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Mathematics' wars - real or imaginary?

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Abstract

This paper investigates different methods to solve a mathematics task attempted by students enrolled for a first-year mathematics module in South Africa (n = 182). The paper examines an expected calculus solution approach and examples of interesting alternative student solutions. Task solutions were analysed according to the number of conceptual and procedural steps used to solve the task. Each step in the task solutions is described by a problem-solving category, based on the knowledge approaches used to solve the task. The results confirm that solution methods are not unique. Task solutions require both procedural and conceptual steps, and problem-solving steps are sometimes iterative. The analyses demonstrate that mathematical questions cannot be uniquely described as mainly conceptual or procedural. The analyses suggest that lecturers could consider explanation and comparison of multiple solution strategies as a way to enhance mathematical proficiency.

Keywords: *Mathematical task solution analysis; conceptual knowledge; procedural knowledge, student work, teaching strategies.*

1. Introduction

The literature describes different perspectives pertaining to how mathematics should be taught – should the focus be on concepts and application or procedures? According to Star (2005, p. 404):

> *Whether developing skills with symbols leads to conceptual understanding, or whether the presence of basic understanding should precede symbolic representation and skill practice, is one of the basic disagreements between the opposing sides of the so-called math wars.*

The concept-driven versus skills-orientated perspectives have led to the so called '*math wars'* between mathematics education researchers globally (Brown, Seidelmann, & Zimmermann, 2002; Sowder, 2007; Star, 2005; Wu, 1999), as well as in South Africa (Engelbrecht, Bergsten, & Kagesten, 2009; Engelbrecht, Harding, & Potgieter, 2005). The concept-driven view accentuate the understanding of mathematics and the use of it to solve problems (Sowder, 2007), emphasizing reasoning, critical thinking and problem-solving skills. In the contrast, the skills-orientated perspective promote the development of skills as necessary vehicles to promote conceptual understanding (Wu, 1999). This paper examines different problem-solving approaches used to solve a mathematics task, focussing on procedural and conceptual steps. The following research questions are explored:

1. *Can we categorise the mathematic task as mainly conceptual or procedural?*
2. *What are the implications of different solution approaches for teaching?*

The content analysis is performed as part of a larger study that investigated the knowledge types required to solve 33 calculus tasks (Hechter, 2020). The study is located within a mathematics module for first year engineering in South Africa. The analysis includes the view of the researcher and written student solutions.

2. Conceptual and procedural knowledge

Conceptual knowledge is described as concepts, and relations between concepts and operations (Kilpatrick, Swafford, & Findell, 2001). Procedural knowledge is the ability to accurately perform step by step procedures to solve problems (Star, Rittle-Johnson, & Durkin, 2016). Procedural flexibility involves both types of knowledge, and is described as knowledge of multiple methods and choosing the most appropriate method based on specific problem properties (Kilpatrick et al., 2001; Rittle-Johnson, 2017; Star, 2005). Mathematical proficiency requires both types of mathematical knowledge and procedural flexibility (Rittle-Johnson, 2017).

A comparative study between South Africa and Sweden investigated teaching emphasis in undergraduate mathematics courses for engineering students (Engelbrecht, Bergsten, & Kågesten, 2012). The study stated that mathematical problem-solving approaches was classified as either mainly conceptual or mainly procedural (Bergsten, Engelbrecht, & Kågesten, 2017):

Conceptual approach: This includes translations between verbal, visual (graphical), numerical, and formal/algebraic mathematical expressions (representations); linking relationships; and interpretations and applications of concepts to mathematical situations.

Procedural approach: This includes symbolic and numerical calculations, employing (given) rules, algorithms, formulae, and symbols.

The description of being 'conceptual' or 'procedural' is not necessarily a property of the task itself, but rather a description of the solution of the task (Engelbrecht et al., 2009). Mathematical solution approaches could be described as bidirectional, causal relations since solution methods show that procedural and conceptual steps alternate, (Rittle-Johnson, Schneider, & Star, 2015), but in no specific order (Rittle-Johnson, Fyfe, & Loehr, 2016). Furthermore, solution methods indicate that some steps repeat, showing iterative relations between concepts and procedures (Rittle-Johnson, 2017). Procedures could be connected to concepts through reasoning and different representations, e.g. graphs (Davis, 2005). Comparing and explaining numerous strategies for solving the same problem promotes student learning (Star et al., 2016). Comparing different methods for the same task develops conceptual and procedural knowledge and advances procedural flexibility among students with some prior knowledge of one of the methods (Durkin, Star, & Rittle-Johnson, 2017).

3. Methodology

This paper examines solutions to a first-year calculus task on application of differentiation:

If a stone is thrown vertically upwards, the position function of the stone is given by
$s(t) = 30t - 5t^2 + 20$, where s is in metres and t is in seconds. Calculate:

a) the time t when the stone will reach its maximum height

b) the maximum height of the stone (before it falls to the ground).

The task was selected since student work (n = 182) presented alternative methods and provided rich data. Students are given the position function of an object and required to answer questions regarding the maximum height that an object will travel. The researcher expected that students would find the extreme value(s) where the derivative function (velocity) is zero, therefore using calculus to solve the task. The paper shares evidence and analyses of the researcher's expected response and student work that provides interesting alternative solutions.

3.1. Data Analysis

The categorisation expands the conceptual and procedural approaches described by Bergsten et al. (2017) for the topics of functions and differentiation. Table 1 shows the categories used to solve a given task, e.g.:

C_{3D}: conceptual category - involves the *interpretation* of concepts related to *differentiation*,
P_{2D}: procedural category that involves use of *differentiation rules.*

Table 1. Conceptual and procedural problem-solving categories

Step	Conceptual and procedural problem-solving categories
C_1	translations between verbal, visual, numerical, and algebraic mathematical expressions
C_{2F}	linking relationships wrt functions: functions $\Leftrightarrow$ inverse functions, equation of a function
C_{2D}	linking relationships wrt differentiation: $f \Leftrightarrow f' \Leftrightarrow f'', D_{f'} \subseteq D_f$, $f'(x)=0 \Rightarrow$ f local extrema, $f'(x)>0 \Rightarrow$ f increasing, $f'(x)<0 \Rightarrow$ f decreasing, $f''(x)=0 \Rightarrow$ possible point of inflection, $f''(x)>0 \Rightarrow$ f concave up, $f''(x)<0 \Rightarrow$ f concave down, link position function (displacement) $\Rightarrow$ velocity (speed) $\Rightarrow$ acceleration
C_{3F}	interpretation of concepts wrt functions: definitions, functions and relations, inverse, domain and range, restrictions, inequalities (quadratic and higher order), incl. concept of intersection and union, turning point of a parabola (min/max), axis of symmetry, *x*-intercepts
C_{3D}	interpretation of concepts wrt differentiation: gradient, continuity, differentiability, point of inflection, concavity
C_4	applications of concepts to mathematical situations
P_1	symbolic and numerical calculations, substitution
P_{2F}	rules wrt functions, expressions e.g. division by zero, equations e.g. $a.b=0 \Rightarrow a=0$ or $b=0$, inequalities e.g. division by -1, exp laws e.g. $a^0 = 1$, log laws, graph of parabola, factorisation
P_{2D}	differentiation rules
P_3	algorithms (set of rules), e.g. long division or completing the square
P_4	formulae, e.g. quadratic formula and turning point formula
P_5	symbols (including notation)

Source: Extracted from Hechter (2020)

Each task solution is analysed according to the number of conceptual and procedural steps used to solve the task. The number of problem-solving categories is coded and counted, resulting in a label for each approach. A problem-solving category is only counted *once* when the *exact* procedure/concept is repeated for a particular approach in a task solution - the category for the repeated step is shaded in grey. A problem-solving category should be counted more than once when the *same* category requires *different thinking skills* for a procedure/concept in a particular step in the solution.

4. Results

The expected solution responses of the given task are shown first. Secondly, the author shares evidence of written student solutions since student work presented alternative methods that can inform teaching practices. Solution approaches are not uniquely described since the task have different solution methods. Table 2 display the content analysis of the task by the author.

Table 2. Approach 1

Approach 1		**Conceptual steps = 5, Procedural steps = 3**
$s(t)=30t-5t^2+20$ (position function)	**$C_4[1]$**	$C_4[1]$[1]: context - position function stone
$s'(t)=30-10t$ (velocity function)	**P_{2D}**	P_{2D}: differentiation rules
$s'(t)=0$ (velocity function = 0)	**$C_4[2]$**	$C_4[2]$[1]: context – velocity: time $_{\text{velocity zero}}$ $\Rightarrow$ time $_{\text{max height}}$
$\Rightarrow 30-10t=0$	**$C_{2D}[1]$**	$C_{2D}[1]$: link $f'(x)=0 \Rightarrow f$ local extrema
$\Rightarrow -10t=-30$	**$P_1[1]$**	$P_1[1]$: numerical calculations
$\Rightarrow t=3s$		$C_4[3]$[1:] context - position function: max height $\Rightarrow$ time $_{\text{velocity zero}}$
$s(3)=30t-5t^2+20$	**$C_4[3]$**	
(position function at t = 3)	**$C_{2D}[2]$**	$C_{2D}[2]$: link position function (max height) and velocity (zero) at t = 3
$\Rightarrow s(3)=90-45+20$	**$P_1[2]$**	$P_1[2]$: substitution into position function
$\Rightarrow s(3)=65m$ (max height)	**$P_1[1]$** **$C_4[1]$**	$P_1[1]$: numerical calculations, $C_4[1]$: context - position function stone

Approach 1 is demonstrated in Figure 1 where student work refers to position and velocity.

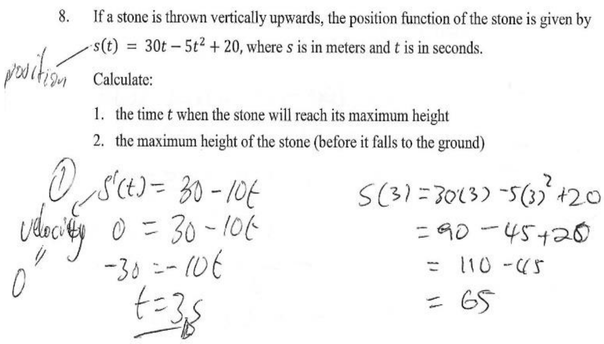

Figure 1 Approach 1 – analysis and evidence of student work.

Students suggested using the turning point (Figure 2, Approach 2) and axis of symmetry of a parabola (Figure 3, Approach 3) methods. The analyses are shared in Table 3.

[1] C_4 are different for position function, velocity function and maximum height, and requires different thinking skills

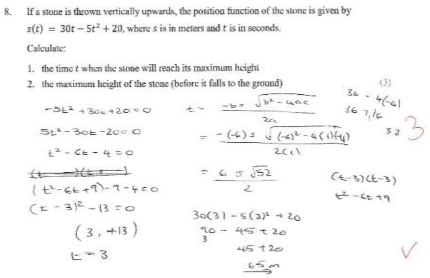

Figure 2 Approach 2 – evidence of student work.

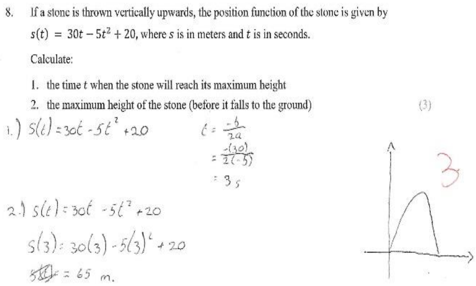

Figure 3 Approach 3 – evidence of student work.

Table 3. Approach 2 and Approach 3

Approach 2: Turning point formula	**Conceptual steps = 2, Procedural steps = 5**
$s(t) = 30t - 5t^2 + 20$ **$P_1[1]$**	$P_1[1]$: numerical calculations (division by -5)
$\Rightarrow t^2 - 6t - 4$ **P_3**	P_3: completing the square
$\Rightarrow (t - 6t + 9) - 4 - 9$ **P_{2F}**	P_{2F}: factorisation
$\Rightarrow (t-3)^2 - 13$	P_4: Turning point formula
TP: $(3,-13)$ **P_4**	C_{3F}: interpretation of turning point $(p;q)$ of a parabola: $p \Rightarrow$ (time; max height)
$\Rightarrow$ maximum height at $t = 3s$ **C_{3F}** **C_4**	C_4: contextual applications: (time, max height)
$\Rightarrow s(3) = 65m$ (max height) **$P_1[2]$** **$P_1[1]$**	$P_1[2]$: substitution, $P_1[1]$: numerical calculations
Approach 3: Axis of symmetry	**Conceptual steps = 2, Procedural steps = 3**
$s(t) = 30t - 5t^2 + 20$	P_4: formula for axis of symmetry
$\Rightarrow x = \frac{-b}{2a}$ **P_4**	$P_1[1]$: substitution into formula
$\Rightarrow x = \frac{-30}{2(-5)}$ **$P_1[1]$**	$P_1[2]$: numerical calculations
$\Rightarrow x = 3$ **$P_1[2]$**	C_{3F}: interpretation of axis of symmetry of a parabola (x; extreme value),
$\Rightarrow$ maximum height at $t = 3s$ **C_{3F}** **C_4**	C_4: context – (time, max height)
$\Rightarrow s(3) = 65m$ (max height) **$P_1[1]$** **$P_1[2]$**	$P_1[1]$: substitution $P_1[2]$: numerical calculations

Table 4 summarises the 182 students' approaches to do the task.

Table 4. Analysis of student strategies to do the task

Method	**n**	**%**
Calculus	172	94.5%
Turning point formula	3	1.7%
Axis of symmetry	7	3.8%

Most students used calculus knowledge to do the task (94.5%), but some used secondary school knowledge to solve it. These students used the turning point formula (1.7%) and the axis of symmetry (3.8%) of a parabola to calculate the time of maximum height. Approach 1 suggested five conceptual and three procedural steps (C > P), however, some student solutions suggest more procedural than conceptual steps steps (P > C).

5. Findings and discussion

The task cannot be described as mainly procedural or mainly conceptual since approach 1 suggests more conceptual than procedural steps (C=5 P=3, C > P), and approach 2 and 3 propose more procedural than conceptual steps (C=2 P=5, C=2 P=3, C < P). Most students used the position and velocity function answer the question (94.5%). However, it is important to note that 5.5% of students used the turning point and the axis of symmetry of the parabola in order to reach the correct answer. Approaches 2 and 3 only could be used since the position function is quadratic – it is not possible if a polynomial of a different degree were chosen. There are two main findings that emerged from the task analysis:

Finding 1: The task cannot be categorised as mainly conceptual or procedural

The analyses confirm the task solutions suggest more than one problem solving approach. Procedural and conceptual steps within task solutions are integrated (Kilpatrick et al., 2001) - knowledge categories alternate (Rittle-Johnson et al., 2015), some repeat (Rittle-Johnson, 2017), in no specific order (Rittle-Johnson et al., 2016). The task analyses provide evidence that disagrees with the statement that the approach used to solve a mathematical task is classified as either mainly conceptual or mainly procedural (Bergsten et al., 2017). The categorisation is further compicated since what is conceptual (and unfamiliar) for one student could be procedural (and familiar) for another, depending on whether the task has been seen before. Many students (procedurally) know that $s'(t) = 0$ indicates where the local extreme value(s) will be found.

Finding 2: Comparing and explaining of multiple solution strategies to promote learning

Students' work suggested additional methods using the turning point or axis of symmetry of a parabola for solving the contextual problem. The derivative of the position function method (where $s'(t)$ represents where the local extreme value(s)) could be connected and compared to the turning point and axis of symmetry methods to enhance conceptual understanding of the contextual problem and promote student learning (Star et al., 2016). This practice could enhance students' procedural flexibility, and development of conceptual and procedural knowledge amid students with prior knowledge of one of the methods (Durkin et al., 2017).

6. Recommendations

The concept-driven and skills-oriented perspectives should not stand in opposition to each other, in fact, teaching and learning strategies should focus on both concepts and procedures. Lecturers should refer to relations between concepts and procedures, and teaching strategies should explain and compare multiple problem-solving methods. Furthermore, I recommend analysing more empirical evidence of student solutions using the defined problem-solving categories to investigate different methods and possibly suggest additional teaching practices.

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