Solving fully randomized first-order linear control systems: Application to study the dynamics of a damped oscillator with parametric noise under stochastic control

J.-C. Cortés[a], A. Navarro-Quiles[b], J.-V. Romero[a], M.-D. Roselló[a,*]

[a]*Instituto Universitario de Matemática Multidisciplinar,*
Universitat Politècnica de València,
Camino de Vera s/n, 46022, Valencia, Spain
[b]*Department of Statistics and Operational Research*
Universitat de València
Dr. Moliner 50, 46100, Burjassot, Spain

Abstract

This paper is devoted to study random linear control systems where the initial condition, the final target, and the elements of matrices defining the coefficients are random variables, while the control is a stochastic process. The so-called Random Variable Transformation technique is adapted to obtain closed-form expressions of the probability density functions of the solution and of the control. The theoretical findings are applied to study the dynamics of a damped oscillator subject to parametric noise.

Keywords: random control systems, Random Variable Transformation technique, first probability density function, random damped linear oscillators.

1. Introduction and motivation

Control theory is an interdisciplinary field of engineering and mathematics, which deals with the behavior of dynamic systems [1, 2]. Stochastic control is a subfield of control theory that studies the existence of uncertainty in the observations or in the noise that drives the evolution of the system [3]. The key role played by randomness in control problems has been extensively studied in a number of scientific fields including mechanics [4], communications [5], neural networks [6], learning control [7], nonlinear neutral stochastic functional integrodifferential equations with infinite delay [8], etc.

A finite dimensional linear control system of dimension $n \in \mathbb{N}$ is given by

$$\left\{ \begin{array}{rcl} x'(t) & = & Ax(t) + B\,u(t), \quad 0 < t \leq T, \\ x(0) & = & x^0, \end{array} \right. \tag{1}$$

*Corresponding author

Email addresses: jccortes@imm.upv.es (J.-C. Cortés), ana.navarro@uv.es (A. Navarro-Quiles), jvromero@imm.upv.es (J.-V. Romero), drosello@imm.upv.es (M.-D. Roselló)

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where $x(t) \in \mathbb{R}^n$ is the solution of the system, $x^0 \in \mathbb{R}^n$ is the initial state, A is a deterministic $n \times n$ matrix containing the free dynamic part, B is a deterministic $n \times m$ matrix, with $m \in \mathbb{N}$ and $m \leq n$, and $u(t)$ is the control vector, which has dimension m. In this paper, we are interested in studying controllable systems, where any final state, $x^1 \in \mathbb{R}^n$, can be reached from every initial state, x^0, in a finite time $T > 0$, i.e. given any initial condition x^0, $x(T) = x^1$.

This contribution is aimed at solving, from a probabilistic point of view, the following control problem with uncertainties

$$\begin{cases} x'(t,\omega) &= A(\omega)x(t,\omega) + B(\omega)\,u(t,\omega), \quad 0 < t \leq T, \\ x(0,\omega) &= x^0(\omega), \end{cases} \tag{2}$$

where all the input parameters, $A_{ij}(\omega)$, $B_{ik}(\omega)$, $1 \leq i,j \leq n$ and $1 \leq k \leq m$, defining the entries of the random matrices $A(\omega)$ and $B(\omega)$, respectively, the starting initial condition, $x^0(\omega) = \left[x_1^0(\omega), \ldots, x_n^0(\omega)\right]^\top$, and the final target, $x^1(\omega) = \left[x_1^1(\omega), \ldots, x_n^1(\omega)\right]^\top$, are assumed to be absolutely continuous random variables (RVs) defined on a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here the superscript $^\top$ stands for the transpose operator. In order to provide as much generality as possible throughout our analysis, hereinafter we will assume that the joint probability density function (PDF) of the random vector $(x^0(\omega), x^1(\omega), A(\omega), B(\omega))$ is $f_{x^0,x^1,A,B}(x^0, x^1, A, B)$. When convenient, hereinafter, we will short the notation of PDFs. For example, the PDF of a RV, say A, will be denoted by f_A instead of $f_A(a)$. So, the above PDF $f_{x^0,x^1,A,B}(x^0, x^1, A, B)$ will be written as $f_{x^0,x^1,A,B}$. As previously indicated, throughout our subsequent study all the entries of matrices $A(\omega)$ and $B(\omega)$ are assumed RVs, but, as it shall be explained later, our analysis can be adapted to study other scenarios where only a few of their components are RVs.

In [9] we solved problem (1) considering a first scenario where only x^0 and/or x^1 was/were absolute continuous RVs, since in this case we can take advantage of the well known Kalman's controllability condition: If A and B are matrices whose elements are deterministic, a necessary and sufficient condition for (A, B) to be controllable is given by

$$\operatorname{rank}(C) = \operatorname{rank}\left(B|AB|\cdots|A^{n-1}B\right) = n.$$

Here, C is called the Kalman's controllability matrix and its dimension is $n \times nm$ [10, 9, 11].

Now, from the proof of this deterministic result [10, pages 88–89], one can straightforwardly establish the following theorem when elements of matrices $A(\omega)$ and $B(\omega)$ are continuous RVs.

Theorem 1 (Random Kalman controllability condition). *Let $A(\omega)$ and $B(\omega)$ be continuous RVs. Then, a necessary and sufficient condition for* (2) *to be controllable, in terms of $A(\omega)$ and $B(\omega)$, $\omega \in \Omega$, is*

$$\mathbb{P}\left[\left\{\omega \in \Omega : rank(C(\omega)) = rank\left(B(\omega)|A(\omega)B(\omega)|\cdots|A^{n-1}(\omega)B(\omega)\right) = n\right\}\right] = 1,$$

where $C(\omega), \omega \in \Omega$, has dimensions $n \times nm$, and we will call $C(\omega)$ the random Kalman's controllability matrix.

Indeed, the proof is based on showing the invertibility of a certain matrix whose entries are absolutely continuous RVs. It is well-known the invertibility of a square matrix is equivalent to prove that its determinant is different from zero. In the case that all elements of the matrix are absolutely continuous RVs, the probability that the determinant is zero is clearly an event whose probability is null since is defined via a condition defined via an equality (=). In other words, the corresponding matrix is invertible with probability one (w.p. 1). The above reasoning can be extended in terms of the rank of a rectangular matrix, since it is computed by means of minors, which are the determinants of smaller matrices contained in the corresponding matrix whose rank need to be computed.

Proposition 1. *If all elements of matrices $A(\omega), B(\omega), \omega \in \Omega$ are continuous RVs, then problem* (2) *is controllable.*

Proof Since all elements of matrices $A(\omega), B(\omega), \omega \in \Omega$ are continuous RVs, then

$$\mathbb{P}\left[\left\{\omega \in \Omega : \operatorname{rank}(C(\omega)) = \operatorname{rank}\left(B(\omega)|A(\omega)B(\omega)|\cdots|A^{n-1}(\omega)B(\omega)\right) = n\right\}\right] = 1,$$

and applying Theorem 1 the result straightforwardly follows.

In contrast to the deterministic control problem (1), when solving its random counterpart, stated in (2), the solution is a stochastic process (SP). In such case, besides seeking for the solution, $x(t)$, is also important to determine its main statistical properties as the mean, $\mu_x(t)$, and the variance-covariance matrix, $\Sigma_x(t)$. However, a more desirable goal is to compute the first probability density function (1-PDF), say $f_1(x, t)$, of the solution SP since from it not only these moments but other statistics can be calculated by integration. For example,

$$\mu_x(t) = \mathbb{E}[x(t, \omega)] = \int_{\mathbb{R}^n} x f_1(x, t)\,\mathrm{d}x, \quad \Sigma_x(t) = \int_{\mathbb{R}^n} (x - \mu_x(t))(x - \mu_x(t))^\top f_1(x, t)\,\mathrm{d}x. \tag{3}$$

Furthermore, the 1-PDF permits calculating the probability that the solution SP lies on a specific set of interest as well,

$$\mathbb{P}[\{\omega \in \Omega : x(t, \omega) \in \mathcal{B}\}] = \int_{\mathcal{B}} f_1(x, t)\,\mathrm{d}x, \qquad \mathcal{B} \subset \mathbb{R}^n.$$

Notice that, fixed t and $\alpha \in (0, 1)$, the 1-PDF also permits constructing confidence regions by determining $z \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} (f_1(x, t) - z)\,\mathrm{d}x = 1 - \alpha, \qquad f_1(x, t) \geq z.$$

The confidence region is the $\mathbb{R}^{n-1}$-manifold defined by $f_1(x, t) = z$. For instance, when $\alpha = 0.05$, it is said that $f_1(x, t) = z$ defines a region with $1 - \alpha = 95\%$ of confidence level.

The main goal of this contribution is to compute the 1-PDF of the control, $u(t, \omega)$, and of the solution SP, $x(t, \omega)$, of the random control problem (2). With this aim, the Random Variable Transformation method (RVT) will be applied. RVT is a powerful technique to determine the joint PDF of a random vector which comes from mapping another random vector whose joint PDF is known. The multidimensional version of the RVT method is stated in the following theorem.

Theorem 2 (RVT (Random Variable Transformation) technique). *[12, pp. 24–25] Let $X(\omega) = (X_1(\omega), \ldots, X_m(\omega))^\top$ and $Z(\omega) = (Z_1(\omega), \ldots, Z_m(\omega))^\top$ be two m-dimensional absolutely continuous random vectors defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $s : \mathbb{R}^m \to \mathbb{R}^m$ be a one-to-one deterministic transformation of $X(\omega)$ onto $Z(\omega)$, i.e., $Z(\omega) = s(X(\omega))$, $\omega \in \Omega$. Assume that s is a continuous mapping with continuous partial derivatives with respect* *to each component x_i, $1 \leq i \leq m$. Then, if $f_X(x_1, \dots, x_m)$ denotes the joint PDF of the vector $X(\omega)$, and $p = s^{-1} = (p_1(z_1, \dots, z_m), \dots, p_m(z_1, \dots, z_m))$ represents the inverse mapping of $s = (s_1(x_1, \dots, x_m), \dots, s_m(x_1, \dots, x_m))$, the joint PDF of the random vector $Z(\omega)$ is given by*

$$f_Z(z_1, \dots, z_m) = f_X\left(p_1(z_1, \dots, z_m), \dots, p_m(z_1, \dots, z_m)\right) |\mathcal{J}_m| \,,$$

where $|\mathcal{J}_m|$, which is assumed to be different from zero, denotes the absolute value of the Jacobian defined by the following determinant

$$\mathcal{J}_m = \det \begin{bmatrix} \dfrac{\partial p_1(z_1, \dots, z_m)}{\partial z_1} & \cdots & \dfrac{\partial p_m(z_1, \dots, z_m)}{\partial z_1} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial p_1(z_1, \dots, z_m)}{\partial z_m} & \cdots & \dfrac{\partial p_m(z_1, \dots, z_m)}{\partial z_m} \end{bmatrix} .$$

We will apply the theoretical results established throughout this paper to study a damped linear oscillator whose resistance and frequency coefficients are assumed to be RVs and whose dynamics is driven by a stochastic control. The analysis of damped oscillators subject to uncertainties has been studied from several points of view. In [13], author studies the long time behaviour of a nonlinear oscillator subject to a random multiplicative noise, which is assumed stationary Gaussian of zero-mean value and with a spectral density that decays as a power law at high frequencies. In [14], authors provide a full probabilistic description of the solution stochastic process to damped pendulum random differential equation assuming different stochastic excitations defined via Gaussian processes, approximations using Karhunen-Loève expansions and random power series. In [15], the problem of suboptimal linear feedback control laws with mean-square criteria for the linear oscillator and the Duffing oscillator under external non-Gaussian excitations is studied. In [16], the forced van der Pol oscillator is analyzed by varying the parameter values, which is a way of perturbing the dynamical behavior of the vibratory system. However, to the best of our knowledge, the approach proposed in our application is a novelty in the extant literature.

This paper is organized as follows. In Section 2 we describe how explicit expressions for the solution SP of problem (2) and for the control SP can be obtained. Then, the 1-PDF of the solution SP of (2) is computed. Computation of the 1-PDF of the control SP is addressed in Section 3. In Section 4, the theoretical results, previously obtained, are applied to study the dynamics of a damped oscillator whose restoring force and resistance coefficients are RVs and the control is a SP. Finally, some conclusions are shown in Section 5.

2. Computing the 1-PDF of the solution SP

We can construct an explicit solution of the random problem (2) following the reasoning described in [9, Section 3] that consists in extending the deterministic solution to the random scenario. Given a stochastic control, $u(t, \omega) \in L^2((0, T] \times \Omega; \mathbb{R}^m)$, applying the formula of variation of parameters, we obtain that the unique solution, $x \in H^1((0, T] \times \Omega; \mathbb{R}^n)$, of random problem (2) is given by

$$x(t, \omega) = \exp\left(A(\omega)t\right) x^0(\omega) + \int_0^t \exp\left(A(\omega)(t - s)\right) B(\omega) u(s, \omega)\, \mathrm{d}s, \qquad t \in [0, T]. \tag{4}$$

However, notice that this is not a closed form-expression since it depends on the control $u(t,\omega)$, which needs to be determined. The stochastic control $u(t,\omega)$ can be obtained using the duality principle [17, p. 51], that reduces the controllability problem (2) into an observability problem. Then, one obtains an explicit formula for the stochastic control in terms of data

$$\begin{aligned} u(t,\omega) &= B^\top(\omega)\exp\left(A^\top(\omega)(T-t)\right)\left(\int_0^T \exp(A(\omega)(T-s))B(\omega)B^\top(\omega)\exp\left(A^\top(\omega)(T-s)\right)\mathrm{d}s\right)^{-1} \\ &\quad \left(x^1(\omega)-\exp(A(\omega)T)x^0(\omega)\right). \end{aligned} \tag{5}$$

In order to simplify the expressions in subsequent developments, we will rewrite expression (4) introducing the following notation

$$F(t,A,B)=\exp(A(T-t))B, \qquad \Lambda(t,A,B)=\int_0^t F(s,A,B)F^\top(s,A,B)\,\mathrm{d}s,$$

$$G(t,A,B)=\Lambda(t,A,B)\Lambda^{-1}(T,A,B), \qquad H(t,A,B)=\exp(A(t-T))G(t,A,B).$$

Remark 1. Notice that $\Lambda(t,A(\omega),B(\omega))$, $0<t\le T$ is an invertible matrix w.p. 1 when all elements of matrices $A(\omega)$ and $B(\omega)$ are absolutely continuous RVs. In other case, i.e. when some element are deterministic, $\Lambda(t,A(\omega),B(\omega))$, $0<t\le T$, is an invertible matrix w.p. 1 provided the random Kalman condition holds (see [9, Remark 1]).

Then, expression (4) can be written as

$$x(t,\omega)=\left(\exp(A(\omega)t)-H(t,A(\omega),B(\omega))\exp(A(\omega)T)\right)x^0(\omega)+H(t,A(\omega),B(\omega))x^1(\omega). \tag{6}$$

As it has been indicated in Section 1, hereinafter, we will assume that all the entries of input data of random problem (2), namely, the initial condition $x^0(\omega)$, the target condition $x^1(\omega)$ and the coefficient matrices $A(\omega)$ and $B(\omega)$ are RVs. So, in total we have $h=n+n+nn+nm=2n+n^2+nm$ RVs, $x_i^0(\omega)$, $x_i^1(\omega)$, $a_{ij}(\omega)$, $b_{ik}(\omega)$, $i=1,\dots,n$; $j=1,\dots,n$ and $k=1,\dots,m$, respectively. For simplicity, in the subsequence presentation all these RVs are conveniently arranged in vectors and matrices,

$$x^0(\omega)=\left[x_1^0(\omega),\dots,x_n^0(\omega)\right]^\top, \qquad x^1(\omega)=\left[x_1^1(\omega),\dots,x_n^1(\omega)\right]^\top,$$

$$A(\omega)=\begin{bmatrix} a_{11}(\omega) & \cdots & a_{1n}(\omega) \\ \vdots & \ddots & \vdots \\ a_{n1}(\omega) & \cdots & a_{nn}(\omega)\end{bmatrix}, \qquad B(\omega)=\begin{bmatrix} b_{11}(\omega) & \cdots & b_{1m}(\omega) \\ \vdots & \ddots & \vdots \\ b_{n1}(\omega) & \cdots & b_{nm}(\omega)\end{bmatrix}.$$

For the sake of generality, we will assume a joint PDF, $f_{x^0,x^1,A,B}$, for these h RVs. Our objective is to compute the 1-PDF of the solution SP, $x(t,\omega)$, $t>0$. To this end, we will take advantage of the RVT technique by fixing $\omega\in\Omega$ and defining the mapping $s:\mathbb{R}^h\longrightarrow\mathbb{R}^h$, whose components, for convenience, are defined by blocks, $s=(s_1,s_2,s_3,s_4)$,

$$s_i:\mathbb{R}^h\longrightarrow\mathbb{R}^n,\ i=1,2,\ s_3:\mathbb{R}^h\longrightarrow\mathbb{R}^{n\times n},\ s_4:\mathbb{R}^h\longrightarrow\mathbb{R}^{n\times m},$$

in the following way

$$\begin{aligned} z^1 &= s_1(x^0,x^1,A,B) = \left(\exp(At)-H(t,A,B)\exp(AT)\right)x^0+H(t,A,B)x^1, \\ z^2 &= s_2(x^0,x^1,A,B) = x^0, \\ Z^3 &= s_3(x^0,x^1,A,B) = A, \\ Z^4 &= s_4(x^0,x^1,A,B) = B. \end{aligned}$$

Notice that, for consistency with the notation used throughout the paper, for the last two blocks we have utilized capital letters since they are matrix mappings. The inverse mapping of s, $s^{-1} = p$, is given by $p = (p_1, p_2, p_3, p_4)$, where

$$\begin{array}{rcccl} x^0 & = & p_1\left(z^1, z^2, Z^3, Z^4\right) & = & z^2, \\ x^1 & = & p_2\left(z^1, z^2, Z^3, Z^4\right) & = & H^{-1}(t, Z^3, Z^4)z^1 - \left(G^{-1}(t, Z^3, Z^4) - I_n\right)\exp(Z^3 t)z^2, \\ A & = & p_3\left(z^1, z^2, Z^3, Z^4\right) & = & Z^3, \\ B & = & p_4\left(z^1, z^2, Z^3, Z^4\right) & = & Z^4. \end{array}$$

Here, I_n denotes the identity matrix of size n, $p_i : \mathbb{R}^h \longrightarrow \mathbb{R}^n$, $i = 1, 2$, $p_3 : \mathbb{R}^h \longrightarrow \mathbb{R}^{n\times n}$, $p_4 : \mathbb{R}^h \longrightarrow \mathbb{R}^{n\times m}$ and

$$\begin{array}{l} G^{-1}(t, A, B) = \Lambda(T, A, B)\Lambda^{-1}(t, A, B), \\ H^{-1}(t, A, B) = G^{-1}(t, A, B)\exp(A(T-t)) = \Lambda(T, A, B)\Lambda^{-1}(t, A, B)\exp(A(T-t)). \end{array}$$

The absolute value of the Jacobian of mapping p is given by

$$\begin{aligned} |\mathcal{J}_h| &= \left| \det \begin{bmatrix} 0_{n\times n} & H^{-1}\left(t, Z^3, Z^4\right) & 0_{n\times n^2} & 0_{n\times nm} \\ I_n & \#_{n\times n} & 0_{n\times n^2} & 0_{n\times nm} \\ 0_{n^2\times n} & \#_{n^2\times n} & I_{n^2} & 0_{n^2\times nm} \\ 0_{nm\times n} & \#_{nm\times n} & 0_{nm\times n^2} & I_{nm} \end{bmatrix} \right| = \left|\det\left(H^{-1}\left(t, Z^3, Z^4\right)\right)\right| \\ &= \left|\det\left(\Lambda(T, A, B)\Lambda^{-1}(t, A, B)\exp(A(T-t))\right)\right| = \frac{|\det\left(\Lambda(T, A, B)\right)|}{|\det\left(\Lambda(t, A, B)\right)|}\left|\det\left(\exp(A(T-t))\right)\right|, \end{aligned}$$

where, as usually, $0_{n_1\times n_2}$ stands for the null matrix of size $n_1 \times n_2$, and $\#_{n_1\times n_2}$ denote certain matrices whose explicit form is unnecessary to compute in order to calculate the value of the Jacobian. Indeed, the null block matrices $0_{n_1\times n_2}$, that appear in the Jacobian matrix, cancel the terms that involve factors of the form $\#_{n_1\times n_2}$. Since we are dealing with the full random case, i.e. where all input parameters are absolute continuous RVs, we can ensure that $|\mathcal{J}_h| \neq 0$, w.p. 1. Then applying Theorem 2, the PDF of random vector $\left(z^1, z^2, Z^3, Z^4\right)$, in terms of the joint PDF of the random vector of input parameters $\left(x^0, x^1, A, B\right)$, is given by

$$\begin{aligned} & f_{z^1,z^2,Z^3,Z^4}\left(z^1, z^2, Z^3, Z^4\right) \\ &= f_{x^0,x^1,A,B}\left(z^2, H^{-1}(t, Z^3, Z^4)z^1 - \left(G^{-1}\left(t, Z^3, Z^4\right) - I_n\right)\exp\left(Z^3 t\right)z^2, Z^3, Z^4\right)\left|\det\left(H^{-1}(t, Z^3, Z^4)\right)\right|. \end{aligned} \tag{7}$$

As the solution of problem (2) corresponds to the first component of the foregoing mapping $s : \mathbb{R}^h \longrightarrow \mathbb{R}^h$, i.e. z^1, the 1-PDF of $x(t, \omega)$ is obtained marginalizing (7) with respect to $z^2 = x^0, Z^3 = A$ and $Z^4 = B$,

$$\begin{aligned} f_1(x, t) = & \int_{\mathbb{R}^{h_1}} f_{x^0,x^1,A,B}\left(x^0, H^{-1}(t, A, B)x - \left(G^{-1}(t, A, B) - I\right)\exp(At)\, x^0, A, B\right) \\ & \cdot\left|\det\left(H^{-1}(t, A, B)\right)\right| \mathrm{d}x^0\, \mathrm{d}A\, \mathrm{d}B, \end{aligned} \tag{8}$$

where $h_1 = n + n^2 + nm$ and

$$\mathrm{d}x^0 \mathrm{d}A\, \mathrm{d}B = \prod_{0\leq i\leq n}\prod_{0\leq j\leq n}\prod_{0\leq k\leq m} \mathrm{d}x_i^0\, \mathrm{d}A_{i,j}\, \mathrm{d}B_{i,k}. \tag{9}$$

Remark 2. The 1-PDF given by (8) is well defined when $0 < t \leq T$, while for $t = 0$ is just the PDF of the initial condition x^0, which is directly obtained by marginalizing the joint PDF of the input data, $f_{x^0,x^1,A,B}$, with respect to the random vector (x^1, A, B). Therefore, the 1-PDF of the solution SP is determined on the whole interval $[0, T]$. Nevertheless, from a computational standpoint, it is worth pointing out that the calculation of the 1-PDF, $f_1(x, t)$, by expression (8) has computational drawbacks because of $\Lambda(t, A, B)$ is quasi-singular about $t = 0$ (observe that $\Lambda(0, A, B)$ is singular and $\Lambda(t, A, B)$ is continuous), so the terms $G(t, A, B)$ and $H(t, A, B)$ are also quasi-singular for t in a neighbourhood of $t = 0$. To overcome this numerical drawback it is better to compute $f_1(x, t)$, for values of t close to $t = 0$, using the following expression

$$\begin{aligned} f_1(x,t) &= \int_{\mathbb{R}^{h_1}} f_{x^0,x^1,A,B}\left((\exp(At) - H(t,A,B)\exp(AT))^{-1}\left(x - H(t,A,B)x^1\right), x^1, A, B\right) \\ &\quad \cdot \left|\det\left(H^{-1}(t,A,B)\right)\right| \mathrm{d}x^1\, \mathrm{d}A\, \mathrm{d}B, \end{aligned}$$

where $\mathrm{d}x^1 = \mathrm{d}x_1^1 \cdots \mathrm{d}\,x_n^1$. This expression is easily obtained changing the second component, s_2, in mapping s, by

$$s_2 = s_2\left(x^0, x^1, A, B\right) = x^1,$$

in the previous reasoning. In this manner, the numerical effects of computing the inverse of quasi-singular matrices is minimized.

Remark 3. In the previous development we have guaranteed that the Jacobian $\mathcal{J}_h$ is different from zero because of all input parameters are absolutely continuous RVs, however expressions (8) and (9) can still be used in case problem (2) is not fully randomized. For example, if only a few components of matrix A, say $A_{1,1}$, $A_{2,3}$ and $A_{n,n}$, are RVs, the 1-PDF given by (8) and (9) can be used considering that $\mathrm{d}A = \mathrm{d}A_{1,1}\, \mathrm{d}A_{2,3}\, \mathrm{d}A_{n,n}$. Notice that in this case the random Kalman condition is fulfilled. (i.e., we are dealing with controllable problems) and, according to Remark 1, $\Lambda(t, A(\omega), B(\omega))$, $0 < t \leq T$, is invertible, so

$$\frac{|\det(\Lambda(T, A(\omega), B(\omega)))|}{|\det(\Lambda(t, A(\omega), B(\omega)))|}\left|\det(\exp(A(\omega)(T-t)))\right| \neq 0, \qquad \text{w.p. } 1,$$

since the exponential matrix is always invertible.

3. Computing the 1-PDF of the control SP

Using the notation introduced in Section 2, the control SP, given in (5), can be written as

$$u(t,\omega) = F^\top(t, A(\omega), B(\omega))\Lambda^{-1}(T, A(\omega), B(\omega))\left(x^1(\omega) - \exp(A(\omega)T)x^0(\omega))\right),$$

i.e.

$$u(t,\omega) = J(t, A(\omega), B(\omega))\left(x^1(\omega) - \exp(A(\omega)T)x^0(\omega))\right),$$

where

$$J(t, A, B) = F^\top(t, A, B)\Lambda^{-1}(T, A, B)$$

is a matrix of size $m \times n$.

As we are assuming that all input parameters are absolutely continuous RVs, the random matrix $B(\omega)$ of size $m \times n$, with $m \leq n$, has maximum rank w.p. 1, i.e.

$$\mathbb{P}[\{\omega \in \Omega : \operatorname{rank}(B(\omega)) = m\}] = 1.$$

This implies that

$$\mathbb{P}[\{\omega \in \Omega : \operatorname{rank}(J(t, A(\omega), B(\omega))) = m\}] = 1,$$

then we can construct an invertible matrix w.p. 1 of dimension $n \times n$,

$$\begin{bmatrix} J(t, A(\omega), B(\omega)) \\ L \end{bmatrix}, \tag{10}$$

where

$$L = \begin{bmatrix} 0_{(n-m)\times m} & I_{n-m} \end{bmatrix}. \tag{11}$$

Notice that the construction of matrix L is not unique. An easy form to construct it ensuring invertibility w.p. 1 of matrix (10)–(11) is to consider zero-vectors to complete the m independent columns of $J(t, A(\omega), B(\omega))$, and to complete the rest of columns with the $n-m$ columns associated to the identity matrix I_{n-m}. Other expressions for matrix L can be obtained keeping the first m columns and completing its last $n-m$ columns by permuting the columns of I_{n-m}.

Now, we will apply Theorem 2 to compute the 1-PDF of $u(t, \omega)$. Let us fix $t > 0$, and define the mapping $s : \mathbb{R}^h \to \mathbb{R}^h$ by

$$\begin{aligned}
z^1 &= s_1(x^0, x^1, A, B) = \begin{bmatrix} J(t, A, B) \\ L \end{bmatrix} x^1 + \begin{bmatrix} -J(t, A, B)\exp(AT)x^0 \\ 0_{(n-m)\times 1} \end{bmatrix}, \\
z^2 &= s_2(x^0, x^1, A, B) = x^0, \\
Z^3 &= s_3(x^0, x^1, A, B) = A, \\
Z^4 &= s_4(x^0, x^1, A, B) = B,
\end{aligned}$$

where $s_i : \mathbb{R}^h \to \mathbb{R}^n$, $i = 1, 2$, $s_3 : \mathbb{R}^h \to \mathbb{R}^{n\times n}$, $s_4 : \mathbb{R}^h \to \mathbb{R}^{n\times m}$ being $h = 2n + n^2 + nm$. The inverse mapping $p = s^{-1}$ is given by

$$\begin{aligned}
x^0 &= p_1(z^1, z^2, Z^3, Z^4) = z^2, \\
x^1 &= p_2(z^1, z^2, Z^3, Z^4) = \begin{bmatrix} J(t, Z^3, Z^4) \\ L \end{bmatrix}^{-1} \left(z^1 + \begin{bmatrix} J(t, Z^3, Z^4)\exp(Z^3 T)z^2 \\ 0_{(n-m)\times 1} \end{bmatrix} \right), \\
A &= p_3(z^1, z^2, Z^3, Z^4) = Z^3, \\
B &= p_4(z^1, z^2, Z^3, Z^4) = Z^4,
\end{aligned}$$

and the absolute value of its Jacobian is

$$|\mathcal{J}_h| = \left| \det \begin{bmatrix} J(t, Z^3, Z^4) \\ L \end{bmatrix}^{-1} \right| = \frac{1}{\left| \det \begin{bmatrix} J(t, Z^3, Z^4) \\ L \end{bmatrix} \right|} \neq 0.$$

Then, applying RVT technique (Theorem 2), the PDF of the random vector (z^1, z^2, Z^3, Z^4) is

$$\begin{aligned}
&f_{z^1,z^2,Z^3,Z^4}(z^1, z^2, Z^3, Z^4) \\
&= f_{x^0,x^1,A,B}\left(z^2, \begin{bmatrix} J(t, Z^3, Z^4) \\ L \end{bmatrix}^{-1} \left(z^1 + \begin{bmatrix} J(t, Z^3, Z^4)\exp(Z^3 T)z^2 \\ 0_{(n-m)\times 1} \end{bmatrix} \right), Z^3, Z^4 \right) \\
&\quad \cdot \left| \det \begin{bmatrix} J(t, Z^3, Z^4) \\ L \end{bmatrix}^{-1} \right|.
\end{aligned} \tag{12}$$

Notice that, for every t fixed, the stochastic control, $u(t, \omega)$, is given by the m first components of vector z^1. To determine the 1-PDF of $u(t, \omega)$, we marginalize expression (12) with respect to the other variables, i.e. $z^2 = x^0$, $Z^3 = A$ and $Z^4 = B$, and the $n-m$ last components of z^1 (corresponding to the $n-m$ last components of x^1, or Lx^1 when L has a different expression of (11)). This leads to

$$f_1(u,t) = \int_{\mathbb{R}^{h_2}} f_{x^0,x^1,A,B}\left(x^0, \begin{bmatrix} J(t,A,B) \\ L \end{bmatrix}^{-1} \begin{bmatrix} u + J(t,A,B)\exp(AT)x^0 \\ q \end{bmatrix}\right) \left|\det \begin{bmatrix} J(t,A,B) \\ L \end{bmatrix}^{-1}\right| \mathrm{d}q\mathrm{d}x^0\mathrm{d}A\mathrm{d}B,$$

where $h_2 = 2n - m + n^2 + nm$, $q = \left(x^1_{m+1}, \ldots, x^1_n\right)^\top$ and

$$\mathrm{d}q\mathrm{d}x^0\mathrm{d}A\,\mathrm{d}B = \prod_{m+1\le l\le n} \prod_{0\le i\le n} \prod_{0\le j\le n} \prod_{0\le k\le m} \mathrm{d}x^1_l\,\mathrm{d}x^0_i\,\mathrm{d}A_{i,j}\,\mathrm{d}B_{i,k}.$$

4. Application to study the dynamics of a damped oscillator with parametric noise

The random differential equation describing a damped oscillator with random inputs and an additive stochastic control is given by:

$$y''(t,\omega) = -\frac{k(\omega)}{m} y(t,\omega) - \frac{R(\omega)}{m} y'(t,\omega) + u(t,\omega), \tag{13}$$

where $y(t,\omega)$ is a SP that determines the position of the mass at the time instant t; $k(\omega)$ is the restoring force random coefficient; the input parameter $R(\omega)$ denotes the resistance random coefficient; m is the mass and $u(t,\omega)$ is the control term described by a SP. All these quantities are defined in a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In practice, the random nature of the restoring force and the resistance coefficients is naturally allocated because of they are obtained via experiments that involve measurement errors. Since these two parameters are treated as RVs, the differential equation (13) is said to have parametric noise.

Our main objective is to determine the 1-PDFs of the solution SP, $y(t,\omega)$, and of the control SP, $u(t,\omega)$, using the theoretical results obtained in Sections 2 and 3, respectively. We consider that the physical system formulated via the differential equation (13) is at an initial state, $\{y(0,\omega), y'(0,\omega)\}$, for position and velocity, respectively, and we want to reach a final target, $\{y(T,\omega), y'(T,\omega)\}$, at a fixed time T. Since initial and target states are RVs, they are not known in a deterministic way but probabilistically because of measurement errors or lacking of knowledge of the physical experiments. In practical scenarios, the distributions of the aforementioned RVs can be established using different information sources, like repeating the physical experiment, using the available knowledge of the oscillator and allocating them plausible distributions, etc.

As expression (13) is a second-order random differential equation, we can rewrite (13) as a first-order linear control system according to the following structure

$$x'(t,\omega) = \begin{pmatrix} 0 & 1 \\ -\frac{k(\omega)}{m} & -\frac{R(\omega)}{m} \end{pmatrix} x(t,\omega) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t,\omega), \tag{14}$$

where

$$x(t,\omega) = \begin{pmatrix} x_1(t,\omega) \\ x_2(t,\omega) \end{pmatrix} = \begin{pmatrix} y(t,\omega) \\ y'(t,\omega) \end{pmatrix}, \quad A(\omega) = \begin{pmatrix} 0 & 1 \\ -\frac{k(\omega)}{m} & -\frac{R(\omega)}{m} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Notice that

$$\operatorname{rank}(C(\omega)) = \operatorname{rank}\left(B|A(\omega)B\right) = \operatorname{rank}\begin{pmatrix} 0 & 1 \\ 1 & -\frac{R(\omega)}{m} \end{pmatrix} = 2, \qquad \forall \omega \in \Omega,$$

so Kalman's controllability condition holds independently of the distributions of the absolutely continuous RVs defining the oscillator behaviour.

To study the influence of randomness in the dynamics of the damped oscillator, we will consider four casuistries. Specifically, we shall analyse the cases where the initial condition is either deterministic or random and, in both cases, we shall consider that matrix A is deterministic or random. In all these scenarios, we will take $m = 1$ and $T = 1$ and, we will assume that the final state, $x^1(\omega)$, has a multinormal distribution with mean and variance-covariance matrix

$$\mu = (1, 0) \qquad \Sigma = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.005 \end{pmatrix}, \tag{15}$$

respectively, i.e. $x^1(\omega) = (x_1^1(\omega), x_2^1(\omega))^\top \sim \mathrm{N}(\mu; \Sigma)$.

Case 1 In this case we choose the following (deterministic) initial condition

$$x^0 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \tag{16}$$

and the parameters involved in (deterministic) matrix A taking the constant values $k = 10$ and $R = 1$.

Case 2 Now the (deterministic) initial condition, x^0, is the same as in Case 1, i.e. given by (16). The random coefficients k and R are assumed to have PDFs whose expected (or mean) values are the same as the ones taken in Case 1. In particular, the following distributions have been considered:

- $k(\omega)$ is a truncated Normal distribution with parameters $\mu_k = 10$ (mean) and $\sigma_k = 0.1$ (standard deviation) on the interval $[9.5, 10.5]$, i.e. $k(\omega) \sim \mathrm{N}_{|[9.5,10.5]}(10; 0.1^2)$.
- $R(\omega)$ is a truncated Normal distribution with parameters $\mu_R = 1$ (mean) and $\sigma_R = 0.05$ (standard deviation) on the interval $[0.75, 1.25]$, i.e. $R(\omega) \sim \mathrm{N}_{|[0.75,1.25]}(1; 0.05^2)$.

Case 3 In this case, we consider a random initial value, $x^0(\omega)$, following a Normal distribution. We assume that its expectation is $\mu_0 = (2, 0)^\top$ (i.e., the same deterministic value given in (16) that has been taken in Cases 1 and 2 too) and that its variance-covariance matrix Σ is given by (15). So, $x^0(\omega) = (x_1^0(\omega), x_2^0(\omega))^\top \sim \mathrm{N}(\mu_0; \Sigma)$. Parameters k and R are assumed deterministic. As in Case 1, we take $k = 10$ and $R = 1$.

Case 4 We choose all parameters as RVs. Their distributions are the ones considered in previous Cases, i.e. $x^0(\omega)$ as in Case 3 and, $k(\omega)$ and $R(\omega)$ as in Case 2.

For all cases we have computed the joint 1-PDF, $f_1(x, t)$, of the solution SP, $x(t, \omega) = (y(t, \omega), y'(t, \omega))^\top$, to the random oscillator control problem (14). Then, from $f_1(x, t)$ confidence regions at certain confidence levels have been determined. Also, $f_1(y, t)$ is obtained marginalizing $f_1(x, t)$. Furthermore, the 1-PDF, $f_1(u, t)$, of the control SP, $u(t, \omega)$, associated to this problem has been obtained.

In Fig. 1 we have represented the joint PDF of the position and velocity, $(x_1(t, \omega), x_2(t, \omega))^\top = (y(t, \omega), y'(t, \omega))^\top$, of the randomized oscillator at $t = 0.2$ in Case 1 (left) and Case 2 (right), where the initial condition is deterministic. We can observe that the analytical computations obtained by applying the RVT method agree with Monte Carlo simulations and that, in Case 2 where parameters are affected by randomness, the 1-PDF is slightly flattened. Similar behaviours are observed when considering other times instants. This issue can be seen in Fig. 2, where we have represented the phase portrait for the random oscillator control problem (14). The expectation vector of the position and velocity adopts the shape of a spiral line (see dotted line). This expectation is highlighted with points at the following time instants, $t \in \{0, 0.2, 0.4, 0.5, 0.6, 0.9, 1\}$. Also, at this specific times, confidence regions at 50% and 90% confidence levels have been plotted in blue and red lines, respectively. We observe that the solution tends to the final target. As the initial point is deterministic, the variability propagates as time increases.

Since the solution $y(t, \omega)$ of the random control problem (13) determines the position of the oscillator at the time instant t, in Fig. 3 we have represented its 1-PDF, $f_1(y, t)$, at the time instants $t \in \{0, 0.2, 0.4, 0.5, 0.6, 0.9, 1\}$ in Case 1 (top) and Case 2 (bottom). Both plots are quite similar, although we see the effect of randomness in model parameters $k(\omega)$ and $R(\omega)$, corresponding to Case 2, induces lower leptokurtic PDFs, as expected. Notice that in the graphical representations, this effect is more apparent at $t = 0.2$. In Fig. 4, we complete the graphical comparison of the aforementioned impact of uncertainty by plotting the mean, $\mu_y(t)$, and the interval centred at this statistic and having two standard deviations as diameter, $[\mu_y(t) - \sigma_y(t), \mu_y(t) + \sigma_y(t)]$. We can see that both graphical representations are similar, so to better compare the graphical results, in Table 1, we collect the figures corresponding to plots shown in Fig. 4 as well as the standard deviation, $\sigma_y(t)$. We then confirm that uncertainty propagates slowly over the time. Notice that the graphical and numerical results commented so far are all in full agreement.

Case 1 / Case 2	$t = 0.2$	$t = 0.4$	$t = 0.5$	$t = 0.6$	$t = 0.9$
$\mu_y(t) + \sigma_y(t)$	1.69745	1.17807	0.992827	0.901246	1.05697
	1.69952	1.18077	0.994808	0.902371	1.05698
$\mu_y(t)$	1.69376	1.15816	0.959047	0.850821	0.961373
	1.69375	1.15814	0.959031	0.850809	0.961372
$\mu_y(t) - \sigma_y(t)$	1.69007	1.13824	0.925268	0.800396	0.865779
	1.68799	1.1355	0.923253	0.799246	0.865769
$\sigma_y(t)$	0.00368999	0.0199181	0.0337797	0.050425	0.095594
	0.00576671	0.0226376	0.0357779	0.051562	0.095603

Table 1: Mean, $\mu_y(t)$, standard deviation, $\sigma_y(t)$, and $\mu_y(t) \pm \sigma_y(t)$ of the solution SP $y(t, \omega)$ at different time instants t. Case 1 and Case 2.

The 1-PDF of control SP, for Case 1 (top) and Case 2 (bottom), are both represented in Fig. 5 at the time instants $t \in \{0, 0.1, 0.5, 0.8, 1\}$. We can observe that, in both cases, they have a sharper form at initial and final times. For a fixed time, we can observe that the 1-PDFs are similar at intermediate times, but they vary near the initial and final times, being a little wider (entailing more variability) when randomness is considered in model parameters as expected.

A similar analysis can be performed to compare Case 3 and Case 4, and analogous conclusions can be obtained. Down below, we briefly report graphical and numerical results.

In Fig. 6, the 1-PDF of the position and velocity at $t = 0.1$ in Case 3 (left) and in Case 4 (right) are represented. In both cases, the initial and final condition are random. Again, we can observe that the analytical computations obtained by applying the RVT method agree with Monte

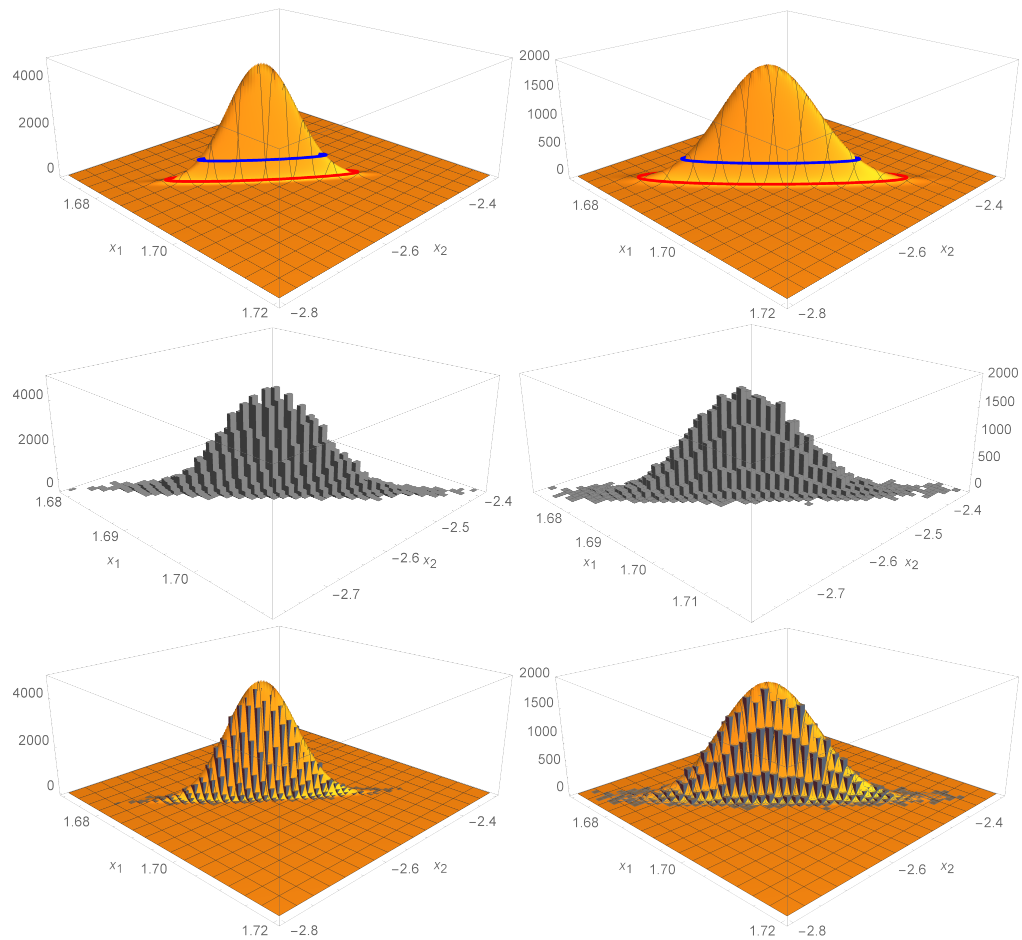
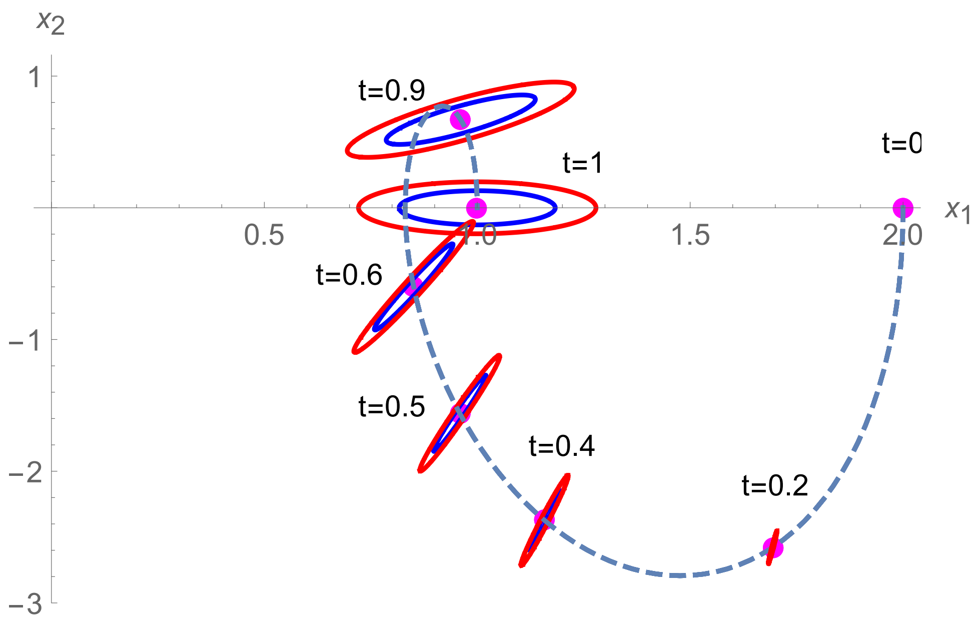


Figure 1: Joint PDF, $f_1(x_1, x_2, t)$, of the position and velocity, $(y(t,\omega), y'(t,\omega))^\top$, of the random oscillator control problem (14) at $t = 0.2$. Left: Case 1; Right: Case 2. Top: Applying the RVT technique and plotting confidence regions for different confidence levels $1-\alpha$ (blue, $1-\alpha = 0.5$ and red, $1-\alpha = 0.9$); Middle: Appying Monte Carlo simulations; Bottom: Comparison between RVT and Monte Carlo simulations.



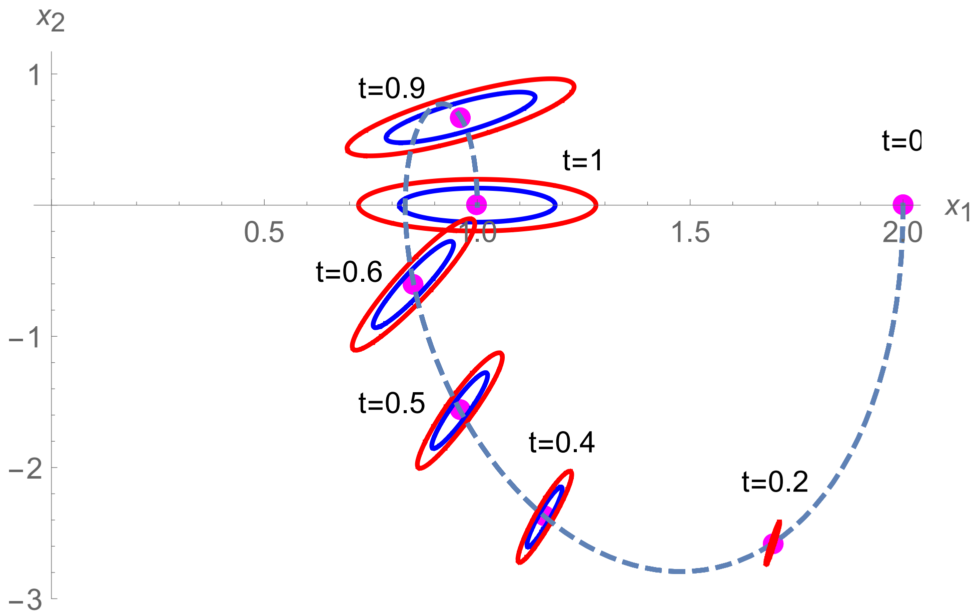


Figure 2: Phase portrait for the random oscillator control problem (14). The expectation of the random vector position-velocity is represented by the spiral line (dotted line). 50% (blue) and 90% (red) confidence regions are plotted at different time instants t. Top: Case 1. Bottom: Case 2.

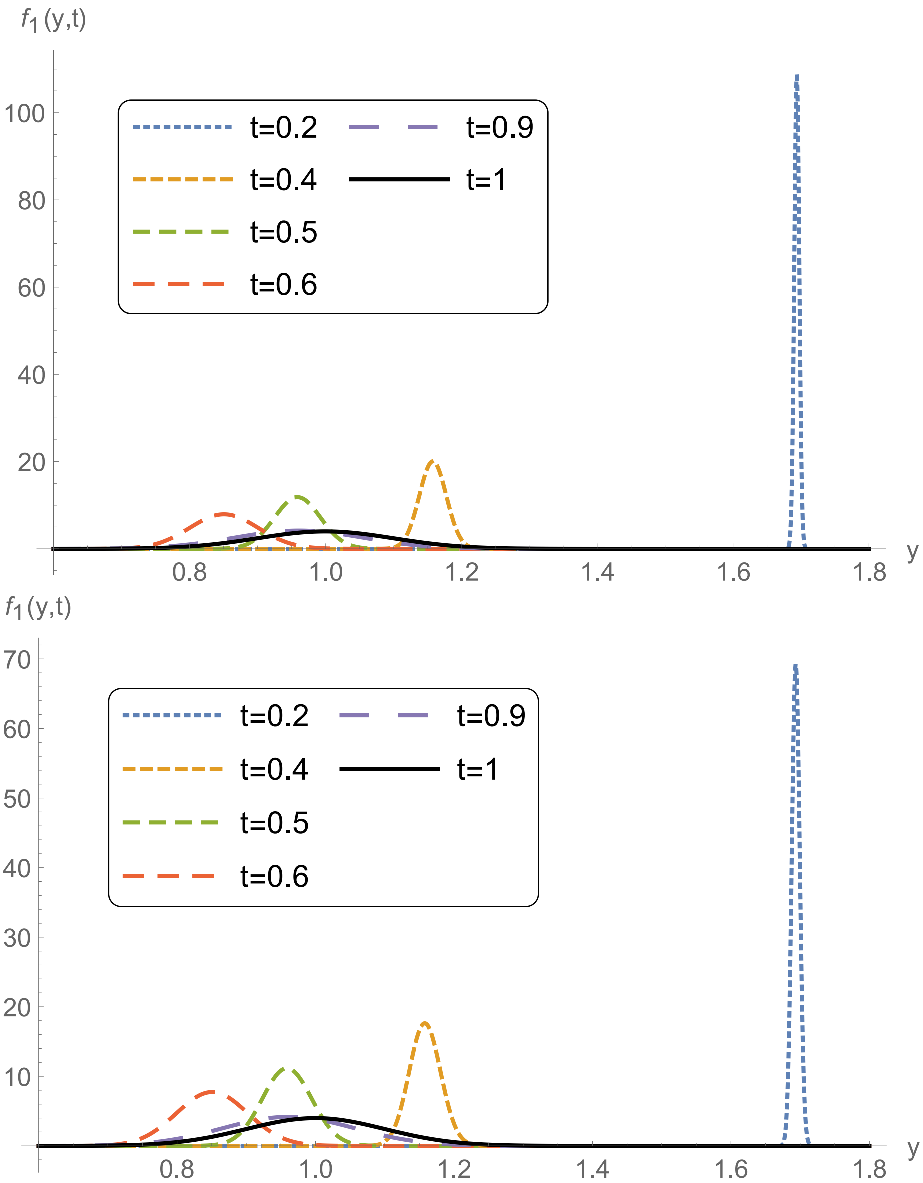


Figure 3: Graphical representation of the 1-PDF, $f_1(y,t)$, of the solution SP, $y(t,\omega)$, at different time instants. Top: Case 1. Bottom: Case 2.

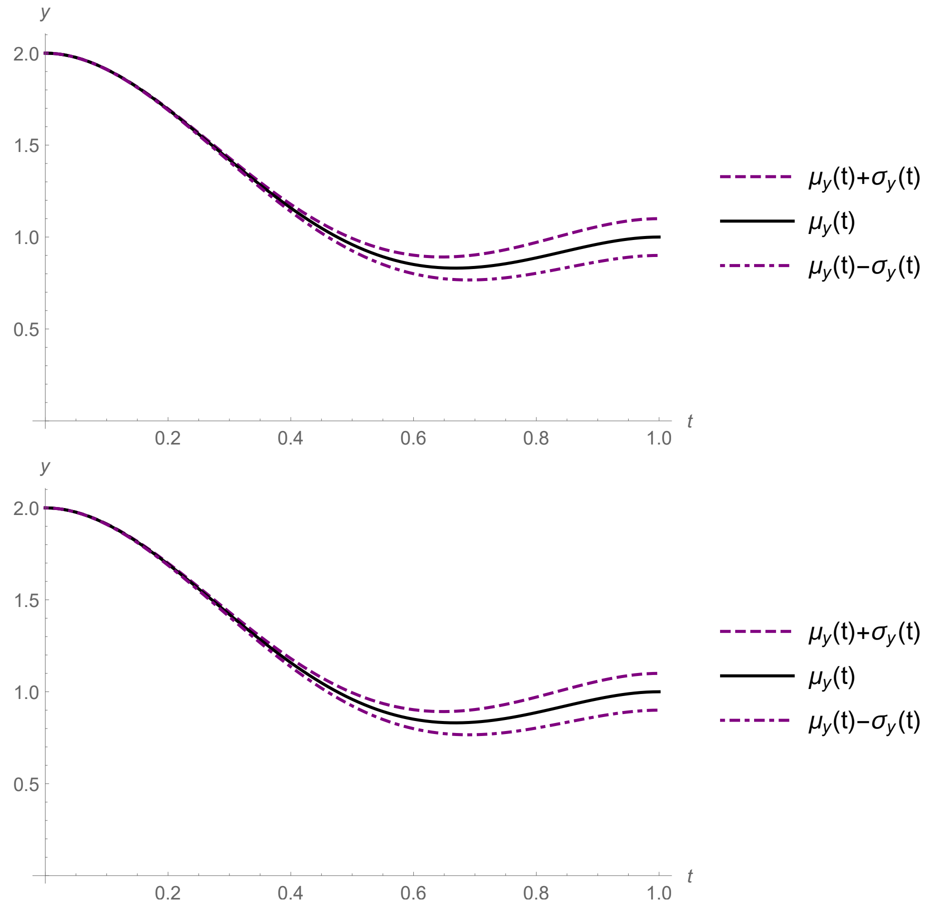


Figure 4: Mean, $\mu_y(t)$, and mean plus/minus standard deviation, $\mu_y(t) \pm \sigma_y(t)$, of the solution SP $y(t, \omega)$. Top: Case 1. Bottom: Case 2.

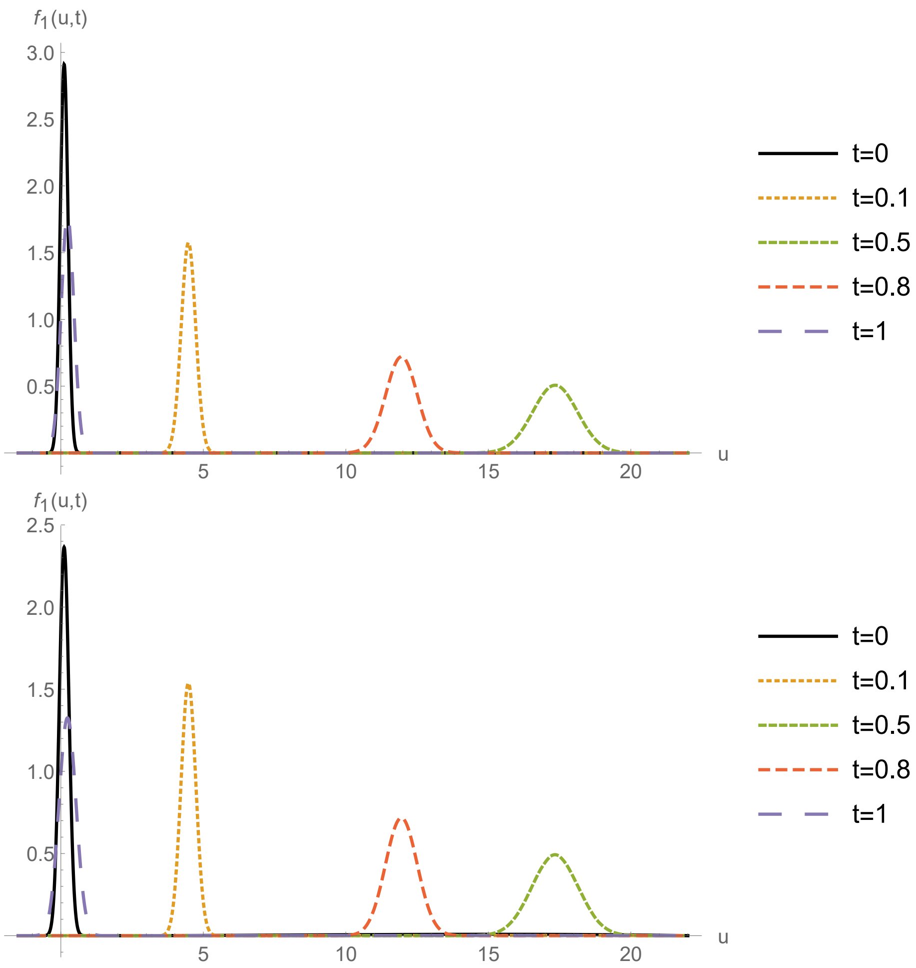


Figure 5: 1-PDF of the control SP, $u(t, \omega)$, associated to the random oscillator control problem (14) at different time instants t. Top: Case 1. Bottom: Case 2.

Carlo simulations and that the 1-PDF is slightly flattened in the presence of randomness in the model parameters. This can be verified if we carefully observe the confidence region at 50% level. Similar conclusions can be obtained from the portrait diagram plotted in Fig. 7, specially if we observe confidence region at $t = 0.1$ and $t = 0.2$.

A zoom of 50% (blue line) and 90% (red line) on the confidence regions for the 1-PDF of the random vector position-velocity to the random oscillator control problem (14), at time instant $t = 0.1$, are drawn in Fig. 8. Case 3 (top) and Case 4 (bottom). Although, these two plots are very similar, if we carefully look at them, we can observe that the confidence regions in Case 4, where all the parameters are RVs, are slightly wider than those corresponding to Case 3.

In Fig. 9, we have plotted the 1-PDF, $f_1(y,t)$, of the solution SP, $y(t,\omega)$, at different time instants in both Cases 3 and 4. In Fig. 10, we compare both cases by plotting the mean, $\mu_y(t)$, and the mean plus/minus standard deviation, $\mu_y(t) \pm \sigma_y(t)$, of $y(t,\omega)$ at different time instants t. From these two graphical representations, we observe the small effect of uncertainty in the position of the damped oscillator since both plots are similar. In Table 2, differences are better highlighted and quantified by means of numerical values. It must be said that the differences between both cases are small since we have chosen small values for the variance associated to input parameters (see the diagonal of variance-covariance matrix Σ in expression (15)). The variability in the model output would increase as variance of $k(\omega)$ and $R(\omega)$ does.

Case 3 / Case 4	$t = 0.1$	$t = 0.2$	$t = 0.4$	$t = 0.5$	$t = 0.6$	$t = 0.9$
$\mu_y(t) + \sigma_y(t)$	2.00717	1.77784	1.21237	1.00682	0.905035	1.05697
	2.00721	1.778	1.21349	1.00831	0.906109	1.057
$\mu_y(t)$	1.91157	1.69376	1.15815	0.959045	0.850819	0.961371
	1.9116	1.6938	1.15822	0.959103	0.850835	0.961399
$\mu_y(t) - \sigma_y(t)$	1.81597	1.60968	1.10394	0.911274	0.796603	0.865774
	1.816	1.6096	1.10295	0.909899	0.795562	0.865794
$\sigma_y(t)$	0.095596	0.0840844	0.0542159	0.0477713	0.0542159	0.0955965
	0.095605	0.0841992	0.0552716	0.0492039	0.0552734	0.0956051

Table 2: Mean, $\mu_y(t)$, and mean plus/minus standard deviation, $\mu_y(t) \pm \sigma_y(t)$, of the solution SP, $y(t,\omega)$, at different time instants t. Case 3 and Case 4.

In Fig. 11, the 1-PDFs of control SP for Case 3 (left) and for Case 4 (right) are represented at the time instants $t \in \{0, 0.1, 0.5, 0.8, 1\}$. We can observe similar behaviours as in Cases 1 and 2, namely, the 1-PDF is sharper at initial and final times. Also, we observe that the 1-PDF in both scenarios are similar at intermediate times, but they vary near the initial and final times, being wider when uncertainty is considered in model parameters.

5. Conclusions

Nowadays modelling in presence of uncertainty is a topic of great interest, particularly in the field of controllability of systems. The main novelty of this contribution is that we have studied autonomous linear control systems assuming full randomness in all model inputs (initial

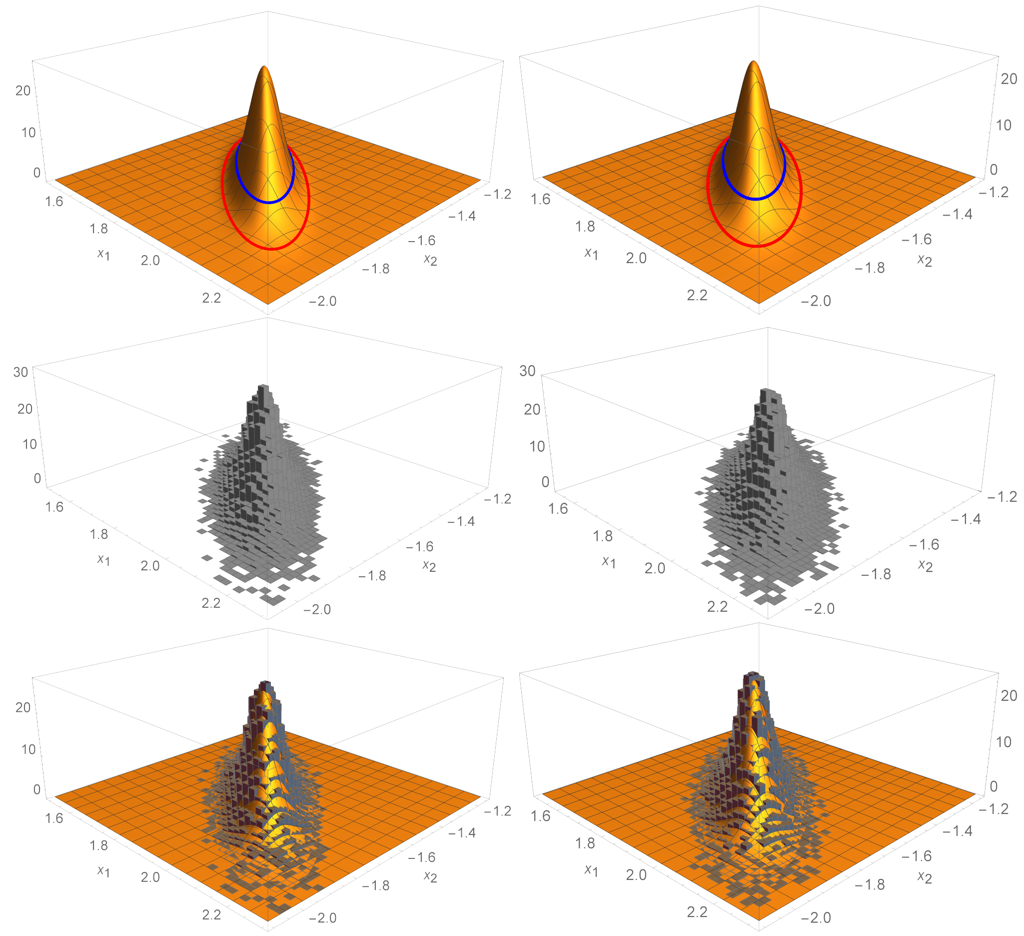


Figure 6: 1-PDF, $f_1(x_1, x_2, t)$, of the random vector position-velocity at the time instante $t = 0.1$ to the random oscillator control problem (14). Left: Case 3; Right: Case 4. Top: Applying RVT and plotting confidence regions for different confidence level $1 - \alpha$ (blue, $1 - \alpha = 0.5$ and red, $1 - \alpha = 0.9$); Middle: Applying Monte Carlo simulations; Bottom: Comparison between Monte Carlo and RVT method.

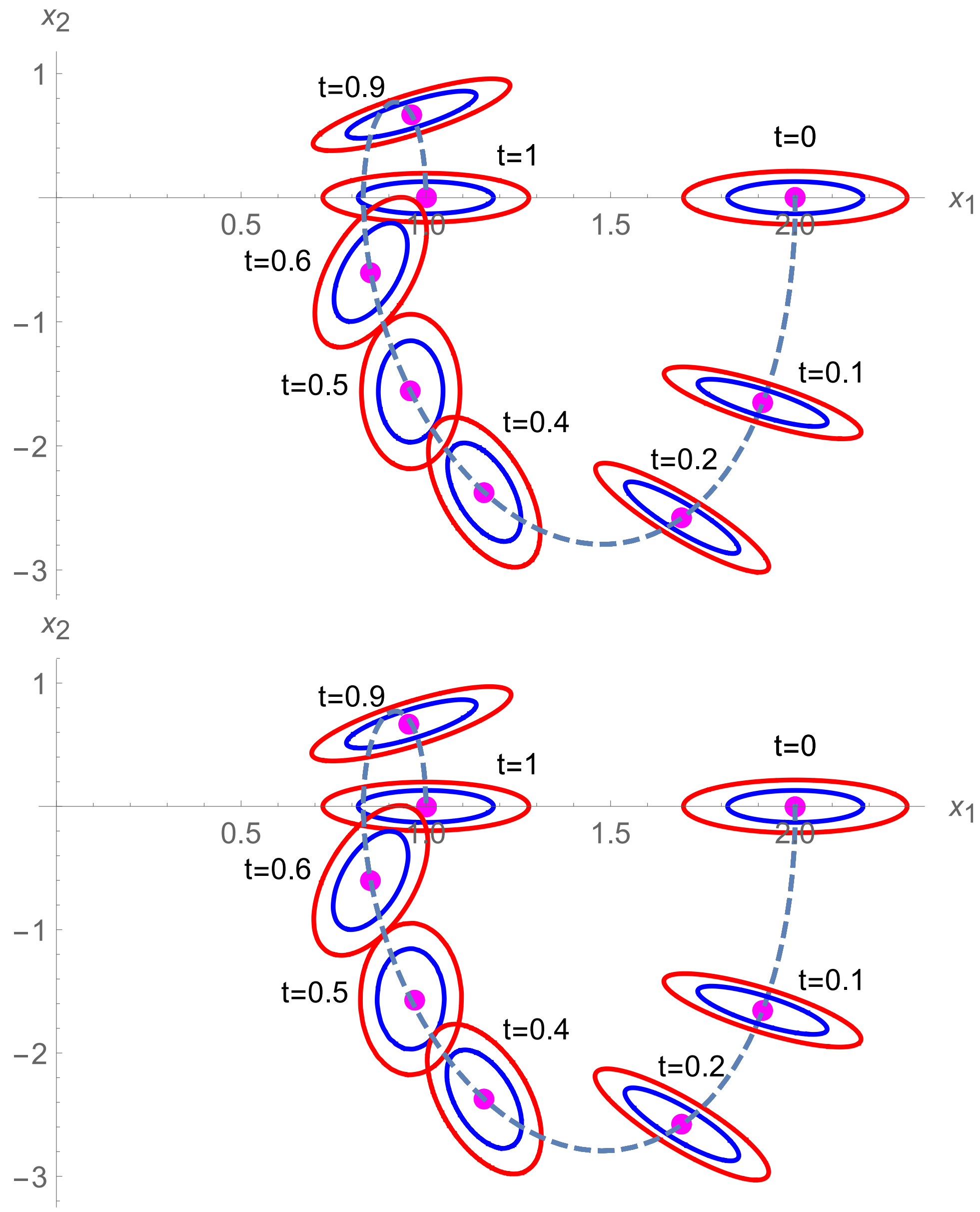


Figure 7: Phase portrait for the random oscillator control problem (14). The expectation of the random vector position-velocity is represented by a spiral line (dotted line). 50% (blue) and 90% (red) confidence regions are plotted at different time instants t. Top: Case 3. Bottom: Case 4.

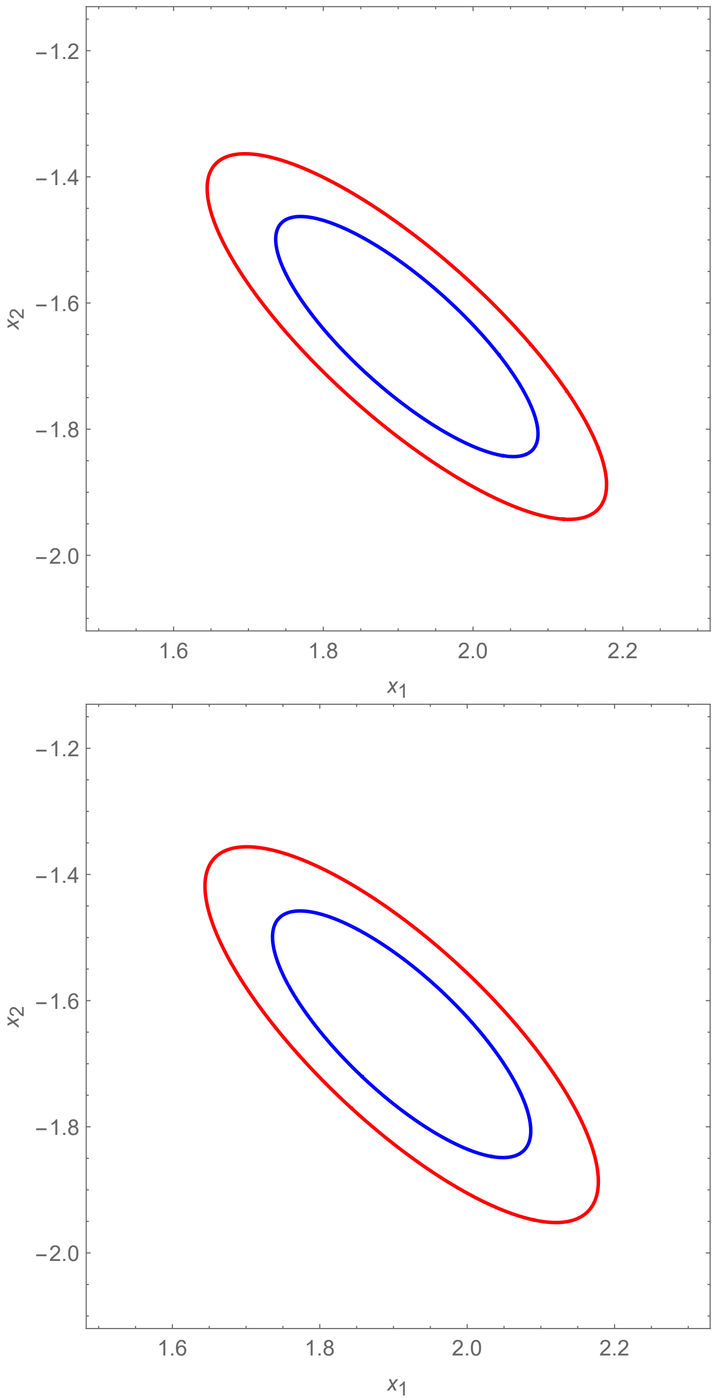


Figure 8: 50% (blue) and 90% (red) confidence regions for the 1-PDF of random vector position-velocity to the random oscillator control problem (14) at the time instant $t = 0.1$. Top: Case 3. Bottom: Case 4.

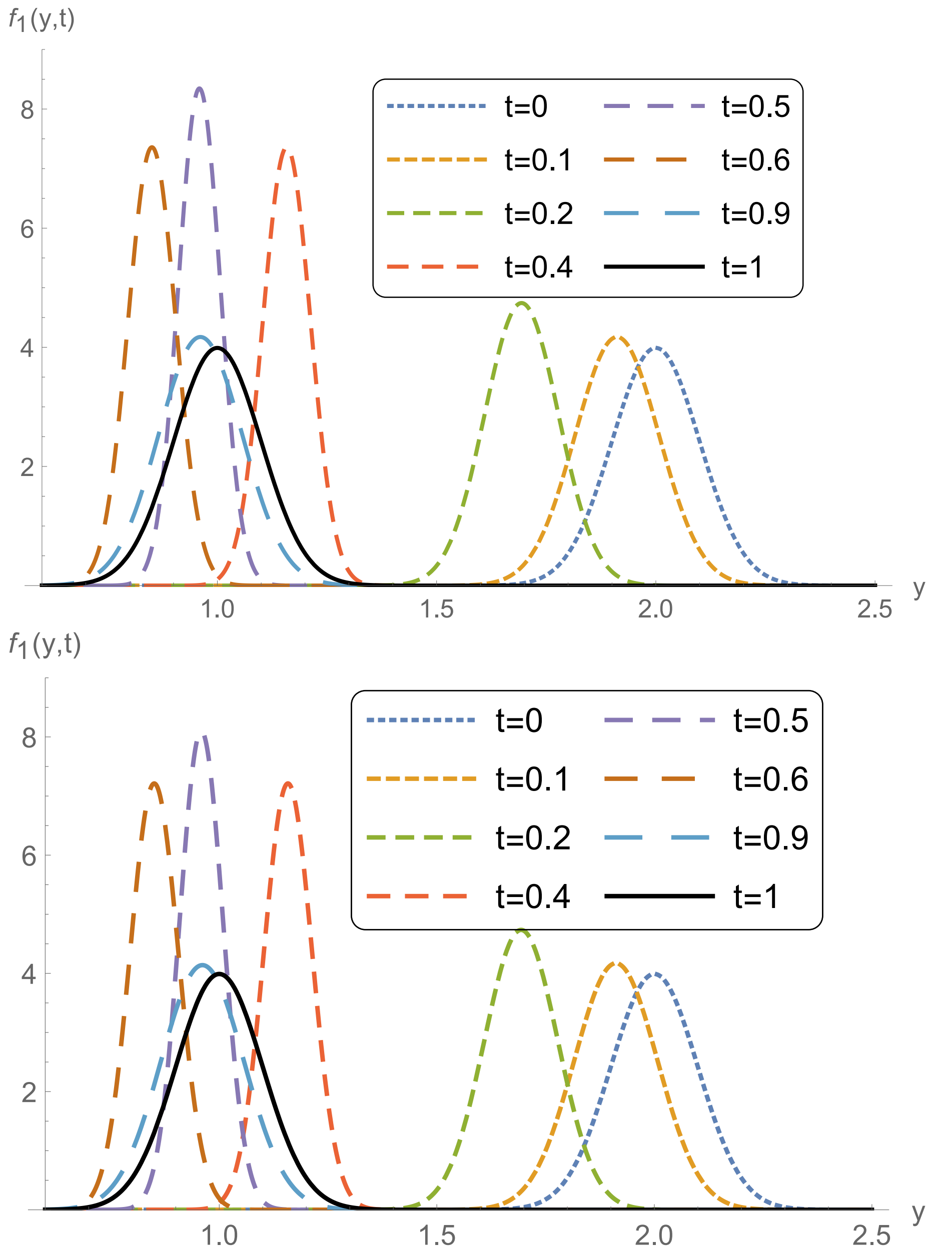


Figure 9: Graphical representation of the 1-PDF, $f_1(y,t)$, of the solution SP, $y(t,\omega)$, at different time instants. Top: Case 3. Bottom: Case 4.

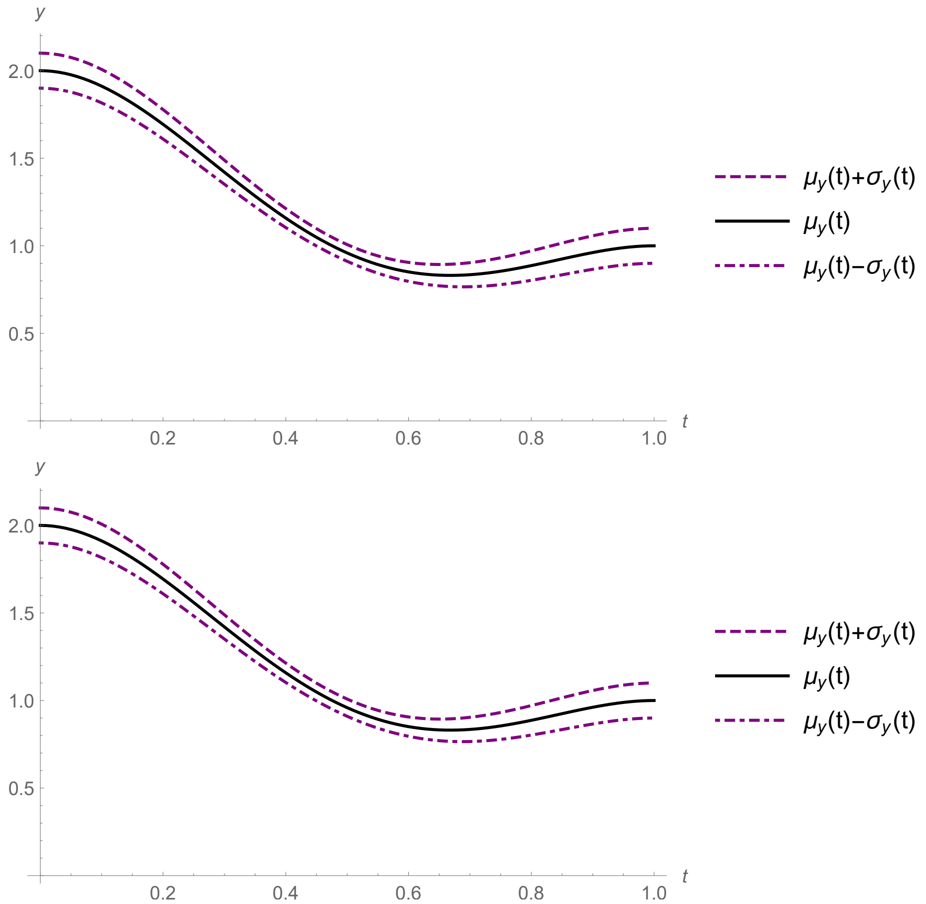


Figure 10: Mean, $\mu_y(t)$, and mean plus/minus standard deviation, $\mu_y(t) \pm \sigma_y(t)$, of the solution SP $y(t, \omega)$. Top: Case 3. Bottom: Case 4.

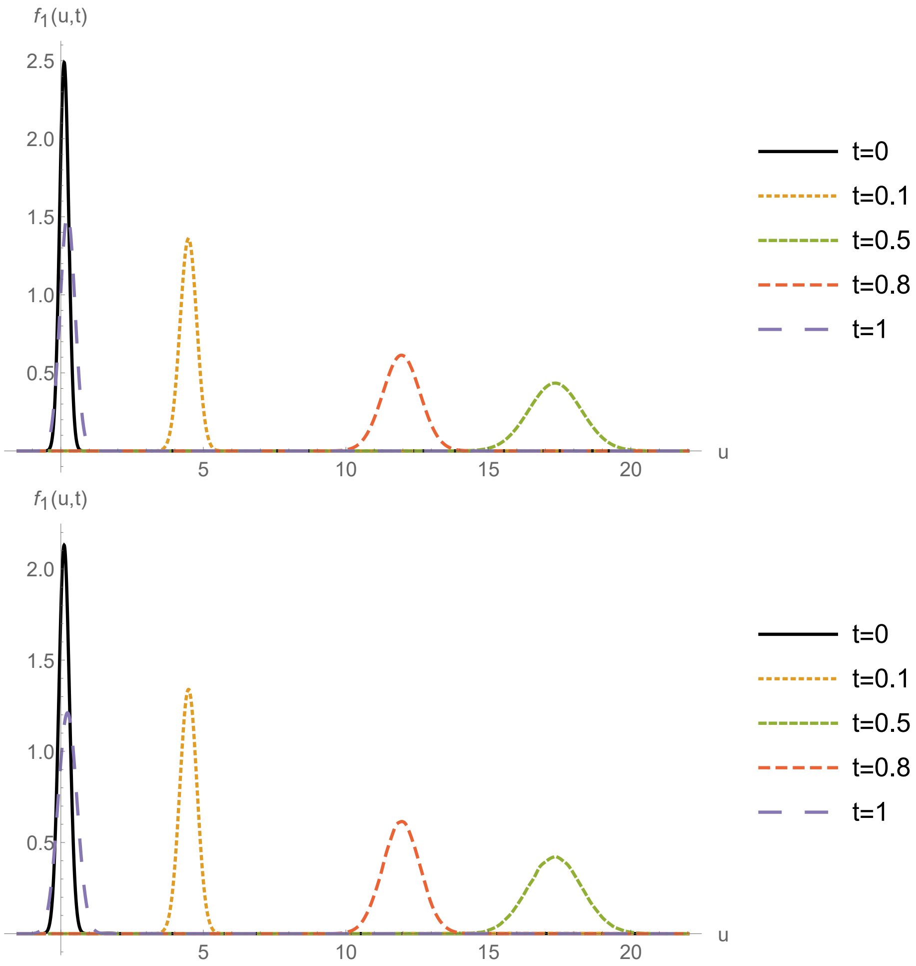


Figure 11: 1-PDF of the control SP $u(t,\omega)$ associated to the random oscillator control problem (14) at different time instants t. Top: Case 3. Bottom: Case 4.

and target conditions, coefficients and control), while other stochastic approaches just consider the associate averaged system or specific forms for the noise (like independent and identically distributed random variables, White noise, etc.). More precisely, in our analysis all model parameters (coefficients and initial and target conditions) are random variables, having arbitrary distributions, instead of deterministic values, and the control is a stochastic process rather than a classical function. Furthermore, for the sake of generality, in our study we have considered the scenario where all model parameters can be dependent random variables with an arbitrary joint distribution. In this general setting, we have provided a complete probabilistic description of the solution stochastic process of the randomized control problem by computing closed-form expressions of the probability density function of the solution and for the control. In this manner, we can calculate, not only the expectation and the variance of the solution and of the control (as is usually done in most contributions dealing with stochastic control systems), but any higher unidimensional moments, confidence intervals as well as the probability that the solution lies within an interval of specific interest.

Our findings can be applied to solve randomized higher order linear differential equations with an additive stochastic control. This can be done using the ideas exhibited in the example dealing with the random damped oscillator, that is based on a second order linear differential equation subject to stochastic control.

Finally, we want to point out that in forthcoming works we plan to extend the present analysis for random non-autonomous linear control systems.

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Conflict of Interest Statement

The authors declare that there is no conflict of interests regarding the publication of this article.

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