Iterative schemes for solving the Chandrasekhar H-equation using the Bernstein polynomials

M.A. Hernández-Verón[a], Eulalia Martínez[b].

[a] Dept. of Mathematics and Computation,
University of La Rioja. Logroño, Spain.
[b]Instituto Universitario de Matemática Multidisciplinar,
Universitat Politècnica de València. València, Spain.
E-mail: mahernan@unirioja.es, eumarti@mat.upv.es

Abstract

In this work, we use Newton-type iterative schemes to obtain a domain of existence of solution, approximate the solution of Chandrasekhar H-equations and deal with the case of nonlinear integral equations with non-separable kernels. A change of variable in the Chandrasekhar H-equation allows us to apply a previous study by describing nonlinear integral equations of Hammerstein-type with non-separable kernel. We use the Bernstein polynomials for approximating the non-separable kernel and then we apply a semilocal converge study done previously to the Chandrasekhar H-equation . Moreover, we apply Newton-type iterative schemes for some specific Chandrasekhar H-equations to approximate the H-function solution and compare our results with others obtained previously.

Keywords: Chandrasekhar H-equation, non-separable kernel, Newton-type iterative scheme, domain of existence of solution, domain of uniqueness of solution.
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1 Introduction

The Chandrasekhar H-equation is a nonlinear integral equation that appears in various physical problems (radioactive transfer, kinetic of gases, etc.) [4, 5, 14]. In addition, this kind of equations describe a great variety of mathematical and physics phenomena where one can transform any ordinary differential equation of second order with boundary conditions into a Hammerstein integral equation by using the Green's function [8, 13]. For

this reason, a great variety of analytical solutions and numerical solutions can be found in the literature for discussing these type of equations (see [1, 2, 3, 7]).

In our study, we consider the space $C[0,1]$ of the continuous functions in $[0,1]$ and endowed with the maximum norm $\|x\| = \max_{t\in[0,1]} |x(t)|$, $x \in C[0,1]$. Fixed the known characteristic function $\psi(t)$, which is a even polynomial in "t", our aim is to find a function $h \in C[0,1]$ verifying the H-equation

$$h(s) = 1 + h(s)\int_0^1 \frac{s}{s+t}\psi(t)h(t)dt. \tag{1}$$

A solution of this equation is called H- function.

For this purpose, we consider the operator $\mathcal{H} : \mathcal{C}[0,1] \longrightarrow \mathcal{C}[0,1]$ with

$$\mathcal{H}(h)(s) = h(s) - 1 - h(s)\int_0^1 \frac{s}{s+t}\psi(t)h(t)dt. \tag{2}$$

Then, a solution of (1) is obtained as a zero of the operator $\mathcal{H}$. In order to approximate this zero of $\mathcal{H}$ we can use the well known Newton method, which is written as follows:

$$\begin{cases} h_0 \text{ given in } \mathcal{C}[0,1] \\ h_{n+1} = h_n - [\mathcal{H}'(h_n)]^{-1}\mathcal{H}(h_n), \quad n \geq 0. \end{cases}$$

It is clear that, the calculus of the inverse of the linear operator $\mathcal{H}'$ is non-trivial. But, it is known [11] that, in the case that (1) is a nonlinear integral equation of Hammerstein type with separable kernel, it is possible to obtain this inverse operator and apply Newton's method to approximate a solution of equation (1). For this reason, we do a change in the integral equation (7) that allows us to transform it into a nonlinear integral equation of Hammerstein type.

Before doing this change, we need information about a solution of (1). So, it is clear that if $h^*(s)$ is a solution of (1), this function cannot be null in the interval $[0,1]$, since in this case it would exists $\tilde{s} \in [0,1]$ such as $h^*(\tilde{s}) = 0$, but by using (1) we have that $h^*(\tilde{s}) = 1$. On the other side, it is evident that $h^*(0) = 1$. Moreover, one can prove that h^* is derivable and verifies

$$(h^*)'(s) = (h^*)'(s)\int_0^1 \frac{s}{s+t}\psi(t)h^*(t)dt + h^*(s)\int_0^1 \frac{t}{(s+t)^2}\psi(t)h^*(t)dt, \tag{3}$$

so that

$$(h^*)'(s) = h^*(s)^2\int_0^1 \frac{t}{(s+t)^2}\psi(t)h^*(t)dt \tag{4}$$

and, from (4), we deduce that $(h^*)'(s) > 0$ and $h^*(s)$ is then a strictly increasing function. On the other hand, by taking norms in (1), we have

$$\begin{aligned} \|h^*\| &\leq 1 + \|h^*\| \max_{s\in[0,1]} \Big| \int_0^1 \frac{t}{(s+t)}dt \Big| \|\psi\|\|h^*\| \\ &. \leq 1 + \log(2)\|\psi\|\|h^*\|^2, \end{aligned}$$

so that

$$1 - \|h^*\| + \log(2)\|\psi\|\|h^*\|^2 \geq 0$$

and, by using $h(0) = 1$ and h is an increasing function, if $\|\psi\| \leqslant \dfrac{1}{4\log(2),}$ it then follows that $\|h^*\| \in \left[1, \frac{1-\sqrt{1-4\|\psi\|\log(2)}}{2\|\psi\|\log(2)}\right]$. Therefore, we can work with the open convex $\mathcal{D} = \{x \in \mathcal{C}[0,1] : x > 0\}$ and consider $\mathcal{H} : \mathcal{D} \subseteq \mathcal{C}[0,1] \longrightarrow \mathcal{C}[0,1]$. In this situation, we propose the change of variable $z(s) = \dfrac{1}{h(s)}$ in (1) and obtain the following integral equation of Hammerstein type:

$$z(s) = 1 - \int_0^1 \mathcal{K}(s,t)\mathcal{N}(z)(t)dt, \tag{5}$$

whose solution $z^*(s)$ allows us to obtain the H-function $h^*(s) = \dfrac{1}{z^*(s)}$, which is a solution of the main problem (1), where the kernel $\mathcal{K}(s,t) = \dfrac{s}{s+t}\psi(t)$ and the Nemystkii operator $\mathcal{N} : \mathcal{D} \longrightarrow \mathcal{C}[0,1]$, such as $\mathcal{N}(z)(t) = \dfrac{1}{z(t)}$, are well defined.

To approximate a solution of (5), we define the nonlinear operator $G : \mathcal{D} \subseteq \mathcal{C}[0,1] \longrightarrow \mathcal{C}[0,1]$ such that

$$G(z)(s) = z(s) - 1 + \int_0^1 \frac{s}{s+t}\psi(t)\frac{1}{z(t)}ds. \tag{6}$$

As

$$[G^{'}(z)]y(s) = y(s) - \int_0^1 \frac{s}{s+t}\psi(t)\frac{1}{z(t)^2}y(t)dt,$$

if the kernel $\mathcal{K}(s,t) = \dfrac{s}{s+t}\psi(t)$ is separable, we can then obtain $[G^{'}(z)]^{-1}$ and apply Newton's method to approximate a zero of $G(z) = 0$. However, in our case, the kernel is non separable. If we approximate the non-separable kernel by a separable kernel $\widetilde{\mathcal{K}}(s,t)$, then $\mathcal{K}(s,t) = \widetilde{\mathcal{K}}(s,t) + \mathcal{R}(s,t)$, with $\widetilde{\mathcal{K}}(s,t) = \sum_{j=1}^m \alpha_j(s)\beta_j(t)$, where α_j and β_j are two real functions in $[0,1]\times[0,1]$ and $\mathcal{R}(s,t)$ is the error made in the approximation. Then, we can obtain $\widetilde{G}^{'}(z)^{-1}$, where $\widetilde{G} : \mathcal{D} \subseteq \mathcal{C}[0,1] \longrightarrow \mathcal{C}[0,1]$, with

$$\widetilde{G}(z)(s) = z(s) - 1 + \int_0^1 \widetilde{\mathcal{K}}(s,t)\frac{1}{z(t)}dt. \tag{7}$$

As in [11], we can consider in the previous study the Newton-type iterative scheme given by:

$$\begin{cases} z_0 \text{ given in } \mathcal{D}, \\ z_{n+1} = z_n - [\widetilde{G}^{'}(z_n)]^{-1}G(z_n), n \geq 0. \end{cases} \tag{8}$$

Now, we analyze how to obtain $[\widetilde{G}'(z)]^{-1}$. For this, we have

$$[\widetilde{G}'(z_n)]y(s) = y(s) - \int_0^1 \widetilde{\mathcal{K}}(s,t)\frac{1}{z_n(t)^2}y(t)dt.$$

By denoting $A_j = -\int_0^1 \beta_j(t)\frac{1}{z_n(t)^2}y(t)dt$, we can then write

$$[\widetilde{G}'(z_n)]^{-1}w(s) = w(s) + \sum_{j=1}^{m}\alpha_j(s)A_j = y(s). \tag{9}$$

In addition, the integrals A_j can be obtained independently of $y(t)$. For this, we consider $i = 1, ..., m$ and multiply in (9) by $\beta_i(s)\frac{(-1)}{z_n(s)^2}$ and integrate with respect to the variable s, so that we have

$$A_i - \sum_{j=1}^{m}\Big(-\int_0^1 \beta_i(s)\frac{1}{z_n(s)^2}\alpha_j(s)ds\Big)A_j = -\int_0^1 \beta_i(s)\frac{1}{z_n(s)^2}w(s)ds.$$

Then, by denoting $a_{ij}(z_n) = -\int_0^1 \beta_i(s)\frac{1}{z_n(s)^2}\alpha_j(s)ds$ and $b_i(z_n) = -\int_0^1 \beta_i(s)\frac{1}{z_n(s)^2}w(s)ds$, we obtain the following linear system:

$$A_i - \sum_{j=1}^{m} a_{ij}(z_n)A_j = b_i(z_n), \quad 1 \le i \le m.$$

Which has an unique solution if the associated matrix has full rank, that is:

$$\begin{vmatrix} a_{11}(z_n)-1 & a_{12}(z_n) & a_{13}(z_n) & \dots & a_{1m}(z_n) \\ a_{21}(z_n) & a_{22}(z_n)-1 & a_{23}(z_n) & \dots & a_{2m}(z_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}(z_n) & a_{m2}(z_n) & a_{m3}(z_n) & \dots & a_{mm}(z_n)-1 \end{vmatrix} \neq 0.$$

Then, we assume that $\lambda = 1$ is not an eigenvalue of the matrix $(a_{ij}(z))_{1\le i,j\le m}$. Thus, by solving the linear system, we have the characterization of the inverse operator of $\widetilde{G}'$:

$$[\widetilde{G}'(z_n)]^{-1}G(z_n)(s) = G(z_n)(s) + \sum_{j=1}^{m}\alpha_j(s)A_j.$$

It is well known that the cost of Newton's method increases slightly if we do a second step without actualizing the derivative (see [10, 12]). So, in order to solve (7), we propose the following two steps of Newton-type iterative scheme:

$$\begin{cases} z_0 \text{ given in } \mathcal{D}, \\ y_n = z_n - [\widetilde{G}'(z_n)]^{-1}G(z_n), \\ z_{n+1} = y_n - [\widetilde{G}'(z_n)]^{-1}G(y_n), \ n \ge 0, \end{cases} \tag{10}$$

which uses the same inverse of the derivative operator in the second step. This iterative scheme achieves an efficient acceleration for the convergence of the Newton-type iterative scheme (8), see [9].

Thus, three are our main goals in this work. First, to approximate the H-function in a more precise and efficient way than by applying the Newton-type iterative scheme (8) used in [11]. Second, locating domains of existence of solution for the Chandrasekhar H-equation. And third, we express explicitly the H-function by means of a linear combination of functions.

The paper is organized as follows. In Section 1 we have introduced the problem to solve. Section 2 is devoted to the Bernstein polynomials that we use to approximate the non-separable kernel. We also include a practical application of Newton-type iterative schemes (8) and (10). In Section 3, we apply a theoretical result of semilocal convergence study, developed in a previous work, to the H-equation of Chandrasekhar. In Section 4, we develop an explicit expression for the H-function and apply it for specific examples. Finally, Section 5 is dedicated to the conclusions.

2 The Bernstein polynomials for the iterative schemes (8) and (10)

Notice that the nonlinear integral equation (6), for obtaining the H-function $h^*(s) = \dfrac{1}{z^*(s)}$, has a non-separable kernel $\mathcal{K}(s,t) = \dfrac{s}{s+t}\psi(t)$. Then, to approximate this kernel by a separable kernel, we consider $f : [0,1] \longrightarrow \mathbb{R}$ given by $f(s) = \dfrac{s}{s+t}\psi(t)$, which is a continuous function in $[0,1]$, for a fixed value $t \in [0,1]$. Now, we use the Bernstein polynomials [15] to approximate the non-separable kernel $\mathcal{K}(s,t)$ by considering different approximations, due to the fact that these polynomials converge uniformly to $f(s)$ on $[0,1]$; $\lim_m \|B_m(f) - f\| = 0$ [14]. So, from the Bernstein polynomials associated to f given by

$$B_m(f)(s,t) = \sum_{j=0}^{m} f(\frac{j}{m})\binom{m}{j}s^j(1-s)^{m-j} = \sum_{j=0}^{m} \frac{j}{j+mt}\psi(t)\binom{m}{j}s^j(1-s)^{m-j},\ m \geqslant 0,$$

we consider $\mathcal{K}(s,t) = \widetilde{\mathcal{K}}(s,t) + \mathcal{R}(s,t)$, with

$$\widetilde{\mathcal{K}}(s,t) = \sum_{j=0}^{m} \alpha_j(s)\beta_j(t), \tag{11}$$

$\alpha_j(s) = \binom{m}{j}s^j(1-s)^{m-j}$, $\beta_j(t) = \dfrac{j}{j+mt}\psi(t)$ and $\mathcal{R}(s,t) = B_m(f)(s,t) - f(s)$.

Thus, we have

$$\widetilde{G}(z)(s) = z(s) - 1 + \sum_{j=0}^{m}\binom{m}{j}s^j(1-s)^{m-j}\int_0^1 \frac{j}{j+mt}\psi(t)\frac{1}{z(t)}dt,$$

and

$$[\widetilde{G}'(z)]y(s) = y(s) - \sum_{j=0}^{m} \binom{m}{j} s^j (1-s)^{m-j} \int_0^1 \frac{j}{j+mt} \psi(t) \frac{1}{z(t)^2} y(t) dt.$$

2.1 Algorithm of iterative schemes (8) and (10)

Now, we apply the iterative schemes (8) and (10) to approximate the solution of equation (5) using the procedure indicated in Section 1 to obtain $[\widetilde{G}'(z)]^{-1}$, we consider a starting guess $z_n(s) \in \mathcal{C}[0,1]$, for $n \geqslant 0$, and apply the following algorithm:

1. **First step:**

$$w(z_n)(s) = G(z_n)(s) = z_n(s) - 1 + \int_0^1 \frac{s}{s+t} \psi(t) \frac{1}{z_n(t)} dt.$$

2. **Second step:** Calculate the following integrals for $i, j = 0, ..., m$.

$$a_{ij}(z_n) = -\binom{m}{j} \int_0^1 \frac{i}{i+ms} \psi(s) \frac{1}{z_n(s)^2} s^j (1-s)^{m-j} ds,$$

and

$$b_i(z_n) = -\int_0^1 \frac{i}{i+ms} \psi(s) \frac{1}{z_n(s)^2} w(z_n)(s) ds.$$

3. **Third step:** To obtain A_j with $j = 0, 1..., m$, we solve the following linear system

$$A_i - \sum_{j=0}^{m} a_{ij}(z_n) A_j = b_i(z_n), \quad i = 0, 1, ..., m.$$

4. **Fourth step:** The final iteration for iterative method (8) is

$$y_n(s) = z_n(s) - w(z_n)(s) - \sum_{j=0}^{m} \binom{m}{j} s^j (1-s)^{m-j} A_j.$$

5. **Fifth step:** For the two steps iterative method (10).

$$w(y_n)(s) = G(y_n(s)) = y_n(s) - 1 + \int_0^1 \frac{s}{s+t} \psi(t) \frac{1}{y_n(t)} dt.$$

6. **Sixth:** Calculate the following integrals

$$\hat{b}_i(y_n) = -\int_0^1 \frac{i}{i+ms} \psi(s) \frac{1}{z_n(s)^2} w(y_n)(s) ds,$$

7. **Seventh step:** To obtain $\hat{A}_j$, with $j = 0, 1, 2, ..., m$, we solve the following linear system

$$\hat{A}_i - \sum_{j=0}^{m} a_{ij}(z_n)\hat{A}_j = \hat{b}_i(y_n), \quad i = 1, 2, ..., m.$$

8. **Eighth step:** Calculate

$$z_{n+1}(s) = y_n(s) - w(y_n)(s) - \sum_{j=0}^{m} \binom{m}{j} s^j (1-s)^{m-j} \hat{A}_j.$$

2.2 Computational cost

To obtain the computational cost of scheme (8), we have to solve a linear system of order $m \times m$. For this, we do a LU factorization for the matrix $(a_{ij}(z_n))$, whose cost in number of products and quotients is $\frac{m(m^2-1)}{3}$, in terms of m the degree of the Bernstein polynomial that we use in the approximation, and then, we need to solve two triangular systems, with cost m^2, that is order 2 in contrast with the order 3 from the LU factorization. But, in case of scheme (10), we have to solve two linear systems with only one LU factorization. So, in order to compare the computational cost for the proposed Newton-type iterative schemes, we obtain that the total computational cost is

$$\left(\frac{m(m^2-1)}{3} + m^2\right) k$$

, when we apply iterative scheme (8), and

$$\left(\frac{m(m^2-1)}{3} + 2m^2\right) k$$

for the iterative scheme (10), where k is the number of iterations done for reaching the tolerance. Notice that, as we can see in [9], the iterative scheme (10) needs,, in general, to perform fewer iterations to approximate the solution with a given tolerance. Thus, in the following section, by applying iterative process (10), we obtain better accuracy in our approach to the solution and with an similar operational cost to the application of iterative scheme (8). We can therefore conclude that the application of iterative scheme (10) is more efficient than the application of iterative scheme (8).

2.3 Practical application of iterative schemes (8) and (10)

First, we use the Bernstein polynomials to approximate the non-separable kernel of equation (5) by using different approximations in (11), $m = 2$ and $m = 3$. We consider the iterative schemes (8) and (10), that are implemented by the previous algorithm. We compute the integrals in the second and sixth steps of the algorithm by applying Gauss-Legendre's formula with 8 nodes.

Second, in order to compare the iterative schemes (8) and (10), we consider the Example 1 given in [6], with $\psi = \frac{1}{2}\psi_0$, for solving the integral equation (5) and approximate the H-function by using iterative schemes (8) and (10), which are denoted by $IS(8)$ and $IS(10)$ in the numerical results. In both schemes we take starting guess $x_0(s) = 2$. In Table 1, the results are obtained by working with 10 significant digits. In the first two rows we can see the number of iterations, k, needed to reach a tolerance at 10^{-6} and the residual error, $\|x_n(s) - x_{n-1}(s)\|$, when the kernel is approximated by the Bernstein polynomial of orders $m = 2$ and $m = 3$. Next row is the computational cost, CC, calculated in Section 2.2, and in the last row, once we have approximated the H-function h^*, we calculate numerically the following integral by Gauss-Legendre formula

$$\int_0^1 h^*(t)dt,$$

which is denoted by I_{app} in Tables 1 and 2 . To measure the accuracy of our approach, we compare this value with the exact value of the previous integral, which is given by:

$$I_{ex} = \frac{2}{\psi_0}[1 - (1 - \psi_0)^{1/2}].$$

In Table (2) we find the same rows we can find , but we work now with 50 significant digits to reach a tolerance of 10^{-15}.

Tables 1 and 2 show that iterative scheme (10) always reaches the stopping criterion quickly. Moreover, this iterative scheme is usually more efficient than the method (8) taking into account the computational cost and the approximation obtained. Obviously, for $m = 3$, both schemes do more operations, so that the computational cost is higher than for $m = 2$.

To finish this practical application, we can see in Table 3, that the evaluation of the iterations $1/z_1(s)$ and $1/z_2(s)$ give approximately the same results for the H-function as those obtained by Chandrasekhar in [5]. Therefore, we check that just two iterations of the iterative scheme (10) provide us with excellent approximations to the H-function.

3 Convergence of iterative scheme (10) for approximating H-functions

In this section we are interested in studying the convergence of iterative scheme (10). For this, we consider the semilocal convergence result given in [9] for the method

$$\begin{cases} z_0 \text{ given in } \Omega, \\ y_n = z_n - A(z_n)^{-1}(z_n - F(z_n)) \\ z_{n+1} = y_n - A(z_n)^{-1}(y_n - F(y_n)), \ \ n \geq 0, \end{cases} \tag{12}$$

where

$$F(z)(s) = f(s) + \lambda \int_a^b \mathcal{K}(s,t)[\mathcal{N}(z)](t)dt$$

ψ_0		IS (8) m=2	IS (10) m=2	IS (8) m=3	IS (10) m=3
	k	4	3	4	3
	$\|x_n - x_{n-1}\|$	2.0360e-07	1.7199e-09	9.8215e-08	8.0487e-10
0.1	CC	24	30	68	78
	$\|I_{ex} - I_{app}\|$	4.7115e-11	3.2994e-15	1.6064e-11	2.5119e-15
	k	5	3	4	3
	$\|x_n - x_{n-1}\|$	9.6324e-09	3.1035e-08	8.6818e-07	1.4540e-08
0.2	CC	30	30	68	78
	$\|I_{ex} - I_{app}\|$	4.7437e-12	8.2330e-14	2.9990e-10	2.0706e-14
	k	5	3	5	3
	$\|x_n - x_{n-1}\|$	5.5241e-08	1.7819e-07	1.9941e-08	8.3600e-08
0.3	CC	30	30	85	78
	$\|I_{ex} - I_{app}\|$	4.3311e-11	1.1469e-12	1.1035e-11	2.8125e-13
	k	5	3	5	3
	$\|x_n - x_{n-1}\|$	1.9913e-07	6.4282e-07	7.2224e-08	3.0212e-07
0.4	CC	. 30	30	85	78
	$\|I_{ex} - I_{app}\|$	2.2166e-10	8.0314e-12	5.6633e-11	1.9689e-12
	k	5	4	5	3
	$\|x_n - x_{n-1}\|$	5.5871e-07	3.8776e-10	2.0376e-07	8.5003e-07
0.5	CC	30	40	85	78
	$\|I_{ex} - I_{app}\|$	8.3065e-10	8.7638e-15	2.1295e-10	9.4620e-12

Table 1: Significant digits 10; tol=10^{-6}; maxiter =30;

ψ_0		IS (8) m=2	IS (10) m=2	IS (8) m=3	IS (10) m=3
	k	8	5	7	5
	$\|x_n - x_{n-1}\|$	9.8731e-18	8.3227e-20	7.4428e-16	1.2007e-20
0.1	CC	48	50	119	130
	$\|I_{ex} - I_{app}\|$	2.2690e-15	2.2690e-15	2.2691e-15	2.2690e-15
	k	9	5	8	5
	$\|x_n - x_{n-1}\|$	8.3139e-18	2.6555e-17	2.2762e-16	3.8065e-18
0.2	CC	54	50	136	130
	$\|I_{ex} - I_{app}\|$	6.3163e-16	6.3163e-16	6.3171e-16	6.3163e-16
	k	9	5	9	5
	$\|x_n - x_{n-1}\|$	2.6803e-16	8.5736e-16	2.9531e-17	1.2211e-16
0.3	CC	54	50	153	130
	$\|I_{ex} - I_{app}\|$	3.5141e-16	3.5162e-16	3.5161e-16	3.5162e-16
	k	42	6	9	6
	$\|x_n - x_{n-1}\|$	3.9034e-17	1.4304e-18	3.7437e-16	1.1149e-19
0.4	CC	252	60	153	156
	$\|I_{ex} - I_{app}\|$	2.3161e-16	2.3156e-16	2.3186e-16	2.3156e-16
	k	10	6	10	6
	$\|x_n - x_{n-1}\|$	3.8688e-16	1.8258e-17	3.1348e-17	1.4081e-18
0.5	CC	60	60	170	156
	$\|I_{ex} - I_{app}\|$	4.0198e-16	4.0140e-16	4.0144e-16	4.0140e-16

Table 2: Significant digits 50; tol=10^{-15}; maxiter =30;

s	$1/z_1(s)$	$1/z_2(s)$	Chandrasekhar [5]
0	1.0	1.0	1.0
0.05	1.00776	1.00780	1.00783
0.1	1.01231	1.01238	1.01238
0.15	1.01575	1.01584	1.01584
0.2	1.01853	1.01864	1.01864
0.25	1.02086	1.02099	1.02099
0.3	1.02286	1.02301	1.02300
0.35	1.02461	1.02476	1.02475
0.4	1.02615	1.02631	1.02630
0.45	1.02751	1.02768	1.02768
0.5	1.02875	1.02892	1.02892
0.55	1.02986	1.03004	1.03004
0.6	1.03087	1.03106	1.03106
0.65	1.03180	1.03199	1.03199
0.7	1.03265	1.03285	1.03284
0.75	1.03343	1.03364	1.03363
0.8	1.03416	1.03437	1.03436
0.85	1.03483	1.03504	1.03504
0.9	1.03546	1.03567	1.03567
0.95	1.03604	1.03626	1.03626
1.0	1.03659	1.03682	1.03682

Table 3: $h^*(x), m = 2, \psi_0 = 0.1$

and

$$A(z) = (\mathcal{I} - \widetilde{F})'(z) \qquad \text{with} \qquad \widetilde{F}(z)(s) = f(s) + \lambda \int_a^b \widetilde{\mathcal{K}}(s,t)[\mathcal{N}(z)](t)dt,$$

being $\widetilde{\mathcal{K}}(s,t)$ a separable approximation of the non-separable kernel $K(s,t)$.

Theorem 1. *[9] Assume*

(I) *The operator $A(z_0)^{-1}$ exists, for some $z_0 \in \Omega \subseteq \mathcal{C}[a,b]$, with $\|A(z_0)^{-1}\| \leq \beta$ and $\|A(z_0)^{-1}(z_0 - F(z_0))\| \leq \eta$.*

(II) *The operator $\mathcal{N}'$ is ω-Lipschitz continuous and*

$$\|\mathcal{N}'(u) - \mathcal{N}'(v)\| \leq \omega(\|u - v\|) \text{ for } u, v \in \Omega, \tag{13}$$

where $\omega : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous and non-decreasing function satisfying $\omega(\alpha x) \leq h(\alpha)\omega(x)$ for $\alpha \in [0,1]$ and $x \in [0, +\infty)$, with $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a continuous and non-decreasing function.

If the equation

$$t = \frac{1 + \phi(\eta, t)}{1 - \xi(\eta, \phi(\eta,t)\eta, t)\phi(\eta,t)}\eta \tag{14}$$

has at least one positive real root and the smallest positive real root, denoted by R, satisfies $\beta\,|\lambda|\,M\,\omega(R) < 1$, $Q = \xi(\eta, \phi(\eta,R)\eta, R)\phi(\eta,R) < 1$ and $B(x_0,R) \subseteq \Omega$, then the sequence generated by (12) converges to the solution z^* of the equation $z - F(z) = 0$, starting at the function z_0, and $z^* \in \overline{B(x_0,R)}$, where $\phi(t,u) = \mu(u)|\lambda|\Big(LD\omega(t) + M(\epsilon + \omega(u))\Big)$, $\xi(t,u,v) = \mu(v)|\lambda|\Big[L\Big(\omega(t) + D\omega(u)\Big) + M(\epsilon + \omega(v))\Big]$, $M = \max_{s\in[a,b]} \int_a^b \|\mathcal{R}(s,t)\|\|dt$, $D = \int_0^1 h(t)dt$ and $\epsilon = \|\mathcal{N}'(x_0)\|$.

In our case, we have

$$F(z)(s) = 1 - \int_0^1 \frac{s}{s+t}\psi(t)\frac{1}{z(t)}dt \qquad \text{and} \qquad A(z) = (\mathcal{I} - \widetilde{F})',$$

with $\widetilde{F}(z)(s) = 1 - \int_0^1 B_m(f)(s,t)\dfrac{1}{z(t)}dt$. So, it is clair that

$$(\mathcal{I} - F)(z)(s) = G(z)(s) \qquad \text{and} \qquad A(z)(s) = \widetilde{G}'(z)(s).$$

Therefore, the iterative scheme (12) coincides with the iterative scheme (10).

Then, from this result, we can study the convergence of iterative scheme (10) when is applied to the Chandrasekhar equation. So, we first consider, if $\|\psi\| \leqslant \dfrac{1}{4\log(2)}$,

$\widetilde{G}, G : \Omega \subseteq \mathcal{C}[0,1] \longrightarrow \mathcal{C}[0,1]$, where the domain $\Omega = \{x \in \mathcal{C}[0,1] : p < \|x\| < q\}$, with $p = \dfrac{2\|\psi\|\log(2)}{1-\sqrt{1-4\|\psi\|\log(2)}}$. and q a positive real number greater than p to be determined.

First, fixed $m \in \mathbb{N}$ and the characteristic function ψ, we consider $z_0(s) = \delta$ with $p < \delta < q$. As $\|I - \widetilde{G}'(z_0)\| \leq \frac{L}{\delta^2}$, if $\dfrac{L}{\delta^2} < 1$, the operator $\widetilde{G}(z_0)^{-1}$ exists with $\|\widetilde{G}'(z_0)^{-1}\| \leq \dfrac{\delta^2}{\delta^2 - L}$ and $L = \max_{s\in[0,1]} \int_0^1 |\widetilde{\mathcal{K}}(s,t)|dt$.

Second, since

$$[\widetilde{G}'(z_0)]^{-1} G(z_0)(s) = G(z_0)](s) + \sum_{j=1}^{m} \alpha_j(s) A_j,$$

then $\|\widetilde{G}'(z_0)^{-1} G(z_0)\| \leqslant |\delta - 1| + \dfrac{\|\psi\|\log(2)}{\delta} + \sum_{j=1}^{m} \|\alpha_j\||A_j|$.

Third, we have $\mathcal{N}'$ is a Lipschitz continuous operator such that

$$\|\mathcal{N}'(u) - \mathcal{N}'(v)\| \leq \left\| \frac{(-1)}{u^2} - \frac{(-1)}{v^2} \right\| \leq \frac{2\delta}{p^4}\|u - v\| \text{ for } u, v \in \Omega.$$

Moreover, it is easy to check that $\|\mathcal{N}'(z_0)\| \leqslant \dfrac{1}{\delta^2}$. So, we have all the necessary parameters to apply the convergence result given in [9]. For this, we define the following auxiliary real functions

$$\begin{aligned} \mu(t) &= \frac{\delta^2 p^4}{p^4(\delta^2 - L) - 2\delta^3 L t}, \\ \phi(t,u) &= \frac{\mu(u)}{\delta^2 p^4}\Big(\delta^3(Lt + 2Mu) + Mp^4\Big), \\ \xi(t,u,v) &= \frac{\mu(v)}{\delta^2 p^4}\Big(\delta^3 L(2t+u) + M(p^4 + 2\delta^3 v)\Big), \end{aligned} \tag{15}$$

where $M = \max\limits_{s\in[0,1]} \displaystyle\int_0^1 \|\mathcal{R}(s,t)\|dt$.

Under these conditions and the convergence result given in [9], we obtain the following particular result

Theorem 2. *Fixed $m = 2$ and $\psi_0(t) = 0.1$, we consider $z_0(t) = 1$ and $\Omega = \{x \in \mathcal{C}[0,1] : 0.9641 < \|x\| < 2\}$. Then, there exists $R = 0.2057$ such that the sequence* (10) *remains in $B(z_0, R) \subseteq \Omega$ and converges to $z^* \in \overline{B(z_0, R)}$.*

Notice that, we can obtain a semilocal convergence result for both fixed values m and $\psi(t)$.

In Tables 4 and 5, we give the semilocal convergence radius for different cases that we consider in (2). Notice that these results have been obtained by fixing the degree of the

m	2	3	2	3	2	3	2	3
δ	1	1	1.1	1.1	1.2	1.2	1.3	1.3
L	0.075	0.0945	0.075	0.0945	0.075	0.0945	0.075	0.0945
R	0.2057	0.2554	0.2899	0.3365	0.3903	0.4395	0.5121	0.5746

Table 4: p= 0.9641, q=2, $M = \psi_0(t) = 0.1$

m	2	3	2	3	2	3	2	3
δ	1	1	1.1	1.1	1.2	1.2	1.3	1.3
L	0.375	0.4722	0.375	0.4722	0.375	0.4722	0.375	0.4722
R	0.0135	0.0546	0.0168	0.0559	0.0248	0.05775	0.03492	0.0603

Table 5: p= 0.7769, q=2, $M = \psi_0(t) = 0.5$

Bernstein polynomial, $m = 2$ and $m = 3$, the values of ψ and the subset Ω. For these values we consider different starting function $z_0(t) = \delta$ and obtain the bonds needed to construct auxiliary functions given in (15) that allow us to obtain the radius R. As we can observe in the results, the radius is always bigger for $m = 3$ and decreases when the value of ψ increases.

4 An explicit expression for the H-functions

Now, we use a quadrature formula to approximate the integral of (1),

$$\int_0^1 \phi(t)\, dt \simeq \sum_{k=1}^{\ell} \gamma_k \phi(\theta_k), \tag{16}$$

where the ℓ, nodes $\{\theta_k\}$ and weights $\{\gamma_k\}$ are known.

Let us check that the approximation of solution of equation (5), given by the iterative schemes (8) and (10), can be expressed by a lineal combination of the functions families $\left\{\dfrac{s}{s+\theta_k}\right\}_{k=1}^{\ell}$ and $\left\{s^j(1-s)^{m-j}\right\}_{j=1}^{m}$. For this, we consider the algorithm that we can see in Section 2.1 and taking into account (16), we obtain the following algorithm to apply the iterative schemes (8) and (10).

1. **First step:**

$$\text{Evaluate } z_n(\theta_k), \text{for } k = 1, ..., \ell,$$

$$w(z_n)(s) = z_n(s) - 1 + \sum_{k=1}^{\ell} \frac{\gamma_k \psi(\theta_k)}{z_n(\theta_k)} \frac{s}{s+\theta_k},$$

$$\text{Evaluate } w(z_n)(\theta_k), \text{for } k = 1, ..., \ell.$$

2. **Second step:** Calculate the following coefficients for $i, j = 0, ..., m$.

$$a_{ij}(z_n) = \binom{m}{j} \sum_{k=1}^{\ell} \frac{i}{i + m\theta_k} \psi(\theta_k) \frac{(-1)}{z_n(\theta_k)^2} \theta_k^j (1 - \theta_k)^{m-j}$$

and

$$b_i(z_n) = \sum_{k=1}^{\ell} \frac{i}{i + m\theta_k} \psi(\theta_k) \frac{(-1)}{z_n(\theta_k)^2} w(z_n)(\theta_k).$$

3. **Third step:** To obtain A_j with $j = 0, 1..., m$, we solve the following linear system

$$A_i - \sum_{j=0}^{m} a_{ij}(z_n) A_j = b_i(z_n), \quad i = 0, 1, ..., m.$$

4. **Fourth step:** Final iteration for iterative method (8).

$$y_n(s) = 1 - \sum_{k=1}^{\ell} \left[\frac{\gamma_k \psi(\theta_k)}{z_n(\theta_k)} \right] \frac{s}{s + \theta_k} - \sum_{j=0}^{m} \left[A_j \binom{m}{j} \right] s^j (1 - s)^{m-j}.$$

5. **Fifth step:** For iterative method (10).

$$\text{Evaluate } y_n(\theta_k), \text{for } k = 1, ..., \ell,$$

$$w(y_n)(s) = y_n(s) - 1 + \sum_{k=1}^{\ell} \frac{\gamma_k \psi(\theta_k)}{y_n(\theta_k)} \frac{s}{s + \theta_k},$$

$$\text{Evaluate } w(y_n)(\theta_k), \text{for } k = 1, ..., \ell.$$

6. **Sixth step:**

$$\hat{b}_i(y_n) = \sum_{k=1}^{\ell} \frac{i}{i + m\theta_k} \psi(\theta_k) \frac{(-1)}{y_n(\theta_k)^2} w(y_n)(\theta_k).$$

7. **Seventh step:** To obtain $\hat{A}_j$ with $j = 0, 1, 2, ..., m$ we solve the following linear system

$$\hat{A}_i - \sum_{j=0}^{m} a_{ij}(z_n) \hat{A}_j = \hat{b}_i(y_n), \quad i = 1, 2, ..., m.$$

8. **Eighth step:**

$$z_{n+1}(s) = 1 - \sum_{k=1}^{\ell} \left[\frac{\gamma_k \psi(\theta_k)}{y_n(\theta_k)} \right] \frac{s}{s + \theta_k} - \sum_{j=0}^{m} \left[\hat{A}_j \binom{m}{j} \right] s^j (1 - s)^{m-j}.$$

As we just tested in the above algorithm, the iterations obtained using iterative schemes (8) and (10) can be obtained by a linear combination of the function families given by $\left\{ \frac{s}{s + \theta_k} \right\}_{k=1}^{\ell}$ and $\left\{ s^j (1 - s)^{m-j} \right\}_{j=1}^{m}$.

4.1 Example 2

Now, for Example 2, the application of iterative scheme (10) and result of Section 4 allow us to compare our results with those obtained in [6] by using Adomian's method. So, we consider $\psi(t) = \frac{3}{8}t^2(1-t^2)$ for solving the integral equation (5) and obtain the H-function explicitly. Notice that done $\|\psi\| < \frac{3}{32} \leq \frac{1}{4\log(2)}$ we can apply the theoretical development done in Section 1. Then, we use the Bernstein polynomials to approximate the non-separable kernel of equation (5) by using $m = 2$ and after two iterations with starting guess $z_0(s) = 1$, the H-function obtained can be expressed as: $H_2(s) = \frac{1}{z_2(s)}$ with

$$\begin{aligned} z_2(s) &= -\frac{7.493e-6s}{s+0.01986} - \frac{0.01474s}{s+0.7628} - \frac{0.0004299s}{s+0.1017} - \frac{0.00965s}{s+0.4083} - \frac{0.0067s}{s+0.8983} \\ &- \frac{0.0031s}{s+0.23723} - \frac{0.00074s}{s+0.9801} - \frac{0.01588s}{s+0.5917} + 7.74701e-8s^2 + 11.22e-8s(1-s) + 1. \end{aligned}$$

In order to compare our results with the previous ones, we calculate the integral cited in [6], pag. 71,

$$\int_0^1 \psi(t)H(t)dt = 1 - \sqrt{0.9} = 0.0513167019,$$

which is the exact value for this example. We approximate the value of this integral by Gauss-Legendre formula with 8 nodes after two iterations

$$\int_0^1 \psi(t)H_2(t)dt = 0.0513166956,$$

while the value obtained after two iterations with Adomian's method is

$$\int_0^1 \Psi(t)H_2(t)dt = 0.05128.$$

This results confirms the competitiveness of the $H-$function that we obtain in this study.

4.2 Example 3

Finally, we consider $\psi(t) = \frac{3}{32}(1-t^2)^2$ for solving the integral equation (5) and obtain the $H-$function explicitly. Under the same conditions as in Example 2, we get after two iterations the following expression for the $H-$function: $H_2(s) = \frac{1}{z_2(s)}$ with

$$\begin{aligned} z_2(s) &= -\frac{0.00477s}{s+0.01986} - \frac{0.01474s}{s+0.7628} - \frac{0.01039s}{s+0.1017} - \frac{0.01219s}{s+0.4083} - \frac{0.0004041s}{s+0.8983} \\ &- \frac{0.01344s}{s+0.2372} - \frac{7.639e-6s}{s+0.9801} - \frac{0.007443s}{s+0.5917} + 4.779e-8s^2 + 7.616e-8s(1-s) + 1. \end{aligned}$$

s	$H_2(s)$ IS (10)	$H_2(s)$ (Adomian)	Chandrasekhar [5]
0	1.0	1	1
0.05	1.0114	1.01120	1.01145
0.1	1.01723	1.01688	1.01724
0.15	1.02133	1.02088	1.02134
0.2	1.02448	1.02393	1.02448
0.25	1.027	1.02639	1.02700
0.3	1.02908	1.02841	1.02909
0.35	1.03085	1.03012	1.03085
0.4	1.03236	1.03159	1.03236
0.45	1.03368	1.03287	1.03368
0.5	1.03483	1.03399	1.03483
0.55	1.03586	1.03499	1.03586
0.6	1.03678	1.03588	1.03679
0.65	1.03761	1.03668	1.03761
0.7	1.03836	1.03741	1.03836
0.75	1.03904	1.03807	1.03904
0.8	1.03966	1.03867	1.03966
0.85	1.04024	1.03923	1.04024
0.9	1.04076	1.03974	1.04076
0.95	1.04125	1.04021	1.04125
1.0	1.0417	1.04065	1.04170

Table 6: $H-$function for Example 3 with different methods.

Resulting in this case the same value for the integral

$$\int_0^1 \psi(t)H(t)dt = 1 - \sqrt{0.9} = 0.0513167019$$

that we approximate by Gauss-Legendre formula with 8 nodes

$$\int_0^1 \psi(t)H_2(t)dt = 0.0513166993$$

while the value obtained after two iterations with Adomian's method is

$$\int_0^1 \psi(t)H_2(t)dt = 0.05128.$$

These results confirm again the competitiveness of the $H-$function obtained by the iterative method (10). Moreover, we evaluate the $H-$function and compare the values with that obtained by Adomian's method developed in [6], see Table 6, where we can appreciate the high accuracy obtained by using the algorithm presented in this work for implementing iterative scheme (10).

5 Conclusions

In this work, we study completely how to obtain the solution of the H-equation that describe a nonlinear integral equation of Hammerstein type with non-separable kernel. Using the Bernstein polynomials, we approximate this non-separable kernel by means a separable kernel so that we can apply Newton-type iterative methods to approximate the H-function solution. The H-function obtained is compared with the results obtained by Chandrasekhar and the Adomian's method, and show the competitiveness of the algorithms presented in this work.

References

[1] I. Argyros, Quadratic equations and applications to Chandrasekhar's and related equations. Bulletin of the Australian Mathematical Society, 32((1985), 275-292.

[2] I. Argyros, On a new iteration for solving Chandrasekhar's H-equation. Mathematical Sciences Research Hot-Line. 7. 10.22199/S07160917.1988.0015.00002. (2001).

[3] H. O. Bakodah, Mohamed Abdalla Darwish, On Discrete Adomian Decomposi- tion Method with Chebyshev Abscissa for Nonlinear Integral Equations of Hammerstein Type. Advances in Pure Mathematics, 2 (2012), 310–313.

[4] I. W. Busbridge, On solutions of Chandrasekhar's integral equation. Trans. Amer. Math. Soc., 105 (1962), 112–117.

[5] S. Chandrasekhar, Radiative Transfer. Dover, New York, 1960.

[6] E. Y. Deeba and S. A. Khuri, The Decomposition Method Applied to Chandrasekhar H-Equation, International conference. Appl. Math. Comput., 77 (1996), 67–78.

[7] L. M. Delves, J. L. Mohamed, Computational methods for integral equations. Cambridge University Press, 1985.

[8] R. Ezzati, K. Shakibi, On Approximation and Numerical Solution of Fredholm- Hammerstein Integral Equations Using Multiquadric Quasi-interpolation. Communication in Numerical Analysis, 112 (2012), 1–10.

[9] M. A. Hernández-Verón, E. Martínez, S. Sing, A reliable treatment to solve nonlinear Fredholm integral equations with non-separable kernel, International conference. Mathematical Methods in Science and Engineering Conference (CMMSE-2020).

[10] M. A. Hernández-Verón, E. Martínez, Carles Teruel, Semilocal convergence of a k-step iterative process and its application for solving a special kind of conservative problems. Numer. Algor., 76 (2017), 309–331.

[11] M. A. Hernández, M. A. Salanova, A Newton-like iterative process for the numerical solution of Fredholm nonlinear integral equations. J. Integral Equations Appl. ,17 (2005), 1–17.

[12] M. A. Hernández-Verón, E. Martínez, On the semilocal convergence of a three steps Newton-type iterative process under mild convergence conditions. Numer. Algor., 70 (2015), 377–392.

[13] K. Maleknejad, H. Derili, The collocation method for Hammerstein equations by Daubechies wavelets. Appl. Math. Comput., 172 (2006), 846–864.

[14] G. M. Phillips, P. J. Taylor, Theory and Applications of Numerical Analysis. Academic Press Inc. (Second Edition), 1996.

[15] V. Sukkrasanti, P. Lerdkasem, An Error Bound on Uniform Approximation of Bounded Function by Bernstein Polynomial. International Mathematical Forum, 3 (2008) 1409–1414