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On a Safety Set for an Epidemic Model with a Bounded Population

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Abstract. Given a class of non-linear SIRS epidemic model, we analyse some useful conditions on the model parameters to determine a safety set for the containment of an epidemic. In addition, once that set is determined, we find control actions so that the epidemic remains within the security set with infection rates below an allowed amount. More specifically, for every initial state in a certain safety set of the state space there exists an adequate control policy maintaining the state of the system in such safety set. Sufficient conditions for the existence of a solution under a feedback are derived in terms of linear inequalities on the input vectors at the vertices of a polytope.

Keywords: epidemiological process, discrete-time non-linear system, positivity, stability, control feedback.

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1 Introduction

In general, epidemic models are based on mathematical models that describe the spread of microorganisms within populations. Several non-linear models have been formulated, mathematically analysed and applied to infectious diseases. For example, one of the most frequent is the SIR model (see [3, 12, 20]), which initially begins with susceptible individuals and assumes that a contagious disease appears in the population. After a contact between a susceptible individual and an infectious individual, the susceptible individual can become infectious (without entering any latency period) and when the period of infection ends, the individual can recover from the disease with immunity or without immunity, [10, 11] and [20]. If the recovered individual does not show immunity then it is again a susceptible individual and the process is represented by a SIRS model (see [1, 9, 12] and the references therein). In recent years, the application of mathematical models to analyse different scenarios in the evolution of infectious epidemics has been a increasing trend. In particular, [2] presents an application of mathematical modelling of an infectious bovine disease, in [14] and [21] the study focuses on the productivity of cattle, in particular in its dairy or in herds of cattle, and in [8] analyses a mathematical model of a human infection.

In general, these types of models incorporate some restrictions on the size of the population due to the special circumstances of the farm. In addition, the continued contacts between susceptible and infected animals in the different farm enclosures contributes to the maintenance of the infection for extended periods of time.

We are more interested in looking for conditions to determine a distribution of population, susceptible, infected and recovered, where the number of infected individuals can be reduced or maintained within the appropriate levels. This distribution is hereafter named as *safety distribution population* or *safety set*.

Reachability problems play an important role in a wide range of problems related to control of biological or engineering systems. In general, a reachability problem evaluates whether a system will attain, in a finite time, a certain set of states by means of trajectories originating from a given initial set. In the literature, several approaches have been proposed to study the reachability property of non-linear systems. For example, in [4] the approach is based on local linearisations of the non-linear system, in [17] a continuous (piecewise) affine state feedback is used to solve this problem and in [23] the structure of the system is exploited to analyse reachability of discrete-time systems. In particular, a reachability analysis is important when designing models to meet some safety or performance specifications. This is the case of the models representing the evolution of an epidemic where it is very important to design control actions so that the evolution of the disease remains within certain levels of security and does not extend to the entire population. Positivity and related mathematical properties are assumptions used in the modelling of the spread of infectious diseases. [16] and [24] are two references of different models commonly used. Positive reachability of non-linear systems has not yet been sufficiently studied. Some recent works related to the characterization of this property for discrete and continuous systems are [5] and [6] and an algorithm for computing reachable states for non-linear biological models is given in [13]. Moreover, an approach based on the fixed point theory can be found in [22]. In [17] the authors consider a reachability problem for an affine system on a full-dimensional polytope and they study how to reach a particular facet of the polytope.

In this paper, we focus on a particular class of non-linear SIRS epidemic models and our goal is to verify if it is possible to maintain the state of an epidemic system inside some safety set by choosing an adequate control policy, i.e. to ensure that the state of the system remains in a certain region of the state space. For that, the spread of a disease is modelled by means of a non-autonomous SIRS model with a bounded population along the time. The difference between a traditional SIRS model and our proposal is the interpretation of the population recruitment as a control action. For example, a situation where this control can be applied is to optimize the resources of a livestock farm. It is important to achieve maximum production optimizing resources. So, it is necessary to maintain a certain population on the farm, replacing non-productive animals. In the our case, the adequate control policy is determined through geometrical methods. Specifically, a polytope that limits the number of infected individuals is considered, as a safety set to ensure an acceptable evolution of the disease. In addition, a class of functions relying on the polytope vertices is constructed. These kind of functions have allowed us to derive the feedback control guaranteeing that the state trajectory of the closed-loop system remains in such a region. This control strategy allow us to increase the susceptible population and maintaining the distribution of individuals in appropriate levels established by such a safety set.

The rest of the paper is organised as follows. The following section includes the notation and those preliminary results used throughout this work. Section 3 gives a description of the epidemic model to be studied and the positivity property is also examined. In the fourth section, an adequate control policy to keep the trajectory of an epidemic system in some safety set is chosen. Finally, some conclusions are given.

2 Preliminaries

For the sake of simplicity, a generalised notation and widely used definitions are adopted. Thus, a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is non-negative (positive) if $a_{ij} \geq 0$ ($a_{ij} > 0$) for all i, j, and it is denoted by $A \geq O$ ($A > O$). The spectrum of A and its spectral radius are denoted by $\sigma(A)$ and $\rho(A)$, respectively. It is known, [7], that a matrix A is stable if $\rho(A) < 1$. An autonomous discrete-time linear system $x(t+1) = Ax(t)$ is asymptotically stable to 0 if and only if A is a stable matrix, that is $\rho(A) < 1$.

Consider $V = \{v_i\}_{i=1,\dots,m}$ with $m \geq n+1$ a set of points in the space $\mathbb{R}^n$ such that there exist no hyperplanes of $\mathbb{R}^n$ containing all these m points. A full-dimensional polytope $P(V)$ is defined as the convex hull of V (see [15]). If a point $v_i,\ i = 1,\dots,m$ cannot be written as a convex combination of $V - v_i$, the point v_i is called a polytope's vertex. Recall that, a full-dimensional polytope is characterised by its set of vertices.

Consider the discrete-time non-linear system given by

$$x(t+1) = Ax(t) + f(x(t)) + Bu(t),\ t \geq 0 \tag{2.1}$$

with $x(\cdot) \in \mathbb{R}^n$, $u(\cdot) \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, and $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ a bound differentiable function satisfying $f(0) = 0$, $f(\lambda e_i) = 0$, with e_i the i-th unit vector $\forall i = 1,\dots,n$, and $\lambda > 0$.

In general, positive systems are those systems whose trajectory from any initial non-negative condition remains in the positive orthant for all future time, that is, a system is positive if from any non-negative initial state and any non-negative input sequence the solution trajectory is non-negative. The notion of local positiveness of a non-linear time-varying system is introduced in [18] and necessary and sufficient conditions for positivity of a class of non-linear systems are established in [19].

Denoting $f(x) = (f_i(x))_{i=1}^n$ and $Ax = ((Ax)_i)_{i=1}^n$, we have the following characterization.

Proposition 1. *System (2.1) is positive if and only if $A \geq O$, $B \geq O$ and for any $x \in \mathbb{R}^n$, $|f_i(x)| \leq (Ax)_i$ for every i such that $f_i(x) \leq 0$.*

Proof. If the system is positive, taking $x(0) = 0$ and $u(0) = 1$, $0 \leq x(1) = B$ since $f(0) = 0$. Then $B \geq 0$. Moreover, taking $x(0) = e_i$ and $u(0) = 0$, $0 \leq x(1) = Ae_i$, since $f(e_i) = 0$. Then, $A \geq O$. And, if $x(0) = x \geq 0$ and $u(0) = 0$, $0 \leq x(1) = Ax + f(x)$. Then, since $A \geq O$, the condition $|f_i(x)| \leq (Ax)_i$ for every i such that $f_i(x) \leq 0$ holds.

Conversely, it is straightforward. □

3 SIRS model

From now on, we consider a SIRS dynamic process for spread of a disease. The individuals are organised in three compartments: Susceptible (x_1), Infected (x_2) and Immune or Recovered (x_3). The mathematical model is described by a system as (2.1) with $x(t) \in \mathbb{R}_+^3$, non-negative input $u(t) \in \mathbb{R}_+$, where the relationship among the individuals is given by

$$A = \begin{pmatrix} p-\mu & 0 & w \\ 0 & q-\gamma & 0 \\ \mu & \gamma & r-w \end{pmatrix}, \; f(x(t)) = \begin{pmatrix} -\alpha x_1(t)x_2(t) \\ \alpha x_1(t)x_2(t) \\ 0 \end{pmatrix}, \; B = e_1, \tag{3.1}$$

where all the parameters involved are described on Table 1. Moreover, in order to the parameters of the mathematical model to fit their epidemiological meaning, it is assumed that they satisfy the following constraints: $\alpha > 0$, $0 \leq \mu < p < 1$, $0 < \gamma < q < 1$ and $0 \leq w < r < 1$.

Table 1. Parameters in SIR model (2.1)–(3.1).

Parameters	Definition
p, q, r	Survival rates of susceptible individuals, infected individuals and recovered individuals, respectively.
α	Exposition rate of susceptible individuals by contact with an infected individual.
γ	Transition rate of infected individuals to recovered individuals.
μ	Transition rate of susceptible individuals to recovered individuals (immune individuals).
w	Transition rate of recovered individuals to susceptible individuals.

Matrix A reflects the interconnections between the three compartments of population. There are recovered (immune) susceptible individuals (with a transition rate μ), recovered infected individuals (with a transition rate γ) and susceptible recovered individuals (with a transition rate w).

Since new susceptible individuals can only be added by modifying the first compartment, then $B = e_1$. This works by increasing the number of susceptible individuals as a recruitment to the population. The newly added population is time-dependent and can be viewed as a control action to achieve the desired objectives, for example the stability property of the model, maintaining an adequate population size or optimising farmers' resources to maximise profit or farm efficiency. In our case, this control action is used to find a safety distribution population or a safety set.

Therefore, using Proposition 1, system (2.1)–(3.1) is positive if and only if

$$\alpha x_1(t)x_2(t) \leq (p-\mu)x_1(t) + wx_3(t), \quad \forall t \geq 0,$$

or equivalently,

$$x_1(t)(p-\mu-\alpha x_2(t)) + wx_3(t) \geq 0, \quad \forall t \geq 0.$$

Then, the condition $x_2(t) \leq \frac{p-\mu}{\alpha}$, for all $t \geq 0$, is a sufficient condition to ensure that system (2.1)–(3.1) is a positive system.

Let us suppose that, for an initial condition $x(0) \in \mathbb{R}^3_+$, system (2.1) represents the dynamic of an initial population $P_0 = x_1(0) + x_2(0) + x_3(0) \in \mathbb{R}_+$, it is evident that system (2.1) must be positive. Moreover, if no controls are considered, due to the survival of individuals, the solution must tend to zero. Note that the state $x(t)$ of an autonomous system can be written as

$$x(t) = A^t x(0) + \begin{pmatrix} I & A & \cdots & A^{t-1} \end{pmatrix} \begin{pmatrix} f(x(t-1)) \\ \vdots \\ f(x(0)) \end{pmatrix}, \;\; t \geq 0.$$

Hence, if matrix A is stable and $f(t)$ is a bounded function then we have assured that the population tends to extinct. Focusing our attention on model (2.1)–(3.1), we observe that, the eigenvalues of matrix A are $q-\gamma < 1$ and

$$\lambda_{1,2} = \frac{1}{2}\left(p-\mu+r-w \pm \sqrt{\mu^2 + (p-r+w)^2 + 2\mu(-p+r+w)}\right). \qquad (3.2)$$

Using the positivity of the parameters μ and w, note that the eigenvalues $\lambda_{1,2}$ are real because they can be written as follow:

$$\lambda_{1,2} = \frac{1}{2}\left(p-\mu+r-w \pm \sqrt{((p-\mu)-(r-w))^2 + 4\mu w}\right).$$

Furthermore, using the Gershgorin circle theorem by columns, it is deduced that the spectrum of this matrix is contained into the line segment $]0,1[$. It is due to the fact that every eigenvalue of A lies within at least one of the Gershgorin disc, which in this case are $D(p-\mu,\mu)$, $D(q-\gamma,\gamma)$, $D(r-w,w)$. Thus, $|\lambda_{1,2}| < 1$, that is, A is stable. Then, considering an initial population $P_0 = x_1(0) + x_2(0) + x_3(0) \in \mathbb{R}_+$ such that $P_0 \leq \frac{p-\mu}{\alpha}$ and zero controls, the autonomous system given by (2.1)–(3.1) is positive and its solution tends to zero.

In the following section we consider a more general case corresponding to a non-autonomous system with a bounded population along the time.

3.1 Equilibrium points for a non-autonomous system with a bounded population

Let us consider henceforth a non-autonomous system with a bounded population, more specifically, $\sum_{i=1}^{3} x_i(t) \leq P,\ t \geq 0$. That is, the population is bounded by P at all times.

As the model represents an epidemiological process, we are interested in analysing the dynamic process when some infected individuals are introduced. That is, the initial state considered is $x(0) = (x_1(0)\ x_2(0)\ 0)^T$. The main aim of this study is to find a safety set and to look for the conditions so that if $x(t)$ (state of the system) belongs to the set, then the trajectory of a system remains in that set forever. Once this set is determined, one could study under what conditions the system trajectory can be led to that set, however that is not the purpose of this study. Nevertheless, in the next development we show a case in which we can search for the set of reachable states that have the structure we are looking for and how we can find a control sequence that leads us to that set if possible. To achieve the latter, given that the state $x(t)$ of the system (2.1) can be written as

$$x(t)=A^t x(0)+ \left(B\ AB \cdots A^{t-1}B\right) \begin{pmatrix} u(t-1) \\ \vdots \\ u(0) \end{pmatrix} + \left(I\ A \cdots A^{t-1}\right) \begin{pmatrix} f(x(t-1)) \\ \vdots \\ f(x(0)) \end{pmatrix},$$

we need to introduce the following concept of positively reachable (non-negative) state at t steps: *A non-negative state $\bar{x}$ is positively reachable from $x(0) \geq 0$ if there exists a non-negative sequence $u(0),\ \ldots, u(t-1) \geq 0$ such that $x(i) \geq 0$ for $i = 1, \ldots, t-1$ and $x(t) = \bar{x}$.* For instance, if $\mu = 0$,

$$\left(B\ AB\ \cdots\ A^{t-1}B\right) \begin{pmatrix} u(t-1) \\ \vdots \\ u(0) \end{pmatrix} = \begin{pmatrix} u(t-1) + \cdots + p^{t-1}u(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then, *the set of positively reachable states from $x(0)$ in t steps* is

$$R_t(x(0))=\{x \in \mathbb{R}^3_+\ /\ \exists \alpha_i \geq 0 : x=A^t x(0)+\sum_{i=0}^{t-1}(p^{t-1-i}\alpha_i e_1+A^{t-1-i}f(x(i)))\}.$$

From here, we would take the control sequence that in t steps takes the path from the initial state $x(0)$ to a final state $x \in R_t(x(0))$ belonging to the safety set.

To approach the construction of the safety set, we have to previously study conditions on the asymptotic stability of the system. For that, we consider that our equilibrium point is of the form $(x^\star, u^\star)$. Notice that the equilibrium points $(x^\star, u^\star)$ are defined by the solution of the system of algebraic equations $x^\star = Ax^\star + f(x^\star) + Bu^\star$ with the condition $\sum_{i=1}^{3}(x^\star)_i \leq P$. These solutions are the disease-free (DFE) equilibrium point E_f and the endemic equilibrium points. So, taking $u^\star$ and centered on the disease-free equilibrium point $E_f = (x^\star, u^\star)$, that is, the point at which no disease is present in the population, then $x_2^\star = 0$. So, E_f is given by

$$E_f = ((x_1^\star, 0, x_3^\star), u^\star) = \left(\frac{u^\star}{(1-p+\mu)(1-r+w)-\mu w}((1-r+w), 0, \mu), u^\star \right). \quad (3.3)$$

Since $x_2^\star = \frac{p-\mu}{\alpha}$, it follows that the point E_f is a positive equilibrium point of system (2.1)–(3.1).

On the other hand, we have a bounded population P, that is, $\Sigma_{i=1}^{3} x_i(t) \leq P$. Then, using the coordinates of E_f given in (3.3), it is satisfied that

$$x_1^\star + x_3^\star = \frac{u^\star(1 - r + w + \mu)}{(1 - p + \mu)(1 - r + w) - \mu w} \leq P.$$

Then,

$$u^\star \leq \frac{P}{(1 - r + w + \mu)}((1 - p + \mu)(1 - r + w) - \mu w).$$

We now study the behaviour of disease-free equilibrium for system (2.1)–(3.1). For that, the model is linearised around the disease-free equilibrium point and we obtain the eigenvalues of the coefficient matrix. Thus, the stability of the disease-free equilibrium point is directly related to the spectral radius of this coefficient matrix. Linearising around E_f we have $x_l(t) = x(t) - x^\star$, $u_l(t) = u(t) - u^\star$ and approximating $f(x(t))$ as follows,

$$f(x(t)) \approx \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} x_1^\star x_2(t) + x_1(t) x_2^\star = \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} x_1^\star x_2(t),$$

the new linearised systems is given by

$$x_l(t+1) = (A + \tilde{A}) x_l(t) + B u_l(t), \ t \geq 0, \quad (3.4)$$

with $\tilde{A} = \begin{pmatrix} 0 & -\alpha x_1^\star & 0 \\ 0 & \alpha x_1^\star & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then, the approximation of the solution of non-linear system is $x_{ap}(t) = x_l(t) + x^\star$. Note that the characteristic polynomial of $A + \tilde{A}$ is given by

$$\det(\lambda I - (A + \tilde{A})) = (\lambda - (q - \gamma + \alpha x_1^\star)) \ \det \begin{pmatrix} \lambda - p + \mu & -w \\ -\mu & \lambda - \gamma + w \end{pmatrix}.$$

Hence, the spectrum of $A + \tilde{A}$ is given by $\lambda_{1,2}$ given in Equation (3.2), which have modulus less than 1, and $\lambda_3 = q - \gamma + \alpha x_1^\star$. Using expression of $x_1^\star$ given in (3.3), the matrix $A + \tilde{A}$ is stable if and only if

$$u^\star < \frac{(1 - q + \gamma)}{\alpha(1 - r + w)} \left((1 - p + \mu)(1 - r + w) - \mu w\right). \quad (3.5)$$

This means that the system stability is directly related to the control action that can be used in the process. This technical result can be interpreted as the maximum population recruitment to fulfil the stability of the model. Thus, in our case the control $u^\star$ must satisfy condition (3.5). Since $\rho(A+\tilde{A})<1$, then system (3.4) is asymptotically stable, $\lim_{t\to\infty} x_l(t)=0$, that is, the disease-free equilibrium point is globally asymptotically stable.

We summarize all the previous comments in the following result.

Theorem 1. *If $u^\star$ satisfy the following relationship*

(i)

$$\frac{u^\star}{(1-p+\mu)(1-r+w)-\mu w}\leq\frac{P}{(1-r+w+\mu)}<\frac{(1-q+\gamma)}{\alpha(1-r+w)},$$

then the condition $\sum_{i=1}^{3}(x^\star)_i\leq P$ *is satisfied and the system is stable.*

(ii)

$$\frac{(1-q+\gamma)}{\alpha(1-r+w)}\leq\frac{u^\star}{(1-p+\mu)(1-r+w)-\mu w}\leq\frac{P}{(1-r+w+\mu)},\quad(3.6)$$

then the condition $\sum_{i=1}^{3}(x^\star)_i\leq P$ *is satisfied and the system is unstable.*

We recall that the goal of this paper is looking for conditions to determine a distribution of population, susceptible, infected and resistant (immune or recovered), where the number of infected individuals can be reduced or maintained within the appropriate levels. This means that there are still some infected individuals in the population while condition (3.6) should hold and therefore, the population satisfies $\alpha P\geq(1-q+\gamma)\,(1+\mu/(1-r+w))$.

In the next section, we obtain some conditions to find control actions so that the epidemic remains within the security set with infection rates below an allowed positive amount ϵ.

4 Safety set

In this section, let us assume that condition (3.6) holds. Then population bound P is such that

$$\alpha P\geq(1-q+\gamma)\,(1+\mu/(1-r+w))\,.\quad(4.1)$$

In this case, the epidemic process is not stable around the disease-free equilibrium point. Despite having instability, we analyse whether it is possible to choose an adequate control policy to keep the trajectory of an epidemic system inside some safety set, that is, to guarantee that the system state $x(t)=(x_1(t)\ x_2(t)\ x_3(t))^T$ remains in a certain subset of the admissible state space such that the infected population, which a percentage of the population, is less than a safety bound $\epsilon = \tau P > 0,\ 0 < \tau < 1$. For that, taking into account the condition that assures a non-negative solution of system (3.1), and the previous consideration, the size of the infected population should satisfy

$$x_2(t) \leq \epsilon \leq \frac{p-\mu}{\alpha}. \tag{4.2}$$

Besides that, we can assume that the susceptible individuals are a suitable percentage of the non-infected individuals $(1-\tau)P$. Thus, for any bound P and any $\epsilon > 0$ satisfying (4.1)–(4.2) we gather all these conditions that make sense for an epidemic model in an admissible set $\mathcal{X}$ given by

$$\begin{aligned} \mathcal{X} = \{x = (x_i)_{i=1}^3 \in \mathbb{R}_+^3 \ / \ \sum_{i=1}^{3} x_i \leq P,\ x_1 \leq Pk(1-\tau), \\ x_2 \leq P\tau,\ \ 0 \leq \tau < k(1-\tau) < 1,\ \ 0 < k \leq 1\}. \end{aligned}$$

In particular, the population of $\mathcal{X}$ has been chosen so that the upper bound for x_1 is greater than the upper bound of x_2. Hence, the constraint $\tau < k(1-\tau) < 1$ with $0 < k \leq 1$. Observe that this set $\mathcal{X} = P(V)$ is a polytope for the set of vertices $V = \{v_i\}_{i=0}^7$,

$$\begin{array}{ll} v_0 = (0\ 0\ 0)^T, & v_4 = P\left(k(1-\tau)\ \tau\ 0\right)^T, \\ v_1 = P\left(k(1-\tau)\ 0\ 0\right)^T, & v_5 = P\left(0\ \tau\ 1-\tau\right)^T, \\ v_2 = P\left(0\ \tau\ 0\right)^T, & v_6 = P\left(k(1-\tau)\ 0\ 1-k(1-\tau)\right)^T, \\ v_3 = P\left(0\ 0\ 1\right)^T, & v_7 = P\left(k(1-\tau)\ P\tau\ 1-\tau-k(1-\tau)\right)^T. \end{array} \tag{4.3}$$

We propose to solve the following problem:

From an initial state $x(0)$ in a subset of $\mathcal{X}$, to construct a feedback control $u(t) = h(x(t))$ keeping the state trajectory $x(t)$ of the closed-loop system in this set, for all $t \geq 0$.

For that, we focus our attention on the function class defined as

$$\Theta = \{\theta : \mathcal{X} \to \mathbb{R}^3 \ / \ \theta(x) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_1x_2,\ c_i \in \mathbb{R}^3, c_i \neq 0\}. \tag{4.4}$$

It is straightforward to prove that the image by a map θ in Θ of any state in $\mathcal{X}$ can be written as a linear combination of the images of the vertices characterizing $\mathcal{X}$.

Proposition 2. *Consider $\theta \in \Theta$. Then, for all $x \in \mathcal{X}$, $\theta(x)$ can be written $\theta(x) = \sum_{i=1}^4 \alpha_i \theta(v_i)$, with $\bar{V} = \{v_1, v_2, v_3, v_4\}$ defined in (4.3) with*

$$\begin{array}{ll} \alpha_1 = \dfrac{x_1}{Pk(1-\tau)}(1 - \dfrac{x_2}{\tau P}), & \alpha_2 = \dfrac{x_2}{\tau P}(1 - \dfrac{x_1}{Pk(1-\tau)}), \\ \alpha_3 = \dfrac{x_3}{P}, & \alpha_4 = \dfrac{x_1x_2}{P^2k(1-\tau)\tau}. \end{array} \tag{4.5}$$

Proof. We consider the image of θ on the vertices of $\mathcal{X}$: $\theta(v_1) = c_1Pk(1-\tau)$, $\theta(v_2) = c_2P\tau$, $\theta(v_3) = c_3P$, $\theta(v_4) = c_1Pk(1-\tau) + c_2P\tau + c_4P^2k(1-\tau)\tau$, $\theta(v_5) = c_2P\tau + c_3P(1-\tau)$, $\theta(v_6) = c_1Pk(1-\tau) + c_3P(1-k(1-\tau))$ and

$\theta(v_7) = c_1Pk(1-\tau) + c_2P\tau + c_3P(1-\tau-k(1-\tau)) + c_4P^2k(1-\tau)\tau$. The result is directly followed by solving the system $\theta(x) = \sum_{i=1}^{6} \alpha_i\theta(v_i)$ and taking a value for the parameters α_5 and α_6, that is

$$Pk(1-\tau)(\alpha_1+\alpha_4+\alpha_6+\alpha_7) = x_1, \quad P\tau(\alpha_2+\alpha_4+\alpha_5+\alpha_7) = x_2,$$
$$P(\alpha_3+\alpha_5(1-\tau)+\alpha_6(1-k(1-\tau))+\alpha_7(1-\tau-k(1-\tau))) = x_3,$$
$$P^2k(1-\tau)\tau(\alpha_4+\alpha_7) = x_1x_2,$$

and consider the solution corresponding to $\alpha_5 = 0$, $\alpha_6 = 0$ and $\alpha_7 = 0$. □

Our goal is to find a subset $\tilde{\mathcal{X}}$ of the state space in which the following statement holds: *Given some map $\theta \in \Theta$ satisfying $\theta(v_i) \in \tilde{\mathcal{X}}$, $v_i \in \bar{V}$, we can ensure that the image of any state of that region is also in it.* In this way, first, we look for a admissible subset of states $\tilde{\mathcal{X}} \subseteq \mathcal{X}$ so that the above linear combination is a convex linear combination, that is, with non-negative coefficients and their coefficient sum less than 1.

Proposition 3. *The coefficients α_i given in (4.5) satisfy $0 \le \alpha_i \le 1$, and $\sum_{i=1}^{4} \alpha_i \le 1$ if and only if $x \in \tilde{\mathcal{X}} \subseteq \mathcal{X}$ with*

$$\tilde{\mathcal{X}} = \Big\{x = (x_i)_{i=1}^3 \in \mathcal{X} \ / \ x_3 \le P(1 - \frac{x_1}{Pk(1-\tau)})\big(1 - x_2/P\tau\big)\Big\}. \quad (4.6)$$

Proof. The condition that the coefficients α_i, $i = 1, \dots, 4$ are between 0 and 1 follows directly from (4.5) since all the α_i can be written as a product of quantities that are between 0 and 1.

In addition, $\sum_{i=1}^{4} \alpha_i \le 1$, if and only if

$$\sum_{i=1}^{4} \alpha_i = \frac{x_3}{P} + \frac{x_1}{Pk(1-\tau)} + \frac{x_2}{P\tau} - \frac{x_1x_2}{P^2k(1-\tau)\tau} \le 1,$$

or equivalently, $x_3 \le P(1 - \frac{x_1}{Pk(1-\tau)})(1 - \frac{x_2}{P\tau})$. So, from $x \in \mathcal{X}$, the condition $\sum_{i=1}^{4} \alpha_i \le 1$ holds if and only if $x \in \tilde{\mathcal{X}} \subseteq \mathcal{X}$ with $\tilde{\mathcal{X}}$ defined in (4.6). □

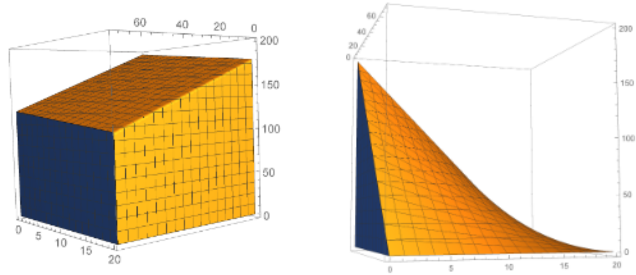
Example 1. Consider a bound of the population $P = 200$ and coefficients $k = 0.2$ and $\tau = 0.05$. Then, the susceptible population has to be less than $Pk(1-\tau) = 38$, the infectious population less than $P\tau = 10$, and the resistant (immune or recovered) population less than $P = 200$. The vertices of the admissible set $\mathcal{X}$ are

$$v_0 = (0,0,0), \ \ v_1 = (38\ 0\ 0)^T, \ \ v_2 = (0\ 10\ 0)^T, \ \ v_3 = (0\ 0\ 200)^T,$$
$$v_4 = (38\ 10\ 0)^T, \ \ v_5 = (0\ 10\ 190)^T, \ \ v_6 = (38\ 0\ 162)^T, \ \ v_7 = (38\ 10\ 152)^T.$$

In order to assure the conditions on the coefficients α_i, $i = 1,2,3,4$ given in Proposition 3, we have to restrict the set of states to

$$\begin{aligned}\tilde{\mathcal{X}} =& \Big\{x = (x_i)_{i=1}^3 \in \mathcal{X} \ /x_3 \le P(1 - \frac{x_1}{Pk(1-\tau)})\big(1 - x_2/P\tau\big)\Big\} \\ =& \Big\{x = (x_i)_{i=1}^3 \in \mathcal{X} \ /x_3 \le 200(1 - x_1/38)(1 - x_2/10)\Big\}.\end{aligned}$$

In Figure 1 we can observe that $\mathcal{X}$ is a polytope and $\tilde{\mathcal{X}} \subseteq \mathcal{X}$ is not it. This fact motivates the need to reduce the set of states a bit more considering a polytope included in $\tilde{\mathcal{X}}$.



(i) Polytope $\mathcal{X}$. (ii) Set $\tilde{\mathcal{X}}$.

Figure 1. Initial admissible polytope $\mathcal{X}$ and set $\tilde{\mathcal{X}}$ under conditions given by Proposition 3, in Example 1.

In order to restrict the region to a polytope contained in $\tilde{\mathcal{X}}$, it is sufficient to consider the region limited by some tangent plane to the surface S: $\frac{x_3}{P} = (1 - \frac{x_1}{Pk(1-\tau)})(1 - \frac{x_2}{P\tau})$, $(x_1\ x_2\ x_3)^T \in \mathcal{X}$, at a point $N = (x_1^0\ x_2^0\ x_3^0)^T \in S$. This tangent plane is defined as

$$(x_1 - x_1^0)\frac{\partial x_3}{\partial x_1}(N) + (x_2 - x_2^0)\frac{\partial x_3}{\partial x_2}(N) - (x_3 - x_3^0) = 0.$$

Since

$$\frac{\partial x_3}{\partial x_1}(N) = -\frac{1}{k(1-\tau)}(1 - \frac{x_2^0}{P\tau}), \quad \frac{\partial x_3}{\partial x_2}(N) = -\frac{1}{\tau}(1 - \frac{x_1^0}{Pk(1-\tau)}),$$

by substituting in the previous equation, we get that

$$-\frac{1}{k(1-\tau)}(\frac{P\tau - x_2^0}{P\tau})(x_1 - x_1^0) - \frac{1}{\tau}(\frac{Pk(1-\tau) - x_1^0}{Pk(1-\tau)})(x_2 - x_2^0) + (x_3 - x_3^0) = 0.$$

Multiplying the above equation by $-Pk(1-\tau)\tau$:

$$(P\tau - x_2^0)(x_1 - x_1^0) + (Pk(1-\tau) - x_1^0)(x_2 - x_2^0) + Pk(1-\tau)\tau(x_3 - x_3^0) = 0.$$

We rewrite the above equation as follows

$$(P\tau - x_2^0)x_1 + (Pk(1-\tau) - x_1^0)x_2 + Pk(1-\tau)\tau x_3 = \mathcal{M},$$

where $\mathcal{M} = (P\tau - x_2^0)x_1^0 + (Pk(1-\tau) - x_1^0)x_2^0 + Pk(1-\tau)\tau x_3^0$.

Now, we consider the polytope $\mathcal{P}_N \subseteq \tilde{\mathcal{X}}$,

$$\mathcal{P}_N = \{x = (x_i)_{i=1}^3 \in \tilde{\mathcal{X}} / (P\tau - x_2^0)x_1 + (Pk(1-\tau) - x_1^0)x_2 + Pk(1-\tau)\tau x_3 \le \mathcal{M}\} \tag{4.7}$$

and we prove the following result.

Proposition 4. *Consider $\mathcal{P}_N \subseteq \tilde{\mathcal{X}}$ given in (4.7) and the set $\bar{V} = \{v_1, v_2, v_3, v_4\} \subset V$ defined in (4.3). If $\theta \in \Theta$ satisfies $\theta(v_i) \in \mathcal{P}_N$, $i = 1, 2, 3, 4$, then $\theta(x) \in \mathcal{P}_N$ for all $x \in \mathcal{P}_N$.*

Proof. Consider $\theta(v_i) = (\theta(v_i)_1\ \theta(v_i)_2\ \theta(v_i)_3)^T$, $i = 1, 2, 3, 4$. From $\theta(v_i) \in \mathcal{P}_N$, $i = 1, 2, 3, 4$, we have that $\sum_{j=1}^4 \theta(v_i)_j \leq P$, $\theta(v_i)_1 \leq kP(1-\tau)$, $\theta(v_i)_2 \leq P\tau$ and $(P\tau - x_2^0)\theta(v_i)_1 + (Pk(1-\tau) - x_1^0)\theta(v_i)_2 + Pk(1-\tau)\tau\theta(v_i)_3) \leq \mathcal{M}$, $i = 1, 2, 3, 4$.

Given $x \in \mathcal{P}_N$ using Propositions 2–3 we can write $\theta(x) = \sum_{i=1}^4 \alpha_i \theta(v_i)$, with $v_i \in \bar{V},\ i = 1, 2, 3, 4$, $0 \leq \alpha_i \leq 1$, and $\sum_{i=1}^4 \alpha_i \leq 1$. Then,

$$\theta(x)_1 = \sum_{i=1}^4 \alpha_i \theta(v_i)_1 \leq \sum_{i=1}^4 \alpha_i kP(1-\tau) \leq kP(1-\tau),$$
$$\theta(x)_2 = \sum_{i=1}^4 \alpha_i \theta(v_i)_2 \leq \sum_{i=1}^4 \alpha_i P\tau \leq P\tau.$$

Moreover,

$$\sum_{j=1}^4 \theta(x)_j = \sum_{j=1}^4 \sum_{i=1}^4 \alpha_i \theta(v_i)_j = \sum_{i=1}^4 \alpha_i \sum_{j=1}^4 \theta(v_i)_j \leq \sum_{i=1}^4 \alpha_i P \leq P,$$

and

$$\begin{aligned}&(P\tau - x_2^0)\theta(x)_1 + (Pk(1-\tau) - x_1^0)\theta(x)_2 + Pk(1-\tau)\tau\theta(x)_3\\ &= \sum_{i=1}^4 \alpha_i \left((P\tau - x_2^0)\theta(v_i)_1 + (Pk(1-\tau) - x_1^0)\theta(v_i)_2 + Pk(1-\tau)\tau\theta(v_i)_3\right)\\ &\leq \sum_{i=1}^4 \alpha_i \mathcal{M} \leq \mathcal{M}.\end{aligned}$$

Hence, $\theta(x) \in \mathcal{P}_N$. □

By the previous proposition, any polytope constructed in (4.7) is valid. We choose the tangent plane to surface at the vertex v_3 in order to simplify the explicit expression of the control that maintains the solution trajectory within the polytope.

Considering the tangent plane to surface S at the point $\bar{N} = v_3 = (0\ 0\ P)^T$ the polytope defined in (4.7) corresponding to this point and denoted by $\mathcal{P}_{\bar{N}}$ is given by

$$\mathcal{P}_{\bar{N}} = \{x = (x_i)_{i=1}^3 \in \tilde{\mathcal{X}}\ /\ \frac{x_1}{k(1-\tau)} + \frac{x_2}{\tau} + x_3 \leq P\}. \tag{4.8}$$

Example 2. Using the parameters given in Example 1 we construct the polytope given in (4.8)

$$\mathcal{P}_{\bar{N}} = \{x = (x_i)_{i=1}^3 \in \tilde{\mathcal{X}}\ /\ \frac{x_1}{0.19} + \frac{x_2}{0.05} + x_3 \leq 200\},$$

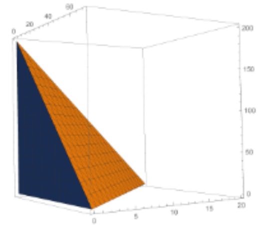


Figure 2. *Polytope $\mathcal{P}_{\bar{N}}$ contained in set $\tilde{\mathcal{X}}$ corresponding to the parameters values given in Example 1.*

whose vertices are $v_0 = (0\ 0\ 0)^T$, $v_1 = (38\ 0\ 0)^T$, $v_2 = (0\ 10\ 0)^T$, $v_3 = (0\ 0\ 200)^T$. This polytope is shown in Figure 2.

As $\mathcal{P}_{\bar{N}}$ is a polytope and the coefficients $\{\alpha_i\}_{i=1}^4$ meet conditions established in Proposition 3, then the image $\theta(x)$ with $x \in \mathcal{P}_{\bar{N}}$ and $\theta \in \Theta$, are also in $\mathcal{P}_{\bar{N}}$ if $\theta(v_i) \in \mathcal{P}_{\bar{N}}$, for all vertices $\{v_i\}_{i=1}^4 = \bar{V}$.

From here, let us assume that the initial state is in $\mathcal{P}_{\bar{N}}$, and that, for all $\theta \in \Theta$, $\theta(x)$ is a combination of the images of the vertices in $\bar{V} = \{v_1,\ v_2,\ v_3,\ v_4\}$ with coefficients $\{\alpha_i\}_{i=1}^4$ given in (4.5).

Going back to system (2.1)–(3.1), we have $x(t+1) = g(x(t)) + Bu(t)$, with $g(x) = Ax + f(x)$ belonging to Θ since $g(x)$ can be written as in (4.4) using the vectors

$$c_1 = \begin{pmatrix} p-\mu \\ 0 \\ \mu \end{pmatrix},\ c_2 = \begin{pmatrix} 0 \\ q-\gamma \\ \gamma \end{pmatrix},\ c_3 = \begin{pmatrix} w \\ 0 \\ r-w \end{pmatrix},\ c_4 = \begin{pmatrix} -\alpha \\ \alpha \\ 0 \end{pmatrix}.$$

We are going to look for a feedback control $u(t) = h(x(t))$, with $\tilde{h} = Bh \in \Theta$, such that the new function g_c of the closed-loop system

$$x(t+1) = g(x(t)) + \tilde{h}(x(t)),\ x(0) \in \mathcal{X},\ t \geq 0, \tag{4.9}$$

given by $g_c(x) = Ax + f(x) + \tilde{h}(x)$ is also in Θ and $\mathcal{P}_{\bar{N}}$ is g_c-invariant, that is, $g_c(\mathcal{P}_{\bar{N}}) \subseteq \mathcal{P}_{\bar{N}}$. To this purpose, and taking into account Proposition 4 we analyse conditions in order to $g(v_i) + Bh(v_i) \in \mathcal{P}_{\bar{N}}$ for $v_i \in \bar{V}$.

Theorem 2. *Consider system (2.1)–(3.1) and the safety polytope $\mathcal{P}_{\bar{N}}$ given in (4.8), $0 < \epsilon = \tau P < P$ satisfying (4.1)–(4.2), and $0 < k \leq 1$ such that $Pk(1-\tau) \leq \frac{1-q+\gamma}{\alpha}$. If a function $h : \mathcal{P}_{\bar{N}} \to \mathbb{R}$ satisfies $Bh(x) \in \Theta$ and the following conditions*

$$\begin{aligned} &h(v_1) \leq Pk(1-\tau)\mathbf{a},\ h(v_2) \leq Pk(1-\tau)\mathbf{b}, h(v_3) \leq Pk(1-\tau)(1-r+w)-Pw, \\ &h(v_4) \leq Pk(1-\tau)(\mathbf{a}+\mathbf{b}-1+\alpha P(\tau-k(1-\tau))),\ v_i \in \bar{V}, i=1,2,3,4, \end{aligned} \tag{4.10}$$

where Θ is given in (4.4), $v_i \in \bar{V}, i = 1, 2.3, 4$, are given in (4.3), $\mathbf{a} = 1 - p + \mu - \mu k(1-\tau)$ and $\mathbf{b} = 1 - q + \gamma(1-\tau)$ then, under the control feedback $u(t) = h(x(t))$, $t \geq 1$, the trajectory of closed-loop system (4.9) remains in $\mathcal{P}_{\bar{N}}$ for all initial state $x(0) \in \mathcal{P}_{\bar{N}}$.

Proof. Since $\tilde{h}(x) = Bh(x) \in \Theta$, according to Proposition 2, $\tilde{h}(x) = B\sum_{i=1}^{4} \alpha_i h(v_i)$ with α_i defined in (4.5). Taking the control feedback $u(t) = h(x(t))$, we have the function of the closed-loop system $g_c(x) = Ax + f(x) + \tilde{h}(x)$. We are going to prove that $g_c(v_i) \in \mathcal{P}_{\bar{N}}$, $v_i \in \bar{V}$, $i = 1, 2, 3, 4$. Note that from (4.8), a state x belongs to $\mathcal{P}_{\bar{N}}$ if and only if $\displaystyle\sum_{i=1}^{3} x_i \leq P$ and

$$x_1 \leq P(1-\tau),\ x_2 \leq P\tau,\ \frac{x_1}{k(1-\tau)} + \frac{x_2}{\tau} + x_3 \leq P. \tag{4.11}$$

Note that if $\frac{x_1}{k(1-\tau)} + \frac{x_2}{\tau} + x_3 \leq P$ holds, then $\sum_{i=1}^{3} x_i \leq P$ also holds since $\frac{1}{k(1-\tau)} \geq 1$ and $\frac{1}{\tau} \geq 1$. Then, we have to prove that the entries of the vectors $g_c(v_i)$, $v_i \in \bar{V}$, $i = 1, 2, 3, 4$ satisfy the conditions given in (4.11). We have that

(a) $g_c(v_1) = ((p-\mu)Pk(1-\tau) + h(v_1)\ 0\ \mu Pk(1-\tau))^T \in \mathcal{P}_{\bar{N}}$ if and only if

$$(p-\mu)Pk(1-\tau) + h(v_1) \leq Pk(1-\tau) \ \Leftrightarrow\ h(v_1) \leq Pk(1-\tau)(1-p+\mu),$$
and
$$P(p-\mu) + \frac{h(v_1)}{k(1-\tau)} + \mu Pk(1-\tau) \leq P.$$

Since $\mu k(1-\tau) > 0$, then $\mathbf{a} < 1-p+\mu$ and using the inequality $h(v_1) \leq Pk(1-\tau)\mathbf{a}$ given in (4.10), the first condition holds. Further, it is straightforward to prove $P(p-\mu) + \frac{h(v_1)}{k(1-\tau)} + \mu Pk(1-\tau) \leq P$.

(b) $g_c(v_2) = (h(v_2)\ (q-\gamma)\tau P\ \gamma\tau P)^T \in \mathcal{P}_{\bar{N}}$ if and only if

$$h(v_2) \leq Pk(1-\tau),\ \ (q-\gamma)\tau P \leq \tau P \ \Leftrightarrow\ (q-\gamma) \leq 1, \text{ and}$$
$$\frac{h(v_2)}{k(1-\tau)} + (q-\gamma)P + \gamma\tau P \leq P \ \Leftrightarrow h(v_2) \leq Pk(1-\tau)(1-q+\gamma(1-\tau)).$$

Using that $0 < \gamma(1-\tau) < \gamma < q < 1$, then $0 < \mathbf{b} = 1-q+\gamma(1-\tau) < 1$ and the first and the second conditions hold, and as $h(v_2) \leq Pk(1-\tau)\mathbf{b}$, the third condition holds too.

(c) $g_c(v_3) = (wP + h(v_3)\ 0\ (r-w)P)^T \in \mathcal{P}_{\bar{N}}$ if and only if

$$wP + h(v_3) \leq Pk(1-\tau), \text{ and}$$
$$\frac{wP + h(v_3)}{k(1-\tau)} + (r-w)P \leq P \Leftrightarrow\ h(v_3) \leq Pk(1-\tau)(1-r+w) - wP.$$

Using $0 < w < r < 1$, we have $0 < r-w < 1$, and $wP + h(v_3) \leq Pk(1-\tau)(1-(r-w)) \leq Pk(1-\tau)$. Further, as $h(v_3) \leq Pk(1-\tau)(1-r+w) - Pw$ the second condition holds.

(d) $g_c(v_4) = (x_1\ x_2\ x_3)^T \in \mathcal{P}_{\bar{N}}$ if and only if $x_1 \leq k(1-\tau)P$, $x_2 \leq \tau P$ and

$$\frac{x_1}{k(1-\tau)} + \frac{x_2}{\tau} + x_3 \leq P,$$

with $x_1 = Pk(1-\tau)(p-\mu) - \alpha k\tau(1-\tau)P^2 + h(v_4)$, $x_2 = (q-\gamma)\tau P + \alpha k\tau(1-\tau)P^2$, $x_3 = \mu k(1-\tau)P + \gamma\tau P$.

To check if $x_1 \leq k(1-\tau)P$, we note that

$$\mu k(1-\tau)+q-\gamma(1-\tau)+\alpha P k(1-\tau) \geq 0$$

and from the condition given on $h(v_4)$ in the hypothesis of the theorem we have that

$$\begin{aligned} h(v_4) &\leq Pk(1-\tau)(1-p+\mu-\mu k(1-\tau)-q+\gamma(1-\tau)+\alpha P(\tau-k(1-\tau))) \\ &= Pk(1-\tau)(1-p+\mu+\alpha P\tau-(\mu k(1-\tau)+q-\gamma(1-\tau)+\alpha Pk(1-\tau)) \\ &\leq Pk(1-\tau)(1-p+\mu+\alpha P\tau). \end{aligned}$$

Using this inequality in the expression of x_1, we derive that $x_1 \leq k(1-\tau)P$. Moreover, $x_2=(q-\gamma)\tau P+\alpha k\tau(1-\tau)P^2=\tau P(q-\gamma+\alpha kP(1-\tau)) \leq \tau P$, since $q-\gamma+\alpha kP(1-\tau) \leq 1$ follows from the condition $Pk(1-\tau) \leq \frac{1-q+\gamma}{\alpha}$. Finally, we check the third condition. From $\frac{h(v_4)}{k(1-\tau)} \leq P(\mathbf{a}+\mathbf{b}-1+\alpha P(\tau-k(1-\tau)))$ we have

$$\begin{aligned} P(p-\mu)-\alpha\tau P^2+\frac{h(v_4)}{k(1-\tau)}+(q-\gamma)P+\alpha k(1-\tau)P^2+\mu k(1-\tau)P \\ +\gamma\tau P=\frac{h(v_4)}{k(1-\tau)}+P(-\mathbf{a}-\mathbf{b}+2-\alpha P(\tau-k(1-\tau)) \leq P. \end{aligned}$$

We have proved that $g_c(v_i) \in \mathcal{P}_{\bar{N}}$, for all $i=1,2,3,4$. Then, from Proposition 4 the polytope $\mathcal{P}_{\bar{N}}$ is g_c-invariant. Then, we can assure that the trajectory of closed-loop system (4.9) remains in $\mathcal{P}_{\bar{N}}$ for all initial state $x(0) \in \mathcal{P}_{\bar{N}}$. □

Remark 1. What is intended with the result of the previous theorem is to determine conditions on the control actions that can be taken to ensure that if the initial condition is within the security set then applying those control actions the trajectory of the model will remain within that set.

If we take the function $h(x)=\sum_{i=1}^{4}\alpha_i h(v_i)$ with α_i given in (4.5) and $h(v_i)$, $i=1,2,3,4$ equal to the upper bounds of the theorem statement given in (4.10), we have that

$$\begin{aligned} h(x) &= \sum_{i=1}^{4}\alpha_i h(v_i)=x_1(1-\frac{x_2}{\tau P})\mathbf{a}+\frac{x_2}{\tau}(1-\frac{x_1}{Pk(1-\tau)})k(1-\tau)\mathbf{b} \\ &+x_3(k(1-\tau)(1-r+w)-w)+\frac{x_1x_2}{P\tau}(\mathbf{a}+\mathbf{b}-1-\alpha P(k(1-\tau)-\tau)) \\ &=\mathbf{a}x_1+\frac{k(1-\tau)\mathbf{b}}{\tau}x_2+(k(1-\tau)(1-r+w)-w)x_3 \\ &-\frac{1+\alpha P(k(1-\tau)-\tau)}{\tau P}x_1x_2. \end{aligned}$$

Hence, the function $\tilde{h}=Bh$ can be written as function of class Θ given in (4.4), $\tilde{h}(x)=\tilde{c}_1x_1+\tilde{c}_2x_2+\tilde{c}_3x_3+\tilde{c}_4x_1x_2$, with the entries of the vectors $\tilde{c}_i$ equal to

$$\begin{aligned} &(\tilde{c}_1)_1=\mathbf{a},\ (\tilde{c}_2)_1=\frac{\mathbf{b}k(1-\tau)}{\tau},\ (\tilde{c}_3)_1=k(1-\tau)(1-r+w)-w, \\ &(\tilde{c}_4)_1=\frac{-1-\alpha P(k(1-\tau)-\tau)}{\tau P},\ (\tilde{c}_i)_2=(\tilde{c}_i)_3=0,\ i=1,2,3,4. \end{aligned}$$

Finally, for this specific control feedback $u(x(t)) = h(x(t))$ the closed-loop system is $x(t+1) = Ax(t) + f(x(t)) + \tilde{h}(x(t)) = \tilde{A}x(t) + F(x(t))$ where

$$\tilde{A} = \begin{pmatrix} p-\mu+\mathbf{a} & (\tilde{c}_2)_1 & (\tilde{c}_3)_1+w \\ 0 & q-\gamma & 0 \\ \mu & \gamma & r-w \end{pmatrix}, \; F(x(t)) = \begin{pmatrix} -\alpha+(\tilde{c}_4)_1 \\ \alpha \\ 0 \end{pmatrix} x_1(t)x_2(t). \tag{4.12}$$

Note that system (4.12) will be stable if the function $F(x(t))$ is bounded and the spectral radius of the matrix $\tilde{A}$ is smaller than the unit. But, the spectrum of $\tilde{A}$ is $\sigma(\tilde{A}) = \{1, q-\gamma, r-w-k\mu(1-\tau)\}$ and the spectral radius of $\tilde{A}$ is $\rho(\tilde{A}) = 1$ since this follows from the epidemiological interpretation of the parameters which implies that $0 \leq q-\gamma \leq 1, 0 \leq r-w-k\mu(1-\tau) \leq 1$.

4.1 An application example

In this section, in order to clarify the results obtained along to work, we go back to focus our attention in the epidemiologic sense of the mathematical model given in (2.1)–(3.1) and we consider a hypothetical infectious disease such that the parameters involved in this epidemiological process take the following values

$$p = 0.7, \;\; q = 0.6, \;\; r = 0.65, \;\; \alpha = 0.008, \;\; \gamma = 0.5, \;\; \mu = 0.1, \;\; w = 0.$$

Further, we suppose the size of the population less than $P = 200$ and the coefficients $k = 0.2$, $\tau = 0.05$ as in Examples 1 and 2. Note that conditions given in (4.1) and (4.2) hold,

$$\alpha P = 1.6 > 1.15714 = (1-q+\gamma)\left(1 + \frac{\mu}{(1-r+w)}\right),$$
$$\epsilon = P\tau = 10 \leq 75 = \frac{p-\mu}{\alpha}.$$

Thus, the system is a positive system and the disease-free equilibrium point is not stable. As $Pk(1-\tau) = 38 \leq 112.5 = (1-q+\gamma)/\alpha$ we can apply Theorem 2.

That is, we have $x(t+1) = Ax(t) + f(x(t)) + Bu(t)$, with $g(x) = Ax + f(x)$ and we construct a feedback control $u(t) = h(x(t))$, with $\tilde{h} = Bh \in \Theta$, such that the new function g_c of the closed-loop system $g_c(x) = Ax + f(x) + \tilde{h}(x)$ is also in Θ and $\mathcal{P}_{\bar{N}}$ is g_c-invariant, that is, $g_c(\mathcal{P}_{\bar{N}}) \subseteq \mathcal{P}_{\bar{N}}$, where the polytope $\mathcal{P}_{\bar{N}}$ is given in Example 2 and Figure 2.

Using the upper bounds of $h(v_i)$, $i = 1, 2, 3, 4$ given in (4.10) and the parameters $\mathbf{a}$ and $\mathbf{b}$ given in the theorem statement, we have the feedback control $u(x(t)) = h(x(t))$ with

$$h(x) = 0.381x_1(1 - \frac{x_2}{10}) + 0.875\frac{x_2}{0.05}(1 - \frac{x_1}{38})0.19 + 0.0665x_3 + 0.032\frac{x_1x_2}{10}.$$

From the process indicated, once we have verified that if the state of the system is in the safety set $\mathcal{P}_{\bar{N}}$ it remains there.

Last, we can note that if the initial state $x(0) = (x_1(0), x_2(0), 0)$ does not belong to that set but $0 \leq Ax(0) + f(x(0)) \in \mathcal{P}_{\bar{N}}$ hold, then we can take a

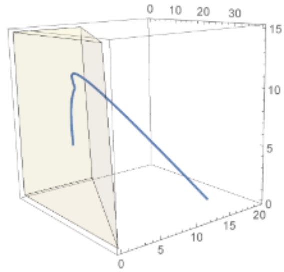


Figure 3. Trajectory solution that reaches the polytope $\mathcal{P}_{\bar{N}}$ and remains within it from the parameters values given in the example of Section 4.1.

control $u(0) \geq 0$ satisfying $x = Ax(0) + u(0)e_1 + f(x(0)) \in \mathcal{P}_{\bar{N}}$. For instance, if we consider the initial state $x(0) = (S_0, I_0, R_0) = (30, 16, 0)$. This satisfies the positivity condition $x_2(0) \leq \dfrac{p-\mu}{\alpha}$ but it is not in the polytope since $x_2(0) = 16$ is greater than $P\tau = 10$. However, we can prove that $Ax(0) + f(x(0)) = (14.16, 5.44, 11) \in \mathcal{P}_{\bar{N}}$. So, using the previous results, there exists a nonzero control $u(0) \geq 0$ such that $Ax(0) + f(x(0)) + u(0)e_1$ is also in the safety set $\mathcal{P}_{\bar{N}}$. Figure 3 shows as the trajectory solution from $x(0) = (30, 16, 0)$ reaches the polytope and considering the feedback control construct using the function $h(x)$ this trajectory remains within this safety set $\mathcal{P}_{\bar{N}}$.

5 Conclusions

One considers a dynamic SIRS process for the spread of a disease. We obtain conditions to determine a population distribution in susceptible, infected and resistant (immune or recovered), where the number of infected can be kept below a safe level. This has allowed us to define the safety set and look for its structure as a polytope determined by its vertices. Furthermore, conditions are obtained so that if a system state is in the safety set then the system trajectory remains in that set forever. Finally, we clarify the results with an example.

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