On the interpretation of the logarithmic strain tensor in an arbitrary system of representation

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Abstract

Logarithmic strains are increasingly used in constitutive modeling because of their advantageous properties. In this paper we study the physical interpretation of the components of the logarithmic strain tensor in any arbitrary system of representation, which is crucial in formulating meaningful constitutive models. We use the path-independence property of total logarithmic strains to propose different fictitious paths which can be interpreted as a sum of infinitesimal engineering strain tensors. We show that the angular (engineering) distortion measure is arguably not a good measure of shear and instead we propose area distortions which are an exact interpretation of the shear terms both for engineering and for logarithmic strains. This new interpretation clearly explains the maximum obtained in some constitutive models for the simple shear load case.

Keywords: Logarithmic strain tensor; Nonlinear behavior; Shear tests;

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Constitutive modelling.

1. Introduction

Traditional constitutive modelling is frequently developed for small (engineering) strains (Bathe, 1996; Kojić and Bathe, 2005). The extension of these models to large strains is not obvious. There are many fundamental issues when extending such models to large strains, as for example objectivity and energy preservation during elastic deformation processes (Eshraghi et al., 2013a, 2013b; Holzapfel, 2000; Ogden, 1984), which do not usually deserve special attention for small strains. One important decision to be made at large strains is which stress and strain measures to employ.

In the small strain kinematically linear context the engineering infinitesimal stress and strain measures are the ones employed because distinction is not relevant among the different measures. Engineers are used to engineering strains, so they have a rather deep understanding of the physical meaning of their components. In the large strain context, unfortunately there are many choices for stress and strain measures and, of course, that choice strongly affects the constitutive equations of the model, which is usually formulated with a given strain measure in mind. Of course one strain measure may always be mapped to any other strain measure, but for example, a constitutive equation linear in one strain measure will not be so in any other measure. Hence, some fundamental conclusions obtained using one measure may not be valid using others.

The Green-Lagrange and Almansi-Euler deformation measures are often used because of two reasons: they are directly obtained from the deformation gradient and they naturally appear in the nonlinear terms of the finite element formulations. However, these deformation measures are not intuitive, even for uniaxial loading, so using them in constitutive equations may bring difficulties interpreting results or material constants of the models.

The large strain measures arguably most intuitive are the logarithmic (Hencky or "true") strain measures. As we will briefly review below, they preserve the physical meaning of the trace operator (and hence the volumetric and deviatoric strains), they are additive in uniaxial situations and they are symmetric respect to the percentage of stretching: doubling the length of an specimen gives the same amount of logarithmic strain than halving the length of the specimen, except for the change of sign. For logarithmic strains, the push-forward and pull-back operations are performed using rotations, so they also preserve the metric. Furthermore, in isotropic metals a linear hyperelastic relationship between logarithmic strains and Kirchhoff stresses has been found to be an accurate representation if the elastic strains are not too large but only moderately large (Anand, 1979, 1986). This fact added to the special structure of the exponential tensor operators on logarithmic strains facilitate enormously the formulation of elastoplastic constitutive models that are physically well grounded, accurate and efficient for finite element implementation, both for the isotropic (Eshraghi et al., 2010; Eterovic and Bathe, 1990; Montáns and Bathe, 2005; Perić et al., 1992; Simó, 1992;

Weber and Anand, 1990) and anisotropic cases (Caminero et al., 2011; Miehe et al., 2002; Papadopoulus and Lu, 1998). It has been shown that logarithmic strains appear naturally as a consequence of the combination of hypoelasticity and hyperelasticity into a single equation in the context of elastoplasticity (Xiao et al., 2007).

Logarithmic strain measures are also increasingly being used in highly nonlinear hyperelasticity to model the behavior of elastomers and living tissues. For example, recent models based on spline interpolation of experimental data are formulated using logarithmic strains, both for isotropic materials (Sussman and Bathe, 2009) and for anisotropic materials (Latorre and Montáns, 2013, 2014). However these models necessitate some experimental data, which must be correctly interpreted. The correct interpretation of the components of the logarithmic strain tensor in any system of representation is a key for obtaining a correct and accurate description for such models. Furthermore, as we show below, if a good understanding of the strain tensor is achieved, some useful expressions involving functions of such tensor may be obtained (Hoger, 1986; Jog, 2008).

The purpose of this paper is to make some progress in the interpretation of the components of the logarithmic strain tensor in any system of representation, paying special attention to the off-diagonal terms, and to link some conclusions with observed phenomena in the literature when these measures are being used. In particular, we are specially interested in elucidating a correct meaning and a correct measure for the shear deformation. This is of crucial importance in constitutive modelling.

The layout of the paper is as follows. First we breafly review some well-known facts about general strains with the objective of properly motivate the definition and the construction of the logarithmic strain tensor in such a way that the components of the tensor may be better understood. Then we analyze some typical shear deformation examples in order to explain the geometrical meaning of the logarithmic strain measures and to understand the limitations of these shear tests when used in constitutive modelling.

2. General strain measures

The strain measure of a uniformly stretched longitudinal rod with initial (time t_0) and current (time t) total lengths L_0 and L, respectively, may be expressed in multiple ways. It is well-known that all those usual strain measures are given by the general Seth-Hill formula (Seth, 1964)

$$E_n = \frac{1}{n}\left(\lambda^n - 1\right) \tag{1}$$

where $\lambda = \partial x\left(X,t\right)/\partial X = L/L_0$ is the current stretch ratio, n is a number that characterizes each uniaxial strain measure and $x\left(X,t\right)$ represents the motion of material points $X \in [0, L_0]$ at time t. The identity $\lambda = L/L_0$ holds due to the homogeneous deformation assumed along the rod. As it is widely known, the general formula given in Eq. (1) can be used to locally define the strains in principal directions of a three-dimensional deformation state. In that way, Eq. (1) is generalized to

$$\begin{cases} \boldsymbol{E}_n = \displaystyle\sum_{i=1}^{3} \frac{1}{n} \left(\lambda_i^n - 1\right) \ \boldsymbol{N}_i \otimes \boldsymbol{N}_i & \text{if} \quad n \neq 0 \\ \boldsymbol{E}_0 = \displaystyle\sum_{i=1}^{3} \ln \lambda_i \ \boldsymbol{N}_i \otimes \boldsymbol{N}_i & \text{if} \quad n = 0 \end{cases} \tag{2}$$

where λ_i are the principal stretches and $\boldsymbol{N}_i$ are the principal directions of the stretch tensor $\boldsymbol{U}$ obtained from the right polar decomposition, or equivalently

$$\begin{cases} \boldsymbol{E}_n = \dfrac{1}{n} \left(\boldsymbol{U}^n - \boldsymbol{I}\right) & \text{if} \quad n \neq 0 \\ \boldsymbol{E}_0 = \ln \boldsymbol{U} & \text{if} \quad n = 0 \end{cases} \tag{3}$$

with $\boldsymbol{I}$ being the second-order identity tensor.

From Eqs. (2), one can easily calculate all the strain tensors $\boldsymbol{E}_n$, including the case $n = 0$, using the principal stretches λ_i and the eigenvectors $\boldsymbol{N}_i$ (previously computed). This way, since $\left(E_n\right)_i = \boldsymbol{N}_i \cdot \boldsymbol{E}_n \boldsymbol{N}_i = \left(\lambda_i^n - 1\right)/n$, any possible physical meaning for unidimensional strains can obviously be interpreted in the same manner along the principal stretching directions in the three-dimensional case. However, from Eqs. (2) expressed in that way, nothing can be said about the components of $\boldsymbol{E}_n$ when these tensors are represented in a general basis.

In order to understand the description of the cases $n \neq 0$ in a general system of representation (not only in principal directions), the general expression given in Eq. $(3)_1$ for $\boldsymbol{E}_n$ can be used. We will use the deformation

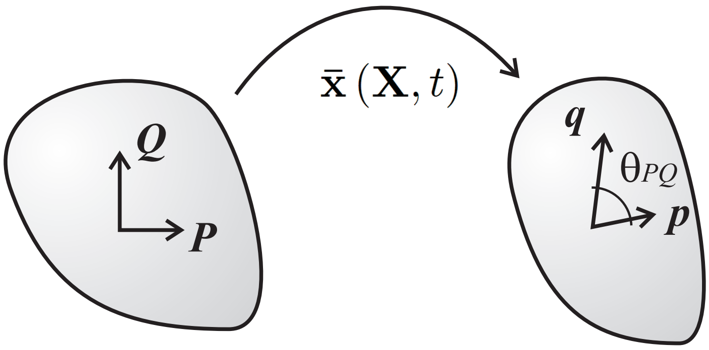

Figure 1: Deformation of two arbitrary orthogonal directions.

gradient $\boldsymbol{U} = \partial \bar{\boldsymbol{x}} (\boldsymbol{X}, t) / \partial \boldsymbol{X}$, where $\bar{\boldsymbol{x}} (\boldsymbol{X}, t)$ represents the motion of material points $\boldsymbol{X}$ with the rotation $\boldsymbol{R}$ removed, which yields a compatible homogeneous rotationless deformation. Hence, for example, the Biot strain tensor, obtained for $n = 1$, is $\boldsymbol{E}_1 = \boldsymbol{U} - \boldsymbol{I} = \partial \bar{\boldsymbol{u}} (\boldsymbol{X}, t) / \partial \boldsymbol{X}$, where it can be seen that $\boldsymbol{E}_1$ represents the material gradient of the displacement field $\bar{\boldsymbol{u}} (\boldsymbol{X}, t) = \bar{\boldsymbol{x}} (\boldsymbol{X}, t) - \boldsymbol{X}$. For any pair of orthogonal unit vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$ in the reference configuration, see Figure 1, we have

$$(E_1)_{PQ} = \boldsymbol{P} \cdot \boldsymbol{E}_1 \boldsymbol{Q} = \boldsymbol{P} \cdot \frac{\partial \bar{\boldsymbol{u}} (\boldsymbol{X}, t)}{\partial \boldsymbol{X}} \boldsymbol{Q} = \boldsymbol{P} \cdot \Delta_{\boldsymbol{Q}} \tag{4}$$

which reveals the meaning of the components of $\boldsymbol{E}_1$ in a reference frame in which $\boldsymbol{P}$ and $\boldsymbol{Q}$ are basis vectors, that is, $(E_1)_{PQ}$ is the projection onto the $\boldsymbol{P}$ direction of the relative displacement $\Delta_{\boldsymbol{Q}} = \bar{\boldsymbol{u}} (\boldsymbol{X} + \boldsymbol{Q}, t) - \bar{\boldsymbol{u}} (\boldsymbol{X}, t)$ when the deformation is assumed to be homogeneous in the solid. Note that if $\boldsymbol{P}$ is not a principal direction of deformation, the diagonal components of $\boldsymbol{E}_1$

can not be understood as in the associated unidimensional case, that is, in general

$$(E_1)_{PP} \neq \lambda_P - 1 \tag{5}$$

where $\lambda_P = |\boldsymbol{p}|$, being $\boldsymbol{p} = \boldsymbol{UP}$ the transformed vector into the current configuration corresponding to the basis vector $\boldsymbol{P}$. Aside, in this case in which the rotation $\boldsymbol{R}$ is removed, $\boldsymbol{E}_1$ is equivalent to the engineering strain tensor $\boldsymbol{\varepsilon} = sym\,(\partial\bar{\boldsymbol{u}}/\partial\boldsymbol{X}) = \partial\bar{\boldsymbol{u}}/\partial\boldsymbol{X} = \boldsymbol{E}_1$. However, the well-known physical descriptions of the diagonal and off-diagonal components of $\boldsymbol{\varepsilon}$ ($\varepsilon_{PP} \approx \lambda_P - 1$ and $\varepsilon_{PQ} \approx \gamma_{PQ}/2$, being γ_{PQ} the angular distortion associated to directions $\boldsymbol{P}$ and $\boldsymbol{Q}$) can only be assigned to $\boldsymbol{E}_1$ if $|\boldsymbol{u}| \ll 1$, that is within the small strain framework.

The values $n = 2$ and $n = -2$ provide the well-known Green-Lagrange and Euler-Almansi strain tensors, respectively. If the first of them is expressed by means of Eq. (3), it results in $\boldsymbol{E}_2 = 1/2(\boldsymbol{U}^2 - \boldsymbol{I})$. As before, one can get a physical interpretation of the PQ-component of $\boldsymbol{E}_2$ when this last expression is pre- and post-multiplied by the orthogonal material basis vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$. Proceeding in that way

$$(E_2)_{PQ} = \boldsymbol{P} \cdot \boldsymbol{E}_2\boldsymbol{Q} = \begin{cases} \frac{1}{2}\,(\lambda_P^2 - 1) & \text{if} \quad P = Q \\ \\ \frac{1}{2}\lambda_P\lambda_Q\cos\theta_{PQ} & \text{if} \quad P \neq Q \end{cases} \tag{6}$$

with $\lambda_P = |\boldsymbol{p}|$ and $\lambda_Q = |\boldsymbol{q}|$. In this case, unlike for $\boldsymbol{E}_1$ –see Eq. (5)– the diagonal terms of $\boldsymbol{E}_2$ correspond to the unidimensional E_2-strain measures of the fibers initially located along the reference frame axes. In a general situation, however, these fibers are not disposed along the reference axes in the current configuration. On the other hand, the off-diagonal term $(E_2)_{PQ}$ gives a measure of the angular deformation corresponding to the initially orthogonal material directions $\boldsymbol{P}$ and $\boldsymbol{Q}$, quantified by means of the angle θ_{PQ} formed between both deformed lines $\boldsymbol{p}$ and $\boldsymbol{q}$, but also affected by the stretching ratios λ_P and λ_Q. In the following section, we will see that in some cases this *shear* measure may lead to some misleading physical interpretations.

However, for the three-dimensional interpretation of the logarithmic strain tensor $\boldsymbol{E}_0$, one cannot turn to the corresponding expression in Eq. (3), as we did for the cases $n = \pm 1$ and $n = \pm 2$. In order to introduce a handy procedure to better grasp the physical meaning of this measure, we define a time-like variable τ, with its domain of definition being $0 \leq \tau \leq t$. The time alike τ (pseudotime) variable is a parameter that continuously maps the reference configuration ($\tau = 0$) to the current configuration ($\tau = t$) following any uniform fictitious motion $\boldsymbol{\xi}(\boldsymbol{X}, \hat{\tau})$, $0 \leq \hat{\tau} = \tau/t \leq 1$, which preserves the principal strain directions of the current configuration $\boldsymbol{N}_i$ for every value of $\hat{\tau}$. The homogeneous deformation gradient (it can also be considered as a local gradient) associated to $\boldsymbol{\xi}(\boldsymbol{X}, \hat{\tau})$ is then

$$\boldsymbol{\Upsilon}(\boldsymbol{X}, \hat{\tau}) = \sum_{i=1}^{3} \Lambda_i(\hat{\tau}) \;\; \boldsymbol{N}_i \otimes \boldsymbol{N}_i \tag{7}$$

where $\Lambda_i(\hat{\tau})$ are the principal stretches associated to the considered fictitious motion at the normalized (pseudo) time $\hat{\tau}$. Then, we perform an integration process from the reference configuration ($\Lambda_i(0) = 1$) to the current configuration ($\Lambda_i(1) = \lambda_i$), with the restrictions $\Lambda_i(\hat{\tau}) > 0$ and the eigenvectors $\boldsymbol{N}_i$ being fixed. Note that the actual motion $\boldsymbol{x}(\boldsymbol{X}, \tau)$ will not necessarily be included in the set $\boldsymbol{\xi}(\boldsymbol{X}, \hat{\tau})$. If we denote the Lagrangian and Eulerian descriptions of the fictitious velocity field by $\boldsymbol{v}(\boldsymbol{X}, \hat{\tau})$ and $\boldsymbol{v}(\boldsymbol{\xi}, \hat{\tau})$ respectively (note the abuse of notation), the spatial velocity gradient associated to the motion $\boldsymbol{\xi}(\boldsymbol{X}, \hat{\tau})$ is

$$\frac{\partial \boldsymbol{v}(\boldsymbol{\xi}, \hat{\tau})}{\partial \boldsymbol{\xi}} = \frac{\partial \boldsymbol{v}(\boldsymbol{X}, \hat{\tau})}{\partial \boldsymbol{X}} \frac{\partial \boldsymbol{X}(\boldsymbol{\xi}, \hat{\tau})}{\partial \boldsymbol{\xi}} = \frac{\partial}{\partial \hat{\tau}} \left(\frac{\partial \boldsymbol{\xi}(\boldsymbol{X}, \hat{\tau})}{\partial \boldsymbol{X}} \right) \frac{\partial \boldsymbol{X}(\boldsymbol{\xi}, \hat{\tau})}{\partial \boldsymbol{\xi}} = \dot{\boldsymbol{\Upsilon}} \boldsymbol{\Upsilon}^{-1} \tag{8}$$

This tensor may be written using the basis of principal directions as

$$\frac{\partial \boldsymbol{v}(\boldsymbol{\xi}, \hat{\tau})}{\partial \boldsymbol{\xi}} = \dot{\boldsymbol{\Upsilon}} \boldsymbol{\Upsilon}^{-1} = \sum_{i=1}^{3} \frac{\dot{\Lambda}_i}{\Lambda_i} \, \boldsymbol{N}_i \otimes \boldsymbol{N}_i \tag{9}$$

where the terms involving time-derivatives of eigenvectors $\boldsymbol{N}_i$ vanish. Using any of these motions, Eq. $(2)_2$ provides

$$\boldsymbol{E}_0 = \ln \boldsymbol{U} = \sum_{i=1}^{3} \ln \lambda_i \, \boldsymbol{N}_i \otimes \boldsymbol{N}_i = \sum_{i=1}^{3} \left(\int_1^{\lambda_i} \frac{d\Lambda_i}{\Lambda_i} \right) \, \boldsymbol{N}_i \otimes \boldsymbol{N}_i \tag{10}$$

$$= \int_0^1 \left(\sum_{i=1}^{3} \frac{\dot{\Lambda}_i}{\Lambda_i} \, \boldsymbol{N}_i \otimes \boldsymbol{N}_i \right) d\hat{\tau} = \int_0^1 \dot{\boldsymbol{\Upsilon}} \boldsymbol{\Upsilon}^{-1} d\hat{\tau} = \int_0^1 \frac{\partial \boldsymbol{v}(\boldsymbol{\xi}, \hat{\tau})}{\partial \boldsymbol{\xi}} d\hat{\tau} \tag{11}$$

Proceeding this way, we observe that $\boldsymbol{E}_0$ represents a direct measure of the sum (integral) of the infinitesimal spatial displacement gradients

$$\frac{\partial \boldsymbol{v}\left(\boldsymbol{\xi},\hat{\tau}\right)}{\partial \boldsymbol{\xi}} d\hat{\tau} =: d\boldsymbol{\varepsilon}_0\left(\boldsymbol{\xi},\hat{\tau}\right) \tag{12}$$

relating positions between two consecutive intermediate configurations at times $\hat{\tau}$ and $\hat{\tau}+d\hat{\tau}$ on any motion $\boldsymbol{\xi}\left(\boldsymbol{X},\hat{\tau}\right)$ evolving from the reference configuration to the current configuration with constant strain eigenvectors, hence the introduced $d\boldsymbol{\varepsilon}_0$ notation. Using this infinitesimal strain tensor with respect to the configuration at time τ, i.e. $d\boldsymbol{\varepsilon}_0$, Eq. (11) can be written symbolically as

$$\boldsymbol{E}_0 = \ln \boldsymbol{U} = \int_0^{\boldsymbol{E}_0} d\boldsymbol{\varepsilon}_0 \tag{13}$$

This symbolic expression is based on the well-known physical meaning of the components of an infinitesimal strain tensor, hence giving a clear meaning to the PQ-component of $\boldsymbol{E}_0$. However, unlike the unidimensional case and similar to what happened with $\boldsymbol{E}_1$, note that in general

$$\left(E_0\right)_{PP} \neq \ln \lambda_P \tag{14}$$

The identity in Eq. (14) only holds when $\boldsymbol{P}$ is a principal direction of deformation. Also, it can be considered as an acceptable approximation if $\boldsymbol{P}$ is not a principal direction but small strains are assumed, that is $\left(E_0\right)_{PP} \approx \ln \lambda_P \approx \lambda_P - 1 \approx \varepsilon_{PP}$. In the following section we give more insight into the components of this tensor in an arbitrary system of representation through a couple of well-known shear tests. Using the interpretation given by Eq. (13) one should not be surprised by some of the special and intuitive properties of the logarithmic strain tensor. For example, the volumetric and isochoric parts are computed as in the small strain context

$$
\begin{aligned}
\boldsymbol{E}_0^V &= \ln\left(J^{1/3}\boldsymbol{I}\right) = \tfrac{1}{3}\left(\ln J\right)\boldsymbol{I} = \tfrac{1}{3}tr\left(\boldsymbol{E}_0\right)\boldsymbol{I} \\
\boldsymbol{E}_0^D &= \ln\left(J^{-1/3}\boldsymbol{U}\right) = \ln\left(\boldsymbol{U}J^{-1/3}\boldsymbol{I}\right) = \boldsymbol{E}_0 - \tfrac{1}{3}tr\left(\boldsymbol{E}_0\right)\boldsymbol{I}
\end{aligned}
\tag{15}
$$

where $J = \det\left(\boldsymbol{U}\right)$, $J^{1/3}\boldsymbol{I}$ is the volumetric part of the deformation gradient from Flory's decomposition and $J^{-1/3}\boldsymbol{U}$ is the isochoric deformation part of the deformation gradient. As a result, they are additive: $\boldsymbol{E}_0 = \boldsymbol{E}_0^V + \boldsymbol{E}_0^D$. It can also be easily shown that superposed deformation gradients result in additive logarithmic strains if principal directions are preserved. These facts have been used in many algorithms for elastoplasticity which preserve the simple and efficient structure of small strain formulations, both in the isotropic case (Eterovic and Bathe, 1990; Montáns and Bathe, 2005; Simó, 1992; Weber and Anand, 1990) and in the anisotropic case (Caminero et al., 2011).

Another interpretation that can be given to the tensor $\boldsymbol{E}_0$ arises when one imposes the specific fictitious spatial velocity gradient $\dot{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}^{-1}$ to be independent of the time-parameter $\hat{\tau}$, that is, the fictitious motion is steady. Then

$$
\boldsymbol{E}_0 = \ln\boldsymbol{U} = \int_0^1 \dot{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}^{-1} d\hat{\tau} = \dot{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}^{-1}\int_0^1 d\hat{\tau} = \dot{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}^{-1} \tag{16}
$$

which tell us that the logarithmic strain tensor $\boldsymbol{E}_0 = \ln \boldsymbol{U}$ can be interpreted as a constant spatial velocity gradient which, acting over the continuum during a unit of time, leads the reference configuration to the current configuration under a steady motion. The specific deformation gradient $\boldsymbol{\Upsilon}$ that fulfills this condition can be obtained from Eq. (16) in principal directions

$$\ln \lambda_i = \frac{\dot{\Lambda}_i}{\Lambda_i} \tag{17}$$

which, integrated between $\hat{\tau} = 0$ and a generic value of $\hat{\tau}$ gives

$$\Lambda_i = (\lambda_i)^{\hat{\tau}} \tag{18}$$

or symbolically

$$\boldsymbol{\Upsilon} = \boldsymbol{U}^{\hat{\tau}} \tag{19}$$

which, effectively, is a monotonically increasing deformation gradient between the reference ($\hat{\tau} = 0$) gradient tensor $\boldsymbol{\Upsilon} = \boldsymbol{U}^0 = \boldsymbol{I}$ and the current ($\hat{\tau} = 1$) gradient tensor $\boldsymbol{\Upsilon} = \boldsymbol{U}^1 = \boldsymbol{U}$.

All the previous derivations yield very interesting tools in order to interpret the shear components in different load cases and experimental procedures. Strain measures are an absolute local measure between two given configurations which do not depend on the specific path that brings one to the other. Hence, an important observation is that we can define any fictitious path to compute and interpret the meaning of the logarithmic strains.

We have seen that those which keep the principal directions are better suited for the physical interpretation, regardless of the time evolution used on each of the principal axes.

3. Examples

In this section we consider some examples and select for them some specific fictitious paths in which the previous concepts are better understood.

3.1. Pure Shear

The deformation gradient of a pure shear state, see Figure 2, represented in a basis $\boldsymbol{X}_e = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ rotated clockwise 45^o with respect to the principal strain directions $\boldsymbol{X}_N = \{\boldsymbol{N}_1, \boldsymbol{N}_2\}$, is

$$\boldsymbol{U} = \sum_{i=1}^{3} \lambda_i \, \boldsymbol{N}_i \otimes \boldsymbol{N}_i = \frac{1}{2\lambda} \begin{bmatrix} \lambda^2 + 1 & \lambda^2 - 1 \\ \lambda^2 - 1 & \lambda^2 + 1 \end{bmatrix}_{\boldsymbol{X}_e} \tag{20}$$

where λ and $1/\lambda$ are the principal stretches with respect to the reference configuration and, for example $\boldsymbol{N}_1 = 1/\sqrt{2}\,[1, 1]^T_{\boldsymbol{X}_e}$. A plain strain state is assumed, so the remaining components $U_{13} = 0$, $U_{23} = 0$ and $U_{33} = 1$ are omitted in Eq. (20). The subscript $\boldsymbol{X}_e$ means that the tensor $\boldsymbol{U}$ is being expressed in the Cartesian basis $\boldsymbol{X}_e$.

In order to obtain the strain tensors $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ we may perform a direct

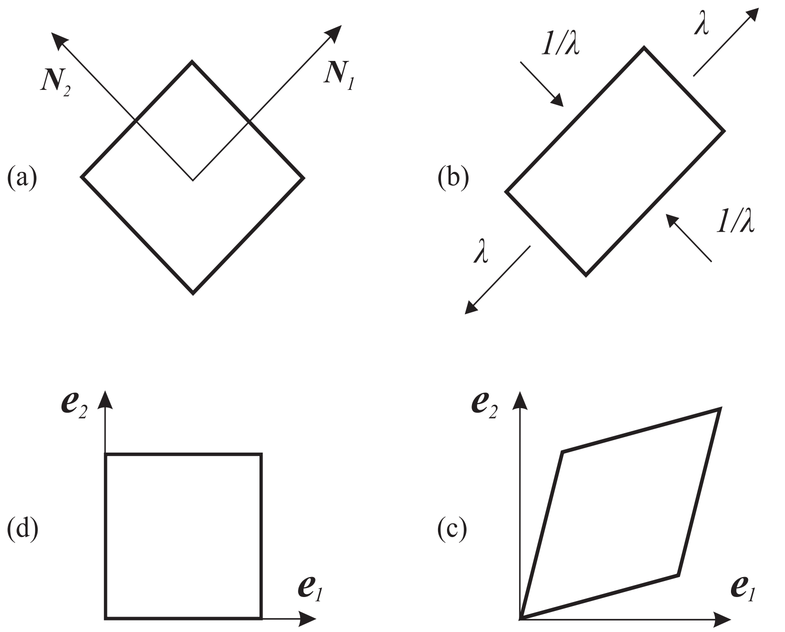

Figure 2: Pure Shear State. From left upper corner clockwise: (a) Reference configuration represented in principal strain basis $\{\boldsymbol{N}_i\}$. (b) Deformed configuration and corresponding principal stretches. (c) Pure shear state represented in the reference basis $\{\boldsymbol{e}_i\}$. (d) Reference configuration represented in $\{\boldsymbol{e}_i\}$.

calculation, i.e.

$$\boldsymbol{E}_1 = \boldsymbol{U} - \boldsymbol{I} = \frac{\lambda - 1}{2\lambda} \begin{bmatrix} \lambda - 1 & \lambda + 1 \\ \lambda + 1 & \lambda - 1 \end{bmatrix}_{\boldsymbol{X}_e} \tag{21}$$

$$\boldsymbol{E}_2 = \frac{1}{2} \left(\boldsymbol{U}^2 - \boldsymbol{I} \right) = \frac{\lambda^2 - 1}{(2\lambda)^2} \begin{bmatrix} \lambda^2 - 1 & \lambda^2 + 1 \\ \lambda^2 + 1 & \lambda^2 - 1 \end{bmatrix}_{\boldsymbol{X}_e} \tag{22}$$

with the interpretation of their components already explained above. Visualizing Figure 2.c, one could have deduced that no component of both tensors could be zero and, furthermore, that all of them are monotonically increasing with λ, as it is effectively apparent from Eqs. (21) and (22).

Unlike $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$, the usual approach to calculate the logarithmic strain tensor is through the spectral decomposition:

$$\boldsymbol{E}_0 = \sum_{i=1}^{3} \ln \lambda_i \ \boldsymbol{N}_i \otimes \boldsymbol{N}_i = \begin{bmatrix} 0 & \ln \lambda \\ \ln \lambda & 0 \end{bmatrix}_{\boldsymbol{X}_e} \tag{23}$$

Both longitudinal logarithmic strains vanish, and the interpretation of these results in the considered reference frame is lost (note the difference with the two previous strain measures). However, we can understand this last result if we compute $\boldsymbol{E}_0$ performing the integration process detailed in the previous Section. In this example, the principal directions of $\boldsymbol{U}$ have always the same orientation, whatever the value of λ (or t). Hence, for each time t, a suitable virtual motion $\boldsymbol{\xi}\left(\boldsymbol{X}, \hat{\tau}\right)$ as defined in the previous section is given by the true path $\boldsymbol{\xi}(\boldsymbol{X},\hat{\tau}) = \boldsymbol{x}(\boldsymbol{X},\tau)$, with $0 \leq \hat{\tau} = \tau/t \leq 1$, and $\boldsymbol{\Upsilon}(\boldsymbol{X},\hat{\tau}) = \boldsymbol{U}(\boldsymbol{X},\tau)$. Denoting the principal stretches at time $\hat{\tau}$ by Λ and $1/\Lambda$, the integrand (spatial velocity gradient $\partial\boldsymbol{v}/\partial\boldsymbol{\xi}$) of Equation $(11)_2$ is

$$\dot{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}^{-1} = \begin{bmatrix} \frac{\Lambda^2-1}{2\Lambda^2} & \frac{\Lambda^2+1}{2\Lambda^2} \\ \frac{\Lambda^2+1}{2\Lambda^2} & \frac{\Lambda^2-1}{2\Lambda^2} \end{bmatrix}_{\boldsymbol{X}_e} \dot{\Lambda} \begin{bmatrix} \frac{\Lambda^2+1}{2\Lambda} & \frac{\Lambda^2-1}{2\Lambda} \\ \frac{\Lambda^2-1}{2\Lambda} & \frac{\Lambda^2+1}{2\Lambda} \end{bmatrix}^{-1}_{\boldsymbol{X}_e} = \begin{bmatrix} 0 & \frac{\dot{\Lambda}}{\Lambda} \\ \frac{\dot{\Lambda}}{\Lambda} & 0 \end{bmatrix}_{\boldsymbol{X}_e} \tag{24}$$

Therefore, tensor $\boldsymbol{E}_0$ is

$$\boldsymbol{E}_0 = \int_0^1 \dot{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}^{-1} d\hat{\tau} = \int_1^\lambda \begin{bmatrix} 0 & 1/\Lambda \\ 1/\Lambda & 0 \end{bmatrix}_{\boldsymbol{X}_e} d\Lambda = \begin{bmatrix} 0 & \ln\lambda \\ \ln\lambda & 0 \end{bmatrix}_{\boldsymbol{X}_e} \tag{25}$$

recovering the previous result, i.e. Eq. (23). Obviously, recalling Eqs. (10)–(13), since the longitudinal (diagonal) components of each infinitesimal spatial displacement gradient relating configurations at times $\hat{\tau}$ and $\hat{\tau} + d\hat{\tau}$ are always identically zero in this case, their sum from the reference to the current configuration results zero as well. Or, in other words, the large strain pure shear state is a deformation state formed from successive spatial small strain pure shear states. Logarithmic strains simply manifest this fact, the other strain measures do not.

As shown in the previous section, Eq. (19), another possible fictitious deformation gradient that can be considered and will be used below is $\boldsymbol{\Upsilon}(\boldsymbol{X},\hat{\tau}) =$

$\left(\boldsymbol{U}\left(\boldsymbol{X},t\right)\right)^{\hat{\tau}}$, so $\Lambda_i\left(\boldsymbol{X},\hat{\tau}\right)=\left(\lambda_i\left(\boldsymbol{X},t\right)\right)^{\hat{\tau}}$. However, in this particular example

$$\boldsymbol{\Upsilon}=\boldsymbol{U}^{\hat{\tau}}=\sum_{i=1}^{3}\lambda_i^{\hat{\tau}}\ \boldsymbol{N}_i\otimes\boldsymbol{N}_i=\frac{1}{2\lambda^{\hat{\tau}}}\begin{bmatrix}\lambda^{2\hat{\tau}}+1 & \lambda^{2\hat{\tau}}-1\\ \lambda^{2\hat{\tau}}-1 & \lambda^{2\hat{\tau}}+1\end{bmatrix}_{\boldsymbol{X}_e} \tag{26}$$

which is only a specific time dependence for $\boldsymbol{\Upsilon}$ which provides a constant spatial velocity gradient

$$\dot{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}^{-1}=\begin{bmatrix}0 & \ln\lambda\\ \ln\lambda & 0\end{bmatrix}_{\boldsymbol{X}_e} \tag{27}$$

included in the more general expression given in Eq. (24), since $\dot{\Lambda}/\Lambda=\ln\lambda$ for this specific fictitious motion.

We remark that the pure shear example is a special case in which the deformation gradient $\boldsymbol{U}$, defined between time $t=0$ and time t, has always the same principal directions of strain. Thus, it can directly be used to define some fictitious deformation gradients $\boldsymbol{\Upsilon}$, defined between $\tau=0$ and $\tau=t$, for each time t. However, other intermediate configurations that preserve the orientation of the principal directions of deformation may be used to perform all the previous integrals. This should not be of surprise since total strain measures are measures of state, not of the path yielding to that deformation state. For instance, the following deformation gradient tensor

$$\boldsymbol{\Upsilon}=\boldsymbol{I}+\hat{\tau}\left(\boldsymbol{U}-\boldsymbol{I}\right) \tag{28}$$

gives the same result for $\boldsymbol{E}_0$ when Eq. $(11)_2$ is applied. As a main difference from that given in Eq. (26), note that the gradient $\boldsymbol{\Upsilon}$ of Eq. (28) provides a non-$\hat{\tau}$-dependent material velocity gradient, i.e $\dot{\boldsymbol{\Upsilon}} = \boldsymbol{U} - \boldsymbol{I}$, associated to a non-steady motion. In this case, the diagonal components of the spatial velocity gradient $\dot{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}^{-1}$ are not zero at each normalized time $\hat{\tau}$, but the total contribution to the integral from the reference to the current configuration vanishes. For example, the results for $\lambda = 5$ are

$$\boldsymbol{U} = \frac{1}{5}\begin{bmatrix} 13 & 12 \\ 12 & 13 \end{bmatrix}_{\boldsymbol{X}_e} \Longrightarrow \boldsymbol{\Upsilon} = \boldsymbol{I} + \hat{\tau}\,(\boldsymbol{U} - \boldsymbol{I}) = \frac{1}{5}\begin{bmatrix} 8\hat{\tau}+5 & 12\hat{\tau} \\ 12\hat{\tau} & 8\hat{\tau}+5 \end{bmatrix}_{\boldsymbol{X}_e} \tag{29}$$

$$\dot{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}^{-1} = \frac{1}{16\hat{\tau}^2 - 16\hat{\tau} - 5}\begin{bmatrix} 16\hat{\tau}-8 & -12 \\ -12 & 16\hat{\tau}-8 \end{bmatrix}_{\boldsymbol{X}_e} \tag{30}$$

$$\boldsymbol{E}_0 = \int_0^1 \dot{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}^{-1} d\hat{\tau} = \begin{bmatrix} 0 & \ln 5 \\ \ln 5 & 0 \end{bmatrix}_{\boldsymbol{X}_e} \tag{31}$$

For a space dimension n, Jog (2008) provides an explicit formula for the logarithm of a tensor exploiting the specific, first-order in $\hat{\tau}$, decomposition given in Eq. (28). Hence, Jog's work can be viewed as an application obtained following a specific fictitious path, which is valid thanks to this path invariance principle.

Other simple possibility is given by the fictitious deformation gradient

$$\boldsymbol{\Upsilon} = \begin{cases} \left[1 + \frac{\hat{\tau}}{T}\left(\frac{1}{2}\left(tr\boldsymbol{U} - 1\right) - 1\right)\right]\boldsymbol{I} & \text{if} \quad 0 \leq \hat{\tau} \leq T \\ \\ \frac{1}{2}\left(tr\boldsymbol{U} - 1\right)\boldsymbol{I} + \frac{\hat{\tau}-T}{1-T}\left(\boldsymbol{U} - \frac{1}{2}\left(tr\boldsymbol{U} - 1\right)\boldsymbol{I}\right) & \text{if} \quad T < \hat{\tau} \leq 1 \end{cases} \tag{32}$$

which first accounts for a pure volumetric contribution ($0 \leq \hat{\tau} \leq T$) and subsequently for the remaining contribution of $\boldsymbol{U}$ ($T < \hat{\tau} \leq 1$). It can be shown that the integral Eq. $(11)_2$ gives the same final result for $\boldsymbol{E}_0$, i.e. Eq. (23), when Eq. (32) is used.

The path followed to compute the integrals for the virtual deformation gradients $\boldsymbol{\Upsilon} = \boldsymbol{U}(\boldsymbol{X}, \tau)$ (or as a particular case, $\boldsymbol{\Upsilon} = \boldsymbol{U}^{\hat{\tau}}$), $\boldsymbol{\Upsilon} = \boldsymbol{I} + \hat{\tau}(\boldsymbol{U} - \boldsymbol{I})$ and that given in Eq. (32) can be seen in Figure 3, where the evolution of $\boldsymbol{\Upsilon}$ from $\hat{\tau} = 0$ to $\hat{\tau} = 1$ is represented in the Mohr's plane corresponding to directions $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. Any other fictitious path from the point which represents the unit tensor $\boldsymbol{I}$ (reference configuration) to the circumference associated to the tensor $\boldsymbol{U}$ (current configuration) which preserves the principal directions (vertical lines are preserved) leads to the same final result, i.e. the expression given in Eq. (23), when the logarithmic strain tensor $\boldsymbol{E}_0$ is calculated using Eq. $(11)_2$.

3.1.1. Geometrical interpretation of the shear logarithmic strain

For deformation states in which strains are small, it is well known that the off-diagonal component PQ of the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ represent

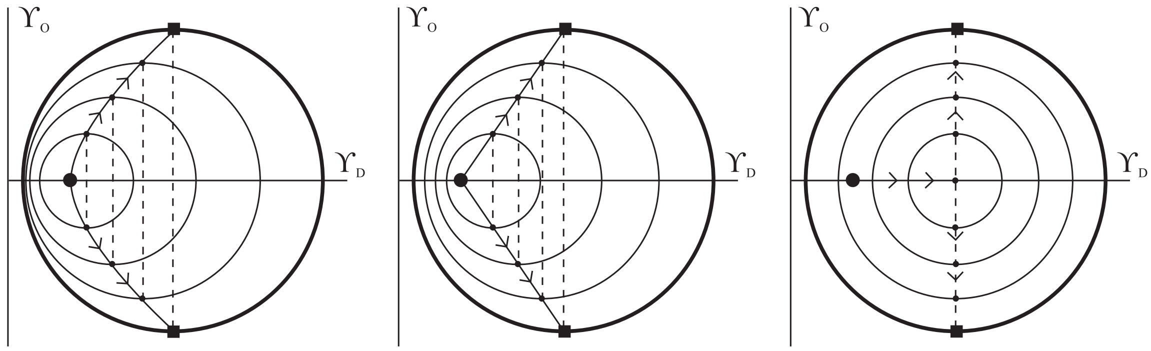

Figure 3: Evolution of the intermediate deformation gradient $\boldsymbol{\Upsilon}$ from the reference configuration $\boldsymbol{I}$ (big solid dot) to the current configuration $\boldsymbol{U}$ (solid squares) of a pure shear state. Left (a): $\boldsymbol{\Upsilon} = \boldsymbol{U}(\boldsymbol{X}, \tau)$. Center (b): $\boldsymbol{\Upsilon} = \boldsymbol{I} + \hat{\tau}(\boldsymbol{U} - \boldsymbol{I})$. Right (c): $\boldsymbol{\Upsilon}$ given in Eq. (32). The orientation of the principal strain directions is conserved in all three cases. Subscript "D" means "Diagonal component" and subscript "O" means "Off-diagonal component".

a measure of the angular distortion associated to the initially orthogonal directions $\boldsymbol{P}$ and $\boldsymbol{Q}$, ie. $\varepsilon_{PQ} = \gamma/2$, see Figure 4.a.

In a more general context in which large strains are considered, the off-diagonal component PQ of the logarithmic strain tensor $\boldsymbol{E}_0$ can be considered as the natural extension of this measure since it accounts for the sum of the

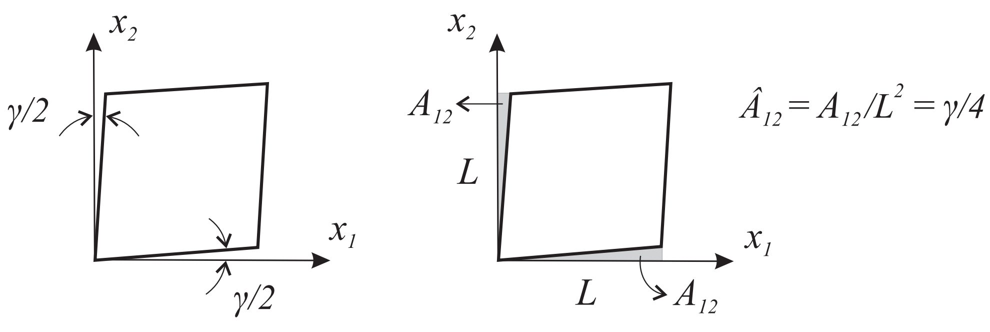

Figure 4: Pure shear state in the small strain case. From left drawing (a): $\varepsilon_{12} = \gamma_{12}/2$. From right drawing (b): $\varepsilon_{12} = 2\hat{A}_{12} = 2(A_{12}/L^2)$.

infinitesimal angular distortions associated to initially orthogonal directions $\boldsymbol{P}$ and $\boldsymbol{Q}$, as we have shown in the previous section, see Eq. (13). However, the geometrical interpretation of this component is apparently lost because, as shown below, $(E_0)_{PQ}$ is *not* the total angular distortion associated to the initially orthogonal directions $\boldsymbol{P}$ and $\boldsymbol{Q}$, ie. $(E_0)_{PQ} \neq \gamma_{PQ}/2$. A more accurate meaning is obtained using the fact that ε_{PQ} also represents twice the dimensionless area indicated in Figure 4.b, i.e. $\varepsilon_{PQ} = 2\hat{A}_{PQ}$. Then, we can give another more general interpretation (*valid for both small and large strains*) of the off-diagonal components of tensor $\boldsymbol{E}_0$, which is based in area distortions rather than angular distortions and that provides a geometrical interpretation for any state of deformation and system of representation.

In order to see the difference between angular and area distortions, consider the example in Fig. 5. As it can be seen in that figure, if $dy_1 = dy_2$, then $dA_1 = dA_2$ without any approximation. However, $d\gamma_1 \neq d\gamma_2$ if we do not consider infinitesimal deformations. Furthermore, $d\gamma/2$ is only an approximation of the engineering shear strain increment $d\varepsilon_{12}$ if we again assume infinitesimal deformations, whereas $2d\hat{A}$ is exactly the amount of engineering strain increment by definition. When integrating engineering strains the correct interpretation is crucial because total deformations may no longer be infinitesimal. This example helps us to understand why area distortions must be considered as the correct interpretation at large strains.

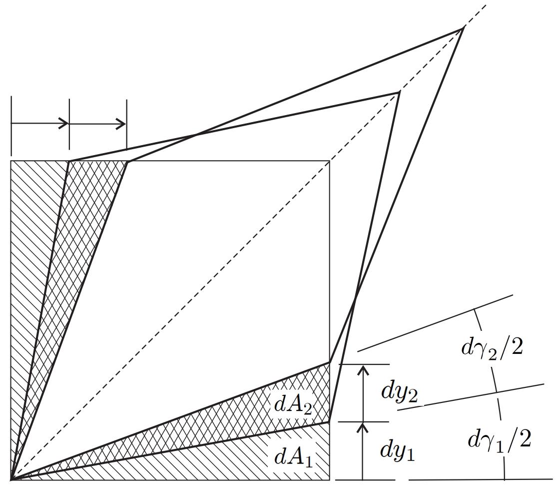

Figure 5: Sketch: incremental engineering shear strains ε_{12} with associated angular distortions $d\gamma$ and area distortions $d\hat{A}$.

Using the surface-like measure, component 12 of Eq. (13) reads

$$(E_0)_{12} = \int_{\hat{\tau}=0}^{\hat{\tau}=1} d\varepsilon_{0_{12}}(\hat{\tau}) = \int_{\hat{\tau}=0}^{\hat{\tau}=1} 2d\hat{A}_{12}(\hat{\tau}) = 2\hat{A}_{12} \tag{33}$$

where $2\hat{A}_{12}$ accounts for the sum of all the infinitesimal dimensionless area distortions (Fig. 4.b) occurring between $\hat{\tau} = 0$ and $\hat{\tau} = 1$, each one of them being measured with respect to the unit differential volume at time $\hat{\tau}$ (Eulerian description). Considering the plain strain condition and that the motion described by the deformation gradient given in Eq. (26) is isochoric, as can be seen by the fact that $\det \boldsymbol{\Upsilon} = 1$ for any value of $\hat{\tau}$, it can be readily deduced that $\hat{A}_{12}$ is coincident with the area swept by each side of the reference unit infinitesimal volume element (Lagrangian description) when the continuum evolves following the constant spatial velocity gradient

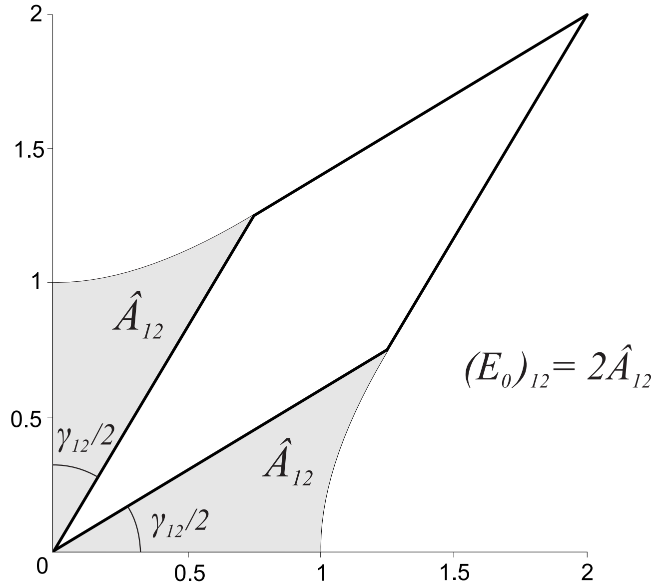

Figure 6: Pure Shear State with $\lambda = 2$, represented in a basis $\{x_1, x_2\}$ rotated clockwise 45^o with respect to the principal strain directions $\{\boldsymbol{N}_1, \boldsymbol{N}_2\}$ (see Figure 2). Geometrical interpretation of the dimensionless areal distortion $\hat{A}_{12}$ and the total angular distortion γ_{12}.

$\dot{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}^{-1} = \ln \boldsymbol{U}$ given in Eq. (27). This area is shown in Figure 6, where the specific pure shear deformation state for $\lambda = 2$ is illustrated. In this example, it is straightforward to obtain that $\hat{A}_{12} = 0.34657$ and $(E_0)_{12} = \ln 2 = 0.69314$, hence effectively $2\hat{A}_{12} = (E_0)_{12}$.

On the other hand, using the angular-like measure γ, component 12 of Eq. (13) reads

$$(E_0)_{12} = \int_{\hat{\tau}=0}^{\hat{\tau}=1} d\varepsilon_{0_{12}}(\hat{\tau}) = \int_{\hat{\tau}=0}^{\hat{\tau}=1} d\gamma_{12}(\hat{\tau})/2 = \Gamma_{12}/2 \tag{34}$$

where Γ_{12} accounts for the sum of all the infinitesimal angular distortions

(Fig. 4.a) occurring between $\hat{\tau} = 0$ and $\hat{\tau} = 1$, each one of them being measured with respect to the unit differential volume at time $\hat{\tau}$. As it has been explained before using Figure 5, equal infinitesimal increments of shear strain $d\varepsilon_{0_{12}}$ generate equal increments of swept area but different increments of swept angle, which implies that

$$(E_0)_{12} = 2\hat{A}_{12} = \Gamma_{12}/2 \neq \gamma_{12}/2 \tag{35}$$

where γ_{12} is the total angular distortion, see Figure 6. For the particular deformation state represented in Figure 6, it is obtained that $\gamma_{12}/2 = 0.54 \neq \ln 2 = (E_0)_{12}$.

We mention that with this geometrical interpretation one can easily understand why the component $(E_0)_{12} = 2\hat{A}_{12}$ increases until infinite with the deformation evolution, which is the result deduced from the analytical calculation, i.e. Eq. (23). This does not happen with γ_{12}.

3.2. Simple Shear

The deformation gradient $\boldsymbol{F}$ and the right stretch tensor $\boldsymbol{U}$ of the simple shear state, under a plain strain condition, shown in Figure 7 are, expressed in the system of representation $\boldsymbol{X}_e = \{\boldsymbol{e}_i\}$ in terms of the angle ψ (cf. Chadwick, 1999)

$$\boldsymbol{F} = \begin{bmatrix} 1 & 0 \\ \bar{\gamma} & 1 \end{bmatrix}_{\boldsymbol{X}_e} = \begin{bmatrix} 1 & 0 \\ \dfrac{2}{\tan(2\psi)} & 1 \end{bmatrix}_{\boldsymbol{X}_e} \tag{36}$$

$$\boldsymbol{U} = \sum_{i=1}^{3} \lambda_i \ \boldsymbol{N}_i \otimes \boldsymbol{N}_i = \begin{bmatrix} \dfrac{1+\cos^2(2\psi)}{\sin(2\psi)} & \cos(2\psi) \\ \cos(2\psi) & \sin(2\psi) \end{bmatrix}_{\boldsymbol{X}_e} \tag{37}$$

where $\psi = (1/2)\arctan(2/\bar{\gamma})$ is the angle between the Lagrangian axes $\{\boldsymbol{N}_i\}$ and the basis $\{\boldsymbol{e}_i\}$, whereas $\bar{\gamma}$ is the so-called amount of shear strain. Both measures, ψ and $\bar{\gamma}$, are shown in Figure 7. The principal stretches have been expressed in terms of the angle ψ, being $\lambda_2 =: \lambda = \tan\psi$ the compressive stretch associated to direction $\boldsymbol{N}_2$ and $\lambda_1 = 1/\lambda$ the stretch in direction $\boldsymbol{N}_1$, as shown in Figure 7.b. We want to note here the difference between the amount of shear measure used in this example, i.e. $\bar{\gamma}$, and the total angular distortion measure used in the pure shear example explained above, denoted by γ. Within the small strain framework they are coincident and no distinction is needed, but for large strains they can differ to a large extent. In this example, the angular distortion γ goes to $\pi/2$ when the amount of shear strain $\bar{\gamma}$ goes to infinite. At the same time, as shown below, $(E_0)_{12}$ goes to zero.

There is a fundamental difference between the Green-Lagrange strain tensor $\boldsymbol{E}_2$ and the logarithmic strain tensor $\boldsymbol{E}_0$ corresponding to this example. This difference has to do with the shear (off-diagonal) components of both tensors when they are projected into the reference frame $\{\boldsymbol{e}_i\}$. On the one hand (not considering the zero-value components involving direction $\boldsymbol{N}_3$),

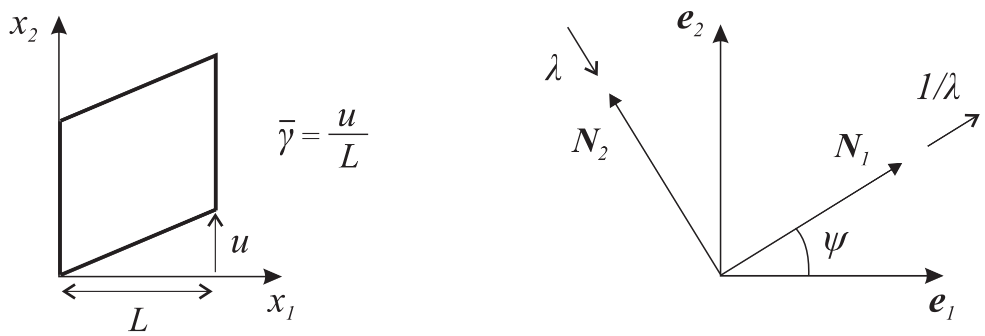

Figure 7: Simple Shear State. *Left* (a): Definition of the amount of shear strain $\bar{\gamma}$ over the current configuration. *Right* (b): Lagrangian principal strain directions, their corresponding stretches and definition of angle ψ.

the tensor $\boldsymbol{E}_2$ is

$$\boldsymbol{E}_2 = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right) = \frac{1}{2}\left(\boldsymbol{U}^2 - \boldsymbol{I}\right) = \begin{bmatrix} \bar{\gamma}^2/2 & \bar{\gamma}/2 \\ \bar{\gamma}/2 & 0 \end{bmatrix}_{\boldsymbol{X}_e} \tag{38}$$

where we note that the component $(E_2)_{12} = \bar{\gamma}/2$ increases monotonically until infinite with the shear deformation. Recalling the expression for this component given above, Eq. $(6)_2$ in which $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ play the role of $\boldsymbol{P}$ and $\boldsymbol{Q}$, respectively, this result can be easily inferred from Figure 7.a, since both λ_{x_1} and $\cos\theta_{x_1x_2}$ increase with $\bar{\gamma}$ (λ_{x_1} from 1 to ∞ and $\cos\theta_{x_1x_2}$ from 0 to 1) while $\lambda_{x_2} = 1$ remains constant.

On the other hand, the tensor $\boldsymbol{E}_0$ is

$$\boldsymbol{E}_0 = \sum_{i=1}^{3} \ln\lambda_i \; \boldsymbol{N}_i \otimes \boldsymbol{N}_i = -\ln(\tan\psi) \begin{bmatrix} \cos(2\psi) & \sin(2\psi) \\ \sin(2\psi) & -\cos(2\psi) \end{bmatrix}_{\boldsymbol{X}_e} \tag{39}$$

where the evolution of $(E_0)_{12} = -\ln(\tan\psi)\sin(2\psi)$ with the shear deformation process have to be analyzed with ψ decreasing from $\psi = \pi/4$ (i.e. $\bar{\gamma} = 0$) to $\psi \to 0$ (i.e. $\bar{\gamma} \to \infty$). In Figure 8, the component $(E_0)_{12}$ is plotted as a function of the amount of shear strain $\bar{\gamma}$. As can be seen, this component reaches a maximum value for $\bar{\gamma} = 3.018$, which corresponds to the principal stretches $\lambda_1 = 3.319$ and $\lambda_2 = 0.301$. This could be regarded an unexpected result because the shear deformation (if represented by $\bar{\gamma}$) increases indefinitely, so one could expect that $(E_0)_{12}$ was an increasing function as well. Moreover, the "direct" extrapolation of the small strain behavior to large strains might lead to the same erroneous conclusion. At this point, the difference between the behavior of the shear components of the two tensors being analyzed is thus apparent. As another appreciation regarding both strain measures, note that although the correspondence between components $(E_0)_{12}$ and $\bar{\gamma}$ (or $(E_2)_{12}$) is not unique in this example, which is clear in Figure 8, the correspondence between tensors $\boldsymbol{E}_0$ and $\boldsymbol{E}_2$ is obviously a one-to-one mapping.

The *a priori* contradictory result mentioned just above, that is, the change from increasing to decreasing tendency of the shear logarithmic strain when the amount of shear strain increases in the simple shear example, can be satisfactorily explained if one computes the tensor $\boldsymbol{E}_0$ by means of the corresponding integration process detailed above. As seen before, for each deformation state, we can define a fictitious motion $\boldsymbol{\xi}(\boldsymbol{X}, \hat{\tau})$ with an associated deformation gradient $\boldsymbol{\Upsilon} = \boldsymbol{U}^{\hat{\tau}}$ and a constant spatial velocity gradient

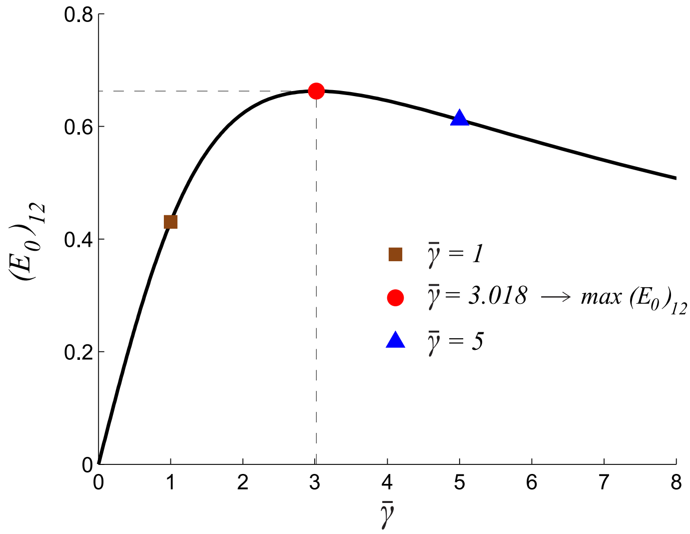

Figure 8: Logarithmic shear strain as a function of the amount of shear strain $\bar{\gamma}$ for the simple shear example, illustrating the maximum value reached by $(E_0)_{12}$. The three points marked over the curve correspond to the deformation states represented in Figure 9. For $\bar{\gamma} > 8$, $(E_0)_{12}$ keeps on decreasing and tends to zero when $\bar{\gamma}$ tends to ∞.

$\dot{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}^{-1} = \ln \boldsymbol{U}$. The integration of this velocity gradient between $\hat{\tau} = 0$ and $\hat{\tau} = 1$ provides the final result (i.e. $\boldsymbol{E}_0$) as an additive contribution of equal infinitesimal strain states acting over the continuum. For any deformation state defined by the angle ψ in the rotated configuration, $\boldsymbol{\Upsilon}$ is

$$\boldsymbol{\Upsilon} = \sum_{i=1}^{3} \lambda_i^{\hat{\tau}}\, \boldsymbol{N}_i \otimes \boldsymbol{N}_i = \begin{bmatrix} \lambda^{\hat{\tau}} \sin^2\psi + \lambda^{-\hat{\tau}} \cos^2\psi & (\lambda^{-\hat{\tau}} - \lambda^{\hat{\tau}}) \cos\psi \sin\psi \\ (\lambda^{-\hat{\tau}} - \lambda^{\hat{\tau}}) \cos\psi \sin\psi & \lambda^{\hat{\tau}} \cos^2\psi + \lambda^{-\hat{\tau}} \sin^2\psi \end{bmatrix}_{\boldsymbol{X}_e} \tag{40}$$

where $\lambda = \tan\psi$ with ψ fixed for $\hat{\tau} \in [0, 1]$. The three deformation states marked over the graph in Figure 8 are represented in Figure 9. For each one of them ($\psi = (1/2) \arctan(2/\bar{\gamma})$), the path followed from the reference configuration ($\hat{\tau} = 0$) to the current configuration ($\hat{\tau} = 1$) by the corresponding fictitious motion with deformation gradient given in Eq. (40) are shown. Focusing on this geometrical interpretation of the shear components of the logarithmic strain tensor, note that, although the amount of shear strain $\bar{\gamma}$ increases, the surface distortion measure $(E_0)_{12}$ (the area swept using constant incremental infinitesimal strains) reaches a maximum. This fact tells us that the contribution of the constant incremental infinitesimal shear strains needed to obtain the deformation state corresponding to $\bar{\gamma} = \bar{\gamma}_{(E_0)_{12max}} \approx 3.018$ is greater than for the other cases: $\bar{\gamma} < \bar{\gamma}_{(E_0)_{12max}}$, where longitudinal strains are of lower or equal order of magnitude than shear strains, and $\bar{\gamma} > \bar{\gamma}_{(E_0)_{12max}}$, where longitudinal strains become more and more relevant than shear strains when $\bar{\gamma}$ increases. Moreover, note that the limit $\bar{\gamma} \to 0$ can also be represented by Figure 4 and that for the limit $\bar{\gamma} \to \infty$, $(E_0)_{12} = 2\hat{A}_{12} \to 0$. Hence, once we have understood how *any simple shear state can be additively generated* and why the shear logarithmic strain reaches a maximum value in this example, one should question if the amount of shear $\bar{\gamma}$ is the most correct variable to effectively measure the *amount of shear* undergone by the continuum. This appreciation is crucial in the choice of the constitutive law which models the mechanical behavior of a material. Obviously, the shear response of a material model will be significantly different if stresses are assumed to be linear in $(E_2)_{12}$ or in $(E_0)_{12}$. That is, arguably the best strain measure to represent the shear behavior may be $(E_0)_{12}$ and so if a linear relationship is to be assumed over a strain measure to compute stresses, $(E_0)_{12}$ is a strong candidate. Furthermore, a linear relation between stress and strain measures would be equivalent to the small strain linear relation if $\boldsymbol{E}_0$ is used as the strain measure, since we have seen that this measure may be considered as the sum of constant infinitesimal engineering strains This is a fact somehow observed experimentally by Anand (1979, 1986) for metals.

The objective rate-form constitutive model presented by Xiao et al. (1997) predicts a maximum value for the shear stress at the simple-shear deformation. Although they directly work over the (Eulerian) stress response rather than over the (Lagrangian) strain behavior, note that the maximum Cauchy shear stress that they calculate is only a consequence of the linear relationship between Cauchy stresses $\boldsymbol{\sigma}$ and logarithmic strains in the spatial configuration $\ln \boldsymbol{V}$ that they obtain, i.e. $\boldsymbol{\sigma} = 2\mu \ln \boldsymbol{V}$ (cf. Eq. (71) in that paper taking into account that the simple shear motion is isochoric). Thus, if the Cauchy stress–logarithmic strain relation is linear and, as explained just above, the shear component of the material logarithmic strain tensor $\boldsymbol{E}_0 = \ln \boldsymbol{U} = \boldsymbol{R}^T (\ln \boldsymbol{V}) \boldsymbol{R}$ reaches the maximum value $(E_0)_{12_{max}} \approx 0.663$ at $\bar{\gamma} = \bar{\gamma}_{(E_0)_{12_{max}}}$, then the shear component of the rotated Cauchy stress tensor $\boldsymbol{\sigma}_R = \boldsymbol{R}^T \boldsymbol{\sigma} \boldsymbol{R}$ has also to reproduce that maximum value at that deformation state, which will be $(\sigma_R)_{12_{max}} = 2\mu (E_0)_{12_{max}} \approx 1.325\mu$. Using $\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1}$, it is straightforward to verify that $\sigma_{12} = (\sigma_R)_{12}$. Hence, Xiao et al. effectively obtain the same result —cf. Eq. (82)— analyzing the Eulerian stress response for their constitutive model based on the rate of spatial logarithmic

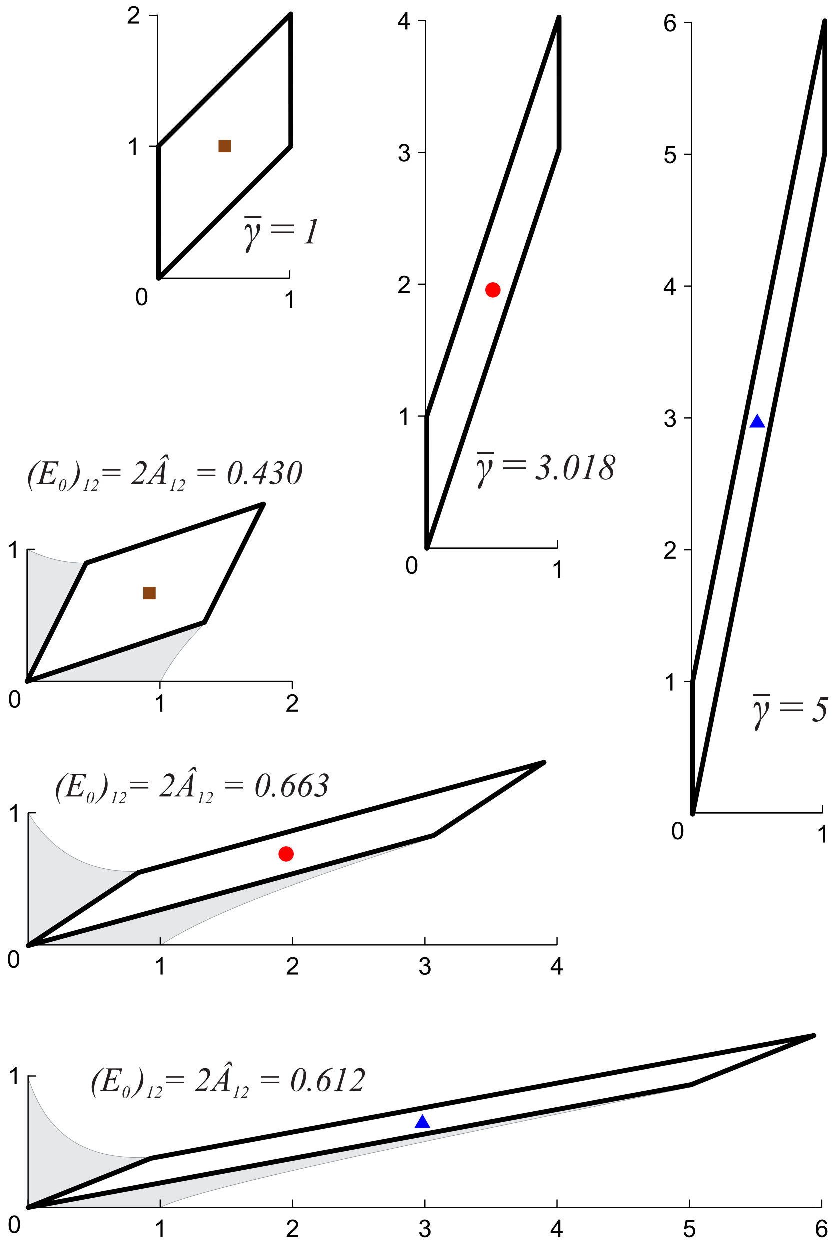

Figure 9: Simple shear states corresponding to the values $\bar{\gamma} = 1$, $\bar{\gamma} = \bar{\gamma}_{(E_0)_{12_{max}}} \approx 3.018$ and $\bar{\gamma} = 5$ (points marked in Figure 8). The deformation states with the rotation removed, with the associated geometrical interpretation of the logarithmic shear strain $(E_0)_{12}$ or surface distortion, are also represented in order to illustrate the maximum value taken by this strain component.

strains, which provides a hyperelastic relation when it is formally integrated (Xiao et al, 1999a, 1999b). Note that for materials fulfilling this specific constitutive law, the total Cauchy shear stress can also be interpreted as an additive contribution of constant Cauchy shear stresses in an analogous way as for shear logarithmic strains.

This maximum value taken by the logarithmic shear strain $(E_0)_{12}$ in the simple shear deformation example may also represent a limitation when defining certain hyperelastic energy functions on uncoupled models. Recently, we have proposed an uncoupled decomposition of the stored energy function in terms of logarithmic strains to model incompressible transversely isotropic hyperelastic materials using a spline-based methodology (Latorre and Montáns, 2013). Since the contribution of the shear logarithmic strain to the strain energy function is considered separately (in an uncoupled way) from the other strain components, the corresponding term of the strain energy function can only be defined up to the maximum amount of the shear deformation $E_{13} = (E_{13})_{max}$ (we use here the index numeration corresponding to that work), or equivalently between $E_1 = 0$ and $E_1 = 1$ which correspond to the same values of $\bar{\gamma} = \bar{\gamma}\,(E_{13_{max}})$, respectively, as can be easily calculated. Otherwise the strain energy function would be a bi-valued function. If logarithmic shear strains larger than $(E_{13})_{max}$ are needed to define the model (even though this value corresponds to a really large shear deformation, see Figure 9), the pure shear test can be used instead to define the energy density term with no limitation in its range, as we properly address

(Latorre and Montáns, 2013). Another possibility is to simply extrapolate the energy function, which is an easy operation due to the intrinsic use of splines by the model.

4. Conclusion

Logarithmic strain measures are increasingly being used in constitutive modelling because of their special properties. One of these properties is the use of the same additive nature of the volumetric-isochoric split as in small strains. Other properties are the also additive nature of strains due to deformation gradients when the principal directions are preserved. Furthermore, some constitutive equations developed for small strains can be naturally extended to large strains simply substituting the small strain tensor by the logarithmic strain tensor. As we have seen in this paper, all these properties seem natural if one considers that the logarithmic strain tensor can be regarded as the sum of infinitesimal engineering strain tensors.

However, the physical interpretation of the components of the logarithmic strain tensor, both diagonal and off-diagonal, are not so evidently inherited from their small strains counterpart. In order to obtain some insight we have used the fact that total strain measures are path independent and, hence, logarithmic strains can be computed using any arbitrary fictitious velocity gradient with the condition that the final deformation gradient is the actual one. We have seen that the angle γ of the engineering shear strain (angular distortion) is arguably not the best measure of the amount of shear strain.

Instead we propose the amount of surface distortion which is also half the off-diagonal component of the logarithmic strain tensor, and which for small strains has the same value as $\gamma/4$. This new physical interpretation of the logarithmic shear strain explains the maximum obtained by this measure and by the stress in some constitutive equations for the simple shear load case.

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