Multi-epoch deformation analysis with geodetic datum invariance

Hiddo Velsink

HU University of Applied Sciences Utrecht, Padualaan 99, 3584 CH Utrecht, Netherlands, (hiddo.velsink@hu.nl)

Key words: *geodetic deformation analysis; multiple epochs; geodetic datum; deformation hypothesis testing; least squares; rank deficient covariance matrix; minimal deformation constraints*

ABSTRACT

Multiple epochs of geodetic deformation observations in 1D, 2D or 3D with their covariance matrices can be adjusted using the least-squares method and tested for deformation hypotheses, using the author's hypothesis constrained multi-epoch analysis method. The method estimates deformations of a subfield relative to points in one or more other subfields. Therefore, the method is invariant for the choice of geodetic datum and does not require stable points. Here it is shown how the geodetic datum is defined at two levels. At the first level position, orientation, size and form of the point field are fixed to enable the use of coordinates (or heights). At the second level the position, orientation, size and form within any epoch interval are fixed, if they cannot be fixed from the observations in the adjustment model. This enables the comparison of epochs that have no predefined points that are stable in the epoch intervals. The stochastic test of the comparison is invariant for the choice of geodetic datum at these two levels. A procedure is described how to find the best deformation hypothesis, taking account of all available statistical information. It is shown that the proposed method is a powerful tool to find the best deformation hypothesis based on geodetic observations and their full stochastic information. It is thus usable for a broad scope of applications of geodetic deformation analysis.

I. Introduction

Geodetic deformation analysis considers the problem of identification of deformations in a geodetic point field (Caspary 2000; Heunecke *et al.*, 2013; Velsink 2018b). Usually, several subfields can be distinguished in a point field, for example the subfield of reference points, and the subfield of object points. This last subfield is often not a single field, but can be divided in several subfields, where each subfield can be subject to different forces, resulting in different deformations. In this paper a geodetic point field will be called a geodetic network alternatively.

As an example a bridge over a river is taken. Usually, the movement of the bridge as a whole, relative to the surroundings is of interest. So, reference points are chosen on constructions and houses whose movements are representative of the surroundings. The bridge itself can consist of two abutments and the bridge deck. The movements of these three parts, relative to each other, and the movements of several points on just one part, relative to each other, are of interest. Therefore the object points of the bridge are separated into three subfields. This results in four subfields to be considered in the geodetic deformation analysis of the bridge.

A new method for geodetic deformation analysis of a point field by means of a time series of 1D, 2D or 3D coordinates, or of geodetic measurements, and their covariance matrices, was proposed by Velsink (2016; 2017). The method is called the "hypothesis constrained multi-epoch analysis method" (Niemeier and Velsink, 2019).

The problem considered in this paper is the geodetic datum invariance at two levels of the hypothesis constrained multi-epoch analysis method. The geodetic datum is here considered to be the coordinate reference system used, including the elements that cannot (or not accurately enough) be derived from the measurements, usually the elements that define the position of the origin, the orientation of the coordinate axes, and the scale. In 3D they comprise seven elements. It is called here the *first level* of the geodetic datum.

In some applications the rates of change in time of these seven elements have to be incorporated into the geodetic datum as well, for example, if a terrestrial reference frame is defined (Altamimi *et al.*, 2002). Likewise, the hypothesis constrained multi-epoch analysis may require an extension of the geodetic datum, which is called here the *second level* of the geodetic datum. In Section VI it will be described, what this second level is, and it will be shown that the hypothesis constrained multi-epoch analysis uses test statistics, which are invariant for the choice of this second level of the geodetic datum.

First, this paper treats in Sections II and III the characteristics of the hypothesis constrained multi-epoch analysis. Subsequently the procedure of deformation tests is shown in Section IV. Then the first and second level of the geodetic datum are treated in

BY-NC-SA 2022, Editorial Universitat Politècnica de València

more detail and it is shown that the stochastic tests, used to perform the analysis, are invariant for the second level (Sections V and VI). Finally, an application example is given in Section VII, which shows, how to perform a hypothesis constrained multi-epoch analysis.

II. HYPOTHESIS CONSTRAINED MULTI-EPOCH ANALYSIS

Niemeier and Velsink (2019) give a concise description of the hypothesis constrained multi-epoch analysis for the case that coordinates and their covariance matrices (regular or singular) are available for *n* epochs, where *n* can be 2 or more. The model used is called the *coordinates model*, see Figure 1, and is elaborated upon by Velsink (2016). A linearized Gauss-Markov model is used with observations $\boldsymbol{\ell}$, residuals $\boldsymbol{v}$, parameters $\boldsymbol{p}$ and a coefficient matrix $\boldsymbol{A}$ (bold letters indicate matrices or vectors) (Eq. 1):

$$\boldsymbol{\ell} + \boldsymbol{v} = \boldsymbol{A}\boldsymbol{p}. \quad (1)$$

In Figure 1 it is shown that the coordinates model consists of two phases.

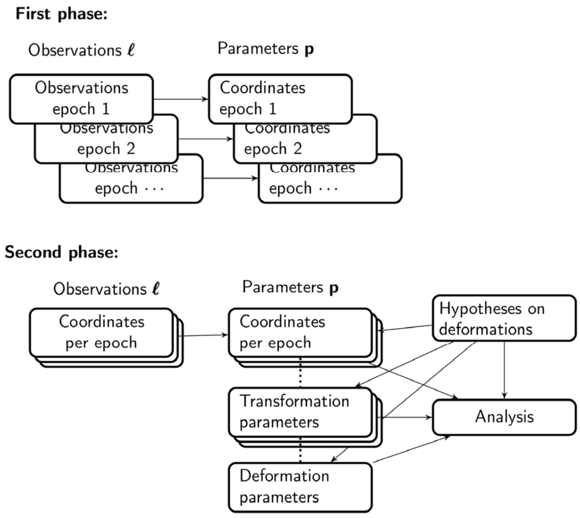

Figure 1. Coordinates model (Velsink 2018b, p. 26).

In the first phase geodetic observations are adjusted for each epoch separately. The results are coordinates for each epoch, including their covariance matrix, which is often singular, because of the geodetic datum used. The second phase takes all coordinates of all epochs, and their covariance matrices, together. Hypotheses on deformations are formulated and tested (this will be treated in Section III A). Figure 1 shows what is contained in the observation vector $\boldsymbol{\ell}$ and what in the parameter vector $\boldsymbol{p}$ in both phases.

If, however, the *measurements* of all *n* epochs are available, including their covariance matrices (regular or singular), use can be made of the *measurements model* (Velsink 2017), see Figure 2. The figure shows likewise as Figure 1, what is contained in $\boldsymbol{\ell}$ and $\boldsymbol{p}$.

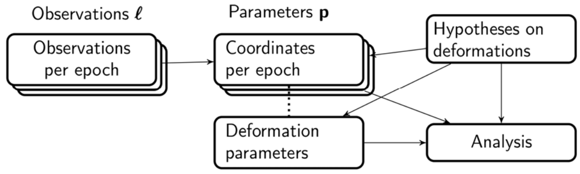

Figure 2. Measurements model (Velsink 2018b, p. 25).

A. *Constraints link epochs*

In both models the vector of parameters $\boldsymbol{p}$ contains for each epoch the coordinates to be estimated. Therefore, every point has different coordinates for each epoch in which it occurs. If we put all epochs in order of time, an epoch interval for a point is defined here as the time between an epoch where a point appears, and the next epoch, where it appears. For each point, and for each epoch interval, one, two or three *constraints* (for 1D, 2D, and 3D respectively) are formulated, which state, how the point is moving in that epoch interval. If a point does not occur in one or more epochs, one or more of its epoch intervals are longer than for other points.

Usually, the null hypothesis is that points don't move, in which case each constraint is that a coordinate (*x*, *y* or *z*) in a certain epoch is equal to the equivalent coordinate of the same point in the next epoch.

B. *Constraints define deformation hypotheses*

Both Figure 1 and Figure 2 mention hypotheses on the deformations, which should be incorporated into the Gauss-Markov model. This is done by the constraints, introduced above. If a point is assumed to be moving in a certain epoch interval, for example linearly with a change rate of ∇, the ∇ is added to the parameter vector. How a constraint can be added to the Gauss-Markov model is treated in Section II D. ∇ may be a vector in itself, for example, if the assumed deformation of one or more points in a certain epoch interval (or even more epoch intervals) is described by a function with more than one parameter.

C. *Geodetic datum – no stable points required*

Figure 1 shows that the parameter vector in the coordinates model contains transformation parameters as well. Each epoch interval (from epoch *i* to epoch *i+1*, where *i* = 1, 2, … *n-1*) involves a transformation. It takes care of the differences in geodetic datum between the coordinates in the *observation vector* that relate to epoch *i* and those that relate to epoch *i+1*. This implies that a geodetic datum has to be fixed for one of the epochs, but not for the others. This, in turn, implies that the analysis method does not require any stable points (*i.e.* points that do not move in one or more epoch intervals).

The coordinates in the *parameter vector* are ordered by epoch as well, but they are all defined relative to the same geodetic datum.

The measurements model does not require stable points either. This is, because in this model a geodetic

2022, Editorial Universitat Politècnica de València BY-NC-SA

datum is needed for the parameter vector, but not for the observations in the observation vector, unless coordinates are used as observations. In the latter case transformation parameters have to be added to the parameter vector, which leads to the same situation as in the coordinates model.

D. *Gauss-Markov model with hard constraints*

Usually, the constraints are considered to be hard constraints, *i.e.* they can be added to the observation vector as observations with a standard deviation of zero. This results in a singular covariance matrix of the observations. Most handbooks on adjustment theory only treat the case that the covariance matrix of the Gauss-Markov model is regular (invertible), see *e.g.* Koch (2013, Section 3.2.1). Maybe, this is the reason that some publications state that only an approximate solution is possible, if observations have a standard deviation of zero (Lehmann and Neitzel, 2013; Shi *et al.*, 2017). This, however, is not true. An exact, rigorous solution has been published already more than fifty years ago (Rao and Mitra, 1971, pp. 147-150).

Even if that solution is not used, it is well-known, how a Gauss-Markov model with hard constraints can be solved rigorously by using Lagrange multipliers (Koch 2013, Section 3.2.7), or by reducing the amount of parameters (Wolf 1982; Velsink 2015, p. 401).

Whether the hard constraints are justified, can be tested with a generalized likelihood ratio test (Velsink 2018a). This test can be used to test any deformation hypothesis, which is formulated by means of constraints in the above described coordinates model or measurements model.

III. ANALYSIS PROCESS

Both the coordinates model of Figure 1 and the measurements model of Figure 2 can be used to perform a geodetic deformation analysis. Both models yield least-squares estimates of the coordinates of all points for all epochs, and the covariance matrix of all these coordinates.

A. *Testing deformation patterns*

However, the estimated coordinates are ***not*** essential for the deformation analysis! It is the testing of the null hypothesis against alternative hypotheses, which is the core of the analysis. Each alternative hypothesis describes one possible deformation pattern of all points through all epochs. It may be, for example, that a subfield is subject to deformation only during the last few epochs, and that the points of another subfield are moving relative to the other subfields during the first few epochs. The deformations of both subfields constitute one intricate deformation pattern, described in the Gauss-Markov model by several constraints. This deformation pattern is considered one alternative hypothesis and is tested by a generalized likelihood ratio test. To do this, the model of the alternative hypothesis is written as an extension of Equation 1 (Eq. 2):

$$\boldsymbol{\ell} + \boldsymbol{v} = \boldsymbol{A}\boldsymbol{p} + \boldsymbol{C}\boldsymbol{\nabla} \qquad (2)$$

vector $\boldsymbol{\nabla}$ contains the deformation parameters of Figure 1 or 2. Matrix $\boldsymbol{C}$ is the corresponding coefficient matrix. The alternative hypothesis is tested against the null hypothesis by using the test statistic (Velsink 2018a) (Eq. 3):

$$F_q = \frac{\boldsymbol{r}^t\boldsymbol{C}(\boldsymbol{C}^t\boldsymbol{Q}_r\boldsymbol{C})^{-1}\boldsymbol{C}^t\boldsymbol{r}}{q\,\sigma_0^2}, \qquad (3)$$

with $\boldsymbol{r}$ the vector of reciprocal residuals, and $\boldsymbol{Q}_r$ its cofactor matrix (Velsink, 2018a), which both can be computed during the adjustment of the null hypothesis. t indicates the transposed of a vector or matrix, and q is an integer that indicates the degrees of freedom of the test and equals the number of parameters in $\boldsymbol{\nabla}$. σ_0^2 is the a priori variance of unit weight, which follows from splitting the covariance matrix of the observations in this factor and the cofactor matrix.

B. *Hypothesis selection problem*

Infinitely many deformation patterns can be formulated for a point field. Many of them can be plausible, considering the physical conditions of the houses, constructions and soil, where the points are located, and the forces that act upon them (*e.g.* earthquakes, landslides, construction works and weather conditions). The approach of the hypothesis constrained multi-epoch analysis method is to test many alternative hypotheses and to find the best one among them. Because different alternative hypotheses can have different degrees of freedom in the generalized likelihood ratio test, the definition of “best” is problematic, and the resulting hypothesis selection problem does not have a unique answer (Velsink 2018b, p. 43). However, a criterion has to be chosen, for example the *test ratio* or the *Akaike information criterion* (Velsink 2018b, p. 43), and applied.

C. *Adaptation of null hypothesis*

When the best alternative hypothesis, according to some criterion of “best”, has been decided upon, this alternative hypothesis becomes the new null hypothesis. The equation of the new null hypothesis is arrived at by writing Equation 2 as (Eq. 4):

$$\boldsymbol{\ell} + \boldsymbol{v} = (\boldsymbol{A} \quad \boldsymbol{C})\begin{pmatrix}\boldsymbol{p}\\ \boldsymbol{\nabla}\end{pmatrix} \qquad (4)$$

The matrix $(\boldsymbol{A} \quad \boldsymbol{C})$ is the new coefficient matrix and the vector $\begin{pmatrix}\boldsymbol{p}\\ \boldsymbol{\nabla}\end{pmatrix}$ is the new parameter vector.

Testing the adjustment of the adapted null hypothesis should result in an accepted overall model test, which confirms that the alternative hypothesis

BY-NC-SA 2022, Editorial Universitat Politècnica de València

yields a good explanation of all measurements in all epochs (or of the coordinates that are derived from those measurements in the separate epoch adjustments).

D. *Least-squares solution by iteration*

Because the used models are linearized ones, both observations $\boldsymbol{\ell}$ and parameters $\boldsymbol{p}$ and $\boldsymbol{\nabla}$ are incremental values relative to approximate values. The approximate values have to comply with the nonlinear equivalent of Equation 4. To get the least-squares solution of Equation 4 an iteration process is needed. In each iteration step the estimated increments for both observations and parameters have to be corrected slightly to let the new approximate values comply with the nonlinear equivalent of Equation 4.

These corrections can be computed by least-squares, using the model of condition equations, which is dual to the Gauss-Markov model and has been published first by Gauss (1828). See Velsink (2018a) for this model and its least-squares solution in matrix equations. The adjusted observations of the previous step are the vector of constant terms in this model and the estimated parameters are the observations. After linearization this yields (Eq. 5):

$$\boldsymbol{t} = (\boldsymbol{A} \quad \boldsymbol{C}) \begin{pmatrix} \boldsymbol{p_0} \\ \boldsymbol{\nabla_0} \end{pmatrix} - \boldsymbol{\ell_0};\ E\{\boldsymbol{t}\} = \boldsymbol{0}, \tag{5}$$

where $\boldsymbol{\ell_0}$, $\boldsymbol{p_0}$ and $\boldsymbol{\nabla_0}$ are the approximate values of $\boldsymbol{\ell}$, $\boldsymbol{p}$ and $\boldsymbol{\nabla}$. The vector $\boldsymbol{t}$ contains the misclosures (computed with the non-linear model) with $E\{\boldsymbol{t}\}$ their mathematical expectations. As covariance matrix of the "observations" $\boldsymbol{p_0}$ and $\boldsymbol{\nabla_0}$ the unit matrix is used. Thus, the values of $\boldsymbol{p_0}$ and $\boldsymbol{\nabla_0}$ are corrected in this "adjustment within an adjustment" and the corrected values comply with the main nonlinear model.

The constantly improved approximated values of both observations and parameters will usually converge to the least-squares solution of the nonlinear model, that is, to the adjusted observations and the estimated parameters respectively. If not, the initially chosen approximate values are not adequate, or the observations contain gross errors.

E. *Estimated deformations*

The adjustment of the adapted null hypothesis yields least-squares estimates of the deformation parameters $\boldsymbol{\nabla}$. Moreover, the constraints that describe the deformations for each point in each epoch, can be included in the vector of observations as observations with standard deviation zero. In that case, the adjusted observations are the least-squares estimates of the deformations of each *x*, *y* or *z* coordinate of each point in each epoch.

It is important to note that the estimated deformations of all points are not relative to the first level of the geodetic datum, but relative to each other, more precisely to points that are held fixed (no deformation) in the model! It may be that such points do not exist. This case will be treated in Section VI A.

F. *Minimal Detectable Deformations*

The hypothesis constrained multi-epoch analysis can be done by the coordinates model or the measurements model. Both models can be adjusted to get a least-squares solution by using a least-squares algorithm that can handle singular covariance matrices, or by using another method to handle hard constraints, see Section II D. Equation 3 can be used to test for deformations. But also formulas are available to compute Minimal Detectable Deformations (MDD) (Velsink 2016; 2017). An MDD of a certain coordinate difference between the coordinates of the same point in an epoch interval gives the deformation in, for example, millimeters that can be detected with a certain probability (chosen beforehand) by a test of a certain alternative hypothesis, described by a matrix $\boldsymbol{C}$.

The MDD's are a powerful tool to describe the ability of a geodetic network to detect deformations (sensitivity analysis). It is an important property of the MDD's that they can be computed before any measurement has been done. Thus, they can be used to formulate *requirements* for deformation networks, and consequently *standards* can be formulated with them.

IV. Procedure of deformation tests

To arrive at the treatment of the invariance for changes in the geodetic datum of the hypothesis constrained multi-epoch analysis, first the *procedure* of deformation tests is reviewed.

First, each epoch is adjusted and tested in itself. If the reliability of the geodetic network of an epoch is good enough, and if testing the epoch does not lead to rejection, it is assumed that the epoch model does not contains errors of any significance.

Subsequently the model of all epochs is tested against the most general alternative hypothesis, which is done by the overall model test (Velsink 2018a, p. 6). If this test rejects the null hypothesis, and all epoch tests did not lead to rejection, it may be that still some errors are present in some epoch or epochs, because the combined adjustment of all epochs is more sensitive to errors in individual epochs than the adjustment of just one epoch.

If it is concluded that no errors in any individual epoch are present any more, but the test of all epochs leads to rejection, the conclusion is that deformations are present.

It may be that physical conditions of the earth and of buildings and structures on it are indicative of possible deformations. But even then, other deformations may be present as well. Therefore, all deformation constraints are tested one by one (that is, for each coordinate difference separately) or one point by one

2022, Editorial Universitat Politècnica de València BY-NC-SA

point or both. To get an impression of the number of deformation constraints: with *n* epochs, *d* dimensions (*d* is 1, 2 or 3), and *k* points in each epoch (although usually the numbers of points in each epoch differ), there are $(n-1)$ epoch intervals and, thus, $(n-1) \times d \times k$ deformation constraints. The test of one point by one point means that the *d* coordinates of a point in an epoch interval are tested for the alternative hypothesis that this point has moved in that epoch interval, and that no other point has moved in that or any other epoch interval.

The test is performed with the test quantity of Equation 3 and is a *d*-dimensional test. The test is repeated for all points in all epoch intervals. Let us call these tests one point by one point the *conventional point tests*. Because the dimension of all conventional point tests is *d*, the F_q's of all these tests can be directly compared, and give an impression, where deformations might be present. It is not possible to deduce directly from these conventional point tests which points in which epoch intervals are subject to a deformation, let alone determine the character and sizes of these deformations. To accomplish this, more complex alternative hypotheses have to be formulated and tested. This is the hypothesis selection problem, treated earlier in Section III B. It will not be treated fully in this paper. It is sufficient here to conclude that the best alternative hypothesis, and thus the best deformation pattern, is selected.

V. First level geodetic datum

In the introduction (Section I) the first level of geodetic datum invariance has been defined. If a deformation hypothesis is defined by constraints on the parameters of the coordinates model or the measurements model, the hypothesis can be tested with Equation 3. This test is ***invariant*** for the first level of the geodetic datum (Velsink 2016; 2018a).

A. *Deformation constraints link epochs*

As mentioned before, the first level can be defined by fixing the coordinates of some points *in only one epoch*. Let us call these points the datum points and let them be fixed in the *first* epoch. The coordinates of the datum points in the other epochs (which are not part of the datum) are linked to those of the first epoch by deformation constraints.

Usually, the null hypothesis assumes the absence of any deformation. This means that the coordinates of the datum points of the second to *n*-th epoch are forced to be equal to the coordinates in the previous epoch by constraints in the coordinates model or measurements model.

B. *Example: just one point moved*

Let us suppose that the overall model test rejects the null hypothesis, and that the solution of the model selection problem is the alternative hypothesis that a point *S* is subject to deformation during just one epoch interval, and no other deformation is present. As explained in Section III A, the alternative hypothesis is defined by a matrix ***C***. In this case, matrix ***C*** has *d* columns (*d* is the dimension: 1, 2 or 3), which contain zeros except for a 1 in the *d* rows that correspond to the *d* observations that represent the deformation constraints of point *S* in that epoch interval.

Let the *k*-th observation be the constraint that the *x* coordinate of point *S* in epoch $i-1$ is equal to the *x* coordinate of point *S* in epoch i. The observation equation reads (Eq. 6):

$$\ell_{\mathrm{k}} + v_{\ell_k} = -x_{S,i-1} + x_{S,i} + \nabla_{S,i-1,i} \qquad (6)$$

The "observed" value of ℓ_{k} is zero. Its standard deviation is zero as well. When Equation 6 is transferred to the matrices of Equation 4, -1 is inserted in ***A*** in the column that corresponds to the x coordinate of point *S* in epoch $i-1$, and 1 is inserted in the corresponding column of epoch i. In matrix ***C*** 1 is inserted in the *k*-th row in column 1 (column 2 is for the y coordinates and column 3 for the z coordinates).

The parameter $\nabla_{S,i-1,i}$ only appears in the *k*-th observation equation and it ensures that the constraint is *disabled*. It is noteworthy that the addition of parameter $\nabla_{S,i-1,i}$ in the adjustment model has the same effect as giving the constraint an infinitely large standard deviation (or a weight of zero). Because the constraint remains in the adjustment model, an estimated value will be computed, which is the estimated deformation of the coordinate difference in the epoch interval concerned.

This example serves as an illustration of the way a hypothesis is formulated that just one point in one epoch interval is subject to deformation. The next section contains an example that treats the opposite case, where all points are suspected to be influenced by deformation. It is there that the second level of the geodetic datum plays an important role.

VI. Second level geodetic datum

A. *Example: All points move*

In professional practice the situation occurs regularly that it is not clear, whether there is any deformation, and that the point by point analysis of Section IV gives an unclear picture, where all points seem to be affected by deformation. In this case it is possible to "disable" all deformation constraints, which is done by adding an additional parameter ∇ to each deformation constraint, as exemplified by Equation 6. This results in *n* separate geodetic networks, when there are *n* epochs. Because the relation between all these networks is not defined, it will result in a parameter vector in Equation 4, which cannot be solved in a least-squares sense uniquely.

To get a unique solution (or, equivalently to make the parameters estimable) every epoch interval has to have

BY-NC-SA 2022, Editorial Universitat Politècnica de València

at least as many "enabled" deformation constraints as needed to get a unique solution. These minimally needed constraints are here called "minimal deformation constraints" (cf. Pope 1971), and define the **second level of the geodetic datum**. We have seen in Section I that the testing quantity of Equation 3 is invariant for a change of the first level of the geodetic datum. In the sequel the invariance of this test quantity for the second level will be considered.

B. *Form and size*

Geodetic deformation analysis is about changes of the form of a geodetic point field, which can clearly be seen in the word *de-form-ation*. It is about size as well, because today our instruments are so good that distances can be measured very precisely and changes of size of a geodetic point field can be determined equally precisely. This means that we are interested in form and size describing elements, and in the changes of these elements. Measurements used for deformation analysis should contain information about form, or about form and size, to be of value. It may be that they contain information about position and orientation relative to the earth, but this is of no value for the deformation analysis, as long as the point field contains all points of relevant objects subject to deformation and of reference objects.

Euclidian or geographic coordinates relative to a reference system contain more than form and size information. They contain information about the position of the origin and the orientation of the coordinate axes relative to the point field under consideration. For this reason coordinates are not suited as form or size elements.

Therefore, it is of importance to know, how form and size elements can be defined in 1D, 2D and 3D. First, form elements in 2D are treated. In the two dimensional Euclidian plane the smallest point field with a form is a triangle (three points). Two triangles have the same form (are *conformal*), when corresponding angles are equal. To fix the form of a triangle, two angles of the triangle are sufficient. Baarda (1966) showed that the form can be defined by two length ratio's as well (cf. Velsink 2018b, pp. 27-28). This means that angles and length ratio's are suited to function as form elements.

In three dimensional Euclidian space the smallest point field with a form that is not in a 2D subspace, is a tetrahedron (four points). To fix the form of a tetrahedron five angles, or length ratio's, or a combination of both are needed (Velsink 2018b, pp. 28-29).

In one dimensional Euclidian space (a straight line) angles are not defined, but length ratio's are. The smallest point field with a meaningful length ratio consists of three points (Velsink 2018b, p. 30).

A size element is a distance between two points. If the form of a point field is defined, the size of it is defined if one distance between two points within the point field is defined.

With the use of these form and size elements, it is possible to consider the invariance of the test quantity of Equation 3 for a change of the second level of the geodetic datum.

C. *Invariance of test quantity*

Let us consider *n* epochs in the coordinates model or the measurements model. They can be viewed as *n* geodetic networks. Let us assume that the measurements of all these networks contain enough information on the form and size of the network. If the networks are linked by minimal deformation constraints, the form and size of each network are solely determined by the adjusted coordinates or measurements of each network, and not by the deformation constraints.

The test quantity of Equation 3 is invariant for a change in the first level of the geodetic datum. Such a change can be performed, in the case of geodetic deformation analysis, by a similarity or congruency (rigid body) transformation, which leaves the form, or the form and size of a point field unchanged. Therefore, the differences (changes) between the form and size elements of the two epochs of an epoch interval, are left unchanged. This means that the test quantity of Equation 3 is ***invariant for a change in the second level of the geodetic datum***, if the second level is defined by minimal deformation constraints.

VII. Application

To describe an application of the invariance of the second level of the geodetic datum, a fictitious example from professional practice is presented.

Suppose a building is close to ongoing roadworks and it is monitored for deformations by 3D tacheometry measurements. Initially the consecutive epoch measurements fit well together: the overall model test is accepted. It seems that no deformation occurs. But then the addition of the next epoch leads to rejection of the test and likewise the subsequent epoch. However, the one-by-one tests of Section IV do not give a clear picture. Maybe, a deformation of several parts of the building started, and maybe already during several epochs. It is decided that all epochs up to the last four epochs will be left out of the model and the last four will be analyzed. All deformation constraints between the four epochs are disabled by introducing a bias ∇, a different one for each epoch interval and each coordinate. This is realized in the software by specifying a suitable matrix $\boldsymbol{C}$, see Equation 2. This makes the parameter vector $\boldsymbol{p}$ not solvable. The software allows the user to specify minimal deformation constraints for each epoch interval by choosing which coordinates are to be fixed. This has the advantage that the adjusted deformation constraints (adjusted from zero to nonzero values) are relative to the fixed coordinates. But

2022, Editorial Universitat Politècnica de València (cc) BY-NC-SA

additionally the software offers the possibility *not* to choose a second level of the geodetic datum. In this case, the software uses a generalized matrix inverse to compute the parameter vector $\boldsymbol{p}$. A generalized matrix inverse of a matrix $\boldsymbol{M}$ is any matrix $\boldsymbol{G}$ that fulfils the Equation 7:

$$\boldsymbol{M} = \boldsymbol{MGM}. \quad (7)$$

A special type of generalized matrix inverse, which is unique for $\boldsymbol{M}$, is the pseudo-inverse as defined by Boullion and Odell (1971), also called the Moore-Penrose inverse (Ben-Israel and Greville, 2003). It minimizes the Euclidian norm of $\boldsymbol{p}$, and thus the adjusted deformation constraints, although this has, in view of the mentioned invariance, no significance: any choice of the minimal deformation constraints will give the same test results. Therefore, the software does not use the pseudo-inverse, but the generalized inverse that is calculated the fastest. In MATLAB this can be done by using the matrix left divide ("\") function to solve the system of normal equations.

The next step is to look at the adjusted deformation constraints, and to identify *deformation patterns* in one or more subfields, during one or more epoch intervals. The result is a collection of possible deformation patterns that might explain the adjusted deformation constraints. Each deformation pattern can consist of several subfields, each of which shows different deformation behavior during different epoch intervals. Each deformation pattern is translated into an alternative hypothesis by specifying a new matrix $\boldsymbol{C}$, and subsequently, after having chosen a level of significance, tested with test statistic (3), which is assumed to have a chi-squared-distribution. This manual process is supported by the software, which has the option to systematically construct many combinations of simple deformation patterns and test them. Research should give more deformation patterns that occur often in professional practice.

Choosing the best alternative hypothesis means solving the hypothesis selection problem as stated in Section III B.

Subsequently all epochs are again added to the model, and the process just described, is repeated for all epochs. Let us assume that it does not lead to noticeably different results, indicating that no deformation was present before the last four epochs.

After determining the best alternative hypothesis, the null hypothesis is adapted (Section III C) to account for the found deformation pattern, and final results are computed. The principal of the roadworks is informed about the deformations.

VIII. Conclusions

In this paper the hypothesis constrained multi-epoch analysis method has been explained, and a new result on its use has been derived. It has been shown, how many epochs of geodetic deformation measurements are included in one adjustment model, for which two options are available, the coordinates model and the measurements model. Deformation hypotheses, or the absence of deformation is introduced in these models by formulating constraints between epochs. The geodetic datum can, thus, be limited to some coordinates in just one epoch. This has the advantage that no stable points are needed to define the geodetic datum.

It has been shown how the least-squares solution of the models is computed with an iterative process of adjustment, which is tested subsequently. If the null hypothesis is rejected, several alternative hypotheses are formulated, which are hypotheses on the deformation pattern.

Each deformation hypothesis can be tested by a test statistic that has a chi-squared distribution and which is invariant for the choice of geodetic datum. One of the deformation hypotheses has to be chosen as the best one. This is the hypothesis selection problem. Subsequently least-squares estimates of the deformations can be computed. It is shown as well, how minimal detectable deformations are determined.

Then a second level of the geodetic datum is defined, which enables the search for possible deformation patterns, when no clear indication is available, what the deformation pattern looks like. The results of the previously mentioned tests are shown to be invariant for this second level.

A fictitious application has been elaborated upon to make it clear what procedure can be used, when the hypothesis constrained multi-epoch analysis method is used.

References

Altamimi, Z., Sillard, P., and Boucher, C. (2002). ITRF2000: A new release of the International Terrestrial Reference Frame for earth science applications. *Journal of Geophysical Research: Solid Earth*, 107(B10), ETG-2.

Baarda, W. (1966). Puntsbepalingsystemen in het platte vlak. Berekening met complexe getallen. Eerste en tweede deel. Veelhoek- en driehoekfiguraties. Heruitgave. Computing Centre of the Delft Geodetic Institute, Delft, stencilled edition.

Ben-Israel, A., and Greville, T. N. (2003). Generalized inverses: theory and applications. *Springer Science & Business Media*.

Boullion, T. L., and Odell, P. L. (1971). Generalized inverse matrices. Wiley-interscience.

Caspary, W. (2000). Concepts of network and deformation analysis, Monograph 11. School of Surveying, The University of New South Wales. Australia, Tech. rep.

Gauss, C. F. (1828). Supplementum theoriae combinationis observationum erroribus minimis obnoxiae. Henricus Dieterich, Göttingen.

Heunecke, O., Kuhlmann, H., Welsch, W., Eichhorn, A., and Neuner, H. (2013). Auswertung geodätischer Überwachungsmessungen. Wichmann, second edition.

BY-NC-SA 2022, Editorial Universitat Politècnica de València

Koch, K.-R. (2013). Parameter Estimation and Hypothesis Testing in Linear Models. Springer, second edition.

Lehmann, R., and Neitzel, F. (2013). Testing the compatibility of constraints for parameters of a geodetic adjustment model. *Journal of Geodesy*, 87(6), pp. 555-566.

Niemeier, W. and Velsink, H. (2019). Strategies and Methods for Multi-Epoch Deformation Analysis based on Geodetic Networks. In *Proceedings of the 4th Joint International Symposium on Deformation Monitoring (JISDM 2019)*, Athens, Greece.

Pope, A. J. (1971). Transformation of Covariance Matrices Due to Changes in Minimal Control. In: AGU Fall Meeting, San Francisco, California, December 9, 1971. *National Ocean Survey Geodetic Research and Development Laboratory*.

Rao, C. R. and Mitra, S. K. (1971). *Generalized Inverse of Matrices and Its Applications*. Wiley Series in Probability and Mathematical Statistics. Wiley

Shi, Y., Xu, P., and Peng, J. (2017). A computational complexity analysis of Tienstra's solution to equality-constrained adjustment. *Studia Geophysica et Geodaetica*, 61(4), pp. 601-615.

Velsink, H. (2015). Extendable linearised adjustment model for deformation analysis. *Survey Review*, Vol 47, No. 345, pp. 397-410.

Velsink, H. (2016). Time Series Analysis of 3D Coordinates Using Nonstochastic Observations. *Journal of Applied Geodesy*, Vol 10, No. 1, pp. 5-16

Velsink, H. (2017). Testing deformation hypotheses by constraints on a time series of geodetic observations. *Journal of Applied Geodesy*. Vol 12, No. 1, pp. 77-93.

Velsink, H. (2018a). Testing Methods for Adjustment Models with Constraints. *Journal of Surveying Engineering*. Vol 144, No. 4 (04018009).

Velsink, H. (2018b). The Elements of Deformation Analysis: Blending Geodetic Observations and Deformation Hypotheses. Doctoral dissertation, Delft University of Technology. DOI: 10.4233/uuid:d7132920-346e-47c6-b754-00dc5672b437

Wolf, H. (1982). Alternate use of restriction equations in Gauss-Markoff adjustments. In: Blotwijk, M.J.; Brouwer, F.J.J.; van Daalen, D.T.; de Heus, H.M.; Gravesteijn, J.T.; van der Hoek, H.C.; Kok, J.J. and Teunissen, P.J.G., editors, Forty Years of Thought. Anniversary edition on the occasion of the 65th birthday of Professor Baarda, volume I, pages 194-200.

2022, Editorial Universitat Politècnica de València (cc) BY-NC-SA