Stress and strain mapping tensors and general work-conjugacy in large strain continuum mechanics

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Abstract

In this paper we show that mapping tensors may be constructed to transform any arbitrary strain measure in any other strain measure. We present the mapping tensors for many usual strain measures in the Seth-Hill family and also for general, user-defined ones. These mapping tensors may also be used to transform their work-conjugate stress measures. These transformations are merely geometric transformations obtained from the deformation gradient and, hence, are valid regardless of any constitutive equation employed for the solid. Then, advantage of this fact may be taken in order to simplify the form of constitutive equations and their numerical implementation and thereafter, perform the proper geometric mappings to convert the results –stresses, strains and constitutive tangents– to usually employed measures and to user-selectable ones for input and output. We herein provide the necessary transformations. Examples are the transformation of small strains formulations and algorithms to large deformations using logarithmic strains.

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1. Introduction

Whereas in small strain continuum mechanics there is no debate about which ones are the stress and strain measures to be used in constitutive equations, at large strains the options are multiple. Regarding large strains, the Seth-Hill [1][2] family of strain measures (see also the previous work [3]) are typically used, although some other deformation measures are being proposed [4]. Different authors have different preferences over the strain measures. For example, in large strain hyperelasticity it is typical to use the Cauchy-Green deformation tensor (see for example [5][6][7]), or alternatively the Green-Lagrange strain tensor. Deformation invariants used in anisotropic hyperelasticity are almost always defined from the Cauchy-Green deformation tensor [5]. The reason for this choice is that the Cauchy-Green deformation tensor and the Green-Lagrange strain tensor are directly obtained from the deformation gradient and the latter from the gradient of the displacements. Hence, they are naturally included in the Updated Lagrangian and Total Lagrangian formulations in finite element codes [8][9]. Logarithmic strains are also a good choice not only for hyperelasticity [10][11][12] and visco-hyperelasticity [13][14][15], but specially for plasticity [16][17][18][19][20][21][22]. It has been shown that a linear relation between logarithmic strains and Kirchhoff stresses yield a rather accurate prediction of the behavior of some metals and polymers [23, 24]. Furthermore, the use of a quadratic hyperelastic energy function of the logarithmic strains and an exponential integration allows for simple, yet accurate stress integration algorithms in large strain elasto-plasticity, where a small strain integration is employed teamed with geometric pre- and postprocessors [17][21][22]. Logarithmic strains have arguably also a more intuitive and meaningful interpretation, not only for uniaxial loading but also for shear terms [25][26].

However, one of the issues usually not well treated in the literature and, hence, which yields some misunderstandings is the fact that the choice of one strain measure over another is essentially a matter of tradition and can be also a matter of convenience. Furthermore, stresses and strains for user input and output should be selectable by the user, independently of the material model being employed. One of the purposes of this paper is to show that any strain measure may be directly related to any other strain measure and then, the proper work-conjugate stress measure must be employed, which remarkably transforms using equivalent relations. Furthermore, generalized strain measures, not only the Seth-Hill bundle [1][2], may be used if they are more convenient for the purpose, for example in order to possibly establish linear constitutive relations between stresses and strains as, for example in [16][17][18][19][20][21][22] and in [4] in a more general context. Then, the transformation from any strain measure (for example the deformation gradient or the Green-Lagrange strain) to the generalized one is simply performed using the proper mapping tensor which we also introduce. In a similar way, the transformation of the resulting generalized stress measure to Cauchy or Piola stresses, or the resulting constitutive tangent, may also be performed using similar mapping tensors. An important point is that these transformations are valid regardless of the constitutive equations for the material and of the material symmetries. In fact, we remark that no constitutive equation will be used throughout the paper except in the examples. In essence, they can be considered as deformation measures in locally transformed bodies. Invariants for constitutive equations may also be defined using these generalized strain measures.

In the following section of the paper we depart from the stress power to establish power conjugacy from scratch. Then we introduce the stress and strain mapping tensors for most of the typically used strain and their work-conjugate stress measures. Finally we introduce generalized strain measures, their work-conjugate stress measures and the mapping between two arbitrary sets. We further derive the transformations for general constitutive equations from any stress/strain couple to any other one. We will assume a Cartesian representation to simplify the exposition, but of course the results are valid regardless the system of representation employed.

2. The stress power and work-conjugacy

Assume we have a body with an original volume ${}^0V$ and a deformed volume ${}^tV$, surrounded respectively by ${}^0S$ and ${}^tS$. A point representing an infinitesimal volume is denoted in the reference volume by ${}^0\boldsymbol{x}$, and in the current volume by

$$ {}^t\boldsymbol{x} = {}^0\boldsymbol{x} + {}^t\boldsymbol{u} \tag{1} $$

where ${}^t\boldsymbol{u}$ are the displacements. The body forces per unit current volume at time t are $\boldsymbol{b}$ and the surface ones (per unit current surface) are $\boldsymbol{t}$. Then by equilibrium of forces

$$ \int_{{}^tV} \boldsymbol{b} \; d\,{}^tV + \int_{{}^tS} \boldsymbol{t} \; d\,{}^tS = \boldsymbol{0} \tag{2} $$

By definition of the *Cauchy stress tensor* $\boldsymbol{\sigma}$ —Cauchy's tetrahedron

$$ \boldsymbol{t}\left({}^t\boldsymbol{x}, \boldsymbol{n}\right) = \boldsymbol{\sigma}({}^t\boldsymbol{x}) \cdot \boldsymbol{n} = \boldsymbol{n} \cdot \boldsymbol{\sigma}({}^t\boldsymbol{x}) \tag{3} $$

where $\boldsymbol{n}$ is the unit vector normal to the plane related to the stress vector $\boldsymbol{t}$ and where the dot implies an index contraction, i.e. a scalar product in the case of vectors. The second identity holds because of equilibrium of angular moments. Then

$$\int_{{}^tV} \boldsymbol{b}\; d\,{}^tV + \int_{{}^tS} \boldsymbol{n}\cdot\boldsymbol{\sigma}\; d\,{}^tS = \boldsymbol{0} \tag{4}$$

and by the Generalized Gauss Theorem —see Eq. (5.1.5) of Reference [27]

$$\int_{{}^tV} (\boldsymbol{b} + \nabla\cdot\boldsymbol{\sigma})\, d\,{}^tV = \boldsymbol{0} \tag{5}$$

where $\nabla\cdot\boldsymbol{\sigma}$ is the divergence of the Cauchy stress tensor respect to the current coordinates. By the Localization Theorem the well known local equilibrium equation is obtained —c.f. Eq. (5.3.5) of Reference [27]

$$\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0} \tag{6}$$

Aside, if $\boldsymbol{v}$ is the velocity field at time t, such that

$$\boldsymbol{v} = {}^t\dot{\boldsymbol{x}} = {}^t\dot{\boldsymbol{u}} \tag{7}$$

the Mechanical Power is

$$\mathcal{P} = \int_{{}^tV} \boldsymbol{b}\cdot\boldsymbol{v}\; d\,{}^tV + \int_{{}^tS} \boldsymbol{t}\cdot\boldsymbol{v}\; d\,{}^tS \tag{8}$$

Then using again Eq. (3) and the Generalized Gauss Theorem

$$\mathcal{P} = \int_{{}^tV} \boldsymbol{b}\cdot\boldsymbol{v}\; d\,{}^tV + \int_{{}^tS} \boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{v}\; d\,{}^tS \tag{9}$$

$$= \int_{{}^tV} \boldsymbol{b}\cdot\boldsymbol{v}\; d\,{}^tV + \int_{{}^tV} \nabla\cdot(\boldsymbol{\sigma}\cdot\boldsymbol{v})\;\; d\,{}^tV \tag{10}$$

Using for example index notation, the integrand of the second addend is

$$\nabla \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{v}) = \frac{\partial}{\partial\, {}^t x_i} (\sigma_{ik} v_k) = \sigma_{ik,i} v_k + \sigma_{ik} v_{k,i} \tag{11}$$

$$= (\nabla \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{v} + \boldsymbol{\sigma} : \nabla \boldsymbol{v} \tag{12}$$

where the double-dot implies a double index contraction and we have used the symmetry of $\boldsymbol{\sigma}$. Then Eq. (10) results in

$$\mathcal{P} = \int_{{}^t V} \boldsymbol{\sigma} : \nabla \boldsymbol{v} \; d\, {}^t V \tag{13}$$

where Eq. (6) has been used. The deformation gradient is defined by —note that frequently this tensor is denoted by $\boldsymbol{F}$ but we use the notation of Reference [8]

$$\boldsymbol{X} = \frac{\partial\, {}^t \boldsymbol{x}}{\partial\, {}^0 \boldsymbol{x}} \tag{14}$$

so

$$\nabla \boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial\, {}^t \boldsymbol{x}} = \frac{\partial \boldsymbol{v}}{\partial\, {}^0 \boldsymbol{x}} : \frac{\partial\, {}^0 \boldsymbol{x}}{\partial\, {}^t \boldsymbol{x}} = \frac{\partial}{\partial t} \left(\frac{\partial\, {}^t \boldsymbol{x}}{\partial\, {}^0 \boldsymbol{x}} \right) : \frac{\partial\, {}^0 \boldsymbol{x}}{\partial\, {}^t \boldsymbol{x}} = \dot{\boldsymbol{X}} \boldsymbol{X}^{-1} \tag{15}$$

We note that since $\boldsymbol{\sigma}$ is a symmetric tensor, the integrand in Eq. (13) is

$$\boldsymbol{\sigma} : \nabla \boldsymbol{v} = \boldsymbol{\sigma} : sym\,(\nabla \boldsymbol{v}) = \boldsymbol{\sigma} : \boldsymbol{d} \tag{16}$$

where we defined the spatial *deformation rate tensor* by

$$\boldsymbol{d} := sym\,(\nabla \boldsymbol{v}) = \frac{1}{2} \left[\dot{\boldsymbol{X}} \boldsymbol{X}^{-1} + \boldsymbol{X}^{-T} \dot{\boldsymbol{X}}^{T} \right] \tag{17}$$

By Euler's formula —see for example Eq. (4.5.24) of Reference [27]

$$d\,^tV = J\ d\,^0V \tag{18}$$

where $J := \det \boldsymbol{X}$. Then, the stress power Eq. (13) may be written in the reference volume as

$$\mathcal{P} = \int_{^tV} \boldsymbol{\sigma} : \boldsymbol{d}\ d\,^tV = \int_{^0V} \boldsymbol{\tau} : \boldsymbol{d}\ d\,^0V \tag{19}$$

where $\boldsymbol{\tau} := J\boldsymbol{\sigma}$ is the spatial *Kirchhoff stress tensor*.

Now consider the material *Green-Lagrange strain tensor*

$$\boldsymbol{A} = \frac{1}{2}\left(\boldsymbol{X}^T\boldsymbol{X} - \boldsymbol{I}\right) \tag{20}$$

Then, its objective time derivative, performed in the reference configuration, is

$$\dot{\boldsymbol{A}} = \frac{1}{2}\left(\dot{\boldsymbol{X}}^T\boldsymbol{X} + \boldsymbol{X}^T\dot{\boldsymbol{X}}\right) \tag{21}$$

The (covariant) push-forward to the spatial configuration of the Green-Lagrange strain tensor is the *Almansi strain tensor*

$$\boldsymbol{a} = \boldsymbol{X}^{-T}\boldsymbol{A}\boldsymbol{X}^{-1} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{X}^{-T}\boldsymbol{X}^{-1}\right) \tag{22}$$

and the (covariant) push-forward to the spatial configuration of the *Green-Lagrange strain rate tensor* is the deformation rate tensor

$$\boldsymbol{X}^{-T}\dot{\boldsymbol{A}}\boldsymbol{X}^{-1} = \frac{1}{2}\left(\boldsymbol{X}^{-T}\dot{\boldsymbol{X}}^T + \dot{\boldsymbol{X}}\boldsymbol{X}^{-1}\right) \equiv \boldsymbol{d} \tag{23}$$

which means that $\boldsymbol{d}$ is the Lie derivative along $\boldsymbol{v}$ of the Almansi strain tensor $\boldsymbol{a}$. In index notation these last two Equations can be written as

$$a_{ij} = X_{ik}^{-T} A_{kl} X_{lj}^{-1} = X_{ik}^{-T} X_{jl}^{-T} A_{kl} \quad \text{and} \quad d_{ij} = X_{ik}^{-T} X_{jl}^{-T} \dot{A}_{kl} \tag{24}$$

Here we note that these are merely kinematic relations which existence should be obvious from physical grounds. This type of relationships has already been used when establishing the equivalence between updated Lagrangian and total Lagrangian finite element formulations, see Example 6.23 of Ref. [8].

3. Stress and Strain mapping tensors

According to the preceding kinematic relations, we can define a fourth-order *mapping tensor* (a merely geometric tensor completely defined from the deformation gradient) with components —to shorten this exposition we omit symmetrization issues

$$(\mathbb{M}_A^a)_{ijkl} = (\mathbb{M}_{\dot{A}}^d)_{ijkl} := (\boldsymbol{X}^{-T} \odot \boldsymbol{X}^{-T})_{ijkl} := X_{ik}^{-T} X_{jl}^{-T} \tag{25}$$

so

$$\boldsymbol{a} = \mathbb{M}_A^a : \boldsymbol{A} \quad \text{and} \quad \boldsymbol{d} = \mathbb{M}_{\dot{A}}^d : \dot{\boldsymbol{A}} \tag{26}$$

Also note that a *geometric mapping tensor* may be established between $\dot{\boldsymbol{X}}$ and $\boldsymbol{d}$, i.e.

$$(\mathbb{M}_{\dot{X}}^d)_{ijkl} := \frac{1}{2}(\boldsymbol{X}^{-T} \boxdot \boldsymbol{I} + \boldsymbol{I} \odot \boldsymbol{X}^{-T})_{ijkl} := \frac{1}{2}(X_{il}^{-T} \delta_{jk} + \delta_{ik} X_{jl}^{-T}) \tag{27}$$

such that

$$\boldsymbol{d} = \mathbb{M}_{\dot{X}}^d : \dot{\boldsymbol{X}} \tag{28}$$

and so on.

At this point we notice an important difference between the mapping tensors present in Eqs. $(26)_2$ and (28). The mapping tensor $\mathbb{M}^{d}_{\dot{A}}$ represents a one-to-one mapping in the sense that if $\dot{\boldsymbol{A}}$ is known then $\boldsymbol{d}$ is given by Eq. $(26)_2$ and, vice versa, if $\boldsymbol{d}$ is known then $\dot{\boldsymbol{A}}$ is given through the inverse relation $\dot{\boldsymbol{A}} = \mathbb{M}^{\dot{A}}_{d} : \boldsymbol{d}$, with

$$\mathbb{M}^{\dot{A}}_{d} = \boldsymbol{X}^T \odot \boldsymbol{X}^T = (\mathbb{M}^{d}_{\dot{A}})^{-1} \tag{29}$$

On the contrary, if we know the rate tensor $\dot{\boldsymbol{X}}$, then $\boldsymbol{d}$ is given by Eq. (28), but the inverse situation is not possible, in general. This is due to the fact that $\dot{\boldsymbol{X}}$ is a two-point tensor that includes information about the spatial rotation through —cf. Eq. (15)

$$\dot{\boldsymbol{X}} = \nabla \boldsymbol{v} \cdot \boldsymbol{X} = \boldsymbol{d} \cdot \boldsymbol{X} + \boldsymbol{w} \cdot \boldsymbol{X} \tag{30}$$

where $\boldsymbol{w}$ is the spatial (antisymmetric) *spin tensor*

$$\boldsymbol{w} := skew\,(\nabla \boldsymbol{v}) = \frac{1}{2}\left[\dot{\boldsymbol{X}}\boldsymbol{X}^{-1} - \boldsymbol{X}^{-T}\dot{\boldsymbol{X}}^{T}\right] \tag{31}$$

It is clear from Eq. (30) that there exist infinite rate tensors $\dot{\boldsymbol{X}}$ for a given tensor $\boldsymbol{d}$, hence the mapping $\mathbb{M}^{d}_{\dot{X}}$ of Eq. (28) is not invertible, in general. For further use, we define herein the *spinless deformation gradient rate tensor* as

$$\dot{\boldsymbol{\chi}} := \boldsymbol{d} \cdot \boldsymbol{X} = \frac{1}{2}\left[\dot{\boldsymbol{X}} + \boldsymbol{X}^{-T}\dot{\boldsymbol{X}}^{T}\boldsymbol{X}\right] \tag{32}$$

such that $\dot{\boldsymbol{\chi}} \cdot \boldsymbol{X}^{-1} = \boldsymbol{d}$. Then, the mapping between the modified rate tensor $\dot{\boldsymbol{\chi}}$ and the deformation rate tensor $\boldsymbol{d}$ becomes invertible, i.e.

$$\dot{\boldsymbol{\chi}} = \boldsymbol{I} \odot \boldsymbol{X}^T : \boldsymbol{d} = \mathbb{M}^{\dot{\chi}}_{d} : \boldsymbol{d} \qquad \text{and} \qquad \boldsymbol{d} = \boldsymbol{I} \odot \boldsymbol{X}^{-T} : \dot{\boldsymbol{\chi}} = \mathbb{M}^{d}_{\dot{\chi}} : \dot{\boldsymbol{\chi}} \tag{33}$$

Now consider the following identities

$$\boldsymbol{\tau} : \boldsymbol{d} = \boldsymbol{\tau} : \boldsymbol{X}^{-T} \dot{\boldsymbol{A}} \boldsymbol{X}^{-1} = \boldsymbol{X}^{-1} \boldsymbol{\tau} \boldsymbol{X}^{-T} : \dot{\boldsymbol{A}} = \boldsymbol{S} : \dot{\boldsymbol{A}} \tag{34}$$

where we identify the *Second Piola-Kirchhoff stress tensor* $\boldsymbol{S} := \boldsymbol{X}^{-1} \boldsymbol{\tau} \boldsymbol{X}^{-T}$. The following *geometric* mapping tensor defines the associated (contravariant) pull-back operation

$$(\mathbb{M}_{\tau}^{S})_{ijkl} := (\boldsymbol{X}^{-1} \odot \boldsymbol{X}^{-1})_{ijkl} = X_{ik}^{-1} X_{jl}^{-1} \tag{35}$$

so

$$\boldsymbol{S} = \mathbb{M}_{\tau}^{S} : \boldsymbol{\tau} \tag{36}$$

Alternatively we can use the transpose

$$(\bar{\mathbb{M}}_{\tau}^{S})_{ijkl} := (\boldsymbol{X}^{-T} \odot \boldsymbol{X}^{-T})_{ijkl} = X_{ki}^{-1} X_{lj}^{-1} = (\mathbb{M}_{\tau}^{S})_{klij} \tag{37}$$

so

$$\boldsymbol{S} = \boldsymbol{\tau} : \bar{\mathbb{M}}_{\tau}^{S} \tag{38}$$

Also note that

$$\boldsymbol{\tau} : \boldsymbol{d} = \boldsymbol{\tau} : (\mathbb{M}_{\dot{A}}^{d} : \dot{\boldsymbol{A}}) = (\boldsymbol{\tau} : \mathbb{M}_{\dot{A}}^{d}) : \dot{\boldsymbol{A}} \tag{39}$$

so

$$\boldsymbol{S} = \boldsymbol{\tau} : \mathbb{M}_{\dot{A}}^{d} \tag{40}$$

which provides the relation (compare to Eq. (38))

$$\bar{\mathbb{M}}_{\tau}^{S} = \mathbb{M}_{\dot{A}}^{d} \tag{41}$$

that is remarkably the same mapping tensor that transforms strain measures.

In a similar way as before

$$\boldsymbol{\tau} : \boldsymbol{d} = (\boldsymbol{\tau} : \mathbb{M}^{d}_{\dot{\chi}}) : \dot{\boldsymbol{\chi}} = (\boldsymbol{\tau} : \boldsymbol{I} \odot \boldsymbol{X}^{-T}) : \dot{\boldsymbol{\chi}} = \boldsymbol{\tau}\boldsymbol{X}^{-T} : \dot{\boldsymbol{\chi}} = \boldsymbol{P} : \dot{\boldsymbol{\chi}} \tag{42}$$

where $\boldsymbol{P} := \boldsymbol{\tau} : \mathbb{M}^{d}_{\dot{\chi}} = \boldsymbol{\tau}\boldsymbol{X}^{-T}$ is the *First Piola-Kirchhoff stress tensor* (transpose of the so-called Nominal stress tensor) and we interpret the rate tensor $\dot{\boldsymbol{\chi}}$ of Eq. (32) as its power-conjugate. We note that

$$\boldsymbol{\tau} : \nabla \boldsymbol{v} = \boldsymbol{\tau} : \dot{\boldsymbol{X}}\boldsymbol{X}^{-1} = \boldsymbol{\tau}\boldsymbol{X}^{-T} : \dot{\boldsymbol{X}} = \boldsymbol{P} : \dot{\boldsymbol{X}} \tag{43}$$

hence the rate tensor $\dot{\boldsymbol{X}}$ is usually defined in the literature as the power conjugate of $\boldsymbol{P}$ as well. However, the product $\boldsymbol{P} : \dot{\boldsymbol{X}}$ inherently includes the addend $\boldsymbol{P} : \boldsymbol{w}\boldsymbol{X} = \boldsymbol{\tau} : \boldsymbol{w} = \boldsymbol{0}$ which gives no stress power.

Consider also the *Right Polar Decomposition* of the deformation gradient

$$\boldsymbol{X} = \boldsymbol{R}\boldsymbol{U} \tag{44}$$

where $\boldsymbol{R}$ is the *rotation tensor* and $\boldsymbol{U}$ is the material *stretch tensor*. It is readily obtained from $\boldsymbol{d} = sym(\dot{\boldsymbol{X}}\boldsymbol{X}^{-1})$ that

$$\boldsymbol{d} = \boldsymbol{R} sym(\dot{\boldsymbol{U}}\boldsymbol{U}^{-1})\boldsymbol{R}^{T} := \boldsymbol{R}\bar{\boldsymbol{d}}\boldsymbol{R}^{T} \tag{45}$$

where $\bar{\boldsymbol{d}} := \boldsymbol{R}^{T}\boldsymbol{d}\boldsymbol{R} = sym(\dot{\boldsymbol{U}}\boldsymbol{U}^{-1})$ is the *rotated deformation rate tensor*. Thus, we may write the spinless rate tensor $\dot{\boldsymbol{\chi}} = \boldsymbol{d} \cdot \boldsymbol{X}$ as

$$\dot{\boldsymbol{\chi}} = \boldsymbol{R}\dot{\boldsymbol{v}} \tag{46}$$

where we define the *spinless stretch rate tensor* as —compare to Eq. (32)

$$\dot{\boldsymbol{\upsilon}} := \frac{1}{2}\left[\dot{\boldsymbol{U}} + \boldsymbol{U}^{-1}\dot{\boldsymbol{U}}\boldsymbol{U}\right] \tag{47}$$

which is a tensor such that $\dot{\boldsymbol{\upsilon}} \cdot \boldsymbol{U}^{-1} = \bar{\boldsymbol{d}}$. In Eq. (46) we recognize the (invertible) mapping

$$\mathbb{M}_{\dot{\upsilon}}^{\dot{\chi}} := \boldsymbol{R} \odot \boldsymbol{I} \quad \Rightarrow \quad \dot{\boldsymbol{\chi}} = \mathbb{M}_{\dot{\upsilon}}^{\dot{\chi}} : \dot{\boldsymbol{\upsilon}} \tag{48}$$

Then

$$\boldsymbol{P} : \dot{\boldsymbol{\chi}} = (\boldsymbol{P} : \mathbb{M}_{\dot{\upsilon}}^{\dot{\chi}}) : \dot{\boldsymbol{\upsilon}} = (\boldsymbol{P} : \boldsymbol{R} \odot \boldsymbol{I}) : \dot{\boldsymbol{\upsilon}} = \boldsymbol{R}^T\boldsymbol{P} : \dot{\boldsymbol{\upsilon}} = \boldsymbol{\beta} : \dot{\boldsymbol{\upsilon}} \tag{49}$$

where we define $\boldsymbol{\beta} := \boldsymbol{P} : \mathbb{M}_{\dot{\upsilon}}^{\dot{\chi}} = \boldsymbol{R}^T\boldsymbol{P}$ as the *Biot stress tensor*, which is power-conjugate of $\dot{\boldsymbol{\upsilon}}$. Inserting Eq. (47) into Eq. (49) we easily arrive at

$$\boldsymbol{\beta} : \dot{\boldsymbol{\upsilon}} = \frac{1}{2}\left(\boldsymbol{\beta} : \dot{\boldsymbol{U}} + \boldsymbol{\beta}^T : \dot{\boldsymbol{U}}\right) = sym(\boldsymbol{\beta}) : \dot{\boldsymbol{U}} \tag{50}$$

so we interpret the symmetric part of the Biot stress tensor, namely $sym(\boldsymbol{\beta})$, as the power-conjugate of $\dot{\boldsymbol{U}}$, which is the usual definition encountered in the literature.

Of course, the identities hold if power-conjugate tensors are consistently rotated by any rotation tensor, in particular by $\boldsymbol{R}$

$$\boldsymbol{\tau} : \boldsymbol{d} = \boldsymbol{R}^T\boldsymbol{\tau}\boldsymbol{R} : \boldsymbol{R}^T\boldsymbol{d}\boldsymbol{R} = \bar{\boldsymbol{\tau}} : \bar{\boldsymbol{d}} \tag{51}$$

where $\bar{\boldsymbol{\tau}} := \boldsymbol{R}^T\boldsymbol{\tau}\boldsymbol{R}$ is the *rotated Kirchhoff stress tensor*. Then we note that a mapping tensor that preserves the metric during the transformation and that may be used for both covariant and contravariant tensors may be also defined in this case

$$(\mathbb{M}_R)_{ijkl} := (\boldsymbol{R} \odot \boldsymbol{R})_{ijkl} = R_{ik}R_{jl} \tag{52}$$

so $\boldsymbol{d} = \mathbb{M}_R : \bar{\boldsymbol{d}}$ and $\boldsymbol{\tau} = \mathbb{M}_R : \bar{\boldsymbol{\tau}}$. Furthermore, we note that the chain rule may be properly applied to (invertible) mapping tensors, e.g.

$$\boldsymbol{S} = \boldsymbol{\tau} : \mathbb{M}_{\dot{A}}^{d} = \boldsymbol{P} : \mathbb{M}_{d}^{\dot{X}} : \mathbb{M}_{\dot{A}}^{d} = \boldsymbol{P} : \mathbb{M}_{\dot{A}}^{\dot{X}} \tag{53}$$

where

$$\mathbb{M}_{\dot{A}}^{\dot{X}} = \boldsymbol{I} \odot \boldsymbol{X}^{T} : \boldsymbol{X}^{-T} \odot \boldsymbol{X}^{-T} = \boldsymbol{X}^{-T} \odot \boldsymbol{I} \tag{54}$$

so $\boldsymbol{S} = \boldsymbol{P} : \boldsymbol{X}^{-T} \odot \boldsymbol{I} = \boldsymbol{X}^{-1}\boldsymbol{P}$.

4. Generalized stress and strain measures

In general, we can define a *Generalized Material Strain Measure* $\boldsymbol{E}^*$ as a function of the Stretch tensor $\boldsymbol{U}$

$$\boldsymbol{E}^* = \mathfrak{f}^*(\boldsymbol{U}) \tag{55}$$

Of course a basic requirement for a strain measure to be valid is that there exist a one-to-one tensorial relation (not necessarily component-to-component) between $\boldsymbol{U}$ and $\boldsymbol{E}^*$ [25]. Examples are the Green-Lagrange strain tensor $\boldsymbol{A} = \frac{1}{2}(\boldsymbol{U}^2 - \boldsymbol{I})$, the Biot strain tensor $(\boldsymbol{U} - \boldsymbol{I})$ and the material logarithmic strains $\boldsymbol{E} = \ln \boldsymbol{U}$. Several requirements need to be fulfilled for general strain and stress measures so the transformation is uniquely defined and is valid for the complete range of deformations; we refer to the work of Curnier and Zysset [4] for further details. We consider herein isotropic transformations of the stretch tensor. Hence, the spectral decomposition of the Stretch tensor is

$$\boldsymbol{U} = \sum_{i=1}^{3} \lambda_i \, \boldsymbol{n}_i \otimes \boldsymbol{n}_i \tag{56}$$

where λ_i are the principal stretches and $\boldsymbol{n}_i$ are the principal strain directions in the reference configuration. Then

$$\dot{\boldsymbol{U}} = \sum_{i=1}^{3} \dot{\lambda}_i \ \boldsymbol{n}_i \otimes \boldsymbol{n}_i + \sum_{i=1}^{3} \lambda_i \ \frac{d\boldsymbol{n}_i}{dt} \otimes \boldsymbol{n}_i + \sum_{i=1}^{3} \lambda_i \ \boldsymbol{n}_i \otimes \frac{d\boldsymbol{n}_i}{dt} \tag{57}$$

but since $\boldsymbol{n}_i$ is a unit vector, its derivative may be written as (see Reference [8], Section 6.2.2, for an alternative derivation)

$$\frac{d\boldsymbol{n}_i}{dt} = \boldsymbol{\Omega} \cdot \boldsymbol{n}_i \tag{58}$$

where

$$\boldsymbol{\Omega} = \sum_{i=1}^{3} \sum_{j=1}^{3} \Omega_{ij} \ \boldsymbol{n}_i \otimes \boldsymbol{n}_j = \sum_{i=1}^{3} \sum_{j \neq i} \Omega_{ij} \ \boldsymbol{n}_i \otimes \boldsymbol{n}_j \tag{59}$$

is the spin of the material principal directions (a skew-symmetric tensor) projected in that basis, so

$$\frac{d\boldsymbol{n}_i}{dt} = \boldsymbol{\Omega} \cdot \boldsymbol{n}_i = \left(\sum_{j=1}^{3} \sum_{k \neq j} \Omega_{jk} \ \boldsymbol{n}_j \otimes \boldsymbol{n}_k \right) \cdot \boldsymbol{n}_i \tag{60}$$

$$= \sum_{j=1}^{3} \sum_{k \neq j} \Omega_{jk} \boldsymbol{n}_j \delta_{ki} = \sum_{j=1}^{3} \sum_{i \neq j} \Omega_{ji} \boldsymbol{n}_j \tag{61}$$

Then Eq. (57) can be written as

$$\dot{\boldsymbol{U}} = \sum_{i=1}^{3} \dot{\lambda}_i \ \boldsymbol{n}_i \otimes \boldsymbol{n}_i + \sum_{i=1}^{3} \sum_{j \neq i} \lambda_i \Omega_{ji} \ \boldsymbol{n}_j \otimes \boldsymbol{n}_i + \sum_{i=1}^{3} \sum_{j \neq i} \lambda_i \ \boldsymbol{n}_i \otimes \Omega_{ji} \boldsymbol{n}_j \tag{62}$$

$$= \sum_{i=1}^{3} \dot{\lambda}_i \ \boldsymbol{n}_i \otimes \boldsymbol{n}_i + \sum_{i=1}^{3} \sum_{j \neq i} (\lambda_j - \lambda_i) \, \Omega_{ij} \ \boldsymbol{n}_i \otimes \boldsymbol{n}_j \tag{63}$$

where the antisymmetry property $\Omega_{ij} = -\Omega_{ji}$ has been used. The spectral decomposition of the Generalized Strain Measure is of the form

$$\boldsymbol{E}^* = \sum_{i=1}^{3} f^*(\lambda_i)\ \boldsymbol{n}_i \otimes \boldsymbol{n}_i \tag{64}$$

so following similar algebra, the rate of that measure is

$$\dot{\boldsymbol{E}}^* = \sum_{i=1}^{3} \frac{df^*(\lambda_i)}{d\lambda_i}\dot{\lambda}_i\ \boldsymbol{n}_i \otimes \boldsymbol{n}_i + \sum_{i=1}^{3}\sum_{j\neq i} \left[f^*(\lambda_j) - f^*(\lambda_i)\right]\Omega_{ij}\ \boldsymbol{n}_i \otimes \boldsymbol{n}_j \tag{65}$$

By inspection of the previous expressions we can establish a *geometric mapping tensor* such that

$$\dot{\boldsymbol{E}}^* = \mathbb{M}_{\dot{U}}^{\dot{E}^*} : \dot{\boldsymbol{U}} \tag{66}$$

which is given in the principal deformation basis as —we use $d(\circ)/d(*)$ to denote *total* differentiation of a single-variable tensor-valued function $(\circ)$ with respect to its tensor-valued argument $(*)$

$$\mathbb{M}_{\dot{U}}^{\dot{E}^*} \equiv \frac{d\boldsymbol{E}^*(\boldsymbol{U})}{d\boldsymbol{U}} = \sum_{i=1}^{3} \frac{df^*(\lambda_i)}{d\lambda_i}\boldsymbol{M}_i \otimes \boldsymbol{M}_i + \sum_{i=1}^{3}\sum_{j\neq i} \frac{f^*(\lambda_j) - f^*(\lambda_i)}{\lambda_j - \lambda_i}\boldsymbol{M}_{ij}^S \otimes \boldsymbol{M}_{ij}^S \tag{67}$$

where we use the (full-symmetric) basis tensors

$$\boldsymbol{M}_{ij}^S = \frac{1}{2}\left(\boldsymbol{n}_i \otimes \boldsymbol{n}_j + \boldsymbol{n}_j \otimes \boldsymbol{n}_i\right) \tag{68}$$

$$\boldsymbol{M}_i = \boldsymbol{M}_{ii}^S = \boldsymbol{n}_i \otimes \boldsymbol{n}_i \quad \text{(no sum on } i\text{)} \tag{69}$$

Moreover, if $\boldsymbol{E}^\dagger$ is another general strain measure

$$\boldsymbol{E}^\dagger = \sum_{i=1}^{3} f^\dagger(\lambda_i)\ \boldsymbol{n}_i \otimes \boldsymbol{n}_i \tag{70}$$

a similar mapping tensor may be established between both general strain measures such that

$$\dot{\boldsymbol{E}}^* = \mathbb{M}_{\dot{E}^\dagger}^{\dot{E}^*} : \dot{\boldsymbol{E}}^\dagger \tag{71}$$

$$\begin{aligned} \mathbb{M}_{\dot{E}^\dagger}^{\dot{E}^*} \equiv \frac{d\boldsymbol{E}^*}{d\boldsymbol{E}^\dagger} &= \sum_{i=1}^{3} \frac{df^*(\lambda_i)/d\lambda_i}{df^\dagger(\lambda_i)/d\lambda_i} \boldsymbol{M}_i \otimes \boldsymbol{M}_i \\ &+ \sum_{i=1}^{3} \sum_{j\neq i} \frac{f^*(\lambda_j) - f^*(\lambda_i)}{f^\dagger(\lambda_j) - f^\dagger(\lambda_i)} \boldsymbol{M}_{ij}^S \otimes \boldsymbol{M}_{ij}^S \end{aligned} \tag{72}$$

which existence should be obvious from physical grounds since the state of deformation of the medium is unique and we required a one-to-one relation between them and the stretch tensor.

We can in general write

$$\boldsymbol{S} : \dot{\boldsymbol{A}} = (\boldsymbol{S} : \mathbb{M}_{\dot{E}^*}^{\dot{A}}) : \dot{\boldsymbol{E}}^* = \boldsymbol{T}^* : \dot{\boldsymbol{E}}^* \tag{73}$$

where we have defined the *Generalized Stress Measure* by the following purely geometric relation

$$\boldsymbol{T}^* := \boldsymbol{S} : \mathbb{M}_{\dot{E}^*}^{\dot{A}} = \boldsymbol{S} : \frac{d\boldsymbol{A}}{d\boldsymbol{E}^*} \tag{74}$$

For example, for the particular case of the material *Logarithmic Strain tensor* $\boldsymbol{E}$ we can write

$$\boldsymbol{T} := \boldsymbol{S} : \mathbb{M}_{\dot{E}}^{\dot{A}} = \boldsymbol{S} : \frac{d\boldsymbol{A}}{d\boldsymbol{E}} \tag{75}$$

with

$$\mathbb{M}_{\dot{E}}^{\dot{A}} \equiv \frac{d\boldsymbol{A}}{d\boldsymbol{E}} = \sum_{i=1}^{3} \lambda_i^2 \boldsymbol{M}_i \otimes \boldsymbol{M}_i + \sum_{i=1}^{3} \sum_{j \neq i} \frac{\lambda_j^2 - \lambda_i^2}{2(\ln \lambda_j - \ln \lambda_i)} \boldsymbol{M}_{ij}^S \otimes \boldsymbol{M}_{ij}^S \tag{76}$$

relating the Second Piola-Kirchhoff stress tensor $\boldsymbol{S}$ (work conjugate of $\boldsymbol{A}$) to the *Generalized Kirchhoff stress tensor* $\boldsymbol{T}$ (work conjugate of $\boldsymbol{E}$). Hence

$$\boldsymbol{\tau} : \boldsymbol{d} = \bar{\boldsymbol{\tau}} : \bar{\boldsymbol{d}} = \boldsymbol{S} : \dot{\boldsymbol{A}} = \boldsymbol{T} : \dot{\boldsymbol{E}} \tag{77}$$

To understand why we call the tensor $\boldsymbol{T}$ Generalized Kirchhoff stress tensor, we show now the relation between this stress tensor and the rotated Kirchhoff stress tensor $\bar{\boldsymbol{\tau}}$. Note that

$$\bar{\boldsymbol{d}} = \boldsymbol{R}^T \boldsymbol{d} \boldsymbol{R} = \boldsymbol{U}^{-1} \dot{\boldsymbol{A}} \boldsymbol{U}^{-1} = (\boldsymbol{U}^{-1} \odot \boldsymbol{U}^{-1}) : \dot{\boldsymbol{A}} = \mathbb{M}_{\dot{A}}^{\bar{d}} : \dot{\boldsymbol{A}} \tag{78}$$

so

$$\bar{\boldsymbol{\tau}} : \bar{\boldsymbol{d}} = \bar{\boldsymbol{\tau}} : \mathbb{M}_{\dot{A}}^{\bar{d}} : \dot{\boldsymbol{A}} \tag{79}$$

$$= \bar{\boldsymbol{\tau}} : (\mathbb{M}_{\dot{A}}^{\bar{d}} : \mathbb{M}_{\dot{E}}^{\dot{A}}) : \dot{\boldsymbol{E}} \tag{80}$$

$$= \bar{\boldsymbol{\tau}} : \mathbb{M}_{\dot{E}}^{\bar{d}} : \dot{\boldsymbol{E}} \tag{81}$$

$$= \boldsymbol{T} : \dot{\boldsymbol{E}} \tag{82}$$

and we obtain the desired relationship

$$\boldsymbol{T} = \bar{\boldsymbol{\tau}} : \mathbb{M}_{\dot{E}}^{\bar{d}} \tag{83}$$

where the geometric mapping tensor $\mathbb{M}_{\dot{E}}^{\bar{d}}$ is

$$\mathbb{M}_{\dot{E}}^{\bar{d}} = \mathbb{M}_{\dot{A}}^{\bar{d}} : \mathbb{M}_{\dot{E}}^{\dot{A}} \tag{84}$$

The tensor $\mathbb{M}_{\dot{A}}^{\bar{d}} = \boldsymbol{U}^{-1} \odot \boldsymbol{U}^{-1}$ projected in principal Lagrangian axes is

$$\boldsymbol{U}^{-1} \odot \boldsymbol{U}^{-1} = \sum_{i=1}^{3} \sum_{j=1}^{3} U_{ii}^{-1} U_{jj}^{-1} \; \boldsymbol{n}_i \otimes \boldsymbol{n}_j \otimes \boldsymbol{n}_i \otimes \boldsymbol{n}_j \tag{85}$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \lambda_i^{-1} \lambda_j^{-1} \; \boldsymbol{n}_i \otimes \boldsymbol{n}_j \otimes \boldsymbol{n}_i \otimes \boldsymbol{n}_j \tag{86}$$

which is clearly a fourth-order "diagonal" (in matrix notation) tensor. Thus, using this last result and Equation (76), Equation (84) can be rewritten as

$$\mathbb{M}_{\dot{E}}^{\bar{d}} = \sum_{i=1}^{3} \boldsymbol{M}_i \otimes \boldsymbol{M}_i + \sum_{i=1}^{3} \sum_{j \neq i} \frac{\lambda_j^2 - \lambda_i^2}{2\lambda_i \lambda_j (\ln \lambda_j - \ln \lambda_i)} \boldsymbol{M}_{ij}^S \otimes \boldsymbol{M}_{ij}^S \tag{87}$$

Projecting now $\boldsymbol{T}$ and $\bar{\boldsymbol{\tau}}$ in the material principal *strain* directions and using Equation (83) and the previous expression for $\mathbb{M}_{\dot{E}}^{\bar{d}}$, we get

$$\boldsymbol{T} = \sum_{i=1}^{3} \sum_{j=1}^{3} T_{ij} \; \boldsymbol{n}_i \otimes \boldsymbol{n}_j \tag{88}$$

$$\bar{\boldsymbol{\tau}} = \sum_{i=1}^{3} \sum_{j=1}^{3} \bar{\tau}_{ij} \; \boldsymbol{n}_i \otimes \boldsymbol{n}_j \tag{89}$$

with components

$$T_{ij} = \bar{\tau}_{ij} \quad \text{if} \quad i = j \tag{90}$$

$$T_{ij} = \frac{\lambda_j^2 - \lambda_i^2}{2\lambda_i\lambda_j(\ln\lambda_j - \ln\lambda_i)}\bar{\tau}_{ij} \quad \text{if} \quad i \neq j \tag{91}$$

which somewhat explain the choice of the name *Generalized Kirchhoff stress tensor* for the tensor $\boldsymbol{T}$, since the diagonal components of $\boldsymbol{T}$ and $\bar{\boldsymbol{\tau}}$ coincide when they are represented in the basis of principal stretches. Moreover, in the case of two principal stretches being equal, the next result holds

$$\lim_{\lambda_i \to \lambda_j} \frac{\lambda_j^2 - \lambda_i^2}{2\lambda_i\lambda_j(\ln\lambda_j - \ln\lambda_i)} = 1 \tag{92}$$

In the special case of the stretches being $\lambda_1 = \lambda_2 = \lambda_3$, then $(\mathbb{M}_{\dot{E}}^{\bar{d}})_{ijkl} = (\mathbb{I}^S)_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$ and $\boldsymbol{T} = \bar{\boldsymbol{\tau}}$ in this particular state of deformation. We also note that Eq. (91) is readily obtained from Eq. (6.61) of Reference [8] using Eq. (82). In fact, some of the previous results are given in Section 6.2.2 of that Reference, but using a different presentation style.

Until now we have mainly worked with material measures made function of the material stretch tensor $\boldsymbol{U}$. However we note that we can also apply the same procedures to spatial measures, where the strain measures are function of the spatial stretch tensor $\boldsymbol{V}$ obtained from the Left Polar Decomposition Theorem of the deformation gradient

$$\boldsymbol{e}^* = \mathfrak{f}^*(\boldsymbol{V}) \tag{93}$$

In this case, similar expressions may be simply obtained with the substitution of the directions of the principal stretches in the reference configuration by the spatial ones (rotated by $\boldsymbol{R}$).

Interestingly, the two-point mapping tensors that relate material measures to spatial ones may also be interpreted as *partial* gradients of one of the strain tensor with respect to the other one. For example, consider the Almansi strain tensor of Eq. (22) as a function of the Green–Lagrange strain tensor and the deformation gradient tensor, i.e.

$$\boldsymbol{a}(\boldsymbol{A},\boldsymbol{X}) = \boldsymbol{X}^{-T}\boldsymbol{A}\boldsymbol{X}^{-1} \tag{94}$$

Taking time derivatives and identifying terms we arrive at —we use $\partial(\bullet)/\partial(*)$ to denote *partial* differentiation of a two-variable tensor-valued function $(\bullet)$ with respect to the tensor-valued argument $(*)$

$$\dot{\boldsymbol{a}} = \dot{\boldsymbol{a}}|_{\dot{\boldsymbol{X}}=\boldsymbol{0}} + \dot{\boldsymbol{a}}|_{\dot{\boldsymbol{A}}=\boldsymbol{0}} \tag{95}$$

$$= \frac{\partial \boldsymbol{a}\left(\boldsymbol{A},\boldsymbol{X}\right)}{\partial \boldsymbol{A}} : \dot{\boldsymbol{A}} + \frac{\partial \boldsymbol{a}\left(\boldsymbol{A},\boldsymbol{X}\right)}{\partial \boldsymbol{X}} : \dot{\boldsymbol{X}} \tag{96}$$

$$= \boldsymbol{X}^{-T} \odot \boldsymbol{X}^{-T} : \dot{\boldsymbol{A}} - \left(\boldsymbol{X}^{-T} \odot \boldsymbol{a} + \boldsymbol{a} \boxdot \boldsymbol{X}^{-T}\right) : \dot{\boldsymbol{X}} \tag{97}$$

where we readily recognize the mapping tensor of Eq. (25)

$$\frac{\partial \boldsymbol{a}\left(\boldsymbol{A},\boldsymbol{X}\right)}{\partial \boldsymbol{A}} = \left.\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{A}}\right|_{\dot{\boldsymbol{X}}=\boldsymbol{0}} = \boldsymbol{X}^{-T} \odot \boldsymbol{X}^{-T} \equiv \mathbb{M}^{d}_{\dot{A}} \tag{98}$$

This last interpretation means that two-point mappings (e.g. $\mathbb{M}^{d}_{\dot{A}}$) between pairs of objective strain rates (e.g. Eq. $(26)_2$) or between the associated power-conjugate stress measures (e.g. Eq. (40)) are performed with the involved configurations remaining fixed. Taking advantage of this concept, we may also interpret the Lie derivative of a spatial measure as its time derivative with the current configuration being frozen, i.e. —recall Eq. (23)

$$\dot{\boldsymbol{a}}|_{\dot{\boldsymbol{X}}=\boldsymbol{0}} = \left.\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{A}}\right|_{\dot{\boldsymbol{X}}=\boldsymbol{0}} : \dot{\boldsymbol{A}} = \boldsymbol{X}^{-T} \odot \boldsymbol{X}^{-T} : \dot{\boldsymbol{A}} = \boldsymbol{X}^{-T} \dot{\boldsymbol{A}} \boldsymbol{X}^{-1} \equiv \boldsymbol{d} \tag{99}$$

Consider now the two-variable function $\boldsymbol{A}(\boldsymbol{a}, \boldsymbol{X}) = \boldsymbol{X}^T \boldsymbol{a} \boldsymbol{X}$. Following analogous steps and invoking mechanical power equivalences we arrive at

$$\boldsymbol{\tau} = \boldsymbol{S} : \left.\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{a}}\right|_{\dot{\boldsymbol{X}}=\boldsymbol{0}} = \boldsymbol{S} : \frac{\partial \boldsymbol{A}\left(\boldsymbol{a}, \boldsymbol{X}\right)}{\partial \boldsymbol{a}} \equiv \boldsymbol{S} : \mathbb{M}_d^{\dot{A}} \tag{100}$$

Consider now that the material is hyperelastic. Then the second Piola–Kirchhoff stress tensor $\boldsymbol{S}$ directly derives from a strain energy function per unit reference volume $\Psi(\boldsymbol{A})$. Then, Eq. (100) let us interpret the Kirchhoff stress tensor as the *partial* gradient of Ψ with respect to the spatial tensor $\boldsymbol{a}$ when its referential configuration (i.e. the spatial configuration) is frozen. That is, the application of the chain rule of differentiation yields —note the abuse of notation $\Psi(\boldsymbol{A}) = \Psi(\boldsymbol{A}(\boldsymbol{a}, \boldsymbol{X})) = \Psi(\boldsymbol{a}, \boldsymbol{X})$

$$\boldsymbol{\tau} = \frac{d\Psi\left(\boldsymbol{A}\right)}{d\boldsymbol{A}} : \frac{\partial \boldsymbol{A}\left(\boldsymbol{a}, \boldsymbol{X}\right)}{\partial \boldsymbol{a}} = \frac{\partial \Psi\left(\boldsymbol{a}, \boldsymbol{X}\right)}{\partial \boldsymbol{a}} \equiv \left.\frac{\partial \Psi}{\partial \boldsymbol{a}}\right|_{\dot{\boldsymbol{X}}=\boldsymbol{0}} \tag{101}$$

Remarkably, unlike the traditional definition of the Kirchhoff stress tensor as the push-forward operation $\boldsymbol{\tau} = \boldsymbol{X}(d\Psi/d\boldsymbol{A})\boldsymbol{X}^T$, this last expression gives a direct, easy-to-interpret, definition of $\boldsymbol{\tau}$ in terms of variations of Ψ.

The understanding of the independent variables from which another variable depends on becomes greatly relevant for constitutive theories based on internal variables. Then, similar partial gradient operations to those introduced just above may be defined between different configurations (reference, intermediate, current or whatever). The interested reader is referred to Refs. [14] and [15] to see the application of this mathematical, physically-based framework to large strain viscoelasticity.

At this point, we want to emphasize that until Eq. (101) no hypothesis has been assumed in order to obtain all the previous results, so they are valid for *any* constitutive equation. In particular, we want to note that the *Generalized Kirchhoff stress tensor* $\boldsymbol{T}$ is work conjugate of the *material Logarithmic Strain tensor* $\boldsymbol{E}$ regardless of the constitutive model being assumed for the material. That is, those tensors are work-conjugate even for the most general anisotropic case. Now, if we assume isotropic behavior, obviously all stress and strain material tensors commute and, as a direct result in Eq. (91), $T_{ij} = \bar{\tau}_{ij} = 0$ for $i \neq j$. Hence, for *isotropic constitutive behavior*, $\boldsymbol{T} = \bar{\boldsymbol{\tau}}$ (another reason for the choice of the name for $\boldsymbol{T}$) and $\boldsymbol{T}$ can be also regarded to be power-conjugate of $\bar{\boldsymbol{d}}$.

In a general constitutive equation, the mapping tensors may be employed to transform the constitutive tangent moduli relating strain increments and stress increments. Assume just as an example that we have derived the constitutive tangent $\mathbb{C}^*$ for a generalized strain measure $\boldsymbol{E}^*$ and its work conjugate stress measure $\boldsymbol{T}^*$ such that (note that time derivatives are objective for Lagrangian measures)

$$\dot{\boldsymbol{T}}^* = \frac{d\boldsymbol{T}^*}{d\boldsymbol{E}^*} : \dot{\boldsymbol{E}}^* = \mathbb{C}^* : \dot{\boldsymbol{E}}^* \tag{102}$$

As usual in finite element codes, assume that we actually need the constitutive tangent tensor $\mathbb{C}$ associated to Green-Lagrange strains $\boldsymbol{A}$ and Second-Piola Kirchhoff stresses $\boldsymbol{S}$. Then

$$\dot{\boldsymbol{E}}^* = \frac{d\boldsymbol{E}^*}{d\boldsymbol{A}} : \dot{\boldsymbol{A}} = \mathbb{M}_{\dot{A}}^{\dot{E}^*} : \dot{\boldsymbol{A}} \tag{103}$$

$$\dot{\boldsymbol{T}}^* = \frac{d\boldsymbol{T}^*}{d\boldsymbol{S}} : \dot{\boldsymbol{S}} = \mathbb{M}_{\dot{S}}^{\dot{T}^*} : \dot{\boldsymbol{S}} \tag{104}$$

Combining the previous equations we arrive to

$$\dot{\boldsymbol{S}} = \left[\left(\mathbb{M}_{\dot{S}}^{\dot{T}^*} \right)^{-1} : \mathbb{C}^* : \mathbb{M}_{\dot{A}}^{\dot{E}^*} \right] : \dot{\boldsymbol{A}} \tag{105}$$

$$= \left[\mathbb{M}_{\dot{T}^*}^{\dot{S}} : \mathbb{C}^* : \mathbb{M}_{\dot{A}}^{\dot{E}^*} \right] : \dot{\boldsymbol{A}} = \mathbb{C} : \dot{\boldsymbol{A}} \tag{106}$$

where the required tangent moduli are given in brackets. The explicit expression for $\mathbb{M}_{\dot{A}}^{\dot{E}^*}$ is readily obtained from Eq. (72). Hence, only $\mathbb{M}_{\dot{T}^*}^{\dot{S}}$ remains to be determined in order to obtain $\mathbb{C}$ in terms of the (known) tangent moduli $\mathbb{C}^*$. From Eq. $(74)_2$, the mapping tensor $\mathbb{M}_{\dot{T}^*}^{\dot{S}} = d\boldsymbol{S}/d\boldsymbol{T}^*$ is obtained as

$$\frac{d\boldsymbol{S}}{d\boldsymbol{T}^*} = \frac{d\boldsymbol{E}^*}{d\boldsymbol{A}} : \mathbb{I}^S + \boldsymbol{T}^* : \frac{d^2\boldsymbol{E}^*}{d\boldsymbol{A}d\boldsymbol{A}} : \frac{d\boldsymbol{A}}{d\boldsymbol{E}^*} : \frac{d\boldsymbol{E}^*}{d\boldsymbol{T}^*} \tag{107}$$

$$= \mathbb{M}_{\dot{A}}^{\dot{E}^*} + \boldsymbol{T}^* : \boldsymbol{\mathcal{L}}_{\dot{A}}^{\dot{E}^*} : (\mathbb{M}_{\dot{A}}^{\dot{E}^*})^{-1} : (\mathbb{C}^*)^{-1} \tag{108}$$

where we have used the major symmetry of $d\boldsymbol{E}^*/d\boldsymbol{A}$ and we have defined $\boldsymbol{\mathcal{L}}_{\dot{A}}^{\dot{E}^*} := d^2\boldsymbol{E}^*/d\boldsymbol{A}d\boldsymbol{A}$ as the *sixth-order geometric tensor* relating the rate of $d\boldsymbol{E}^*/d\boldsymbol{A}$ and the rate of $\boldsymbol{A}$. Therefore

$$\mathbb{C} = \mathbb{M}_{\dot{T}^*}^{\dot{S}} : \mathbb{C}^* : \mathbb{M}_{\dot{A}}^{\dot{E}^*} \tag{109}$$

$$= \mathbb{M}_{\dot{A}}^{\dot{E}^*} : \mathbb{C}^* : \mathbb{M}_{\dot{A}}^{\dot{E}^*} + \boldsymbol{T}^* : \boldsymbol{\mathcal{L}}_{\dot{A}}^{\dot{E}^*} \tag{110}$$

and both geometrical mapping tensors $\mathbb{M}_{\dot{A}}^{\dot{E}^*} = d\boldsymbol{E}^*/d\boldsymbol{A}$ and $\boldsymbol{\mathcal{L}}_{\dot{A}}^{\dot{E}^*} = d^2\boldsymbol{E}^*/d\boldsymbol{A}d\boldsymbol{A}$ are required in order to formally map the tangent moduli associated to one strain measure to the tangent moduli associated to the other strain measure. However, in practice, computing the fourth-order tensor $\boldsymbol{T}^* : \boldsymbol{\mathcal{L}}_{\dot{A}}^{\dot{E}^*}$ is computationally more efficient than computing the sixth-order tensor $\boldsymbol{\mathcal{L}}_{\dot{A}}^{\dot{E}^*}$ and then perform the two-index contraction. Following similar lines as above, i.e. by inspection of the spectral decompositions of the rate of $d\boldsymbol{E}^*/d\boldsymbol{A}$ and the rate of $\boldsymbol{A}$ (see Ref. [12] for the particular case of logarithmic stress and strain measures), the explicit expression for $\boldsymbol{T}^* : \mathcal{L}_{\dot{A}}^{\dot{E}^*}$ with minor and major symmetries is found to be

$$\boldsymbol{T}^* : \frac{d^2\boldsymbol{E}^*}{d\boldsymbol{A}d\boldsymbol{A}} = \sum_{i=1}^{3} F\left(\lambda_i\right) T_{ii}^* \; \boldsymbol{M}_i \otimes \boldsymbol{M}_i \tag{111}$$

$$+ \sum_{i=1}^{3} \sum_{j \neq i} G\left(\lambda_i, \lambda_j\right) T_{ii}^* \; \boldsymbol{M}_{ij}^S \otimes \boldsymbol{M}_{ij}^S \tag{112}$$

$$+ \sum_{i=1}^{3} \sum_{j \neq i} G\left(\lambda_i, \lambda_j\right) T_{ij}^* \left(\boldsymbol{M}_i \otimes \boldsymbol{M}_{ij}^S + \boldsymbol{M}_{ij}^S \otimes \boldsymbol{M}_i\right) \tag{113}$$

$$+ \sum_{i=1}^{3} \sum_{j \neq i} \sum_{j \neq k \neq i} \frac{1}{2} H\left(\lambda_i, \lambda_j, \lambda_k\right) T_{ik}^* \left(\boldsymbol{M}_{ij}^S \otimes \boldsymbol{M}_{jk}^S + \boldsymbol{M}_{jk}^S \otimes \boldsymbol{M}_{ij}^S\right) \tag{114}$$

where

$$F\left(\lambda_i\right) = -\frac{2}{\lambda_i^4} \tag{115}$$

$$G\left(\lambda_i, \lambda_j\right) = \frac{8\left(f^*(\lambda_j) - f^*(\lambda_i)\right) - 4\Lambda_{ij}/\lambda_i^2}{\Lambda_{ij}^2} \tag{116}$$

$$H\left(\lambda_i, \lambda_j, \lambda_k\right) = 8\frac{-\Lambda_{jk} f^*(\lambda_i) - \Lambda_{ki} f^*(\lambda_j) - \Lambda_{ij} f^*(\lambda_k)}{\Lambda_{ij}\Lambda_{jk}\Lambda_{ki}} \tag{117}$$

with $\Lambda_{ij} = \lambda_j^2 - \lambda_i^2$. Note that $H\left(\lambda_i, \lambda_j, \lambda_k\right) = H\left(\lambda_j, \lambda_i, \lambda_k\right) = H\left(\lambda_i, \lambda_k, \lambda_j\right) = H\left(\lambda_k, \lambda_j, \lambda_i\right)$ but that $G\left(\lambda_i, \lambda_j\right) \neq G\left(\lambda_j, \lambda_i\right)$. Furthermore, when two or three principal stretches converge to the same value we obtain

$$H\left(\lambda_i, \lambda_j, \lambda_k \to \lambda_i\right) = G\left(\lambda_i, \lambda_j\right) \tag{118}$$

$$H\left(\lambda_i, \lambda_j \to \lambda_i, \lambda_k \to \lambda_i\right) = G\left(\lambda_i, \lambda_j \to \lambda_i\right) = F\left(\lambda_i\right) \tag{119}$$

and so forth.

5. Example

In this example we see how work-conjugate stress and strain measures may be employed to naturally extend the small strains plasticity theory to large strains.

As usually done, we decompose the total small strains *rate* tensor into elastic and plastic parts $\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p$ so the stress power per unit volume is

$$\mathcal{P} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^e + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p \tag{120}$$

We can interpret the strain rate split as

$$\dot{\boldsymbol{\varepsilon}}^e \left(\dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{\varepsilon}}^p \right) = \dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^p = \left. \dot{\boldsymbol{\varepsilon}}^e \right|_{\dot{\boldsymbol{\varepsilon}}^p = 0} + \left. \dot{\boldsymbol{\varepsilon}}^e \right|_{\dot{\boldsymbol{\varepsilon}} = 0} \tag{121}$$

where the first addend is the elastic predictor rate and the last addend is the plastic corrector rate. By subscript $\dot{\boldsymbol{\varepsilon}}^p = \mathbf{0}$ we generically imply that no plastic flow is taking place when performing the derivative and by subscript $\dot{\boldsymbol{\varepsilon}} = \mathbf{0}$ we imply that the system is mechanically isolated. This last contribution may be divided into two parts accounting for microstructural elastic strain rates and microstructural plastic dissipation

$$\dot{\boldsymbol{\varepsilon}}^p \equiv - \left. \dot{\boldsymbol{\varepsilon}}^e \right|_{\dot{\boldsymbol{\varepsilon}} = 0} = \dot{\boldsymbol{\varepsilon}}^{pe} + \dot{\boldsymbol{\varepsilon}}^{pd} = - \left. \dot{\boldsymbol{\varepsilon}}^e \right|_{\dot{\boldsymbol{\varepsilon}} = 0, conservative} - \left. \dot{\boldsymbol{\varepsilon}}^e \right|_{\dot{\boldsymbol{\varepsilon}} = 0, dissipative} \tag{122}$$

Then, the internal power may be enlarged with two terms which cancel out each other

$$\mathcal{P} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^e + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^e + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + \underbrace{\boldsymbol{s}^{pe} : \dot{\boldsymbol{\varepsilon}}^{pe} + \boldsymbol{s}^{pd} : \dot{\boldsymbol{\varepsilon}}^{pd}}_{=0} \tag{123}$$

where $\boldsymbol{s}^{pe}$ and $\boldsymbol{s}^{pd}$ are stress-like internal variables. The free energy is written as $\psi = \Psi(\boldsymbol{\varepsilon}^e) + \mathcal{H}(\boldsymbol{\varepsilon}^{pe})$. Then the dissipation equation takes the form

$$\mathcal{D} = \mathcal{P} - \dot{\psi} = \left(\boldsymbol{\sigma} - \frac{d\Psi}{d\boldsymbol{\varepsilon}^e}\right) : \dot{\boldsymbol{\varepsilon}}^e + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + \left(\boldsymbol{s}^{pe} - \frac{d\mathcal{H}}{d\boldsymbol{\varepsilon}^{pe}}\right) : \dot{\boldsymbol{\varepsilon}}^{pe} + \boldsymbol{s}^{pd} : \dot{\boldsymbol{\varepsilon}}^{pd} \geq 0 \tag{124}$$

Following the Coleman-Noll procedure, since the equality holds for purely elastic responses $\boldsymbol{\sigma} = d\Psi/d\boldsymbol{\varepsilon}^e$. During plastic flow there is an additional part of the internal energy which is not dissipated taking (these are proportional to the backstresses)

$$\boldsymbol{s}^{pe} = \frac{d\mathcal{H}}{d\boldsymbol{\varepsilon}^{pe}} \equiv \boldsymbol{\beta}, \tag{125}$$

The plastic dissipation is now

$$\begin{aligned} \mathcal{D}^p &= \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + \boldsymbol{s}^{pd} : \dot{\boldsymbol{\varepsilon}}^{pd} \\ &= \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p - \boldsymbol{s}^{pe} : \dot{\boldsymbol{\varepsilon}}^{pe} \geq 0 \end{aligned} \tag{126}$$

If one considers, by convention, the typical form of the yield function $f_y(\boldsymbol{\sigma} - \boldsymbol{\beta}, \kappa)$ where κ is a material parameter, then we can establish the Lagrangian

$$\mathcal{L} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p - \boldsymbol{\beta} : \dot{\boldsymbol{\varepsilon}}^{pe} - \dot{t} f_y \tag{127}$$

so $\nabla\mathcal{L} = 0$ implies

$$\dot{\boldsymbol{\varepsilon}}^p = -\left.\dot{\boldsymbol{\varepsilon}}^e\right|_{\dot{\boldsymbol{\varepsilon}}=0} = \dot{t}\ \partial f_y/\partial\boldsymbol{\sigma} \tag{128}$$

$$\dot{\boldsymbol{\varepsilon}}^{pe} = -\dot{t}\ \partial f_y/\partial\boldsymbol{\beta} = \dot{t}\ \partial f_y/\partial\boldsymbol{\sigma} \tag{129}$$

i.e. $\dot{\boldsymbol{\varepsilon}}^{pe} = \dot{\boldsymbol{\varepsilon}}^p$, which emerges as a result of the usually adopted convention for $f_y\left(\boldsymbol{\sigma} - \boldsymbol{\beta}, \kappa\right)$. The associated hardening rule results into

$$\dot{\boldsymbol{s}}^{pe} = \frac{d^2\mathcal{H}}{d\boldsymbol{\varepsilon}^{pe} d\boldsymbol{\varepsilon}^{pe}} : \dot{\boldsymbol{\varepsilon}}^{pe} = \tfrac{2}{3}H\dot{t}\ \partial f_y/\partial\boldsymbol{\sigma} \tag{130}$$

where H is the usual uniaxial kinematic hardening modulus. Note that

$$\mathcal{D}^p = (\boldsymbol{\sigma} - \boldsymbol{\beta}) : \dot{\boldsymbol{\varepsilon}}^p = -\left(\boldsymbol{\sigma} - \boldsymbol{\beta}\right) : \left.\dot{\boldsymbol{\varepsilon}}^e\right|_{\dot{\boldsymbol{\varepsilon}}=0} \geq 0 \tag{131}$$

In References [25][26] we have shown that logarithmic strains may be interpreted as the integral of engineering strains through a fictitious path. Furthermore, Anand [23, 24] has shown that engineering constants for small strains may be applied to large logarithmic strains to obtain a good prediction of the material behavior up to moderate large strains. Hence, the choice of logarithmic strain measures is somehow justified for finite elastoplasticity modelling. From the Lee decomposition, we readily obtain the dependences —cf. Ref. [14] for the analogous case of the Sidoroff decomposition

$$\boldsymbol{E}^e = \boldsymbol{E}^e(\boldsymbol{E}, \boldsymbol{X}^p) \tag{132}$$

Hence, taking the total logarithmic strain tensor $\boldsymbol{E}$ and the plastic part of the deformation gradient $\boldsymbol{X}^p$ as the independent variables of the problem at hand, the time derivative of the elastic logarithmic strain tensor $\boldsymbol{E}^e$ yields

$$\dot{\boldsymbol{E}}^e = \dot{\boldsymbol{E}}^e\Big|_{\dot{\boldsymbol{X}}^p=\boldsymbol{0}} + \dot{\boldsymbol{E}}^e\Big|_{\dot{\boldsymbol{E}}=\boldsymbol{0}} = {}^{tr}\dot{\boldsymbol{E}}^e - \dot{\boldsymbol{E}}^p \tag{133}$$

which is to be compared to Eq. (121). The tensor ${}^{tr}\dot{\boldsymbol{E}}^e$ is the rate of elastic strains when plastic flow is frozen (i.e. the rate of the trial elastic strains) and $\dot{\boldsymbol{E}}^p$ is the rate of the plastic correction. Then, using the work-conjugate stress measures, we can write

$$\mathcal{L} = \boldsymbol{T} : \dot{\boldsymbol{E}}^p - \boldsymbol{B} : \dot{\boldsymbol{E}}^{pe} - \dot{t}\bar{f}_y \tag{134}$$

where $\bar{f}_y\left(\boldsymbol{T} - \boldsymbol{B}, \bar{\kappa}\right)$ is the yield function. It can be seen that a parallel frame to that of small strains is recovered. We here note that Eq. (133) does not imply the use of a plastic metric because it is evaluated in rate form. For example, the (incremental) integration of Eq. (133) using a backward-Euler scheme becomes

$${}^{t+\Delta t}_{\ \ t}\boldsymbol{E}^e \equiv {}^{t+\Delta t}_{\ \ 0}\boldsymbol{E}^e - {}^{t}_{0}\boldsymbol{E}^e = {}^{tr}\boldsymbol{E}^e - {}^{t}_{0}\boldsymbol{E}^e - \Delta t\, \frac{\partial \bar{f}_y}{\partial \boldsymbol{T}}\bigg|_{t+\Delta t} \tag{135}$$

which is the typical update expression used in the integration algorithms for large strains computational plasticity, see form example [17] and [21] and in finite viscoelasticity [13][14][15]. However, we note that Expression (135) is employed here as a consequence of Eq. (133) whereas in the mentioned references for elastoplasticity it is employed as a consequence of an algorithmic approximation of the integration of the plastic deformation gradient using the exponential mapping (see [16])

$${}^{t+\Delta t}_{\ \ 0}\boldsymbol{X}^p = \exp\left(\Delta t\, {}^{t+\Delta t}\boldsymbol{L}^p\right) {}^{t}_{0}\boldsymbol{X}^p \tag{136}$$

so

$$ {}^{t+\Delta t}_{\ \ \ \ \ 0}\boldsymbol{X}^{e} = {}^{t+\Delta t}_{\ \ \ \ \ 0}\boldsymbol{X}\, {}^{t+\Delta t}_{\ \ \ \ \ 0}\boldsymbol{X}^{p-1} = {}^{t+\Delta t}_{\ \ \ \ \ 0}\boldsymbol{X}\, {}^{t}_{0}\boldsymbol{X}^{p-1} \exp\left(-\Delta t\, {}^{t+\Delta t}\boldsymbol{L}^{p}\right) \tag{137} $$

Upon the usual assumption of vanishing plastic spin and the assumption for the flow rule

$$ \boldsymbol{D}^{p} = \dot{t}\frac{\partial \bar{f}_{y}}{\partial \boldsymbol{T}} \;\Rightarrow\; \exp\left(-\Delta t\, {}^{t+\Delta t}\boldsymbol{D}^{p}\right) \simeq \boldsymbol{I} - \Delta t\, {}^{t+\Delta t}\boldsymbol{D}^{p} \tag{138} $$

and the definitions

$$ \boldsymbol{E}^{e} = \tfrac{1}{2}\log\left(\boldsymbol{C}^{e}\right) = \tfrac{1}{2}\ln\left(\boldsymbol{X}^{eT}\boldsymbol{X}^{e}\right) \tag{139} $$

$$ {}^{tr}\boldsymbol{E}^{e} = \tfrac{1}{2}\log\left({}^{tr}\boldsymbol{C}^{e}\right) = \tfrac{1}{2}\ln\left({}^{tr}\boldsymbol{X}^{eT}\, {}^{tr}\boldsymbol{X}^{e}\right) \tag{140} $$

the plastic correction results into Eq. (135), see [16][17][21]. We note also that although $\boldsymbol{T}$ is not coincident with the symmetric part of the Mandel stress tensor, the difference may be neglected for practical purposes [21]. Similar frameworks using plastic metrics can be found in [19] and [20].

Then, the development of a large strains algorithm which keeps the structure of the small strains parent algorithm becomes a simple task. Furthermore, the algorithmic tangent can also be computed using the general Eq. (110) employing logarithmic strains and generalized Kirchhoff stresses as the $(*)$ stress and strain measures. This expression is simpler than that given in [21] and does not employ any further approximation.

Other similar examples may be found in Refs. [14] and [15] for the case of anisotropic finite non-linear viscoelasticity based on logarithmic stress and strain measures.

6. Conclusions

The purpose of this paper is to remark that a mapping tensor may be constructed to transform any arbitrary strain measure in any other strain measure. We present the mapping tensors for many usual strain measures and also for general ones. These same mapping tensors may also be used to transform the work-conjugate stress measures and the corresponding constitutive tensors. An important point is that the transformations are valid regardless of any constitutive equation employed for the solid. Then, as a result the choice of the particular stress and strain measures may be considered simply a matter of convenience. Advantage of this fact may be taken in order to simplify the form of constitutive equations and their numerical implementation and thereafter, perform the proper geometric mappings to convert the results to usually employed measures. In fact, this procedure has already been applied in the past to large-strain anisotropic computational elasto-plasticity using logarithmic stress and strain measures and recently to anisotropic computational finite strain viscoelasticity. Stress, strain and moduli transformations may also be used to select the input and output measures at the user convenience in a finite element program, where the deformation gradient is readily available to perform such operations.

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