}{\partial x} = 0 \tag{6}$$

This model fulfils the conditions for applying the method of separation of variables (all the equations, except one, are homogeneous and of constant coefficients),

$$\psi(x,t) = F(x)G(t) \rightarrow \frac{1}{F(x)}\frac{d^2F(x)}{dx^2} = \frac{1}{G(t)}\frac{dG(t)}{dt} = -\omega^2 \tag{7}$$

The equation $G(t)$ is a first order equation and $F(x)$ is a second order equation, so integrating them results in:

$$\begin{cases} G(t) = C_1 e^{-\alpha\omega^2 t} \\ F(x) = C_2\,sin(\omega x) + C_3\,cos(\omega x) \end{cases} \tag{8}$$

Now, substituting equation (8) into equation (7), we obtain:

$$\psi(x,t) = e^{-\alpha\omega^2 t}[A\,sin(\omega x) + B\,cos(\omega x)] \tag{9}$$

where A and B are integration constants obtained from the boundary conditions. Imposing the symmetry boundary condition on equation we obtain,

$$\frac{\partial\psi(0,t)}{\partial x} = \left\{e^{-\alpha\omega^2 t}[A\omega\,cos(\omega x) - B\omega\,sin\,(\omega x)]\right\}_{x=0} = 0 \tag{10}$$

Then, A = 0 is obtained and equation (10) is rewritten as

$$\psi(x,t) = e^{-\alpha\omega^2 t}[\ B\,cos(\omega x)] \tag{11}$$

The conduction heat flux that arrives to the surface of the system is interchanged by convection with the fluid that around the system (Figure 1b). Therefore, it could be stated that:

$$\frac{\partial\psi(L,t)}{\partial x} = \left\{e^{-\alpha\omega^2 t}[\ -B\omega\,sin(\omega x)]\right\}_{x=L} = -\frac{h}{k}e^{-\alpha\omega^2 t}[\ B\,cos(\omega L)] \tag{12a}$$

$$or\ \psi(L,t) = e^{-\alpha\omega^2 t}[\ B\,cos(\omega x)] = 0, \tag{12b}$$

Where h, is the convective heat transfer coefficient. If h is very high, which it turns out to be:

$$tan(\omega L) = \frac{hL}{k\omega L} = \frac{N_{Bi}}{\omega L} \tag{13}$$

Where N_{Bi}, is the Biot number and is equal to hL/k. The eigenvalues ω_n are the positive roots of equation (13).

$$\omega_n L = f(N_{Bi});\quad n = 1,2,\ldots \tag{14}$$

Therefore, the function $\psi(x,t)$ will be a linear combination of all possible solutions:

$$\psi(x,t) = \sum_{n=1}^{\infty} B_n e^{-\alpha\omega_n^2 t}\ cos(\omega_n x) \tag{15}$$

To solve for the coefficients B_n, the initial condition is applied,

$$\psi(x,0) = \sum_{n=1}^{\infty} B_n \; cos(\omega_n x) = 1 \qquad (16)$$

And, by orthogonality of the eigenfunctions:

$$\int_0^L \cos(\omega_n x) \cos\,(\omega_m x)\text{dx} == \begin{cases} 0 \; if \; n \neq m \\ \frac{L}{2} + \frac{1}{4\omega_n}\text{sen}(2\omega_n L) \; if \; n = m \end{cases} \qquad (17)$$

At this point, introducing $\cos\,(\omega_m x)$ is on both sides of equation (17) and integrating gives (18)*,* where the eigenvalues ω_n are the positive roots of the transcendental equation *when h is very high:*

$$B_n\left(\frac{L}{2} + \frac{1}{4\omega_n}\sin(2\omega_n L)\right) = \frac{1}{\omega_n}\text{sen}(\omega_n L) \rightarrow B_n = \frac{2\,\text{sen}(\omega_n L)}{\omega_n L + \text{sen}(\omega_n L)\cos(\omega_n L)} = \frac{2(-1)^{n-1}}{\omega_n L} \qquad (18)$$

Finally, the result of the equation is*:*

$$\psi(x,t) = 2\sum_{n=1}^{\infty} \frac{sen(\omega_n L)}{\omega_n L + sen(\omega_n L)\,cos(\omega_n L)} e^{-\alpha\omega_n^2\,t}\,cos\;\;(\omega_n x) \qquad (19a)$$

where ω_n *is* $tan(\omega_n L) = \frac{N_{Bi}}{\omega_n L}$*, or*

$$\psi(x,t) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\omega_n L} e^{-\alpha\omega_n^2\,t}\;cos(\omega_n x) \qquad (19b)$$

with $\omega_n L = (2\text{n} - 1)\frac{\pi}{2}$, when h is very high.

2.2 Finite Elements Method modelling

One of the most widely used methods for simulations is the finite element method (FEM). It is a flexible method that can be used in multiple areas of science and engineering [9]. Using the FEM, it is possible to generate numerical solutions for problems with different complexity. To do so, it is necessary to know the initial and boundary conditions of the problem to be solved. The commercial software COMSOL Multiphysics allows the application of FEM, working with problems of various dimensions and degrees of freedom. To simulate a non-steady-state heat problem, the geometry (1D, 2D or 3D) must first be defined. To reduce the degrees of freedom, and therefore the computational cost, it is important to start with simple problem approaches that allow obtaining valid solutions. Once the geometry of the problem has been defined, when selecting the module (in this case the one containing the heat equation), the physical conditions of the problem are defined (boundary and initial conditions). Now, once the physics of the problem has been configured, the mesh size and type are selected. The mesh generates a finite number of points called nodes where the equations with partial derivatives will be solved to obtain the numerical solution of the problem. In the 1D case, the mesh is formed by several points. When the problems become two-dimensional, different geometries can be used, such as triangular geometry. The number of nodes to be solved must meet certain conditions for the solution to be convergent and robust; it needs a minimum number of elements that limits the maximum element size. In short, it is a compromise between the number of points and the computational cost (the higher the number of points, the higher the computational cost). Once the mesh type and the number of elements is established, the solver is defined. For non-steady state problems, a temporary solver is used. Therefore, both the time range and the time steps must be configured. It is important to define a suitable time step in order not to increase the computational cost excessively and, at the same time, to obtain a good time solution that allows to see and understand the physics of the problem.

3 RESULTS

Given that the diameter of the slice is much larger than its thickness, the heat that the slice exchanges with the water through its peripheral surface could be neglected with respect to the heat it exchanges through its other surfaces. Therefore, the problem is simplified to a one-way heat transfer. Due to the high convective heat transfer coefficient (h) and thus a high value of NBi, and with the objective of determining the time required for the center of the carrot slice to reach 1ºC, equation (19b) has been used as follows:

$$t/\psi(0,t) = \frac{\mathbf{1-(-1)}}{\mathbf{25-(-1)}} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\omega_n L}e^{-\alpha\omega^2 t} \qquad (20)$$

The time required for the center of the carrot slice to reach 1°C is obtained by iterating equation (20). To determine an initial value of $\boldsymbol{t}$, it has been assumed that it is sufficient to consider only the first term:

$$\frac{\mathbf{1}}{\mathbf{13}} = \frac{\mathbf{4}}{\boldsymbol{\pi}}\boldsymbol{e}^{-\mathbf{0.002}\left(\frac{\boldsymbol{\pi}}{\mathbf{0.15}}\right)^2 \boldsymbol{t}}$$

$$\boldsymbol{t = 3.2s}$$

Considering that the Fourier number at the calculated time ($\boldsymbol{Fo} = \frac{\propto \boldsymbol{t}}{\mathbf{L}^2} = \mathbf{1.138}$) is higher than 0.2 [3], the error produced by considering the first term of the series, and disregarding all the others, is less than 2%, which is a margin more than valid to perform the calculations. Therefore, the result will be:

$$\boldsymbol{t = 3.2s}$$

Once the analytical solution has been obtained, we proceed to obtain the numerical solutions. Given the type of problem, a one-dimensional system is proposed to obtain the solution initially. First, it is necessary to define the geometry of the problem. In this case, it is a line of length equal to half the thickness of the carrot. The next step is to define the physics of the problem. Using the heat modulus, the initial and boundary conditions must be established. In this regard, the initial condition for all points on the initial temperature line is the proposed one (25°C). The boundary conditions are defined as follows: the heat equation must be set with a diffusion coefficient of $2 \cdot 10^{-7}$ W/(mK); the symmetry at the 0 coordinate of the straight line is acting as the center of the carrot; the Dirichlet condition at the other end of the straight line behaving as a fixed temperature value, in this case -1°C. As for the meshing, several equidistant and separated nodes along the 25 μm line is selected. Finally, to obtain the results of Figure 2, a step of 0.1 seconds between 0 and 8 seconds has been selected in the solver.

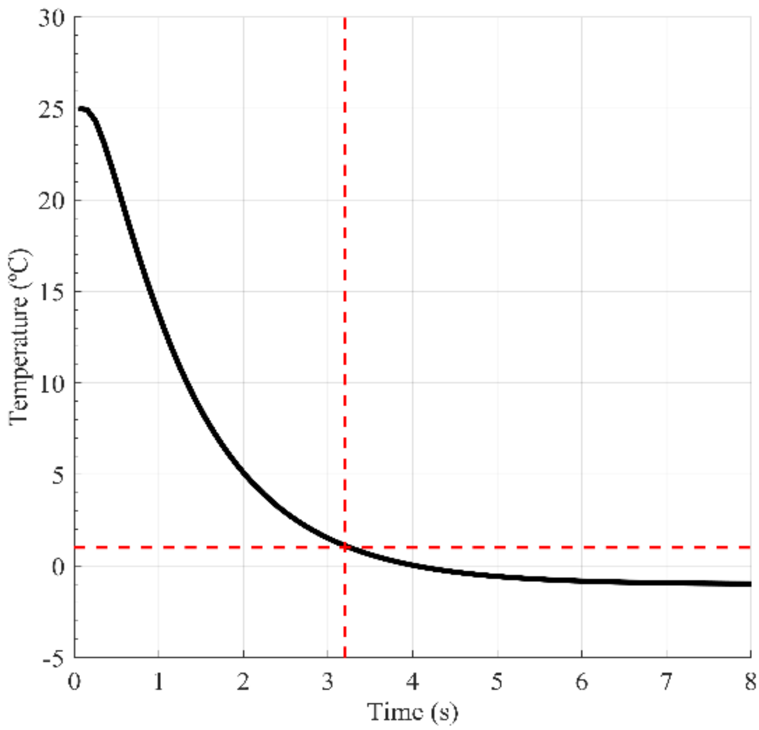

Figure 2. Numerical solution for the 1D FEM model T(t).

Figure 2 shows the solution of the first one-dimensional model. In this case, the temperature is shown as a function of time, and it can be stated that the solution is consistent in obtaining the value of t = 3.2 seconds to reach the target temperature. Therefore, the analytical and numerical results are in good agreement and, therefore, the model is validated. One of the advantages of the FEM is that the degrees of freedom can be easily increased. The solution for a two-dimensional model is shown below. In this case, to reduce the degrees of freedom, a simplification using the geometry of the carrot slice is used. The boundary conditions remain the same as in the previous model, except that a single point on the line is no longer defined, but a contour in each zone that satisfies the problem conditions. Figure 3 shows the 2D solution of the proposed problem for time t = 3.2 seconds. Solutions of how the heat is distributed at each instant of time can be obtained.

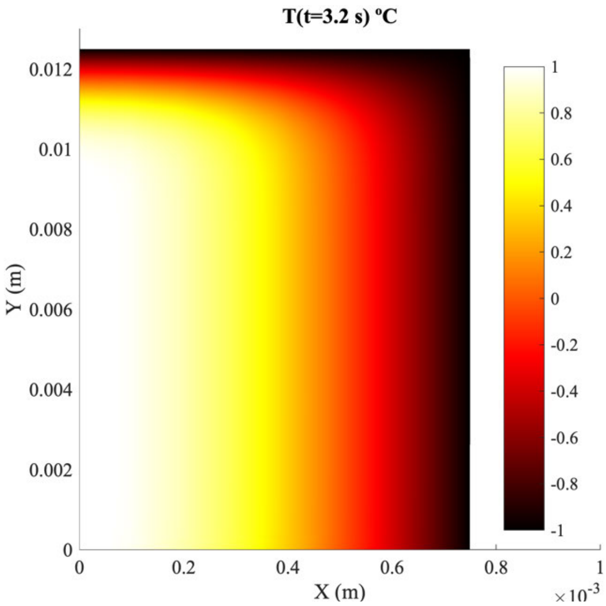

Figure. 3 Numerical solutions using the two-dimensional FEM for time t = 3.2 seconds.

Finally, we want to show the possibility of solving the same problem in a three-dimensional way. In this case, we must take advantage of the geometrical characteristics of the carrot, for example, using its revolution symmetry properties. Therefore, it can be stated that a cross section can be reproduced by generating a revolution. That is, it has a 2D-axisymmetric geometry. To define the proposed model, a rectangle has been prepared which, when rotated 360°, generates a carrot slice. In this case, Figure 4 shows a two-dimensional XY slice for the solution time.

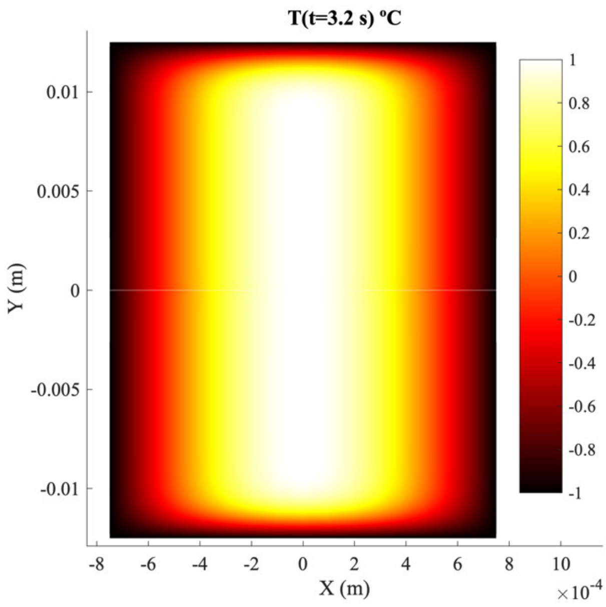

Figure. 4 Numerical solutions generating a 3D solution using a 2D axis FEM model for time t = 3.2 seconds.

Once these results have been completed and the numerical models have been validated, further simulations can be carried out by simply changing the boundary conditions. For example, the initial temperature could be modified. By changing one variable or performing a parametric simulation, the student can obtain all the solutions. Table 1 shows the results by modifying T_0. The solutions shown are consistent, because the higher the initial temperature, the longer the cooling time.

Table 1. Numerical solutions for different initial temperatures (T_0).

Temperature (°C)	*Time (s)*
20	3.00
25	3.20
30	3.45
35	3.60
40	3.75
45	3.85

Another possible option, for example, is to modify the thickness of the vegetable slice. By modifying a geometrical parameter in a simple way, the solutions can be recalculated and the values in Table 2 can be obtained.

Table 2. Numerical solutions for different thicknesses.

Thickness (mm)	*Time (s)*
1.5	3.20
2.0	5.70
2.5	8.90

In this way, the student can see which parameters influence most the problem. For example, if we consider the values in Table 1 and 2, we can state that the most influential parameter is the thickness. Not as much as the initial temperature.

4 CONCLUSIONS

A non-steady state heat transport problem has been proposed in order to understand a very important phenomenon in the field of engineering and science. The underlying physics has been explained for the case of an infinite sheet. A proposed problem related to this type of physics has been solved analytically. Using FEM, we have solved the same problem, both one-dimensionally, 2D and 3D, obtaining a consistent solution that validates the model. One of the advantages of using FEM is that it is possible to start from a simple model explained theoretically, as proposed in this work. Subsequently, 2D or 3D models can be generated, whose solution is analytically more complicated to demonstrate and validate. For a student, it is a very complete way to see the time evolution of such a problem and to understand the physical behavior of the problem. In addition, the initial and boundary conditions can be added or changed, being the use of FEM a procedure that allows, at an academic level, the understanding of the physical phenomena involved in a problem of these characteristics. Digital tools such as PoliformaT provided by the Universitat Politècnica de València, allow teachers to upload example numerical models, videos and other tools and resources. Therefore, these tools and our work can be implemented for teaching physical phenomena in a dynamic way.

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