A new higher-order optimal derivative free scheme for multiple roots*

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Abstract

In this paper, we presented a novel and efficient fourth order derivative free optimal family of iterative methods for approximating the multiple roots of nonlinear equations. Initially the convergence analysis is performed for particular values of multiple roots afterward it concludes in general form. In addition, we study several numerical experiments on real life problems in order to confirm the efficiency and accuracy of our methods. We illustrate the applicability and comparisons of our methods on eigenvalue problem, Van der Waals equation of state, continuous stirred tank reactor (CSTR), Plank's radiation and clustering problem of roots with earlier robust iterative methods. Finally, on the basis of obtained computational results, we conclude that our methods perform better than the existing ones in terms of CPU timing, absolute residual errors, asymptotic error constants, absolute error difference between two last consecutive iterations and approximated roots as compared to the existing ones.

Keywords: nonlinear equation; iterative method; multiple root; efficiency index; convergence.

1. Introduction

Finding the novel higher-order derivative free iterative methods for nonlinear equations of the form

$$f(x) = 0, \tag{1}$$

where $f : \mathbb{D} \subset \mathbb{C} \to \mathbb{C}$ is an analytic function defined in $\mathbb{D}$, neighborhood of the required zero ξ), is one of the most fascinating and hard problems in the area of numerical analysis and computational mathematics. Some of the main reasons are: the non-existence of analytical methods that give us the exact solution, the non-differentiability of function f, the complexity of the derivative of f, etc. More details can be found in some of the standard books [1, 2, 3].

In the recent and past years, some researchers suggested derivative free methods for the simple roots of expression (1), in their research articles [4, 5, 6, 7, 8] but not for multiple roots. Finding derivative free techniques that can handle multiple roots is more tougher and harder problem than the simple roots. So, higher-order derivative free techniques are in high demand and attracting the scholars in the case of multiple roots. In the recent years, some scholars proposed the following new iterative methods in this direction.

In 2015, Hueso et al. [9], proposed a new derivative free method for multiple roots (when the multiplicity $m \geq 2$ is known in advance) of nonlinear equations, which is given by

$$\begin{aligned} y_j &= x_j - m\frac{f(x_j)}{f[\mu_j, x_j]}, \\ x_{j+1} &= x_j - \Big(a_1 + a_2 h(y_j, x_j) + a_3 h(x_j, y_j) + a_4 h(y_j, x_j)^2\Big)\frac{f(x_j)}{f[\mu_j, x_j]}, \end{aligned} \tag{2}$$

where $\mu_j = x_j + f(x_j)^q,\ q \in \mathbb{R}$ with $h(x_j, y_j) = \dfrac{f[y_j + f(y_j)^q,\ y_j]}{f[x_j + f(x_j)^q,\ x_j]}$ and $f[x, y] = \dfrac{f(x) - f(y)}{x - y}$. The scheme (2) attains the non-optimal fourth-order convergence for $\begin{cases} q = 1, & \text{for each } m \geq 4 \\ q \geq 2, & \text{for all } \ m \geq 2 \end{cases}$.

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In 2019 and 2020, Sharma et al. [10, 11], suggested the following two optimal fourth-order iterative schemes

$$\begin{aligned} z_j &= t_j - m\frac{f(t_j)}{f[s_j,t_j]}, \\ t_{j+1} &= z_j - H(x_j,y_j)\frac{f(t_j)}{f[s_j,t_j]}, \end{aligned} \tag{3}$$

and

$$\begin{aligned} z_j &= u_j - m\frac{f(u_j)}{f[v_j,u_j]}, \\ u_{j+1} &= z_j - G(h_j)\left(\frac{1}{y_j}+1\right)\frac{f(u_j)}{f[v_j,u_j]}, \end{aligned} \tag{4}$$

where $s_j = t_j + \beta f(t_j)$, $\beta \in \mathbb{R}$, $\beta \neq 0$; $x_j = \left(\frac{f(z_j)}{f(t_j)}\right)^{\frac{1}{m}}$, $y_j = \left(\frac{f(z_j)}{f(s_j)}\right)^{\frac{1}{m}}$, $v_j = u_j + \beta f(u_j)$, $x_j = \left(\frac{f(z_j)}{f(u_j)}\right)^{\frac{1}{m}}$, $y_j = \left(\frac{f(v_j)}{f(u_j)}\right)^{\frac{1}{m}}$ and $h_j = \frac{x_j}{x_j+1}$. The conditions on weight functions H and G, in order to reach order four, can be found in [10] and [11], respectively.

Recently, in 2020, Kumar et al. [12], introduced the following optimal fourth-order derivative free iterative method for approximating multiple roots

$$\begin{aligned} w_j &= u_j - m\frac{f(u_j)}{f[v_j,u_j]}, \\ u_{j+1} &= w_j - \frac{s_j}{\eta_1+\eta_2 s_j}\frac{f(u_j)}{\eta_3 f[v_j,u_j]+\eta_4 f[w_j,v_j]}, \end{aligned} \tag{5}$$

where $\eta_1, \eta_2, \eta_3, \eta_4$ are disposable parameters with $v_j = u_j + \beta f(u_j)$ and $s_j = \left(\frac{f(w_j)}{f(u_j)}\right)^{\frac{1}{m}}$.

In this manuscript, we introduce a novel and efficient derivative free family of iterative methods for multiple roots $(m \geq 2)$. The derivation of our scheme is establish on the weight function approach. The new family consume only three evaluations of the involved function f and attaining the optimal order of convergence in the sense of classical Kung-Traub conjecture [13]. The members of our families have the simple body structure as well as consume the lowest CPU timing in comparison to the existing ones. In addition, we propose a main theorem which illustrate the fourth-order convergence when the multiplicity of roots (m) is known in advance. A numerical exhibition of our family is also illustrated on several numerical experiments which are based on the real life problems like: eigenvalue, Van der Waals equation of state, continuous stirred tank reactor (CSTR), Plank's radiation and roots clustering problems.

2. Construction of higher-order scheme

Here, we construct an optimal fourth-order family of iterative method for multiple zeros $m \geq 2$ with simple and compact body structure, which is defined by

$$\begin{aligned} \eta_j &= x_j + \beta f(x_j), \\ y_j &= x_j - m\frac{f(x_j)}{f[\eta_j,x_j]}, \\ x_{j+1} &= y_j + (y_j - x_j)\left[\frac{1}{2}\mu + Q(\nu)\right], \end{aligned} \tag{6}$$

where $\beta \in \mathbb{R}$is a nonzero real parameter and $m \geq 2$ is the known multiplicity of the required zero. In addition, function $Q : \mathbb{C} \to \mathbb{C}$ is analytic in the neighborhood of origin. Moreover, we considered $\mu = \left(\frac{f(y_j)}{f(\eta_j)}\right)^{\frac{1}{m}}$ and $\nu = \left(\frac{f(y_j)}{f(x_j)}\right)^{\frac{1}{m}}$ two multi-valued functions.

Suppose their principal analytic branches (see [14]), ν as a principal root given by $\nu = \exp\left[\frac{1}{m}\log\left(\frac{f(y_j)}{f(x_j)}\right)\right]$, with

$$\log\left(\frac{f(y_j)}{f(x_j)}\right) = \log\left|\frac{f(y_j)}{f(x_j)}\right| + i\arg\left(\frac{f(y_j)}{f(x_j)}\right), \text{ for } -\pi < \arg\left(\frac{f(y_j)}{f(x_j)}\right) \leq \pi.$$

The choice of $\arg(z)$, for $z \in \mathbb{C}$, agrees with that of $\log(z)$ to be employed later in the section of numerical experiments. We have an analogous way

$$\nu = \left|\frac{f(y_j)}{f(x_j)}\right|^{\frac{1}{m}} \exp\left[\frac{1}{m}\arg\left(\frac{f(y_j)}{f(x_j)}\right)\right] = O(e_j).$$

In the following result, we illustrate that the constructed scheme (6) attains maximum fourth-order of convergence for all $\beta \in \mathbb{R}$, $\beta \neq 0$, without adopting any supplementary evaluation of function or its derivative.

Theorem 1. *Let $x = \xi$ be a multiple zero, of multiplicity $m = 2$, of function f. Consider that $f : \mathbb{D} \subset \mathbb{C} \to \mathbb{C}$ is an analytic function in $\mathbb{D}$ surrounding the required zero ξ. Then, the presented scheme (6) has fourth-order convergence, provided*

$$Q(0) = 0,\ Q'(0) = \frac{1}{2},\ Q''(0) = 4,\ Q'''(0) = \kappa \in \mathbb{R}, \tag{7}$$

and satisfies the following error equation

$$e_{j+1} = -\frac{\left(\beta f''(\xi) + 2c_1\right)}{384}\left[\beta^2\big(f''(\xi)\big)^2\kappa + 4\beta f''(\xi)\big(\kappa - 9\big)c_1 + 4\big(\kappa - 33\big)c_1^2 + 48c_2\right]e_j^4 + O(e_j^5).$$

Proof. Let us consider that $e_j = x_j - \xi$ and $c_k = \dfrac{2!}{(2+k)!}\dfrac{f^{(2+k)}(\xi)}{f^{(2)}(\xi)},\ k = 1, 2, 3, 4$, are the error in jth iteration and asymptotic error constant numbers, respectively. Now, we adopt Taylor's series expansions for functions $f(x_j)$ and $f(\eta_j)$ around $x = \xi$ with the assumption $f(\xi) = f'(\xi) = 0$ and $f''(\xi) \neq 0$, which are given by

$$f(x_j) = \frac{f''(\xi)}{2!}e_j^2\left(1 + c_1e_j + c_2e_j^2 + c_3e_j^3 + c_4e_j^4 + O(e_j^5)\right) \tag{8}$$

and

$$\begin{aligned} f(\eta_j) =& \frac{f''(\xi)}{2!}e_j^2\Bigg[1 + \left(\beta f''(\xi) + c_1\right)e_j + \frac{1}{4}\left(\beta^2\big(f''(\xi)\big)^2 + 10\beta f''(\xi)c_1 + 4c_2\right)e_j^2 + \frac{1}{4}\Big(5\beta^2\big(f''(\xi)\big)^2c_1 \\ &+ 6\beta f''(\xi)c_1^2 + 12\beta f''(\xi)c_2 + 4c_3\Big)e_j^3 + \frac{1}{8}\Big(\beta^3\big(f''(\xi)\big)^3c_1 + 14\beta^2\big(f''(\xi)\big)^2c_1^2 + 16\beta^2\big(f''(\xi)\big)^2c_2 \\ &+ 28\beta f''(\xi)c_1c_2 + 28\beta f''(\xi)c_3 + 8c_4\Big)e_j^4 + O(e_j^5)\Bigg], \end{aligned} \tag{9}$$

respectively.

By using expressions (8) and (9) in scheme (6), we get

$$y_j - \xi = \frac{1}{4}\left(\beta f''(\xi) + 2c_1\right)e_j^2 + \theta_0e_j^3 + \theta_1e_j^4 + O(e_j^5), \tag{10}$$

where

$$\begin{aligned} \theta_0 &= -\frac{1}{16}\beta^2\big(f''(\xi)\big)^2 + \frac{1}{2}\beta f''(\xi)c_1 + c_2 - \frac{3c_1^2}{4}, \\ \theta_1 &= \frac{1}{64}\left[\beta^3\big(f''(\xi)\big)^3 - 10c_1\left(\beta^2\big(f''(\xi)\big)^2 + 16c_2\right) - 20\beta f''(\xi)c_1^2 + 64\beta f''(\xi)c_2 + 72c_1^3 + 96c_3\right]. \end{aligned}$$

Expression (10) and Taylor series expansion, leads us to

$$\begin{aligned} f(y_j) =& \frac{f''(\xi)}{2!}e_j^2\Bigg[\frac{1}{16}\left(\beta f''(\xi) + 2c_1\right)^2e_j^2 - \frac{1}{32}\left(\beta f''(\xi) + 2c_1\right)\left(\beta^2\big(f''(\xi)\big)^2 - 8\beta f''(\xi)c_1 + 12c_1^2 - 16c_2\right)e_j^3 \\ &+ \frac{1}{256}\Big(3\beta^4\big(f''(\xi)\big)^4 - 4c_1\left(7\beta^3\big(f''(\xi)\big)^3 - 48\beta f''(\xi)c_2 - 96c_3\right) + 96\beta^2\big(f''(\xi)\big)^2c_2 \\ &+ 32c_1^2\left(\beta^2\big(f''(\xi)\big)^2 - 32c_2\right) - 80\beta f''(\xi)c_1^3 + 192\beta f''(\xi)c_3 + 464c_1^4 + 256c_2^2\Big)e_j^4 + O(e_j^5)\Bigg]. \end{aligned} \tag{11}$$

From expressions (8), (9) and (11), we obtain

$$\mu = \left(\frac{f(y_j)}{f(\eta_j)}\right)^{\frac{1}{m}} = \frac{1}{4}\left(\beta f^{''}(\xi) + 2c_1\right)e_j + \bar{\theta}_0 e_j^2 + \bar{\theta}_1 e_j^3 + O(e_j^4) \tag{12}$$

and

$$\nu = \left(\frac{f(y_j)}{f(x_j)}\right)^{\frac{1}{m}} = \frac{1}{4}\left(\beta f^{''}(\xi) + 2c_1\right)e_j + \Theta_0 e_j^2 + \Theta_1 e_j^3 + O(e_j^4), \tag{13}$$

where

$$\begin{aligned}
\bar{\theta}_0 &= \frac{-3\beta^3\big(f''(\xi)\big)^3 - 4\beta^2\big(f''(\xi)\big)^2 c_1 - 12\beta f^{''}(\xi)c_1^2 + 16\beta f^{''}(\xi)c_2 - 32c_1^3 + 32c_1c_2}{16\left(\beta f''(\xi) + 2c_1\right)},\\
\bar{\theta}_1 &= \frac{1}{64}\left(7\beta^3\big(f''(\xi)\big)^3 - 22\beta^2\big(f''(\xi)\big)^2 c_1 - 14\beta f^{''}(\xi)c_1^2 + 24\beta f^{''}(\xi)c_2 + 116c_1^3 - 208c_1c_2 + 96c_3\right),\\
\Theta_0 &= \frac{-\beta^3\big(f''(\xi)\big)^3 + 4\beta^2\big(f''(\xi)\big)^2 c_1 - 4\beta f^{''}(\xi)c_1^2 + 16\beta f^{''}(\xi)c_2 - 32c_1^3 + 32c_1c_2}{16\left(\beta f''(\xi) + 2c_1\right)},\\
\Theta_1 &= \frac{1}{64}\left(\beta^3\big(f''(\xi)\big)^3 - 6\beta^2\big(f''(\xi)\big)^2 c_1 - 22\beta f^{''}(\xi)c_1^2 + 56\beta f^{''}(\xi)c_2 + 116c_1^3 - 208c_1c_2 + 96c_3\right).
\end{aligned}$$

Next, from expression (13), $\nu = O(e_j)$. Then, we expand the weight function $Q(\nu)$ in the neighborhood of origin (0) as:

$$Q(\nu) \approx Q(0) + Q'(0)\nu + \frac{1}{2!}Q''(0)\nu^2 + \frac{1}{3!}Q'''(0)\nu^3. \tag{14}$$

By using expressions (8)–(14) in (6), we have

$$e_{j+1} = -Q(0)e_j + \sum_{i=0}^{2} A_i e_j^{i+2} + O(e_j^5), \tag{15}$$

where $A_i = A_i(f^{''}(\xi), \beta, c_1, c_2, c_3, c_4, Q(0), Q'(0), Q''(0), Q'''(0))$, $i = 0, 1, 2$. For example, first coefficient explicitly written as

$$A_0 = \frac{1}{8}\Big(2Q(0) - 2Q'(0) + 1\Big)\left(\beta f^{''}(\xi) + 2c_1\right).$$

By (15), we deduce at least second-order convergence, provided

$$Q(0) = 0. \tag{16}$$

From expression (16) and $A_0 = 0$, we have

$$\frac{1}{8}\Big(1 - 2Q'(0)\Big)\left(\beta f^{''}(\xi) + 2c_1\right) = 0, \tag{17}$$

which further yield

$$Q'(0) = \frac{1}{2}. \tag{18}$$

By using expressions (16) and (18) in $A_1 = 0$, we get

$$-\frac{1}{32}\Big(Q''(0) - 4\Big)\left(\beta f^{''}(\xi) + 2c_1\right)^2 = 0, \tag{19}$$

which further have

$$Q''(0) = 4. \tag{20}$$

The asymptotic error constant term is obtained if we insert (16), (18) and (20) in (15). Then, we have

$$e_{j+1} = -\frac{\Big(\beta f''(\xi) + 2c_1\Big)}{384}\Big[\beta^2\big(f''(\xi)\big)^2\kappa + 4\beta f''(\xi)\big(\kappa - 9\big)c_1 + 4\big(\kappa - 33\big)c_1^2 + 48c_2\Big]e_j^4 + O(e_j^5), \tag{21}$$

where $\kappa = Q'''(0) \in \mathbb{R}$.

Expression (21) demonstrates maximum fourth-order convergence for all β, $\beta \neq 0$, with three evaluations of function f. Hence, our scheme (6) has an optimal convergence order as stated in conjecture given by Kung and Traub. ■

Theorem 2. *Adopting the same hypotheses of Theorem 1, then the proposed class (6) has fourth-order convergence for multiple roots of multiplicity $m = 3$.*

Proof. We adopt Taylor's series expansions for functions $f(x_j)$ and $f(\eta_j)$ around $x = \xi$ with the assumption $f(\xi) = f'(\xi) = f''(\xi) = 0$ and $f'''(\xi) \neq 0$, which are defined as follow:

$$f(x_j) = \frac{f^{'''}(\xi)}{3!} e_j^3 \Big(1 + b_1 e_j + b_2 e_j^2 + b_3 e_j^3 + b_4 e_j^4 + O(e_j^5) \Big) \tag{22}$$

and

$$f(\eta_j) = \frac{f^{'''}(\xi)}{3!} e_j^3 \Big[1 + b_1 e_j + \frac{1}{2} \left(\beta f^{'''}(\xi) + 2b_2 \right) e_j^2 + \left(\frac{7}{6} \beta f^{'''}(\xi) b_1 + b_3 \right) e_j^3 + O(e_j^4) \Big], \tag{23}$$

where $b_i = \dfrac{3!}{(3+i)!} \dfrac{f^{(3+i)}(\xi)}{f^{(3)}(\xi)}$, $i = 1, 2, 3, 4$, are asymptotic error constant numbers.

By using expressions (22) and (23) in scheme (6), we get

$$y_j - \xi = \frac{b_1}{3} e_j^2 + \frac{1}{18} \left(3\beta f^{'''}(\xi) - 8b_1^2 + 12b_2 \right) e_j^3 + \left(\frac{1}{9} b_1 \left(2\beta f^{'''}(\xi) - 13b_2 \right) + \frac{16b_1^3}{27} + b_3 \right) e_j^4 + O(e_j^5). \tag{24}$$

Expression (24) and Taylor series expansion, leads us to

$$f(y_j) = \frac{f^{'''}(\xi)}{3!} e_j^3 \Big[\frac{b_1^3}{27} e_j^3 + \frac{1}{54} b_1^2 \left(3\beta f^{'''}(\xi) - 8b_1^2 + 12b_2 \right) e_j^4 + O(e_j^5) \Big]. \tag{25}$$

By adopting expressions (22), (23) and (25), we obtain

$$\mu = \left(\frac{f(y_j)}{f(\eta_j)} \right)^{\frac{1}{m}} = \frac{b_1}{3} e_j + \frac{1}{18b_1^2} \Big(3\beta f^{'''}(\xi) b_1^2 - 10b_1^4 + 12b_2 b_1^2 \Big) e_j^2 + \frac{1}{27} \Big(3\beta f^{'''}(\xi) b_1 + 23b_1^3 - 48b_2 b_1 + 27b_3 \Big) e_j^3 + O(e_j^4) \tag{26}$$

and

$$\nu = \left(\frac{f(y_j)}{f(x_j)} \right)^{\frac{1}{m}} = \frac{b_1}{3} e_j + \frac{1}{18b_1^2} \Big(3\beta f^{'''}(\xi) b_1^2 - 10b_1^4 + 12b_2 b_1^2 \Big) e_j^2 + \frac{1}{54} \Big(9\beta f^{'''}(\xi) b_1 + 46b_1^3 - 96b_2 b_1 + 54b_3 \Big) e_j^3 + O(e_j^4). \tag{27}$$

Next, from expression (27), we have $\nu = O(e_\sigma)$. Then, we expand the weight function $Q(\nu)$ in the neighborhood of origin (0) as:

$$Q(\nu) = Q(0) + Q'(0)\nu + \frac{1}{2!} Q''(0)\nu^2 + \frac{1}{3!} Q'''(0)\nu^3. \tag{28}$$

By using expressions (22)–(28) in equation (6), we have

$$e_{j+1} = -Q(0) e_j + \sum_{i=0}^{2} B_i e_j^{i+2} + O(e_j^5), \tag{29}$$

where $B_i = B_i(f^{'''}(\xi), \beta, b_1, b_2, b_3, b_4, Q(0), Q'(0), Q''(0), Q'''(0))$, $i = 0, 1, 2$. For example, first coefficient explicitly written as

$$B_0 = \frac{1}{6} (2Q(0) - 2Q'(0) + 1) b_1.$$

By (29), we deduce at least second-order convergence, provided

$$Q(0) = 0. \tag{30}$$

From equation (30) and $B_0 = 0$, we have

$$\frac{1}{6} \Big(1 - 2Q'(0) \Big) b_1 = 0, \tag{31}$$

which further yield

$$Q'(0) = \frac{1}{2}. \tag{32}$$

By using expressions (30) and (32) in $B_1 = 0$, we get

$$-\frac{1}{18}\Big(Q''(0) - 4\Big)b_1^2 = 0, \tag{33}$$

which further have

$$Q''(0) = 4. \tag{34}$$

The asymptotic error constant term is obtained if we insert (30), (32) and (34) in (29). Then, we have

$$e_{n+1} = -\frac{b_1}{324}\Big[2(\kappa - 36)b_1^2 + 9\beta f^{'''}(\xi) + 36b_2\Big]e_j^4 + O(e_j^5), \tag{35}$$

where $\kappa = Q'''(0) \in \mathbb{R}$. Hence, we have proved that scheme (6) has fourth-order convergence for $m = 3$. ■

In a same way as before the following result can be established.

Theorem 3. *Adopting the same hypotheses of Theorem 1, the family of iterative methods given by (6) is of fourth-order convergence for* $m = 4, 5, 6$. *It satisfies the error equations for* $m = 4, 5$ *and* $m = 6$, *which are given by, respectively*

$$e_{j+1} = -\frac{1}{384}\left((\kappa - 39)p_1^3 + 24p_1p_2\right)e_j^4 + O(e_j^5),$$

$$e_{j+1} = -\frac{1}{750}\left((\kappa - 42)q_1^3 + 30q_1q_2\right)e_j^4 + O(e_j^5),$$

and

$$e_{j+1} = -\frac{1}{1296}\left((\kappa - 45)r_1^3 + 36r_1r_2\right)e_j^4 + O(e_j^5),$$

where $p_i = \dfrac{4!}{(4+i)!}\dfrac{f^{(4+i)}(\xi)}{f^{(4)}(\xi)}$, $q_i = \dfrac{5!}{(5+i)!}\dfrac{f^{(5+i)}(\xi)}{f^{(5)}(\xi)}$ *and* $r_i = \dfrac{6!}{(6+i)!}\dfrac{f^{(6+i)}(\xi)}{f^{(6)}(\xi)}$, $i = 1, 2, 3, 4$.

2.1. *Error for the general form of class* (6)

Now, let us prove a result similar to Theorem 1 for an arbitrary multiplicity m, $m \geq 4$.

Theorem 4. *Adopting the same hypotheses of Theorem 1, the iterative schemes given by (6) are of fourth-order convergence for* $m \geq 4$. *In this case, its error equation is*

$$e_{j+1} = -\frac{1}{6m^3}\Big[\Big(\kappa - 3(m+9)\Big)s_1^3 + 6ms_1s_2\Big]e_j^4 + O(e_j^5).$$

Proof. Let us consider that $e_j = x_j - \xi$ and $s_k = \dfrac{m!}{(m+k)!}\dfrac{f^{(m+k)}(\xi)}{f^{(m)}(\xi)}$, $k = 1, 2, 3, 4$ are the error in jth iteration and asymptotic error constant numbers, respectively. Now, we adopt Taylor's series expansions for functions $f(x_j)$ and $f(\eta_j)$ around $x = \xi$ with the assumption $f(\xi) = f'(\xi) = \ldots = f^{(m-1)}(\xi) = 0$ and $f^{(m)}(\xi) \neq 0$, which are given by, respectively,

$$f(x_j) = \frac{f^{(m)}(\xi)}{m!}e_j^m\Big(1 + s_1e_j + s_2e_j^2 + s_3e_j^3 + s_4e_j^4 + O(e_j^5)\Big) \tag{36}$$

and

$$f(\eta_j) = \frac{f^m(\xi)}{m!}e_j^m\Big[1 + \sum_{i=0}^{2}\Delta_i e_j^{i+2}O(e_j^5)\Big], \tag{37}$$

where $\Delta_i = \Delta_i(m, f^{(m)}(\xi), \beta, s_1, s_2, s_3, s_4)$, $i = 0, 1, 2$. For example, the first three coefficients explicitly written are $\Delta_0 = s_1$, $\Delta_1 = s_2$ and

$$\Delta_2 = \begin{cases} \frac{1}{6}\left(\beta f^{(4)}(\xi) + 6\Delta_3\right), & m = 4 \\ s_3, & m \geq 5 \end{cases}.$$

By using expressions (36) and (37) in (6), we get

$$\begin{aligned} e_{y_j} = y_j - \xi = \frac{s_1}{m} e_j^2 + \frac{1}{m^2}\left(2ms_2 - (1+m)s_1^2\right) e_j^3 + \frac{1}{m^3}\left(3m^2 s_3 + (m+1)^2 s_1^3 - m(3m+4)s_2 s_1\right) e_j^4 \\ + O(e_j^5). \end{aligned} \tag{38}$$

Expression (38) and Taylor series expansion, leads us to

$$f(y_j) = \frac{f^{(m)}(\xi)}{m!} e_{y_j}^m \left[1 + s_1 e_{y_j} + s_2 e_{y_j}^2 + s_3 e_{y_j}^3 + s_4 e_{y_j}^4 + O(e_j^5)\right]. \tag{39}$$

By adopting expressions (36), (37) and (39), we obtain

$$\begin{aligned} \mu = \left(\frac{f(y_j)}{f(\eta_j)}\right)^{\frac{1}{m}} = & \frac{s_1}{m} e_j + \frac{1}{m^2}\left(2ms_2 - (m+2)s_1^2\right) e_j^2 + \frac{1}{2m^3}\Big((2m^2 + 7m + 7)s_1^3 - 2m(3m+7)s_1 s_2 \\ & + 6m^2 s_3\Big) e_j^3 + O(e_j^4) \end{aligned} \tag{40}$$

and

$$\begin{aligned} \nu = \left(\frac{f(y_j)}{f(x_j)}\right)^{\frac{1}{m}} = & \frac{s_1}{m} e_j + \left(\frac{2}{m} s_2 - \frac{(m+2)}{m^2} s_1^2\right) e_j^2 + \frac{1}{2m^3}\Big[(2m^2 + 7m + 7)s_1^3 - 2m(3m+7)s_1 s_2 \\ & + 6m^2 s_3\Big] e_j^3 + \frac{1}{3m^4}\Big[-\left(\frac{11m^2}{2} + \frac{33m}{2} + 14\right) s_1^4 - 12m^2 s_3 s_1 + 3m(6m+11)s_2 s_1^2 \\ & - 6m^2 s_2^2\Big] e_j^4 + O(e_j^5). \end{aligned} \tag{41}$$

Next, from expression (41), $\nu = O(e_\sigma)$. Then, we expand the weight function $Q(\nu)$ in the neighborhood of origin (0) as:

$$Q(\nu) = Q(0) + Q'(0)\nu + \frac{1}{2!} Q''(0)\nu^2 + \frac{1}{3!} Q'''(0)\nu^3. \tag{42}$$

By using expressions (36)–(42) in (6), we have

$$e_{j+1} = -Q(0)e_j + \sum_{i=0}^{2} \Omega_i e_j^{i+2} + O(e_j^5), \tag{43}$$

where $\Omega_i = \Omega_i(m, f^{(m)}(\xi), \beta, s_1, s_2, s_3, s_4, Q(0), Q'(0), Q''(0), Q'''(0))$, $i = 0, 1, 2$. For example, the first coefficient explicitly written is

$$\Omega_0 = \frac{1}{2m}\Big(2Q(0) - 2Q'(0) + 1\Big).$$

By (43), we deduce at least second-order convergence, provided

$$Q(0) = 0. \tag{44}$$

From the expression (44) and $\Omega_0 = 0$, we have

$$\frac{1}{2m}\Big(1 - 2Q'(0)\Big) = 0, \tag{45}$$

which further yield

$$Q'(0) = \frac{1}{2}. \tag{46}$$

By using expressions (44) and (46) in $\Omega_1 = 0$, we get

$$-\frac{1}{2m^2}\Big(Q''(0) - 4\Big) s_1^2 = 0, \tag{47}$$

which further have

$$Q''(0) = 4. \tag{48}$$

The asymptotic error constant term is obtained if we insert (44), (46) and (48) in (43). Then, we have

$$e_{j+1} = -\frac{1}{6m^3}\left[\Big(\kappa - 3(m+9)\Big)s_1^3 + 6ms_1s_2\right]e_j^4 + O(e_j^5). \tag{49}$$

The expression (49) demonstrates the maximum fourth-order convergence for all $\beta, \kappa \in \mathbb{R}$ by consuming only three distinct evaluations of f. Hence, the proposed schemes (6) have an optimal convergence order as stated in conjecture given by Kung-Traub. ■

Remark 1. *No doubts from the expression* (49) *(for* $m \geq 4$*) that* β *is not appearing there. Actually, it appears with the coefficient of* e_j^5*. But, we don't need the calculation of the coefficient of* e_j^5 *in order to prove optimal fourth-order convergence. On the other hand, it is quite hard to calculate and time consuming task. However, the role of* β *and* κ *can be found in the expressions* (21) *and* (35) *for* $m = 2$ *and* $m = 3$*, respectively.*

3. Some Special cases of class (6)

Here, some of the special cases are generated from proposed class (6), by using different weight functions $Q(\nu)$ that satisfies the conditions of Theorem 1–4.

1. Consider $Q(\nu) = 2\nu^2 + \dfrac{\nu}{2}$, then, we have the following iterative method

$$\begin{aligned} y_j &= x_j - m\frac{f(x_j)}{f[\eta_j, x_j]}, \\ x_{j+1} &= y_j + (y_j - x_j)\left[\frac{1}{2}\mu + 2\nu^2 + \frac{\nu}{2}\right]. \end{aligned} \tag{50}$$

2. Assume $Q(\nu) = -\dfrac{\nu}{2(4\nu - 1)}$, which further leads to

$$\begin{aligned} y_j &= x_j - m\frac{f(x_j)}{f[\eta_j, x_j]}, \\ x_{j+1} &= y_j + (y_j - x_j)\left[\frac{1}{2}\mu - \frac{\nu}{2(4\nu - 1)}\right], \end{aligned} \tag{51}$$

   another new iterative scheme for multiple roots.

3. Suppose $Q(\nu) = \dfrac{\nu}{2\left(a_1\nu^2 - 4\nu + 1\right)}$, then, we have the new scheme

$$\begin{aligned} y_j &= x_j - m\frac{f(x_j)}{f[\eta_j, x_j]}, \\ x_{j+1} &= y_j + (y_j - x_j)\left[\frac{1}{2}\mu + \frac{\nu}{2\left(a_1\nu^2 - 4\nu + 1\right)}\right], \ a_1 \in \mathbb{R}. \end{aligned} \tag{52}$$

4. Let us assume another weight function $Q(\nu) = \dfrac{\nu(2a_2\nu + 1)}{4(a_2 - 2)\nu + 2}$, which further yields

$$\begin{aligned} y_j &= x_j - m\frac{f(x_j)}{f[\eta_j, x_j]}, \\ x_{j+1} &= y_j + (y_j - x_j)\left[\frac{1}{2}\mu + \frac{\nu(2a_2\nu + 1)}{4(a_2 - 2)\nu + 2}\right], \ a_2 \in \mathbb{R}, \end{aligned} \tag{53}$$

   an another new parametric family of iterative methods for multiple roots.
   Similarly, many more new methods can be introduced by adopting different weight functions $Q(\nu)$ that satisfy the conditions of Theorems in 1–4.

4. Numerical experimentation

This section is devoted to confirm the efficiency and convergence of our suggested family (6). Therefore, we consider the particular methods (50), (51) and (53) $\left(\text{for } a_2 = \dfrac{7-m}{8}\right)$, denoted by $OM1$, $OM2$ and $OM3$, respectively, with $\left(\beta = \frac{1}{2}\right)$ in all cases. Here, we consider several numerical experiments based on real life problems like eigenvalue problem, Van der Waals equation of state, continuous stirred tank reactor (CSTR), Plank's radiation and clustering of roots. The details of these numerical examples can be found in examples (1)–(5).

In Tables 1 – 5, we depict the approximated root (only up to 15 significant digits), absolute residual errors (only up to two significant digits with exponent), asymptotic error constants $\dfrac{\|x_{j+1}-x_j\|}{\|x_j - x_{j-1}\|^4}$ (only up to 6 significant digits with/without exponent), absolute error differences between two consecutive iteration (only up to two significant digits with exponent), due to the page restriction. However, we consider several number of significant digits (minimum 3000 significant digits) in order to minimize the rounding off errors. In addition, we also calculate the computational order of convergence based on the following formula

$$\rho = \frac{\ln \frac{\|x_{j+1}-\xi\|}{|x_j-\xi\|}}{\ln \frac{\|x_j-\xi\|}{\|x_{j-1}-\xi\|}}, \quad \text{for each } j = 1, 2, \ldots \tag{54}$$

Finally, we calculate the CPU timing and results are depicted in Table 6. These results are obtained by adopting the command "AbsoluteTiming[]" in $Mathematica$ 9. We run the same program for five times and mentioned their average in Table 6. We adopted $Mathematica$ 9 with multiple precision arithmetic for calculating the required values. In the Tables 1–5, the meaning of $b_1(\pm b_2)$ is $b_1 \times 10^{(\pm b_2)}$.

The configurations of the used computer are given below:
Processor: Intel(R) Core(TM) i7-4790 CPU @ 3.60GHz,
Made: HP,
RAM: 8:00GB,
System type: 64-bit-Operating System, x64-based processor.

We choose five existing robust optimal fourth-order methods for comparisons. Firstly, we make a contrast of our methods with the following fourth-order optimal scheme suggested by Kumar et al. [12]:

$$\begin{aligned} w_j &= u_j - m\frac{f(u_j)}{f[v_j, u_j]}, \\ u_{j+1} &= w_j - \frac{(m+2)s_j}{1-2s_j}\frac{f(u_j)}{f[v_j,u_j]+2f[w_j,v_j]} \end{aligned} \tag{55}$$

where $v_j = u_j + \beta f(u_j),\ s_j = \left(\dfrac{f(w_j)}{f(u_j)}\right)^{\frac{1}{m}}$, is called by KS (for $\beta = 0.5$).

Further, we contrast the same with the optimal schemes given by Sharma et al. [10], which are defined as follow:

$$\begin{aligned} z_j &= t_j - m\frac{f(t_j)}{f[s_j, t_j]}, \\ t_{j+1} &= z_j - \left(mx_jy_j + mx_j^2 + (m-1)y_j + x_j\right)\frac{f(t_j)}{f[s_j,t_j]} \end{aligned} \tag{56}$$

and

$$\begin{aligned} z_j &= t_j - m\frac{f(t_j)}{f[s_j, t_j]}, \\ t_{j+1} &= z_j - \left(\frac{x_j - y_j + my_j - m^2x_jy_j + 2mx_jy_j}{-mx_j + x_j^2 + 1}\right)\frac{f(t_j)}{f[s_j,t_j]}, \end{aligned} \tag{57}$$

where $s_j = t_j + \beta f(t_j),\ x_j = \left(\dfrac{f(z_j)}{f(t_j)}\right)^{\frac{1}{m}}$ and $y_j = \left(\dfrac{f(z_j)}{f(s_j)}\right)^{\frac{1}{m}}$. The above expressions are one of their best schemes claimed by Sharma et al. [10]. For particular $\beta = 0.5$, the previous schemes are denoted by $SS1$ and $SS2$, respectively.

Furthermore, we compare them with the following two optimal methods constructed by Kumar et al. [11]:

$$
\begin{aligned}
z_j &= u_j - m\frac{f(u_j)}{f[v_j,u_j]},\\
u_{j+1} &= z_j - \frac{mh_j(m-2h_j)}{2\left(2mh_j^2 - h_j(3m+2)+m\right)}\left(\frac{1}{y_j}+1\right)\frac{f(u_j)}{f[v_j,u_j]}
\end{aligned} \tag{58}
$$

and

$$
\begin{aligned}
z_j &= u_j - m\frac{f(u_j)}{f[v_j,u_j]},\\
u_{j+1} &= z_j - \frac{h_j(3-h_j)m}{6-20h_j}\left(\frac{1}{y_j}+1\right)\frac{f(u_j)}{f[v_j,u_j]}
\end{aligned} \tag{59}
$$

where $v_j = u_j + \beta f(u_j),\ x_j = \left(\frac{f(z_j)}{f(u_j)}\right)^{\frac{1}{m}}$ and $y_j = \left(\frac{f(v_j)}{f(u_j)}\right)^{\frac{1}{m}}$, with $h_j = \frac{x_j}{x_j+1}$. The expressions (58) and (59) are one of their best methods claimed by them. We called them by $KS1$ and $KS2$, respectively, with $\beta = 0.5$.

Example 1. Eigenvalue problem

Finding the eigenvalues of a large matrix is a difficult task in the field of linear algebra. The linear algebra approach is not always feasible. So, one of the best way is to use numerical techniques. Here, we consider the following square matrix of order 9:

$$
A = \frac{1}{8}\begin{pmatrix}
-12 & 0 & 0 & 19 & -19 & 76 & -19 & 18 & 437\\
-64 & 24 & 0 & -24 & 24 & 64 & -8 & 32 & 376\\
-16 & 0 & 24 & 4 & -4 & 16 & -4 & 8 & 92\\
-40 & 0 & 0 & -10 & 50 & 40 & 2 & 20 & 242\\
-4 & 0 & 0 & -1 & 41 & 4 & 1 & 2 & 25\\
-40 & 0 & 0 & 18 & -18 & 104 & -18 & 20 & 462\\
-84 & 0 & 0 & -29 & 29 & 84 & 21 & 42 & 501\\
16 & 0 & 0 & -4 & 4 & -16 & 4 & 16 & -92\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24
\end{pmatrix},
$$

whose characteristic equation is modeled into nonlinear equation as

$$
f_1(x) = x^9 - 29x^8 + 349x^7 - 2261x^6 + 8455x^5 - 17663x^4 + 15927x^3 + 6993x^2 - 24732^x + 12960.
$$

A root of this equation is $\xi = 3$ with multiplicity $m = 4$. Table 1 depicts the computational results by taking initial guess $x_0 = 3.5$.

Our methods $OM1, OM2$ and $OM3$ have same approximated roots and same absolute error difference between two consecutive iterations in compare to the existing schemes. In addition, methods $OM2$ and $OM3$ have higher computational order of convergence among all the mentioned methods.

Example 2. Van der Waals equation of state

$$
\left(P + \frac{a_1n^2}{V^2}\right)(V - na_2) = nRT,
$$

describes the nature of a real gas between two gases namely, a_1 and a_2 when we introduce the ideal gas equations. For calculating the volume V of gases, we need the solution of preceding expression in terms of remaining constants

$$
PV^3 - (na_2P + nRT)V^2 + \alpha_1n^2V - \alpha_1\alpha_2n^2 = 0.
$$

For choosing the particular values of gases α_1 and α_2, we can easily obtain the values for n, P and T. Then, we yield

$$
f_2(x) = x^3 - 5.22x^2 + 9.0825x - 5.2675.
$$

The function f_2 having 3 zeros and among them: $\xi = 1.75$ is a multiple zero of multiplicity $m = 2$ and $\xi = 1.72$ is a simple zero. The computational results by adopting starting guess $x_0 = 2.0$ for the required zero $\xi = 1.75$, are given in the Table 2.

Based on the obtained results, we conclude that our method $OM3$ has the least absolute residual error and absolute error difference between two consecutive iterations as compared to the other mentioned methods. In addition, it also shows the stable computational order of convergence.

Table 1: Convergence behavior of different methods on eigenvalue problem f_1

Methods	j	x_j	$\|x_{j+1}-x_j\|$	$\|f(x_j)\|$	ρ	$\frac{\|x_{j+1}-x_j\|}{\|x_j-x_{j-1}\|^4}$
	1	2.98003995753693	$2.0(-2)$	$1.3(-5)$		
KS	2	3.00000000783350	$7.8(-9)$	$3.0(-31)$		$9.63863(+1)$
	3	3.00000000000000	$2.9(-17)$	$5.8(-65)$	1.316	$4.93525(-2)$
	1	2.98048789485049	$2.0(-2)$	$1.2(-5)$		
$SS1$	2	3.00000000492928	$4.9(-9)$	$4.7(-32)$		$1.14425(+2)$
	3	3.00000000000000	$1.2(-17)$	$1.4(-66)$	1.308	2.03699
	1	2.97977318565461	$2.0(-2)$	$1.4(-5)$		
$SS2$	2	3.00000002569087‘	$2.6(-8)$	$3.5(-29)$		$1.20509(+2)$
	3	3.00000000000000	$3.1(-16)$	$7.7(-61)$	1.342	8.99188
	1	2.98013951815471	$2.0(-2)$	$1.3(-5)$		
$KS1$	2	3.00000002608467	$2.6(-8)$	$3.7(-29)$		$1.17369(+2)$
	3	3.00000000000000	$3.7(-29)$	$8.7(-61)$	1.344	9.93570
	1	2.98028470211794	$2.0(-2)$	$1.2(-5)$		
$KS2$	2	3.00000002141616	$2.1(-8)$	$2.1(-8)$		$1.16134(+2)$
	3	3.00000000000000	$2.2(-16)$	$1.8(-61)$	1.340	8.43820
	1	2.98060140187230	$1.9(-2)$	$1.2(-5)$		
$OM1$	2	3.00000001183796	$1.2(-8)$	$1.6(-30)$		$9.54495(+1)$
	3	3.00000000000000	$6.7(-17)$	$1.6(-63)$	1.328	$8.35977(-2)$
	1	2.97976872816216	$2.0(-2)$	$1.4(-5)$		
$OM2$	2	3.00000003737290	$3.7(-8)$	$1.6(-28)$		$9.68176(+1)$
	3	3.00000000000000	$6.6(-16)$	$1.5(-59)$	1.352	$2.23080(-1)$
	1	2.97998520031377	$2.0(-2)$	$1.3(-5)$		
$OM3$	2	3.00000003134086	$3.1(-8)$	$7.7(-29)$		$9.64745(+1)$
	3	3.00000000000000	$4.7(-16)$	$3.8(-60)$	1.348	$1.95300(-1)$

Table 2: Convergence behavior of different methods on Van der waals problem f_2

Methods	j	x_j	$\|x_{j+1} - x_j\|$	$\|f(x_j)\|$	ρ	$\frac{\|x_{j+1}-x_j\|}{\|x_j-x_{j-1}\|^4}$
	1	1.75173443192182	$1.7(-3)$	$9.5(-8)$		
KS	2	1.75000003709553	$3.7(-8)$	$4.1(-17)$		$3.19594(+2)$
	3	1.75000000000000	$8.8(-27)$	$2.3(-54)$	3.989	$4.09949(+3)$
	1	1.75310030071776	$3.1(-3)$	$3.2(-7)$		
$SS1$	2	1.75000147880784	$1.5(-6)$	$6.6(-14)$		$1.46364(+2)$
	3	1.75000000000000	$1.2(-19)$	$4.5(-40)$	3.939	7.42318
	1	1.75233138059653	$2.3(-3)$	$1.8(-7)$		
$SS2$	2	1.75000025958081	$2.6(-7)$	$2.0(-15)$		$8.78843(+1)$
	3	1.75000000000000	$5.3(-23)$	$8.3(-47)$	3.970	3.40317
	1	1.75145379531864	$1.5(-3)$	$6.6(-8)$		
$KS1$	2	1.75000001017186	$1.0(-8)$	$3.1(-18)$		$4.21235(+1)$
	3	1.75000000000000	$2.5(-29)$	$1.8(-59)$	3.999	$7.13151(-1)$
	1	1.75140121500918	$1.4(-3)$	$6.2(-8)$		
$KS2$	2	1.75000000858279	$8.6(-9)$	$2.2(-18)$		$3.99562(+1)$
	3	1.75000000000000	$1.3(-29)$	$4.7(-60)$	3.997	$6.88266(-1)$
	1	1.75309730578006	$3.1(-3)$	$3.2(-7)$		
$OM1$	2	1.75000147342676	$1.5(-6)$	$6.5(-14)$		$6.39715(+2)$
	3	1.75000000000000	$1.2(-19)$	$4.3(-40)$	3.940	$1.60405(+4)$
	1	1.74792039281209	$2.3(-3)$	$1.2(-7)$		
$OM2$	2	1.75017553497394	$1.8(-4)$	$9.3(-10)$		$3.06551(+2)$
	3	1.74999999998922	$1.1(-11)$	$3.5(-24)$	6.504	$6.78685(+6)$
	1	1.75101278063150	$1.0(-3)$	$3.2(-8)$		
$OM3$	2	1.75000000018765	$1.9(-10)$	$1.1(-21)$		$1.75867(+2)$
	3	1.75000000000000	$2.1(-38)$	$1.3(-77)$	4.153	$1.78359(+2)$

Example 3. Continuous stirred tank reactor (CSTR)

Here, we assume an isothermal continuous stirred tank reactor (CSTR) problem. Let us consider that components M_1 and M_2 stands for fed rates to the reactors B_1 and $B_2 - B_1$, respectively. Then, we obtain the following reaction scheme in the reactor (for the details see [15]):

$$\begin{aligned} M_1 + M_2 &\to B_1, \\ B_1 + M_2 &\to C_1, \\ C_1 + M_2 &\to D_1, \\ C_1 + M_2 &\to E_1. \end{aligned}$$

Douglas [16] studied the above model, when he was designing a simple model for feedback control systems. He converted the above model in to the following mathematical expression:

$$R_{C_1} \frac{2.98(x + 2.25)}{(x + 1.45)(x + 2.85)^2(x + 4.35)} = -1,$$

where R_{C_1} is the gain of proportional controller. The expression (60) is balanced for the negative real values of values of R_{C_1}. In particular, by choosing $R_{C_1} = 0$, we yield

$$f_3(x) = x^4 + 11.50x^3 + 47.49x^2 + 83.06325x + 51.23266875. \tag{60}$$

The zeros of function f_3 are known as the poles of the open-loop transfer function. The function f_3 has four zeros $\gamma = -1.45, -2.85, -2.85, -4.35$. But, we choose $\gamma = -2.85$ as the required zero with multiplicity $m = 2$. We assume $x_0 = -2.8$ as the starting point for f_3 and results are mentioned in the Table 3.

Table 3: Convergence behavior of different methods on CSTR problem f_3

Methods	j	x_j	$\|x_{j+1} - x_j\|$	$\|f(x_j)\|$	ρ	$\frac{\|x_{j+1}-x_j\|}{\|x_j-x_{j-1}\|^4}$
	1	-2.85309218936353	$3.1(-3)$	$2.0(-5)$		
KS	2	-2.84999999997525	$2.5(-11)$	$1.3(-21)$		$3.89174(+2)$
	3	-2.85000000000000	$6.7(-22)$	$9.5(-43)$	1.305	$2.70769(-1)$
	1	-2.85308590593325	$3.1(-3)$	$2.0(-5)$		
$SS1$	2	-2.85000000004208	$4.2(-11)$	$3.7(-21)$		$2.96502(+3)$
	3	-2.85000000000000	$1.5(-42)$	$4.5(-84)$	3.999	2.85278
	1	-2.85308999319490	$3.1(-3)$	$2.0(-5)$		
$SS2$	2	-2.84999999999768	$2.3(-12)$	$1.1(-23)$		$2.96996(+3)$
	3	-2.85000000000000	$5.9(-24)$	$7.3(-47)$	1.271	$1.56209(-1)$
	1	-2.85309291853761	$3.1(-3)$	$2.0(-5)$		
$KS1$	2	-2.84999999996770	$3.2(-11)$	$2.2(-21)$		$2.97345(+3)$
	3	-2.85000000000000	$1.1(-21)$	$2.8(-42)$	1.309	2.17017
	1	-2.85309294673949	$3.1(-3)$	$2.0(-5)$		
$KS2$	2	-2.84999999996768	$3.2(-11)$	$2.2(-21)$		$2.97348(+3)$
	3	-2.85000000000000	$1.1(-21)$	$2.8(-42)$	1.309	2.17130
	1	-2.85308349814459	$3.1(-3)$	$2.0(-5)$		
$OM1$	2	-2.85000000001372	$1.4(-11)$	$4.0(-22)$		$3.88334(+2)$
	3	-2.85000000000000	$5.4(-45)$	$6.2(-89)$	4.000	$1.51755(-1)$
	1	-2.85309503996439	$3.1(-3)$	$2.0(-5)$		
$OM2$	2	-2.84999999989344	$1.1(-10)$	$2.4(-20)$		$3.89449(+2)$
	3	-2.85000000000000	$1.2(-20)$	$3.3(-40)$	1.331	1.16129
	1	-2.85309111881677	$3.1(-3)$	$2.0(-5)$		
$OM3$	2	-2.84999999993161	$6.8(-11)$	$9.8(-21)$		$3.89070(+2)$
	3	-2.85000000000000	$5.1(-21)$	$5.5(-41)$	1.323	$7.49503(-1)$

From the results presented in Table 3, we deduce that our method $OM1$ perform far better than the existing ones in terms of absolute errors difference between two iterations, absolute residual error and computational order of convergence. We can clearly see from the approximated root that $OM1$ converges faster to the required root than the other depicted methods.

Example 4. Planck's radiation problem

Consider the Planck's radiation equation that determines the spectral density of electromagnetic radiations released by a black-body at a given temperature, and at thermal equilibrium [17] as

$$G(y) = \frac{8\pi chy^{-5}}{e^{\frac{ch}{ykT}} - 1},$$

where T, y, k, h, and c denotes the absolute temperature of the black-body, wavelength of radiation, Boltzmann constant, Plank's constant, and speed of light in the medium (vacuum), respectively. To evaluate the wavelength y which results to the maximum energy density $G(y)$, set $G'(y) = 0$. We obtained the equation

$$\frac{\left(\frac{ch}{ykT}\right) e^{\frac{ch}{ykT}}}{e^{\frac{ch}{ykT}} - 1} = 5.$$

Further, the nonlinear equation is formulated by setting $x = \dfrac{ch}{ykT}$ as follows:

$$f_4(x) = \left(e^{-x} - 1 + \frac{x}{5}\right)^3.$$

The root of this equation is $\gamma \approx 4.96511423174428$ of multiplicity $m = 3$ and with this root one can easily find the wave length y from the relation $x = \dfrac{ch}{ykT}$. The Planck's problem is tested with initial guess $x_0 = 5.4$ and computational results are depicted in Table 4.

The results shown in Table 4 confirm that our method $OM2$ perform far better than the existing ones in the terms of absolute errors difference between two iterations, absolute residual error and computational order of convergence. In addition, it have the least asymptotic error constant as compare to other mentioned methods.

Example 5. Root clustering problem

We picked a root clustering problem (the details can be found in Zeng [18])

$$f_5(x) = (x-2)^{15}(x-4)^5(x-3)^{10}(x-1)^{20}. \tag{61}$$

In the above function, we have four multiple zeros $x = 1,\ 2,\ 3$ and 4 with multiplicities $20,\ 15,\ 10$ and 5, respectively. All multiple zero are quite close to each other. We chose $x = 2$ multiple zero of multiplicity 15 for the computational point of view. The computational results by adopting the initial approximation $x_0 = 2.1$ are depicted in Table 5.

From Table 5, we deduce that our method $OM2$ perform far better than the existing ones in the terms of absolute errors difference between two iterations, absolute residual error and computational order of convergence. In addition, it have the lowest asymptotic error constant as compare to the all mentioned methods.

Finally, in Table 6 we show the computational time of each of the methods used in the different examples above. In this table, the meaning of T.T. and A.T. are total time and average time, respectively. We can observe that the best times, both total and average, are provided by the methods proposed in this manuscript.

5. Concluding Remarks

In this study, we have proposed a novel and efficient fourth-order derivative free family of iterative methods for multiple roots ($m \geq 2$) of nonlinear equations. The presented schemes are stand on the weight function approach. By choosing new weight functions, we can easily construct several new or existing iterative methods.

It is important to note down that our class (6), uses only three distinct functional evaluations at each iteration. Thus, it has optimal fourth-order convergence in sense of the classic Kung–Traub conjecture.

We find from the numerical experimentation that our methods $OM1, OM2$ and $OM3$ have better numerical results in contrast to $KS, SS1, SS2, KS1$ and $KS2$ in Examples 2–5, in the terms of absolute errors difference between two iterations, absolute residual error and computational order of convergence. Our methods $OM1, OM2$ and $OM3$ have

Table 4: Convergence behavior of different methods on Planck's radiation problem f_4

Methods	j	x_j	$\|x_{j+1}-x_j\|$	$\|f(x_j)\|$	ρ	$\frac{\|x_{j+1}-x_j\|}{\|x_j-x_{j-1}\|^4}$
	1	4.96511652308559	$2.3(-6)$	$8.7(-20)$		
KS	2	4.96511423174428	$2.8(-27)$	$1.6(-82)$		$6.40617(-5)$
	3	4.96511423174428	$6.3(-111)$	$1.8(-333)$	4.000	$1.01708(-4)$
	1	4.96511673344157	$2.5(-6)$	$1.1(-19)$		
$SS1$	2	4.96511423174428	$4.6(-27)$	$6.9(-82)$		$2.54544(+1)$
	3	4.96511423174428	$5.1(-110)$	$9.6(-331)$	4.000	$197740(+1)$
	1	4.96511613241687	$1.9(-6)$	$4.9(-20)$		
$SS2$	2	4.96511423174428	$1.1(-27)$	$1.1(-83)$		$1.93389(+1)$
	3	4.96511423174428	$1.5(-112)$	$2.3(-338)$	4.000	$1.47339(+1)$
	1	4.96511580759512	$1.6(-6)$	$2.8(-20)$		
$KS1$	2	4.96511423174428	$4.7(-28)$	$7.5(-85)$		$1.60339(+1)$
	3	4.96511423174428	$3.8(-114)$	$3.8(-343)$	4.000	$1.29392(+1)$
	1	4.96511592838008	$1.7(-6)$	$3.5(-20)$		
$KS2$	2	4.96511423174428	$6.8(-28)$	$2.3(-84)$		$1.72629(+1)$
	3	4.96511423174428	$1.8(-113)$	$4.0(-341)$	4.000	$1.39392(+1)$
	1	4.96511639458599	$2.2(-6)$	$7.3(-20)$		
$OM1$	2	4.96511423174428	$2.3(-27)$	$9.0(-83)$		$6.04690(-5)$
	3	4.96511423174428	$3.1(-111)$	$2.1(-334)$	4.000	$1.05906(-4)$
	1	4.96511542365886	$1.2(-6)$	$1.2(-20)$		
$OM2$	2	4.96511423174428	$1.2(-28)$	$1.2(-86)$		$3.33234(-5)$
	3	4.96511423174428	$1.2(-116)$	$1.1(-350)$	4.000	$5.86773(-5)$
	1	4.96511567121202	$1.4(-6)$	$2.1(-20)$		
$OM3$	2	4.96511423174428	$3.0(-28)$	$2.0(-85)$		$4.02445(-5)$
	3	4.96511423174428	$5.9(-115)$	$1.5(-345)$	4.0000	$7.04845(-5)$

Table 5: Convergence behavior of different methods on root clustering problem f_5

Methods	j	x_j	$\|x_{j+1}-x_j\|$	$\|f(x_j)\|$	ρ	$\frac{\|x_{j+1}-x_j\|}{\|x_j-x_{j-1}\|^4}$
	1	2.00003020108641	$3.0(-5)$	$5.1(-67)$		
KS	2	2.00000000000000	$4.9(-19)$	$6.2(-274)$		$3.02376(-1)$
	3	2.00000000000000	$3.2(-74)$	$1.4(-1101)$	4.000	$5.83228(-1)$
	1	2.00003890701229	$3.9(-5)$	$2.3(-65)$		
$SS1$	2	2.00000000000000	$2.5(-18)$	$2.7(-263)$		$3.33065(+1)$
	3	2.00000000000000	$4.1(-71)$	$5.1(-1055)$	4.000	8.67022
	1	1.99993731903336	$6.3(-5)$	$2.9(-62)$		
$SS2$	2	$2.00000000388735 + 4.1(-10)i$	$3.9(-9)$	$2.4(-125)$		$5.37703(+1)$
	3	$2.00000000000000 + 2.41(-34)i$	$5.9(-34)$	$1.3(-497)$	5.902	$2.02363(+9)$
	1	2.00002793705549	$2.8(-5)$	$1.6(-67)$		
$KS1$	2	2.00000000000000	$2.8(-19)$	$1.6(-277)$		$2.38980(+1)$
	3	2.00000000000000	$2.8(-75)$	$1.5(-1117)$	4.000	3.66841
	1	2.00003379654677	$3.4(-5)$	$2.7(-66)$		
$KS2$	2	2.00000000000000	$1.0(-18)$	$4.4(-269)$		$2.89217(+1)$
	3	2.00000000000000	$8.5(-73)$	$3.0(-1080)$	4.000	6.26933
	1	2.00003890701229	$3.9(-5)$	$2.3(-65)$		
$OM1$	2	2.00000000000000	$2.5(-18)$	$2.7(-263)$		$3.89676(-1)$
	3	2.00000000000000	$4.1(-71)$	$5.1(-1055)$	4.000	1.08291
	1	2.00002041197111	$2.0(-5)$	$1.4(-69)$		
$OM2$	2	2.00000000000000	$1.4(-20)$	$8.2(-297)$		$2.04286(-1)$
	3	2.00000000000000	$3.7(-81)$	$9.1(-1206)$	4.000	$8.33951(-2)$
	1	2.00000932410079	$9.3(-6)$	$1.1(-74)$		
$OM3$	2	2.00000000000000	$3.1(-21)$	$9.5(-307)$		$9.32758(-2)$
	3	$2.00000000000000 + 1.0(-42)i$	$9.9(-42)$	$2.6(-614)$	1.325	$4.16587(-1)$

Table 6: CPU timing of distinct iterative schemes

I.M.	$Ex.$(1)	$Ex.$(2)	Ex. (3)	Ex. (4)	Ex. (5)	$T.T.$	$A.T.$
KS	0.059216	0.00100	0.001000	0.071049	0.021157	0.153422	0.0306844
$SS1$	0.052139	0.000998	0.001001	0.056041	0.019011	0.12919	0.025838
$SS2$	0.054030	0.001000	0.001000	0.053054	0.043776	0.15286	0.030572
$KS1$	0.054752	0.000998	0.001001	0.051036	0.031698	0.139485	0.027897
$KS2$	0.050770	0.001000	0.000999	0.062369	0.018012	0.13315	0.02663
$OM1$	0.050717	0.000983	0.001000	0.054038	0.017688	0.124426	0.0248852
$OM2$	0.046783	0.000992	0.000984	0.049828	0.004056	0.102643	0.0205286
$OM3$	0.048770	0.000989	0.000998	0.052038	0.030021	0.132816	0.0265632

same approximated roots and same error difference between two consecutive iterations in compare to the existing methods, in Example 1.

The lowest CPU time (total and average time on the examples) is taken by our methods in order to execute the computational results as compared to all mentioned schemes.

Finally, we deduce on the basis of Tables 1–6, that $OM1, OM2$ and $OM3$ are more effective and could be a better alternative to the earlier existing methods.

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