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Iterative schemes for finding all roots simultaneously of nonlinear equations

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ABSTRACT

In this paper, we propose a procedure that can be added to any iterative scheme in order to turn it into an iterative method for approximating all roots simultaneously of any nonlinear equations. By applying this procedure to any iterative method of order p, we obtain a new scheme of order of convergence $2p$. Some numerical tests allow us to confirm the theoretical results and to compare the proposed schemes with other known methods for simultaneous roots of polynomial and non-polynomial functions.

1. Introduction

In many problems of Sciences and Engineering we need to obtain the solutions of nonlinear equations $f(x) = 0$, $f : D \subseteq \mathbb{C} \to \mathbb{C}$, but in general, solving these type of equations is usually neither easy nor possible. For this reason, iterative methods are useful, which, given an initial estimate of the solution, generate a sequence of iterations that, under certain conditions, converge to the root of nonlinear equation $f(x) = 0$. These schemes can be classified in to single and simultaneous root finding methods. This paper addresses both types of iterative schemes.

The simultaneous root finding schemes obtain, from a set of initial approximations, sequences of iterations, which, under certain conditions, converge to all the roots of the equation simultaneously. See, for example, [1–6] and the references therein.

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In [7] Ehrlich presented his third-order method for simultaneous computation of all zeros of a polynomial f. It is defined by the fixed point iteration

$$x^{(k+1)} = \Phi(x^{(k)}) = (\phi_1(x^{(k)}), \phi_2(x^{(k)}), \ldots, \phi_n(x^{(k)})),$$

where $\Phi : \mathcal{D} \subseteq \mathbb{C}^n \to \mathbb{C}^n$ and

$$\phi_i(x^{(k)}) = x_i^{(k)} - \frac{f(x_i^{(k)})}{f'(x_i^{(k)}) - f(x_i^{(k)}) \sum_{j\neq i} 1/(x_i^{(k)} - x_j^{(k)})},\ i = 1, 2, \ldots, n.$$

Given a fixed point iterative scheme of order p and by using a Ehrlich-type method, we design in Section 2 a simultaneous root-finding scheme of order $2p$. The convergence of this method is also proved in Section 2. In Section 3, we perform several numerical experiments to compare the results obtained by the proposed methods and other known schemes.

2. Design and convergence analysis

Let us consider a nonlinear equation $f(x) = 0$ with n simple roots, which we denote by α_i for $i = 1, \ldots, n$. We have a fixed point iterative method of the form $x^{(k+1)} = \phi(x^{(k)})$. From Ehrlich's method, we consider the following two-steps iterative method for approximating simultaneous roots of $f(x) = 0$, ϕ_s, obtained from ϕ, as follows:

$$\begin{cases} \bar{x}_i^{(k+1)} = \phi(x_i^{(k)}),\ i = 1, \ldots, n \\ x_i^{(k+1)} = x_i^{(k+1)} - \dfrac{1}{\dfrac{f'(\bar{x}_i^{(k+1)})}{f(\bar{x}_i^{(k+1)})} - \sum_{j=1, j\neq i}^{n} \dfrac{1}{\bar{x}_i^{(k+1)} - x_j^{(k)}}},\ i, j = 1, \ldots, n. \end{cases} \tag{1}$$

In the following result, we establish that method ϕ_s has order $2p$.

Theorem 1. *Let $f : D \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a sufficiently differentiable function in a neighborhood D of α_i for $i = 1, 2, \ldots, n$, such that $f(\alpha_i) = 0$ for $i = 1, 2, \ldots, n$. We assume that $f'(\alpha_i) \neq 0$ for $i = 1, 2, \ldots, n$. If ϕ is an iterative method of order p, then, taking an initial estimation $x^{(0)}$ close enough to $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, the sequence of iterates $\{x^{(k)}\}$ generated by the method ϕ_S(1) converges to α with order $2p$.*

Proof. We denote by $e_{i,k} = x_i^{(k)} - \alpha_i$ the error of ith component of iterate k. Since ϕ is an iterative scheme that has order of convergence p, then we know that $e_{i,k+1} \sim e_{i,k}^p$.

Applying Taylor's expansion to $f(\bar{x}_i^{(k+1)})$ and $f'(\bar{x}_i^{(k+1)})$ around α_i, we obtain

$$f(\bar{x}_i^{(k+1)}) = f'(\alpha_i)\left(e_{i,k+1} + C_2 e_{i,k+1}^2 + O(e_{i,k+1}^3)\right),$$

$$f'(\bar{x}_i^{(k+1)}) = f'(\alpha_i)\left(1 + 2C_2 e_{i,k+1} + O(e_{i,k+1}^2)\right).$$

From the above expressions, we have

$$\begin{aligned} &f'(\bar{x}_i^{(k+1)}) - f(\bar{x}_i^{(k+1)}) \sum_{j=1, j\neq i}^{n} \frac{1}{\bar{x}_i^{(k+1)} - x_j^{(k)}} \\ &\quad = f'(\alpha_i)\left(1 + \left(2C_2 - \sum_{j=1, j\neq i}^{n} \frac{1}{e_{i,k+1} + \alpha_i - \alpha_j - e_{j,k}}\right) e_{i,k+1} + O(e_{i,k+1}^2)\right). \end{aligned}$$

Then,

$$x_i^{(k+1)}-\alpha_i=\bar{x}_i^{(k+1)}-\alpha_i-\frac{f(\bar{x}_i^{(k+1)})}{f'(\bar{x}_i^{(k+1)})-f(\bar{x}_i^{(k+1)})\sum_{j=1,j\neq i}^{n}\dfrac{1}{e_{i,k+1}+\alpha_i-\alpha_j-e_{j,k}}}$$
$$=e_{i,k+1}-\frac{e_{i,k+1}+C_2e_{i,k+1}^2+O(e_{i,k+1}^3)}{1+\left(2C_2-\sum_{j=1,j\neq i}^{n}\dfrac{1}{e_{i,k+1}+\alpha_i-\alpha_j-e_{j,k}}\right)e_{i,k+1}+O(e_{i,k+1}^2)}$$
$$=\frac{\left(C_2-\sum_{j=1,j\neq i}^{n}\dfrac{1}{e_{i,k+1}+\alpha_i-\alpha_j-e_{j,k}}\right)e_{i,k+1}^2+O(e_{i,k+1}^3)}{1+\left(2C_2-\sum_{j=1,j\neq i}^{n}\dfrac{1}{e_{i,k+1}+\alpha_i-\alpha_j-e_{j,k}}\right)e_{i,k+1}+O(e_{i,k+1}^2)}.$$

Thus, by the previous relation and given that ϕ has order p, we affirm that

$$e_{i,k+2}\sim e_{i,k+1}^2\sim (e_{i,k}^p)^2\sim e_{i,k}^{2p}.$$

Thus, it is proven that the method ϕ_s has order of convergence $2p$. □

3. Numerical experiments

In this section, we perform different numerical experiments in order to observe the behavior of the proposed methods. In this case, we modify, as discussed in the previous section, Newton's method, Steffensen's method [8], the N_4 and N_8 methods designed in [9], and the M_4 and M_6 schemes constructed in [10]. We denote these methods in the same way as in the previous section, that is, if the method is denoted by ϕ, then its variant with the added step is denoted by ϕ_s. Furthermore, we compare the results obtained by these modified schemes with those of the following well-known methods for simultaneous roots: Ehrlich's method [7] (denoted by E), Shams' method [11] (denoted by SH) and Petkovic's method [2](denoted by P).

Matlab 2020b has been used to carry out the numerical experiments, with variable precision arithmetics with 2000 digits. As stopping criterion we choose that $\left\|x^{(k+1)}-x^{(k)}\right\|_2+\left\|F(x^{(k+1)})\right\|_2$ is less than a chosen tolerance, in that case 10^{-200}, where $F(x^{(k+1)})=\left(f(x_1^{(k+1)}),\ldots,f(x_n^{(k+1)})\right)$. We use also a maximum of 100 iterations.

In the different tables we show the following data:

- the norm of the function evaluated in the last iteration, $\left\|F(x^{(k+1)})\right\|_2$,
- the norm of the distance between the last two approximations, $\left\|x^{(k+1)}-x^{(k)}\right\|_2$,
- the number of iterations necessary to satisfy the required tolerance,
- and the approximated computational order of convergence (ACOC), defined by Cordero and Torregrosa in [12], which has the following expression:

$$p\approx ACOC=\frac{\ln\left(\left\|x^{(k+1)}-x^{(k)}\right\|_2/\left\|x^{(k)}-x^{(k-1)}\right\|_2\right)}{\ln\left(\left\|x^{(k)}-x^{(k-1)}\right\|_2/\left\|x^{(k-1)}-x^{(k-2)}\right\|_2\right)}.$$

The first numerical experiment we perform is to solve all the roots of the polynomial of degree 10 $x^{10}-1=0$. As an initial estimate we choose vector

$$x^{(0)}=(-2,2,0.5+i,0.5-i,-0.5+i,-0.5-i,-1+0.5i,-1-0.5i,1+0.5i,1-0.5i)\,.$$

We analyze the results obtained by the proposed iterative methods and those known for the polynomial of 10th-degree. The results obtained are shown in Table 1. Let us observe that for this initial estimation the only

Table 1
Results for $x^{10} - 1 = 0$.

Method	$\left\|x^{(k+1)} - x^{(k)}\right\|_2$	$\left\|F(x^{(k+1)})\right\|_2$	iter	ACOC
N_s	8.8667e−359	6.9362e−1790	6	5.0
S_s	5.1494e−683	1.5542e−3408	10	5.0
$N_{4,s}$	1.2183e−706	8.5302e−2008	5	9.3669
$N_{8,s}$	1.8392e−266	2.4078e−4657	4	17.669
$M_{4,s}$	2.1418e−277	7.2963e−2544	5	9.2664
$M_{6,s}$	1.3057e−1577	1.1268e−6007	5	13.057
P	n.c.	n.c.	n.c.	n.c.
SH	7.9468e−511	2.0186e−2007	6	4.9999
E	2.9015e−553	3.1822e−1657	8	3.0

Table 2
Results for $e^{x^2} - x = 0$.

Method	$\left\|x^{(k+1)} - x^{(k)}\right\|_2$	$\left\|F(x^{(k+1)})\right\|_2$	iter	ACOC
N_s	1.2767e−427	1.3179e−1708	6	4
S_s	1.0824e−224	1.9281e−896	6	4
$N_{4,s}$	6.4008e−215	6.6603e−1716	4	8.0
$N_{8,s}$	8.9419e−1739	3.4781e−15008	4	16.0
$M_{4,s}$	7.756e−274	5.8622e−2187	4	8
$M_{6,s}$	4.4876e−829	1.1485e−9943	4	12
P	1.8968e−217	5.0848e−434	12	2.0
SH	2.3099e−274	7.5408e−548	12	2.0
E	2.6495e−371	9.9211e−742	12	2.0

method that fails to converge to the roots is P scheme. We also notice that ACOC of the designed methods is higher than expected in all cases. Observing the number of iterations, most of the proposed methods need fewer iterations than known ones, being method N_{8_s} the one that performs the fewest iterations. In the second and third columns, we see that the methods that N_{8_s} and M_{6_s} obtain the best results taking into account the number of iterations needed.

Now, we calculate all the roots of the nonlinear equation $e^{x^2} - x = 0$. As initial estimations we choose $x^{(0)} = (-i, i)$. The results obtained by the proposed a known iterative methods are shown in Table 2. We see that for this initial estimation all the methods converge to the roots. We also notice that the ACOC of the designed schemes is identical to the expected one, that is, twice the order of the original method. Observing the number of iterations, all the proposed methods perform significantly fewer iterations than known ones. Regarding second and third columns of Table 2, we see that the methods that obtain the best results are obtained by $N8S$ and $M6S$ methods, taking into account the number of iterations they perform.

3.1. An example of real dynamics

In what follows, we generate several dynamical planes for some of the methods discussed in order to be able to compare them from the perspective of the wideness of their basins of attraction. We apply this idea on Newton', Steffensen's and M_4 method, both the original and the variants by adding the step to obtain all the roots simultaneously. In this case, we only show the dynamical planes associated with each of the methods when they are applied to a simple quadratic polynomial $p(x) = x^2 - 1$, whose roots are 1 and -1.

To generate the dynamical planes, we have chosen a mesh of 400×400 points, then we apply the iterative schemes to each of these points used as the initial estimate. For the non-simultaneous methods, one of the axes is the real part of the initial point, and the other is the imaginary part. For simultaneous methods, one of the axes is the initial estimate x_0 and the other is x_1, both real.

We have also fixed that the maximum number of iterations in 80, and we conclude that the scheme converges to one of the solutions when the distance between the iterate and the solution is lower than 10^{-3}. For the original methods, we paint in orange the initial points converging to the root -1, in green the initial

(a) Method N_s (b) Newton

Fig. 1. Dynamical planes of Newton's method and N_S.

(a) Method S_s (b) Steffensen

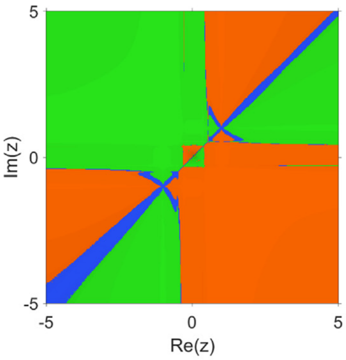
Fig. 2. Dynamical planes of Steffensen's method and S_S.

points that converge to the root 1 and in blue the initial points that do not converge to any root in the maximum of iterations. For the modified procedures, we paint the initial point in green color if the seed (x_1, x_2) converges to the set of roots $(-1, 1)$ and we paint the point in orange color if (x_1, x_2) converges to the set of roots $(1, -1)$. In case of non-convergence in the maximum of iterations we paint the initial point in blue color.

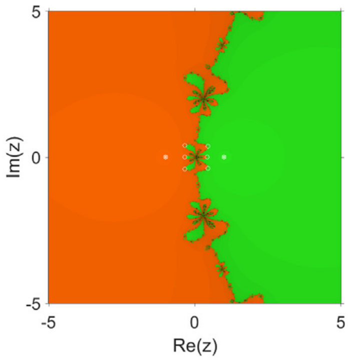
In Fig. 1, we present the dynamical planes obtained for the quadratic polynomial of Newton's and N_s methods. As we can see, the basins of attraction show global convergence in Newton's method, as they do for its variant to find roots simultaneously.

In Fig. 2, we show the dynamical planes obtained by Steffensen and S_s methods. In this case, Steffensen's scheme does not converge in some areas, as for example at the point $z = -5$, although we can observe that its variant S_s does converge to the roots at any point of this mesh, except in a small area around $x_1^{(0)} = x_2^{(0)} = 0$.

We also show in Fig. 3 the dynamical planes of M_4 and M_{4_s} methods. As we can observe, in this case we obtain that the dynamical plane of M_{4_s} has blue zones of non-convergence to the roots. This behavior corresponds to the higher order of convergence, as the denominator of the second step is closer to zero. This is solved by using more digits in the calculation, but the conditions have been held, for the sake of consistency.



(a) Method M_{4_s}



(b) Method M_4

Fig. 3. Dynamical planes of methods M_4 and M_{4_s}.

4. Conclusions

In this manuscript, we have defined a general procedure that can be used on any iterative scheme. With this process, an iterative step that is added to the original iterative method in such a way that the new procedure is able to find the roots simultaneously. Moreover this new method has duplicated the order of convergence of the original scheme.

We have selected several known iterative methods to which we have applied this procedure, and we have performed several numerical experiments to check the behavior of these new iterative methods. We have observed that the ACOC is similar to the expected order of convergence. We have also compared the results obtained from these iterative methods with those of other methods that find the roots simultaneously, and we have observed that the proposed methods in general perform in fewer number of iterations, since the order is much higher, so they reach the stopping criterion earlier, especially in the nonlinear equation that is not a polynomial.

We have also concluded, with the aim of some dynamical planes, that adding this iterative step changes the basins of attraction. However, in general the basins of attraction are similar or better than in the original schemes, in terms of the wideness of the basins of attraction of the roots.

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