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Results in Mathematics

Global Wave Front Sets in Ultradifferentiable Classes

Vicente Asensio, Chiara Boiti, David Jornet, and Alessandro Oliaro

Abstract. We introduce a global wave front set using Weyl quantizations of pseudodifferential operators of infinite order in the ultradifferentiable setting. We see that in many cases it coincides with the Gabor wave front set already studied by the last three authors of the present work. In this sense, we also extend, to the ultradifferentiable setting, previous work by Rodino and Wahlberg. Finally, we give applications to the study of propagation of singularities of pseudodifferential operators.

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1. Introduction

In the theory of partial differential equations, the *wave front set* locates the singularities of a distribution and, at the same time, describes the directions of the high frequencies (in terms of the Fourier transform) responsible for those singularities. In the classical context of Schwartz distributions theory, it was originally defined by Hörmander [27]. There is a huge literature on wave front sets for the study of the regularity of linear partial differential operators in spaces of distributions or ultradistributions in a local sense; see, for instance, [1,2,9,10,23,36,37] and the references therein.

In *global* classes of functions and distributions (like the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ and its dual) the concept of singular support does not make sense, since we require the information on the whole $\mathbb{R}^d$. However, we can still define a global wave front set to describe the micro-regularity of a distribution, where the cones are taken with respect to the whole of the phase space variables. In fact, in [28] Hörmander introduces two different types of global wave front sets addressed to the study of quadratic hyperbolic operators: the C^∞ wave front set, in the Beurling setting, for temperate distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ using Weyl quantizations, and the analytic wave front set, in the Roumieu setting, for ultradistributions $\mathcal{S}'_A(\mathbb{R}^d)$ of Gelfand-Shilov type, defined in terms of a very general known version of the FBI transform as introduced originally by Sjöstrand [40]. Unfortunately, these global versions of wave front set have been almost ignored in the literature. Only very recently, Rodino and Wahlberg [37] recover the concept of C^∞ wave front set of [28] and show that it can be reformulated in terms of the short-time Fourier transform (or Gabor transform), very related to the FBI transform. Moreover, in [37] the authors show also that the original wave front set coincides with the Beurling version of the analytic wave front set introduced by Hörmander and that it can be described merely by a Gabor frame, i.e. with the information of the decay of the Gabor coefficients in a sufficiently dense lattice. The latter is what the authors in [37] call the *Gabor wave front set*. On the other hand, Nakamura [30] introduces the homogenous wave front set for the study of propagation of micro-singularities for Schrödinger equations, and it turns out to be equal to the Gabor wave front set [38]. Cappiello and Schulz [17] recover the analytic wave front set of [28] and show that it can be written using the Gabor transform (with Gaussian window) and study some cases not treated by Hörmander for Gelfand-Shilov ultradistributions of Gevrey type.

Here, we work in the classes of ultradifferentiable functions $\mathcal{S}_\omega(\mathbb{R}^d)$, where ω is a weight function in the sense of Braun, Meise, and Taylor [16], originally introduced by Björck [7] as follows: a function $u \in L^1(\mathbb{R}^d)$ is in $\mathcal{S}_\omega(\mathbb{R}^d)$ if (u and $\hat{u} \in C^\infty(\mathbb{R}^d)$ and)

$$\forall \lambda > 0, \alpha \in \mathbb{N}_0^d : \sup_{x \in \mathbb{R}^d} e^{\lambda\omega(x)} \max\{|D^\alpha u(x)|, |D^\alpha \widehat{u}(x)|\} < +\infty,$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$. The classes under consideration are suitable for our purposes, since they are invariant under Fourier transform and provide a big scale of spaces that contain as a particular case the Schwartz class when the weight function is $\omega(t) = \log(1+t)$ for $t \geq 0$ (example of weight function that we do not consider in this paper). We have seen in the literature the benefits of time-frequency analysis when applied to such classes (see [26]), even in combination with the global theory of (pseudo)differential operators (see e.g. [11] and the references therein, or [33, 35] when the classes are defined by sequences in the sense of Denjoy–Carleman). We have to mention also that our classes always contain compactly supported functions (they are non-quasianalytic) and we recover Gelfand–Shilov spaces of Beurling type of index $s > 1$ when the weight function is $\omega(t) = t^{1/s}$, i.e. a Gevrey weight.

In [12] the last three authors of the present paper introduce the ultradifferentiable version of the analytic wave front set found in [17,28,37] in the (Beurling) setting for $\mathcal{S}'_\omega(\mathbb{R}^d)$-ultradistributions, show that it can be described also in terms of Gabor frames for subadditive weight functions (as it is done in the setting of temperate distributions in [37]) and apply it to the study of the global regularity of (pseudo)differential operators of infinite order (in [37] the authors cannot treat operators of infinite order, since they consider symbols with polynomial growth only). However, the question if the latter wave front set can also be described in terms of Weyl quantizations, as in [28,37], remained open in the ultradifferentiable setting.

The first author in [4] studies the change of quantization in the class of global pseudodifferential operators introduced in [6] and gives sufficient conditions to obtain parametrices for any quantization. This is the starting point to define a new wave front set in terms of Weyl quantizations for $\mathcal{S}'_\omega(\mathbb{R}^d)$. The purpose of the present paper is twofold: on the one hand, we define a *Weyl wave front set* and study when it coincides with the (continuous version of the) Gabor wave front set of [12] for the ultradifferentiable setting; on the other hand, we give applications of this set to the regularity of pseudodifferential operators in the very general setting of [6].

The paper is organized as follows: in the next section we give some preliminaries, in Sect. 3 we study the kernel of some operators given by Weyl quantizations for symbols as in [6]. We already observe in this section that, just to give examples of symbols with a prescribed exponential growth from above and from below, the range of weight functions ω we need is quite restrictive; see Example 3.3 (in fact, to give an example for the Gevrey weight $\omega(t) = t^a$, we need that $0 < a < 1/2$). In Sect. 4 we introduce the Weyl wave front set and see that it can be characterized in terms of symbols of order zero. In Sect. 5, we extend the inclusion on the Gabor wave front set [12, Theorem 4.13] to any differential operator with variable coefficients and, later, we compare the Gabor wave front set, given in terms of the short-time Fourier transform in the continuous form, as in [12, Definition 3.1] (see Definition 5.1) with the Weyl wave front set. We need also here to impose that our weight functions be smaller than some Gevrey weight (see Remark 5.8). We could not circumvent this restriction, since we use similar techniques as in [37]. Finally, in Sect. 6 we study the propagation of singularities of Weyl quantizations with respect to the Weyl wave front set. For instance, for a suitable weight function ω, any $0 < \rho \leq 1$ and a symbol $a(x, \xi)$ as in [6], we are able to prove that (Theorem 6.8)

$$\begin{aligned} \mathrm{WF}^\omega_\rho(a^w(x, D)u) &\subset \mathrm{WF}^\omega_\rho(u) \cap \mathrm{conesupp}(a) \subset \mathrm{WF}^\omega_\rho(u) \\ &\subset \mathrm{WF}^\omega_\rho(a^w(x, D)u) \cup \mathrm{char}(a), \end{aligned}$$

where $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$, $a^w(x, D)u$ is the Weyl quantization of u, conesupp(a) is the conic support of $a(x, \xi)$ (Definition 4.4), char(a) is the characteristic set of

$a(x,\xi)$ (the set of points which are characteristic for $a(x,\xi)$; see Definition 4.1) and $\mathrm{WF}^{\omega}_{\rho}(u)$ is the Weyl wave front set of u.

2. Preliminaries

We denote the *Fourier transform* of $f \in L^1(\mathbb{R}^d)$ by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx,$$

with standard extensions to more general spaces of functions and distributions. We work with weight functions as in Braun, Meise, and Taylor [16].

Definition 2.1. A (non-quasianalytic) weight function $\omega : [0,+\infty[\to [0,+\infty[$ is an increasing and continuous function that satisfies

(α) There exists $L \geq 1$ such that $\omega(2t) \leq L(\omega(t)+1), \quad t \geq 0$;
(β) $\displaystyle\int_1^{+\infty} \frac{\omega(t)}{t^2}dt < +\infty$;
(γ) $\log(t) = o(\omega(t))$ as $t \to \infty$;
(δ) $\varphi_\omega : t \mapsto \omega(e^t)$ is convex.

For $z \in \mathbb{C}^d$ we denote $\omega(z) = \omega(|z|)$, where $|z|$ denotes the Euclidean norm of z. These weight functions satisfy from Definition 2.1(α),

$$\omega(x+y) \leq L\omega(x) + L\omega(y) + L; \qquad \omega(x,y) \leq L\omega(x) + L\omega(y) + L \tag{2.1}$$

for all $x, y \in \mathbb{R}^d$. We also recall that

$$\omega\Big(\frac{x+y}{2}\Big) \leq \omega(\max\{|x|,|y|\}) \leq \omega(x) + \omega(y), \qquad x, y \in \mathbb{R}^d. \tag{2.2}$$

We shall assume that the weight functions vanish on the interval $[0,1]$. Then it holds, for $\langle x\rangle^2 = 1 + |x|^2$, $x \in \mathbb{R}^d$,

$$\omega(\langle x\rangle) \leq L\omega(x) + L, \qquad x \in \mathbb{R}^d.$$

We define the Young conjugate as follows:

$$\varphi^*_\omega(t) := \sup_{s\geq 0}\{st - \varphi_\omega(s)\}, \qquad t \geq 0.$$

When the choice on the weight is clear, we will write φ and φ^* for short. From the convexity of φ (Definition 2.1(δ)), we have that φ^* is a convex function, $\varphi^*(t)/t$ is increasing and tends to infinity as $t \to \infty$, and $\varphi^{**} = \varphi$. Furthermore, since $\omega|_{[0,1]} \equiv 0$, we have $\varphi^*(0) = 0$. Here we gather some well-known facts and estimates involving the Young conjugate. See for instance [13, Appendix A] for their proofs, in a more general context.

Lemma 2.2. *Given a weight ω as in Definition 2.1, we have*

(i) For all $\lambda > 0$ and $k \in \mathbb{N}$,

$$t^k \leq e^{\lambda\varphi^*\left(\frac{k}{\lambda}\right)} e^{\lambda\omega(t)}, \qquad t \geq 1.$$

(ii) For all $\sigma > 0$ there exists $C_\sigma > 0$ such that for each $\lambda > 0$,

$$\inf_{j\in\mathbb{N}_0} t^{-\sigma j} e^{\lambda\varphi^*\left(\frac{\sigma j}{\lambda}\right)} \leq C_\sigma e^{-(\lambda-1)\omega(t)}, \qquad t \geq 1.$$

(iii) Let $L \geq 1$ such that $\omega(et) \leq L\omega(t) + L$, $t \geq 0$. Then, for all $\lambda > 0$ and $n \in \mathbb{N}$,

$$\lambda L^n \varphi^*\Big(\frac{t}{\lambda L^n}\Big) + nt \leq \lambda\varphi^*\Big(\frac{t}{\lambda}\Big) + \lambda\sum_{j=1}^{n} L^j, \qquad t \geq 0.$$

(iv) For all $\lambda > 0$,

$$2\lambda\varphi^*\Big(\frac{s+t}{2\lambda}\Big) \leq \lambda\varphi^*\Big(\frac{s}{\lambda}\Big) + \lambda\varphi^*\Big(\frac{t}{\lambda}\Big) \leq \lambda\varphi^*\Big(\frac{s+t}{\lambda}\Big), \qquad s, t \geq 0.$$

(v) For all $\lambda, B > 0$ there exists $C > 0$ such that

$$B^{|\alpha|}\alpha! \leq C e^{\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)}, \qquad \alpha \in \mathbb{N}_0^d.$$

From now on $L \geq 1$ stands for the constant in Lemma 2.2(*iii*). When considering a suitable change of weights, the following estimates included in [6, Lemma 2.9] will appear on the stage.

Lemma 2.3. *Let $a \geq 1$. If ω and σ are weight functions such that*

(i) *$\omega(t^a) = o(\sigma(t))$, $t \to \infty$, then for all $\lambda, \mu > 0$ there exists $C_{\lambda,\mu} > 0$ such that*

$$\lambda\varphi_\sigma^*\Big(\frac{j}{\lambda}\Big) \leq C_{\lambda,\mu} + \mu\frac{1}{a}\varphi_\omega^*\Big(\frac{j}{\mu}\Big), \qquad j \in \mathbb{N}_0.$$

(ii) *$\omega(t^a) = O(\sigma(t))$, $t \to \infty$, then there exists $C > 0$ such that for each $\lambda > 0$,*

$$\lambda\varphi_\sigma^*\Big(\frac{j}{\lambda}\Big) \leq \lambda + \lambda C\frac{1}{a}\varphi_\omega^*\Big(\frac{j}{\lambda C}\Big), \qquad j \in \mathbb{N}_0.$$

We have that ω is (equivalent to) a subadditive weight function if and only if ω satisfies [32, Proposition 1.1]:

(α_0) $\exists\, C_1 > 0\ \exists\, t_0 > 0\ \forall\, \lambda \geq 1\ \forall\, t \geq t_0 \,:\, \omega(\lambda t) \leq \lambda C_1\omega(t)$.

See [8,21,32] for more information on property (α_0).

The setting of this work is the space of ultradifferentiable functions defined by Björck [7]. This space is characterized by different systems of seminorms (see, e.g. [11,13,26]). In fact, $\mathcal{S}_\omega(\mathbb{R}^d)$ consists of all $u \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\forall\, \lambda, \mu > 0,\ \exists\, C_{\lambda,\mu} > 0, \forall\, \alpha \in \mathbb{N}_0^d,\, x \in \mathbb{R}^d :\ |D^\alpha u(x)| \leq C_{\lambda,\mu} e^{\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)} e^{-\mu\omega(x)},$$

or, equivalently,

$$\forall\,\lambda > 0\ \exists\, C_\lambda > 0,\ \forall\, \alpha,\beta \in \mathbb{N}_0^d,\ x \in \mathbb{R}^d:\ |x^\beta D^\alpha u(x)| \le C_\lambda e^{\lambda\varphi^*\left(\frac{|\alpha+\beta|}{\lambda}\right)}, \tag{2.3}$$

and also

$$\forall\,\lambda > 0,\ \sup_{x\in\mathbb{R}^d} \left(\max\{|u(x)|, |\widehat{u}(x)|\} e^{\lambda\omega(x)}\right) < +\infty. \tag{2.4}$$

The strong dual of $\mathcal{S}_\omega(\mathbb{R}^d)$ is denoted by $\mathcal{S}'_\omega(\mathbb{R}^d)$.

We use the following notation for the translation, modulation and phase-shift operators:

$$T_x f(y) = f(y-x); \qquad M_\xi f(y) = e^{iy\cdot\xi} f(y); \qquad \Pi(z) f(y) = e^{iy\cdot\xi} f(y-x),$$

for all $x, y, \xi \in \mathbb{R}^d$ and $z = (x, \xi)$.

Definition 2.4. Let $\psi \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$ be a window function. The short-time Fourier transform of $f \in \mathcal{S}'_\omega(\mathbb{R}^d)$ is defined, for $z = (x, \xi) \in \mathbb{R}^{2d}$, by

$$\begin{aligned} V_\psi f(z) &:= \langle f, \Pi(z)\psi \rangle \\ &= \int_{\mathbb{R}^d} f(y)\overline{\psi(y-x)} e^{-iy\cdot\xi} dy, \qquad z = (x,\xi) \in \mathbb{R}^{2d}. \end{aligned}$$

We note that the conjugate linear action of $\mathcal{S}'_\omega(\mathbb{R}^d)$ on $\mathcal{S}_\omega(\mathbb{R}^d)$, $\langle\cdot,\cdot\rangle$, is consistent with $\langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^d)}$. We also observe that

$$V_\psi f(z) = \widehat{f \cdot T_x\overline{\psi}}(\xi), \qquad z = (x,\xi) \in \mathbb{R}^{2d}. \tag{2.5}$$

If $\psi \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$ is a window function and F is measurable on $\mathbb{R}^{2d}$, the adjoint operator is

$$V_\psi^* F = \int_{\mathbb{R}^{2d}} F(z)\Pi(z)\psi dz, \tag{2.6}$$

and it follows from [12, (2.25)] that

$$V_\psi^* V_\psi = (2\pi)^d \left\|\psi\right\|^2_{L^2(\mathbb{R}^d)} I_{\mathcal{S}'_\omega(\mathbb{R}^d)}. \tag{2.7}$$

We recall from [26, Theorem 2.7] that given $\psi \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$ and $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$ we have $u \in \mathcal{S}_\omega(\mathbb{R}^d)$ if and only if for all $\lambda > 0$ there exists $C_\lambda > 0$ such that

$$|V_\psi u(z)| \le C_\lambda e^{-\lambda\omega(z)}, \qquad z \in \mathbb{R}^{2d}.$$

Some results involving the short-time Fourier transform for the Schwartz class are known. See e.g. [25, Chapter 3]. The proofs are the same for the ultradifferentiable setting.

Lemma 2.5. *If $T \in \mathcal{S}'_\omega(\mathbb{R}^d)$ and $g \in \mathcal{S}_\omega(\mathbb{R}^d)$, then*

$$\widehat{gT} = (2\pi)^{-d}\big(\widehat{g} * \widehat{T}\big), \qquad \widehat{g * T} = \widehat{g}\cdot\widehat{T}.$$

Lemma 2.6. *If* $f, g \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$, *then*

$$V_g f(x,\xi) = e^{-ix\cdot\xi}\overline{V_f g(-x,-\xi)}, \qquad x,\xi \in \mathbb{R}^d.$$

Lemma 2.7. [25, (1.8),(1.9)] *If* $\psi \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$, *then*

$$M_y\widehat{\psi}(\eta) = \widehat{T_{-y}\psi}(\eta), \qquad \widehat{M_y\psi}(\eta) = T_y\widehat{\psi}(\eta), \qquad y,\eta \in \mathbb{R}^d.$$

By applying formula (2.5) and Lemmas 2.5 and 2.7 , we get

Lemma 2.8. *If* $f \in \mathcal{S}'_\omega(\mathbb{R}^d)$ *and* $\psi \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$, *then*

$$V_\psi f(x,\xi) = (2\pi)^{-d}\big(\widehat{f} * M_{-x}\widehat{\overline{\psi}}\big)(\xi) \qquad x,\xi \in \mathbb{R}^d.$$

3. The Weyl Quantization

In this section we study properties of the kernel of an operator given by a Weyl quantization and the short-time Fourier transform. First, we recall the definition of the global symbols defined in [4,6]. From now on, $m \in \mathbb{R}$ and $0 < \rho \leq 1$.

Definition 3.1. We define $\mathrm{GS}^{m,\omega}_\rho$ as the set of symbols $p(x,\xi) \in C^\infty(\mathbb{R}^{2d})$ such that for all $\lambda > 0$ there exists $C_\lambda > 0$ with

$$|D^\alpha_x D^\beta_\xi p(x,\xi)| \leq C_\lambda \langle (x,\xi)\rangle^{-\rho|\alpha+\beta|} e^{\lambda\rho\varphi^*\left(\frac{|\alpha+\beta|}{\lambda}\right)} e^{m\omega(x,\xi)}, \qquad \alpha,\beta \in \mathbb{N}^d_0,\ x,\xi \in \mathbb{R}^d.$$

We observe that the only difference with the global symbols in [6, Definition 3.1] is the factor $e^{m\omega(x,\xi)}$ instead of $e^{m\omega(x)}e^{m\omega(\xi)}$, which is more convenient for our purposes here, but the corresponding theory of pseudodifferential operators remains the same. The constant m is called the *order* of the symbol.

For $b \in \mathrm{GS}^{m,\omega}_\rho$, we consider the Weyl quantization for $u \in \mathcal{S}_\omega(\mathbb{R}^d)$ (see for example [39, Definition 23.5] or [37, page 631]):

$$b^w(x,D)u = (2\pi)^{-d}\int_{\mathbb{R}^{2d}} e^{i(x-s)\cdot\xi} b\Big(\frac{x+s}{2},\xi\Big)u(s)ds d\xi, \qquad x \in \mathbb{R}^d.$$

By [4, Lemma 3.3] and [6, Theorem 3.7], given a global symbol in $\mathrm{GS}^{m,\omega}_\rho$, the corresponding Weyl quantization is well defined and continuous from $\mathcal{S}_\omega(\mathbb{R}^d)$ into itself. Given two symbols $a(x,\xi)$ and $b(x,\xi)$, we write $a\#b(x,\xi)$ to denote the *Weyl product* of the two symbols, i.e. the symbol corresponding to the composition of the Weyl quantizations of $a(x,\xi)$ and $b(x,\xi)$:

$$(a\#b)^w(x,D) = a^w(x,D)b^w(x,D).$$

By [39, Theorem 23.6 and Problem 23.2] (cf. [4, Corollary 4.5]), the Weyl product of a and b has the following asymptotic expansion:

$$a\#b(x,\xi) \sim \sum_{j=0}^\infty \sum_{|\beta+\gamma|=j} \frac{(-1)^{|\beta|}}{\gamma!\beta!} 2^{-|\beta+\gamma|} \partial^\gamma_\xi D^\beta_x a(x,\xi)\, \partial^\beta_\xi D^\gamma_x b(x,\xi). \qquad (3.1)$$

When dealing with asymptotic expansions, a special type of operator appears, usually denoted by R, and called globally ω-regularizing operator, which acts $R:\mathcal{S}'_\omega(\mathbb{R}^d)\to\mathcal{S}_\omega(\mathbb{R}^d)$, see [6, Proposition 3.11]. We also have that given a global symbol $r(x,\xi)\in \mathrm{GS}^{m,\omega}_\rho$, the pseudodifferential operator associated $R=r^w(x,D)$ is *globally ω-regularizing* if and only if $r\in\mathcal{S}_\omega(\mathbb{R}^{2d})$ (see [6, Proposition 3.11], [31, Proposition 1.2.1]).

The following result provides sufficient conditions for a symbol to admit a left parametrix and is an extension of [22, Theorem 3.4] for global pseudodifferential operators. These conditions are the basis to define the *Weyl wave front set*. By [4, Theorems 3.11 and 5.4], it is easy to see that the same conditions are valid for any global quantization. In particular, for Weyl quantizations.

Theorem 3.2. *Let ω be a weight function and let σ be a subadditive weight function with $\omega(t^{1/\rho})=o(\sigma(t))$ as $t\to\infty$. Let $p\in \mathrm{GS}^{|m|,\omega}_\rho$ be such that, for some $R\geq 1$:*

(i) *There exists $C_1>0$ such that $|p(z)|\geq C_1 e^{-|m|\omega(z)}$ for $|z|\geq R$;*
(ii) *There exist $C_2>0$ and $n\in\mathbb{N}$ such that*

$$|D^\alpha p(z)|\leq C_2^{|\alpha|}\langle z\rangle^{-\rho|\alpha|}e^{\frac{1}{n}\varphi^*_\sigma(n|\alpha|)}|p(z)|,$$

for $\alpha\in\mathbb{N}_0^{2d}$, $|z|\geq R$.

Then, there exists $q\in\mathrm{GS}^{|m|,\omega}_\rho$ such that $q\#p=1+r$, for some $r\in\mathcal{S}_\omega(\mathbb{R}^{2d})$.

The following example is inspired by [3, Capitolo 4]. [39]

Example 3.3. Let $\omega(t)=t^a$ be a Gevrey weight, for some $0<a<1/2$. For $m\in\mathbb{R}$ we consider

$$p(z):=e^{|m|\langle z\rangle^a},\qquad z\in\mathbb{R}^{2d}.$$

We want to show that (i) and (ii) in Theorem 3.2 hold. It is clear that

$$|p(z)|=e^{|m|\omega(\langle z\rangle)}\geq e^{|m|\omega(z)}\geq e^{-|m|\omega(z)},\qquad z\in\mathbb{R}^{2d}.$$

On the other hand, by using Faà di Bruno formula for several variables (see, for example [29, Page 234]), we obtain that there exists $C>0$ such that, for $\rho=1-a$,

$$|D^\alpha p(z)|\leq C^{|\alpha|}\alpha!\langle z\rangle^{-\rho|\alpha|}p(z),\qquad \alpha\in\mathbb{N}_0^{2d},\ z\in\mathbb{R}^{2d}. \tag{3.2}$$

Let σ be as in Theorem 3.2. By Lemma 2.2(v) there exists $C'\geq 1$ such that

$$\alpha!\leq C'e^{\varphi^*_\sigma(|\alpha|)}.$$

Therefore

$$|D^\alpha p(z)|\leq (CC')^{|\alpha|}e^{\varphi^*_\sigma(|\alpha|)}\langle z\rangle^{-\rho|\alpha|}|p(z)|,$$

for all $\alpha\in\mathbb{N}_0^{2d}$ and $z\in\mathbb{R}^{2d}$. We claim that $p\in\mathrm{GS}^{|m|,\omega}_\rho$. Indeed, from Lemmas 2.2(v) and 2.3(i), for all $\lambda>0$ there exist $C_\lambda, C'_\lambda>0$ such that

$$C^{|\alpha|}\alpha!\leq C_\lambda e^{\lambda\varphi^*_\sigma\left(\frac{|\alpha|}{\lambda}\right)}\leq C'_\lambda e^{\lambda\rho\varphi^*_\omega\left(\frac{|\alpha|}{\lambda}\right)},\qquad \alpha\in\mathbb{N}_0^{2d}.$$

So, from (3.2) and the subadditivity of the weight ω, for all $\lambda > 0$ there exists $C'_\lambda > 0$ such that

$$\begin{aligned}|D^\alpha p(z)| &\le C'_\lambda e^{\lambda\rho\varphi^*_\omega\left(\frac{|\alpha|}{\lambda}\right)}\langle z\rangle^{-\rho|\alpha|}e^{|m|\omega(\langle z\rangle)}\\ &\le C'_\lambda e^{|m|}e^{\lambda\rho\varphi^*_\omega\left(\frac{|\alpha|}{\lambda}\right)}\langle z\rangle^{-\rho|\alpha|}e^{|m|\omega(z)},\end{aligned}$$

for all $\alpha \in \mathbb{N}_0^{2d}$ and $z \in \mathbb{R}^{2d}$.

We observe that in this example we have the restriction $0 < a < 1/2$, since $a = 1-\rho$, $\omega(t^{1/\rho}) = t^{\frac{a}{1-a}}$, and ω is non-quasianalytic.

Since $\mathcal{S}_\omega(\mathbb{R}^{4d})$ is nuclear [14,15,19,20], for $b \in \mathrm{GS}^{m,\omega}_\rho$, there exists $K \in \mathcal{S}'_\omega(\mathbb{R}^{4d})$ such that

$$V_\psi b^w(x,D)V_\psi^* : \mathcal{S}_\omega(\mathbb{R}^{2d}) \to \mathcal{S}'_\omega(\mathbb{R}^{2d}),$$

where

$$V_\psi(b^w(x,D)V_\psi^* F)(y',\eta') = (2\pi)^d\int_{\mathbb{R}^{2d}} K(y',\eta',y,\eta)F(y,\eta)dyd\eta, \qquad F \in \mathcal{S}_\omega(\mathbb{R}^{2d}),$$

in the sense that

$$\langle V_\psi b^w(x,D)V_\psi^* F, G\rangle = (2\pi)^d\langle K(y',\eta',y,\eta), G(y',\eta')F(y,\eta)\rangle, \qquad G \in \mathcal{S}_\omega(\mathbb{R}^{2d}).$$

For $u \in \mathcal{S}_\omega(\mathbb{R}^d)$ and $\psi \in \mathcal{S}_\omega(\mathbb{R}^d)$, $\|\psi\|_{L^2(\mathbb{R}^d)} = 1$, we denote $F = V_\psi u$, which belongs to $\mathcal{S}_\omega(\mathbb{R}^{2d})$ by [26, Theorem 2.7]. We have by (2.7) (see [12])

$$V_\psi(b^w(x,D)u)(y',\eta') = \int_{\mathbb{R}^{2d}} K(y',\eta',y,\eta)V_\psi u(y,\eta)dyd\eta, \qquad (y',\eta') \in \mathbb{R}^{2d}. \tag{3.3}$$

We analyse the kernel of this operator:

Theorem 3.4. *Let $b \in \mathrm{GS}^{m,\omega}_\rho$ and $\psi \in \mathcal{S}_\omega(\mathbb{R}^d)$ such that $\|\psi\|_{L^2(\mathbb{R}^d)} = 1$. If $u \in \mathcal{S}_\omega(\mathbb{R}^d)$, then we have*

$$\begin{aligned}&K(y',\eta',y,\eta)\\ &\quad = (2\pi)^{-2d}\int_{\mathbb{R}^{2d}}\left(\int_{\mathbb{R}^d} e^{ix\cdot(\xi-\eta')}e^{is\cdot(\eta-\xi)}b\Big(\frac{x+s}{2},\xi\Big)\overline{\psi(x-y')}\psi(s-y)ds\right)d\xi dx,\end{aligned} \tag{3.4}$$

for all $(y',\eta',y,\eta) \in \mathbb{R}^{4d}$, where K is the kernel in (3.3).

Proof. Let us consider $V_\psi^* : \mathcal{S}_\omega(\mathbb{R}^{2d}) \to \mathcal{S}_\omega(\mathbb{R}^d)$ as in (2.6). For $F \in \mathcal{S}_\omega(\mathbb{R}^{2d})$ we have

$$\begin{aligned}&b^w(x,D)V_\psi^* F(x)\\ &\quad = (2\pi)^{-d}\int_{\mathbb{R}^{2d}} e^{i(x-s)\cdot\xi}b\Big(\frac{x+s}{2},\xi\Big)V_\psi^* F(s)dsd\xi, \qquad x \in \mathbb{R}^d.\end{aligned}$$

Then, by using the definition of $V_\psi^* F$

$$\begin{aligned} &V_\psi(b^w(x,D)V_\psi^* F)(y',\eta') \\ &\quad = (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{2d}} e^{-ix\cdot\eta'}\overline{\psi(x-y')}e^{i(x-s)\cdot\xi} b\Big(\frac{x+s}{2},\xi\Big)V_\psi^* F(s)dsd\xi dx \\ &\quad = (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} e^{-ix\cdot\eta'}\overline{\psi(x-y')}e^{i(x-s)\cdot\xi} b\Big(\frac{x+s}{2},\xi\Big) \\ &\qquad \times F(y,\eta)e^{is\cdot\eta}\psi(s-y)dyd\eta dsd\xi dx \end{aligned}$$

for all $(y',\eta')\in\mathbb{R}^{2d}$. We shall assume that $m\geq 0$; otherwise, the proof is easier. We want to use Fubini's theorem for the variables y,η,s. To this aim, we estimate the modulus of the integrand as follows: Since $\psi\in\mathcal{S}_\omega(\mathbb{R}^d)$, $b\in \mathrm{GS}_\rho^{m,\omega}$, and $F\in\mathcal{S}_\omega(\mathbb{R}^{2d})$, for all $\lambda_1,\lambda_2>0$ there exists $C_{\lambda_1,\lambda_2}>0$ such that, by (2.2), we have

$$\begin{aligned} &\Big|\overline{\psi(x-y')}b\Big(\frac{x+s}{2},\xi\Big)F(y,\eta)\psi(s-y)\Big| \\ &\quad\leq C_{\lambda_1,\lambda_2}e^{m\omega\left(\frac{x+s}{2},\xi\right)}e^{-\lambda_1\omega(y,\eta)}e^{-\lambda_2\omega(s-y)} \\ &\quad\leq C_{\lambda_1,\lambda_2}e^{mL\omega(x)}e^{mL\omega(s)}e^{mL\omega(\xi)}e^{mL}e^{-\frac{\lambda_1}{2}\omega(y)}e^{-\frac{\lambda_1}{2}\omega(\eta)}e^{-\frac{\lambda_2}{L}\omega(s)}e^{\lambda_2\omega(y)}e^{\lambda_2}, \end{aligned}$$

which belongs to $L^1(\mathbb{R}^{3d}_{y,\eta,s})$ if we choose $\lambda_2>mL^2$ (the integral depending on s converges) and $\lambda_1>2\lambda_2$ (the integrals depending on y and η converge). Therefore, we use Fubini's theorem, and we obtain

$$\begin{aligned} V_\psi(b^w(x,D)V_\psi^* F)(y',\eta') = (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{2d}} e^{ix\cdot\xi}F(y,\eta)\overline{\psi(x-y')}e^{-ix\cdot\eta'} \\ \times\Big(\int_{\mathbb{R}^d} e^{is\cdot(\eta-\xi)}b\Big(\frac{x+s}{2},\xi\Big)\psi(s-y)ds\Big)dyd\eta d\xi dx. \end{aligned} \tag{3.5}$$

We want to use again Fubini's theorem, now in $dyd\eta d\xi dx$. To that aim, we need some preparation for

$$I(y,\eta,\xi,x) := \int_{\mathbb{R}^d} e^{is\cdot(\eta-\xi)}b\Big(\frac{x+s}{2},\xi\Big)\psi(s-y)ds.$$

Similarly as before, since $b\in\mathrm{GS}_\rho^{m,\omega}$ and $\psi\in\mathcal{S}_\omega(\mathbb{R}^d)$, for all $\lambda>0$ there exists $C_\lambda>0$ such that

$$\begin{aligned} \Big|b\Big(\frac{x+s}{2},\xi\Big)\psi(s-y)\Big| &\leq C_\lambda e^{m\omega(\frac{x+s}{2},\xi)}e^{-\lambda\omega(s-y)} \\ &\leq C_\lambda e^{mL\omega(x)}e^{mL\omega(s)}e^{mL\omega(\xi)}e^{mL}e^{-\frac{\lambda}{L}\omega(s)}e^{\lambda\omega(y)}e^{\lambda}, \end{aligned}$$

which belongs to $L^1(\mathbb{R}^d_s)$ if $\lambda>mL^2$. Moreover, it tends to 0 as $|s|\to+\infty$.

Let us assume $|\eta-\xi|_\infty := \max_{1\leq h\leq d}|\eta_h-\xi_h| = |\eta_j-\xi_j|\geq 1$ for some $1\leq j\leq d$. For any $N\in\mathbb{N}_0$, we integrate by parts as follows:

$$\begin{aligned}|I| &= \Big|\int_{\mathbb{R}^d} \frac{1}{(\eta_j-\xi_j)^N}(D^N_{s_j} e^{is\cdot(\eta-\xi)}) b\Big(\frac{x+s}{2},\xi\Big)\psi(s-y)ds\Big| \\ &= \Big|\int_{\mathbb{R}^d} \frac{(-1)^N}{(\eta_j-\xi_j)^N} e^{is\cdot(\eta-\xi)} D^N_{s_j}\Big(b\Big(\frac{x+s}{2},\xi\Big)\psi(s-y)\Big)ds\Big| \\ &\le \frac{1}{|\eta_j-\xi_j|^N}\sum_{k=0}^N \binom{N}{k}\int_{\mathbb{R}^d}\Big|D^{N-k}_{s_j} b\Big(\frac{x+s}{2},\xi\Big)\Big||D^k_{s_j}\psi(s-y)|ds.\end{aligned}$$

We observe that $|\eta-\xi| \le \sqrt{d}|\eta-\xi|_\infty = \sqrt{d}|\eta_j-\xi_j|$. We put $p\in\mathbb{N}$ so that $2\sqrt{d}\le e^p$, and since $b\in \mathrm{GS}^{m,\omega}_\rho$ and $\psi\in\mathcal{S}_\omega(\mathbb{R}^d)$, by Lemma 2.2($iv$) and (2.2) we have that for all $n\in\mathbb{N}$ there exist $C_n, C'_n>0$ such that

$$\begin{aligned}|I| \le{}& \frac{(\sqrt{d})^N}{|\eta-\xi|^N}\sum_{k=0}^N\binom{N}{k} \\ &\times \int_{\mathbb{R}^d} C_n\Big\langle\Big(\frac{x+s}{2},\xi\Big)\Big\rangle^{-\rho(N-k)} e^{(n+1)L^p\rho\varphi^*\left(\frac{N-k}{(n+1)L^p}\right)} e^{m\omega(\frac{x+s}{2},\xi)} \\ &\times C'_n e^{(n+1)L^p\varphi^*\left(\frac{k}{(n+1)L^p}\right)} e^{-(mL^2+L)\omega(s-y)}ds \\ \le{}& C_nC'_n\frac{(\sqrt{d})^N}{|\eta-\xi|^N} e^{(n+1)L^p\varphi^*\left(\frac{N}{(n+1)L^p}\right)}\sum_{k=0}^N\binom{N}{k}\int_{\mathbb{R}^d} e^{m\omega(\frac{x+s}{2},\xi)}e^{-(mL^2+L)\omega(s-y)}ds \\ \le{}& C_nC'_n e^{mL}e^{mL^2+L}\frac{(2\sqrt{d})^N}{|\eta-\xi|^N}e^{(n+1)L^p\varphi^*\left(\frac{N}{(n+1)L^p}\right)} \\ &\times\int_{\mathbb{R}^d} e^{mL\omega(x)+mL\omega(s)+mL\omega(\xi)}e^{-\frac{mL^2+L}{L}\omega(s)+(mL^2+L)\omega(y)}ds.\end{aligned}$$

From the choice of $p\in\mathbb{N}$, Lemma 2.2(iii) gives

$$\begin{aligned}|I| \le{}& C_nC'_n e^{mL}e^{mL^2+L}e^{(n+1)\sum_{j=1}^p L^j}|\eta-\xi|^{-N}e^{(n+1)\varphi^*\left(\frac{N}{n+1}\right)} \\ &\times e^{mL\omega(x)}e^{mL\omega(\xi)}e^{(mL^2+L)\omega(y)}\int_{\mathbb{R}^d}e^{-\omega(s)}ds.\end{aligned}$$

The integral depending on s converges by property (γ). We take the infimum on $N\in\mathbb{N}_0$ and we use Lemma 2.2(ii) (for $\sigma=1$) as follows: for each $n\in\mathbb{N}$ there exists $C''_n>0$ such that

$$\begin{aligned}|I| &\le C''_n e^{-n\omega(\xi-\eta)}e^{mL\omega(x)}e^{mL\omega(\xi)}e^{(mL^2+L)\omega(y)} \\ &\le C''_n e^n e^{(-\frac{n}{L}+mL)\omega(\xi)}e^{n\omega(\eta)}e^{mL\omega(x)}e^{(mL^2+L)\omega(y)}.\end{aligned}$$

Thus, for all $x,y,y',\xi,\eta\in\mathbb{R}^d$ satisfying $|\xi-\eta|_\infty\ge 1$, we have that for all $\lambda,\lambda_1,\lambda_2>0$ there exists $C_{\lambda,\lambda_1,\lambda_2}>0$ so that

$$\begin{aligned}&|F(y,\eta)\overline{\psi(x-y')}I| \\ &\le C_{\lambda,\lambda_1,\lambda_2}e^{-\lambda_1\omega(y,\eta)}e^{-\lambda_2\omega(x-y')}e^{(-\frac{\lambda}{L}+mL)\omega(\xi)}e^{\lambda\omega(\eta)}e^{mL\omega(x)}e^{(mL^2+L)\omega(y)} \\ &\le C_{\lambda,\lambda_1,\lambda_2}e^{-\frac{\lambda_1}{2}\omega(y)}e^{-\frac{\lambda_1}{2}\omega(\eta)}e^{-\frac{\lambda_2}{L}\omega(x)}e^{\lambda_2\omega(y')}\end{aligned}$$

$$
\begin{aligned}
&\times e^{\lambda_2} e^{\left(-\frac{\lambda}{L}+mL\right)\omega(\xi)} e^{\lambda\omega(\eta)} e^{mL\omega(x)} e^{(mL^2+L)\omega(y)} \\
&= C_{\lambda,\lambda_1,\lambda_2} e^{\lambda_2} e^{\left(-\frac{\lambda_1}{2}+mL^2+L\right)\omega(y)} e^{\left(-\frac{\lambda_1}{2}+\lambda\right)\omega(\eta)} e^{\left(-\frac{\lambda}{L}+mL\right)\omega(\xi)} \\
&\times e^{\left(-\frac{\lambda_2}{L}+mL\right)\omega(x)} e^{\lambda_2\omega(y')},
\end{aligned}
$$

which belongs to $L^1(\mathbb{R}^{4d}_{y,\eta,\xi,x})$ provided $\lambda > mL^2$ (the integral in $d\xi$ converges), $\lambda_2 > mL^2$ (the integral in dx converges), and $\lambda_1 > \max\{2(mL^2+L), 2\lambda\}$ (the integrals in dy and $d\eta$ converge).

On the other hand, if $|\xi-\eta|_\infty \le 1$, then $|\xi|-|\eta| \le |\xi-\eta| \le \sqrt{d}|\xi-\eta|_\infty \le \sqrt{d}$, so $|\xi| \le |\eta| + \sqrt{d}$. Hence

$$
\omega(\xi) \le \omega(|\eta| + \sqrt{d}) \le L\omega(\eta) + L\omega(\sqrt{d}) + L.
$$

Then, as before, for all $\lambda, \lambda_1, \lambda_2 > 0$ there exists $C_{\lambda,\lambda_1,\lambda_2} > 0$ such that

$$
\begin{aligned}
&\left| F(y,\eta)\overline{\psi(x-y')} b\Big(\frac{x+s}{2},\xi\Big)\psi(s-y)\right| \\
&\quad\le C_{\lambda,\lambda_1,\lambda_2} e^{-\lambda\omega(y,\eta)} e^{-\lambda_1\omega(x-y')} e^{m\omega\left(\frac{x+s}{2},\xi\right)} e^{-\lambda_2\omega(s-y)} \\
&\quad\le C_{\lambda,\lambda_1,\lambda_2} e^{-\frac{\lambda}{2}\omega(y)} e^{-\frac{\lambda}{2}\omega(\eta)} e^{-\frac{\lambda_1}{L}\omega(x)} e^{\lambda_1\omega(y')} e^{\lambda_1} \\
&\qquad\times e^{mL\omega(x)} e^{mL\omega(s)} e^{(mL+1)\omega(\xi)} e^{-\omega(\xi)} e^{mL} e^{-\frac{\lambda_2}{L}\omega(s)} e^{\lambda_2\omega(y)} e^{\lambda_2} \\
&\quad\le C'_{\lambda,\lambda_1,\lambda_2} e^{\left(-\frac{\lambda_2}{L}+mL\right)\omega(s)} e^{\left(-\frac{\lambda}{2}+\lambda_2\right)\omega(y)} \\
&\qquad\times e^{\left(-\frac{\lambda}{2}+mL^2+L\right)\omega(\eta)} e^{-\omega(\xi)} e^{\left(-\frac{\lambda_1}{L}+mL\right)\omega(x)} e^{\lambda_1\omega(y')},
\end{aligned}
$$

for some $C'_{\lambda,\lambda_1,\lambda_2} > 0$. It belongs to $L^1(\mathbb{R}^{5d}_{s,y,\eta,\xi,x})$ if $\lambda_2 > mL^2$ (the integral depending on s converges), $\lambda > \max\{2\lambda_2, 2mL^2+2L\}$ (the integrals depending on y and η converge), and $\lambda_1 > mL^2$ (the integral depending on x converges).

We can therefore use Fubini's theorem in (3.5), and we obtain

$$
\begin{aligned}
&V_\psi(b^w(x,D)V_\psi^* F)(y',\eta') \\
&\quad= (2\pi)^{-d}\int_{\mathbb{R}^{2d}}\left(\int_{\mathbb{R}^{3d}} e^{ix\cdot(\xi-\eta')} e^{is\cdot(\eta-\xi)}\right. \\
&\qquad\times\left. b\Big(\frac{x+s}{2},\xi\Big)\overline{\psi(x-y')}\psi(s-y)\,ds\,d\xi\,dx\right) F(y,\eta)\,dy\,d\eta.
\end{aligned}
$$

We put $F = V_\psi u$, with $u \in \mathcal{S}_\omega(\mathbb{R}^d)$. Then, by (2.7) it follows by assumption $V_\psi^* F = V_\psi^* V_\psi u = (2\pi)^d u$. Hence

$$
V_\psi(b^w(x,D)u)(y',\eta') = \int_{\mathbb{R}^{2d}} K(y',\eta',y,\eta)V_\psi u(y,\eta)\,dy\,d\eta,
$$

for all $(y',\eta') \in \mathbb{R}^{2d}$, where the kernel $K(y',\eta',y,\eta)$ is as in (3.4). □

Now we recall some facts about entire functions [6, Theorem 2.16, (2.18), (2.19)]: for any weight function ω, there exists an entire function

$$G(z)=\sum_{\alpha\in\mathbb{N}_0^d} a_\alpha z^\alpha, \qquad z\in\mathbb{C}^d, \tag{3.6}$$

with

$$|a_\alpha|\le e^C e^{-C\varphi^*_\omega\left(\frac{|\alpha|}{C}\right)}, \qquad \alpha\in\mathbb{N}_0^d, \tag{3.7}$$

for some $C>0$, satisfying

$$\log|G(z)|\le\omega(z)+C_1, \qquad z\in\mathbb{C}^d$$
$$\log|G(z)|\ge C_2\omega(z)-C_4, \qquad \text{for } z\in\widetilde{U}=\{z\in\mathbb{C}^d:|\operatorname{Im}(z)|\le C_3(|\operatorname{Re}(z)|+1)\},$$

for some $C_1,C_2,C_3,C_4>0$. Moreover, for $n\in\mathbb{N}$, the n-th power of G, G^n, is a power series

$$G^n(z)=\sum_{\alpha\in\mathbb{N}_0^d} b_\alpha z^\alpha, \qquad z\in\mathbb{C}^d \tag{3.8}$$

satisfying

$$|b_\alpha|\le e^{nC} e^{-nC\varphi^*_\omega\left(\frac{|\alpha|}{nC}\right)}, \qquad \alpha\in\mathbb{N}_0^d, \tag{3.9}$$

for the same constant $C>0$ as in (3.7). For this $C>0$, by using Lemma 2.2(v) we have that for all $\lambda>0$ there exists $C_\lambda>0$ such that

$$\left|D^\beta\frac{1}{G^n(\xi)}\right|\le C^n C_\lambda e^{\lambda\varphi^*\left(\frac{|\beta|}{\lambda}\right)}e^{-nK\omega(\xi)}, \qquad \beta\in\mathbb{N}_0^d,\ \xi\in\mathbb{R}^d, \tag{3.10}$$

for some $K>0$. We integrate by parts using the ultradifferential operator of (ω)-class associated to G (see [6, p. 3483] for the definition) in (3.6) with the next formula, which follows from [6, (3.3)]:

$$e^{i(x-y)\cdot\xi}=\frac{1}{G^n(y-x)}G^n(-D_\xi)e^{i(x-y)\cdot\xi}. \tag{3.11}$$

Under the assumptions in Theorem 3.4, we estimate the kernel (3.4) as done in [12, Proposition 4.4], with the corresponding modifications for the general symbols of [4,6].

Theorem 3.5. *Let $b\in\mathrm{GS}_\rho^{m,\omega}$ and $\psi\in\mathcal{S}_\omega(\mathbb{R}^d)$ with $\|\psi\|_{L^2(\mathbb{R}^d)}=1$. If $u\in\mathcal{S}_\omega(\mathbb{R}^d)$ and $K(y',\eta',y,\eta)$ is as in (3.4), then for all $\lambda>0$ there exist $C_\lambda,\mu_\lambda>0$ such that*

$$|K(y',\eta',y,\eta)|\le C_\lambda e^{-\lambda\omega(y-y')}e^{-\lambda\omega(\eta-\eta')}e^{\mu_\lambda\omega(\eta')}e^{\max\{0,mL^2\}(\omega(y')+\omega(y))} \tag{3.12}$$

for all $(y',\eta',y,\eta)\in\mathbb{R}^{4d}$.

Moreover, if $b(z)=0$ for $z\in\Gamma\setminus\overline{B(0,R)}$ for an open conic set $\Gamma\subseteq\mathbb{R}^{2d}\setminus\{0\}$ and for some $R>0$, then for every open conic set $\Gamma'\subseteq\mathbb{R}^{2d}\setminus\{0\}$

such that $\overline{\Gamma' \cap S_{2d-1}} \subseteq \Gamma$ *(where* S_{2d-1} *denotes the unit sphere in* $\mathbb{R}^{2d}$*) we have that for all* $\lambda > 0$ *there exists* $C_\lambda > 0$ *such that*

$$|K(y', \eta', y, \eta)| \leq C_\lambda e^{-\lambda\omega(y-y')} e^{-\lambda\omega(\eta-\eta')} e^{-2\lambda\omega(y')} e^{-2\lambda\omega(\eta')} \tag{3.13}$$

for all $(y', \eta') \in \Gamma'$, $(y, \eta) \in \mathbb{R}^{2d}$.

Proof. Let $m > 0$. We use the following change of variables in the kernel (3.4)

$$x - y' = x', \qquad s - y = s'.$$

By abuse of notation, we will denote x' by x and s' by s. We have, by Theorem 3.4,

$$\begin{aligned} K(y', \eta', y, \eta) &= (2\pi)^{-2d} \int_{\mathbb{R}^{3d}} e^{i(x+y')\cdot(\xi-\eta')+i(s+y)\cdot(\eta-\xi)} \\ &\quad \times b\Big(\frac{x+y'+s+y}{2}, \xi\Big)\overline{\psi(x)}\psi(s) ds dx d\xi \\ &= (2\pi)^{-2d} e^{-iy'\cdot\eta'+iy\cdot\eta} \int_{\mathbb{R}^{3d}} b\Big(\frac{x+y'+s+y}{2}, \xi\Big)\overline{\psi(x)}\psi(s) \\ &\quad \times e^{is\cdot(\eta-\xi)} e^{ix\cdot(\xi-\eta')} e^{i\xi\cdot(y'-y)} ds dx d\xi. \end{aligned} \tag{3.14}$$

Let $G \in \mathcal{H}(\mathbb{C}^d)$ be as in (3.6). For $\ell, h \in \mathbb{N}$, $k \in \mathbb{N}_0$, we use (3.11) as follows:

$$\begin{aligned} &e^{i(s\cdot(\eta-\xi)+x\cdot(\xi-\eta')+\xi\cdot(y'-y))} \\ &= \frac{1}{G^\ell(\xi-\eta)} G^\ell(-D_s)\big[e^{i(s\cdot(\eta-\xi)+x\cdot(\xi-\eta')+\xi\cdot(y'-y))}\big] \\ &= \frac{1}{G^\ell(\xi-\eta)G^h(\eta'-\xi)} G^\ell(-D_s) e^{is\cdot(\eta-\xi)} G^h(-D_x)\big[e^{i(x\cdot(\xi-\eta')+\xi\cdot(y'-y))}\big] \\ &= \frac{1}{G^\ell(\xi-\eta)G^h(\eta'-\xi)\langle y-y'\rangle^{2k}} \\ &\quad \times G^\ell(-D_s) e^{is\cdot(\eta-\xi)} G^h(-D_x) e^{ix\cdot(\xi-\eta')} (1-\Delta_\xi)^k e^{i\xi\cdot(y'-y)}. \end{aligned}$$

We want to apply this into (3.14) and then integrate by parts in order to write

$$|K(y', \eta', y, \eta)| = (2\pi)^{-2d} \langle y-y'\rangle^{-2k} \Big| \int_{\mathbb{R}^{3d}} e^{i\xi\cdot(y'-y)} \lambda_{\ell,h,k}(y', \eta', y, \eta, s, x, \xi) ds dx d\xi \Big| \tag{3.15}$$

with

$$\begin{aligned} \lambda_{\ell,h,k}(y', \eta', y, \eta, s, x, \xi) &= (1-\Delta_\xi)^k \big[G^{-\ell}(\xi-\eta) G^{-h}(\eta'-\xi) e^{ix\cdot(\xi-\eta')} e^{is\cdot(\eta-\xi)} \\ &\quad \times G^h(D_x) G^\ell(D_s)\Big\{b\Big(\frac{x+y'+s+y}{2}, \xi\Big)\overline{\psi(x)}\psi(s)\Big\}\Big]. \end{aligned}$$

We can integrate by parts in ds and dx since $\psi \in \mathcal{S}_\omega(\mathbb{R}^d)$. To check if we can integrate by parts also in $d\xi$, $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$, we estimate

$$
\begin{aligned}
|\lambda_{\ell,h,k}| &= |(1-\partial^2_{\xi_1}-\cdots-\partial^2_{\xi_d})^k[G^{-\ell}(\xi-\eta)G^{-h}(\eta'-\xi)e^{ix\cdot(\xi-\eta')}e^{is\cdot(\eta-\xi)} \\
&\quad\times G^h(D_x)G^\ell(D_s)\Big\{b\Big(\frac{x+y'+s+y}{2},\xi\Big)\overline{\psi(x)}\psi(s)\Big\}\Big]\Big| \\
&\leq \sum_{j_1+\cdots+j_d+j_{d+1}=k}\Big|\frac{k!}{j_1!\cdots j_d! j_{d+1}!}\partial^{2j_1}_{\xi_1}\cdots\partial^{2j_d}_{\xi_d}[G^{-\ell}(\xi-\eta)G^{-h}(\eta'-\xi) \\
&\quad\times e^{ix\cdot(\xi-\eta')}e^{is\cdot(\eta-\xi)}G^h(D_x)G^\ell(D_s)\Big\{b\Big(\frac{x+y'+s+y}{2},\xi\Big)\overline{\psi(x)}\psi(s)\Big\}\Big]\Big| \\
&= \sum_{j'=0}^{k}\binom{k}{j'}\sum_{j_1+\cdots+j_d=k-j'}\frac{(k-j')!}{j_1!\cdots j_d!}\Big|\partial^{2j_1}_{\xi_1}\cdots\partial^{2j_d}_{\xi_d}[G^{-\ell}(\xi-\eta)G^{-h}(\eta'-\xi) \\
&\quad\times e^{ix\cdot(\xi-\eta')}e^{is\cdot(\eta-\xi)}G^h(D_x)G^\ell(D_s)\Big\{b\Big(\frac{x+y'+s+y}{2},\xi\Big)\overline{\psi(x)}\psi(s)\Big\}\Big]\Big|.
\end{aligned}
$$

Then, for $j=(j_1,\ldots,j_d)\in\mathbb{N}_0^d$,

$$
\begin{aligned}
|\lambda_{\ell,h,k}| &\leq \sum_{j'=0}^{k}\binom{k}{j'}\sum_{|j|=k-j'}\frac{(k-j')!}{j_1!\cdots j_d!}\sum_{\sigma_1+\cdots+\sigma_5=2j}\frac{(2j)!}{\sigma_1!\cdots\sigma_5!} \\
&\quad\times|\partial^{\sigma_1}_\xi G^{-\ell}(\xi-\eta)||\partial^{\sigma_2}_\xi G^{-h}(\eta'-\xi)||\partial^{\sigma_3}_\xi e^{ix\cdot(\xi-\eta')}||\partial^{\sigma_4}_\xi e^{is\cdot(\eta-\xi)}| \\
&\quad\times\Big|\partial^{\sigma_5}_\xi G^h(D_x)G^\ell(D_s)\Big\{b\Big(\frac{x+y'+s+y}{2},\xi\Big)\overline{\psi(x)}\psi(s)\Big\}\Big| \\
&\leq \sum_{j'=0}^{k}\binom{k}{j'}\sum_{|j|=k-j'}\frac{(2(k-j'))!}{(2j_1)!\cdots(2j_d)!}\sum_{\sigma_1+\cdots+\sigma_5=2j}\frac{(2j_1)!\cdots(2j_d)!}{\sigma_1!\cdots\sigma_5!} \\
&\quad\times|\partial^{\sigma_1}_\xi G^{-\ell}(\xi-\eta)||\partial^{\sigma_2}_\xi G^{-h}(\eta'-\xi)||x|^{|\sigma_3|}|s|^{|\sigma_4|} \\
&\quad\times\Big|\partial^{\sigma_5}_\xi G^h(D_x)G^\ell(D_s)\Big\{b\Big(\frac{x+y'+s+y}{2},\xi\Big)\overline{\psi(x)}\psi(s)\Big\}\Big| \\
&\leq \sum_{j'=0}^{k}\binom{k}{j'}\sum_{|\sigma_1+\cdots+\sigma_5|=2(k-j')}\frac{(2(k-j'))!}{\sigma_1!\cdots\sigma_5!}|\partial^{\sigma_1}_\xi G^{-\ell}(\xi-\eta)||\partial^{\sigma_2}_\xi G^{-h}(\eta'-\xi)| \\
&\quad\times|x|^{|\sigma_3|}|s|^{|\sigma_4|}\Big|\partial^{\sigma_5}_\xi G^h(D_x)G^\ell(D_s)\Big\{b\Big(\frac{x+y'+s+y}{2},\xi\Big)\overline{\psi(x)}\psi(s)\Big\}\Big|.
\end{aligned}
$$

We take $M\in\mathbb{N}$, to be determined later. By (3.10) there exist $C_M, C_1, C_3>0$ so that

$$
|\partial^{\sigma_1}_\xi G^{-\ell}(\xi-\eta)|\leq C_1^\ell C_M e^{(M+1)L^2\varphi^*\left(\frac{|\sigma_1|}{(M+1)L^2}\right)}e^{-\ell C_3\omega(\xi-\eta)};
$$

$$
|\partial^{\sigma_2}_\xi G^{-h}(\eta'-\xi)|\leq C_1^h C_M e^{(M+1)L^2\varphi^*\left(\frac{|\sigma_2|}{(M+1)L^2}\right)}e^{-hC_3\omega(\eta'-\xi)}.
$$

By Lemma 2.2(i), it holds that for $M\in\mathbb{N}$,

$$
\begin{aligned}
|x|^{|\sigma_3|} &\leq e^{(M+1)L^2\varphi^*\left(\frac{|\sigma_3|}{(M+1)L^2}\right)}e^{(M+1)L^2\omega(\langle x\rangle)} \\
&\leq e^{(M+1)L^2\varphi^*\left(\frac{|\sigma_3|}{(M+1)L^2}\right)}e^{(M+1)L^3\omega(x)}e^{(M+1)L^3}.
\end{aligned}
$$

Analogously,

$$
|s|^{|\sigma_4|}\leq e^{(M+1)L^2\varphi^*\left(\frac{|\sigma_4|}{(M+1)L^2}\right)}e^{(M+1)L^3\omega(s)}e^{(M+1)L^3}.
$$

We also have

$$\Big|\partial_\xi^{\sigma_5} G^h(D_x)G^\ell(D_s)\Big\{b\Big(\frac{x+y'+s+y}{2},\xi\Big)\overline{\psi(x)}\psi(s)\Big\}\Big| \le \sum_{\delta,\tau\in\mathbb{N}_0^d} |a_\delta||b_\tau|$$
$$\times \sum_{\substack{\delta_1+\delta_2=\delta\\ \tau_1+\tau_2=\tau}} \frac{\delta!}{\delta_1!\delta_2!}\frac{\tau!}{\tau_1!\tau_2!}\Big|D_x^{\delta_1}D_s^{\tau_1}\partial_\xi^{\sigma_5} b\Big(\frac{x+y'+s+y}{2},\xi\Big)\Big||D_x^{\delta_2}\overline{\psi(x)}||D_s^{\tau_2}\psi(s)|,$$

where a_δ and b_τ correspond to the coefficients of G^h and of G^ℓ in (3.8). Moreover, by (3.9) there exists $C_4>0$ so that they can be estimated by

$$|a_\delta| \le e^{hC_4}e^{-hC_4\varphi^*\left(\frac{|\delta|}{hC_4}\right)};$$
$$|b_\tau| \le e^{\ell C_4}e^{-\ell C_4\varphi^*\left(\frac{|\tau|}{\ell C_4}\right)}.$$

Since $b\in \mathrm{GS}_\rho^{m,\omega}$ and $\psi\in\mathcal{S}_\omega(\mathbb{R}^d)$, for the above $M\in\mathbb{N}$ and for all $\mu>0$ there exists $C_{M,\mu}>0$ such that

$$\begin{aligned}\Big|D_x^{\delta_1}D_s^{\tau_1}\partial_\xi^{\sigma_5} b\Big(\frac{x+y'+s+y}{2},\xi\Big)\Big||D_x^{\delta_2}\overline{\psi(x)}||D_s^{\tau_2}\psi(s)|\\ \le C_{M,\mu}e^{4(M+1)L^2\rho\varphi^*\left(\frac{|\delta_1+\tau_1+\sigma_5|}{4(M+1)L^2}\right)}e^{m\omega(\frac{x+y'+s+y}{2},\xi)}\\ \times e^{ML^2\varphi^*\left(\frac{|\delta_2|}{ML^2}\right)}e^{-\mu\omega(x)}e^{ML^2\varphi^*\left(\frac{|\tau_2|}{ML^2}\right)}e^{-\mu\omega(s)}.\end{aligned}$$

Similarly as in (2.2), we have

$$\begin{aligned}e^{m\omega(\frac{x+y'+s+y}{2},\xi)} &\le e^{mL\omega(\frac{x+y'+s+y}{2})}e^{mL\omega(\xi)}e^{mL}\\ &\le e^{mL\omega(2\max\{|x|,|y'|,|s|,|y|\})}e^{mL\omega(\xi)}e^{mL}\\ &\le e^{mL^2\omega(x)}e^{mL^2\omega(y')}e^{mL^2\omega(s)}e^{mL^2\omega(y)}e^{mL\omega(\xi)}e^{mL^2+mL}.\end{aligned}$$

Since $0<\rho\le 1$ and $\varphi^*(x)/x$ is increasing, we obtain

$$e^{4(M+1)L^2\rho\varphi^*\left(\frac{|\delta_1+\tau_1+\sigma_5|}{4(M+1)L^2}\right)} \le e^{ML^2\varphi^*\left(\frac{|\delta_1|}{ML^2}\right)}e^{ML^2\varphi^*\left(\frac{|\tau_1|}{ML^2}\right)}e^{(M+1)L^2\varphi^*\left(\frac{|\sigma_5|}{(M+1)L^2}\right)}.$$

Then, as $|\sigma_1+\cdots+\sigma_5| = 2(k-j')\le 2k$,

$$e^{(M+1)L^2\varphi^*\left(\frac{|\sigma_1|}{(M+1)L^2}\right)}\cdots e^{(M+1)L^2\varphi^*\left(\frac{|\sigma_5|}{(M+1)L^2}\right)} \le e^{(M+1)L^2\varphi^*\left(\frac{2k}{(M+1)L^2}\right)},$$

and also

$$\begin{aligned}\sum_{\substack{\delta_1+\delta_2=\delta\\ \tau_1+\tau_2=\tau}} \frac{\delta!}{\delta_1!\delta_2!}\frac{\tau!}{\tau_1!\tau_2!}e^{ML^2\varphi^*\left(\frac{|\delta_1|}{ML^2}\right)}e^{ML^2\varphi^*\left(\frac{|\delta_2|}{ML^2}\right)}e^{ML^2\varphi^*\left(\frac{|\tau_1|}{ML^2}\right)}e^{ML^2\varphi^*\left(\frac{|\tau_2|}{ML^2}\right)}\\ \le 2^{|\delta+\tau|}e^{ML^2\varphi^*\left(\frac{|\delta|}{ML^2}\right)}e^{ML^2\varphi^*\left(\frac{|\tau|}{ML^2}\right)}.\end{aligned}$$

On the other hand, we have

$$\sum_{j'=0}^{k}\binom{k}{j'}\sum_{|\sigma_1+\cdots+\sigma_5|=2(k-j')}\frac{(2(k-j'))!}{\sigma_1!\cdots\sigma_5!}$$
$$=\sum_{j'=0}^{k}\binom{k}{j'}5^{2(k-j')}=5^{2k}\left(1+\frac{1}{5^2}\right)^k=(26)^k.$$

By all these estimates, we have that for all $M\in\mathbb{N}$ and $\mu>0$ there exists $C'_{M,\mu}>0$ such that

$$\begin{aligned}|\lambda_{\ell,h,k}|\le{}& C'_{M,\mu}(26)^k e^{(M+1)L^2\varphi^*\left(\frac{2k}{(M+1)L^2}\right)}(C_1e^{C_4})^{\ell+h}e^{-\ell C_3\omega(\xi-\eta)}e^{-hC_3\omega(\eta'-\xi)}\\ &\times e^{((M+1)L^3+mL^2-\mu)(\omega(x)+\omega(s))}e^{mL^2\omega(y')}e^{mL^2\omega(y)}e^{mL\omega(\xi)}\\ &\times\sum_{\delta,\tau\in\mathbb{N}_0^d}2^{|\delta|}e^{ML^2\varphi^*\left(\frac{|\delta|}{ML^2}\right)-hC_4\varphi^*\left(\frac{|\delta|}{hC_4}\right)}2^{|\tau|}e^{ML^2\varphi^*\left(\frac{|\tau|}{ML^2}\right)-\ell C_4\varphi^*\left(\frac{|\tau|}{\ell C_4}\right)}.\end{aligned}$$

For any $\ell,h\in\mathbb{N}$, let $M\in\mathbb{N}$ satisfy

$$M\ge C_4\max\{\ell,h\}.$$

Then, by Lemma 2.2(iii), we obtain

$$\begin{aligned}e^{ML^2\varphi^*\left(\frac{|\delta|}{ML^2}\right)}e^{-hC_4\varphi^*\left(\frac{|\delta|}{hC_4}\right)}&\le e^{hC_4L^2\varphi^*\left(\frac{|\delta|}{hC_4L^2}\right)}e^{-hC_4\varphi^*\left(\frac{|\delta|}{hC_4}\right)}\\ &=\left(\frac{1}{e^2}\right)^{|\delta|}e^{2|\delta|+hC_4L^2\varphi^*\left(\frac{|\delta|}{hC_4L^2}\right)}e^{-hC_4\varphi^*\left(\frac{|\delta|}{hC_4}\right)}\\ &\le\left(\frac{1}{e^2}\right)^{|\delta|}e^{hC_4L^2+hC_4L}.\end{aligned}$$

Analogously,

$$e^{ML^2\varphi^*\left(\frac{|\tau|}{ML^2}\right)}e^{-\ell C_4\varphi^*\left(\frac{|\tau|}{\ell C_4}\right)}\le\left(\frac{1}{e^2}\right)^{|\tau|}e^{\ell C_4L^2+\ell C_4L}.$$

The series

$$\sum_{\delta,\tau\in\mathbb{N}_0^d}\left(\frac{2}{e^2}\right)^{|\delta+\tau|}\le\sum_{\delta,\tau\in\mathbb{N}_0^d}\left(\frac{1}{e}\right)^{|\delta+\tau|}$$

converges (see for instance [6, (3.6)]). Again Lemma 2.2(iii) yields, as $26<e^4$, that

$$(26)^k e^{(M+1)L^2\varphi^*\left(\frac{2k}{(M+1)L^2}\right)}\le e^{(M+1)L^2+(M+1)L}e^{(M+1)\varphi^*\left(\frac{2k}{M+1}\right)}.$$

Then, for all $\mu>0$ there exists $C''_{\mu,M}>0$ such that (for M so that $M\ge C_4\max\{\ell,h\}$)

$$\begin{aligned} |\lambda_{\ell,h,k}| \leq \ & C''_{\mu,M} e^{(M+1)\varphi^*\left(\frac{2k}{M+1}\right)} (C_1 e^{C_4+C_4L+C_4L^2})^{\ell+h} \\ & \times e^{((M+1)L^3+mL^2-\mu)(\omega(x)+\omega(s))} e^{mL^2\omega(y')} \\ & \times e^{mL^2\omega(y)} e^{mL\omega(\xi)} e^{-\ell C_3\omega(\xi-\eta)} e^{-hC_3\omega(\eta'-\xi)}. \end{aligned} \tag{3.16}$$

By estimating

$$e^{-\ell C_3\omega(\xi-\eta)} e^{-hC_3\omega(\eta'-\xi)} \leq e^{-\ell\frac{C_3}{L}\omega(\eta)} e^{\left(-h\frac{C_3}{L}+\ell C_3\right)\omega(\xi)} e^{hC_3\omega(\eta')} e^{(\ell+h)C_3},$$

given $\ell \in \mathbb{N}$ we take $h > \ell L + mL^2/C_3$, and we choose $\mu > (M+1)L^3 + mL^2$ so that $|\lambda_{\ell,h,k}|$ is estimated by a function that belongs to $L^1(\mathbb{R}^{2d}_{s,x})$ for all $k \in \mathbb{N}_0$, and that goes to 0 as $|\xi| \to +\infty$. From this, it follows that we can integrate by parts also in $d\xi$, putting the ∂_ξ-derivatives inside the integral in $dsdx$. Therefore, since

$$e^{-\ell\frac{C_3}{L}\omega(\eta)} e^{hC_3\omega(\eta')} \leq e^{-\ell\frac{C_3}{L^2}\omega(\eta-\eta')} e^{(\ell\frac{C_3}{L}+hC_3)\omega(\eta')} e^{\ell\frac{C_3}{L}},$$

there exists $C_{M,\ell,h,\mu} > 0$ such that, by (3.15)

$$\begin{aligned} |K(y',\eta',y,\eta)| \leq\ & C_{M,\ell,h,\mu} \langle y-y'\rangle^{-2k} e^{(M+1)\varphi^*\left(\frac{2k}{M+1}\right)} e^{mL^2(\omega(y)+\omega(y'))} \\ & \times e^{-\ell\frac{C_3}{L^2}\omega(\eta-\eta')} e^{(\ell\frac{C_3}{L}+hC_3)\omega(\eta')} \\ & \times \int_{\mathbb{R}^{3d}} e^{((M+1)L^3+mL^2-\mu)(\omega(x)+\omega(s))} e^{(mL+\ell C_3-h\frac{C_3}{L})\omega(\xi)} dsdxd\xi. \end{aligned}$$

We take the infimum on $k \in \mathbb{N}_0$ and we use Lemma 2.2(ii) (with $\sigma = 2$) to get for some $C'_{M,\ell,h,\mu} > 0$,

$$\begin{aligned} |K(y',\eta',y,\eta)| \leq\ & C'_{M,\ell,h,\mu} e^{-M\omega(\langle y-y'\rangle)} e^{mL^2(\omega(y)+\omega(y'))} e^{-\ell\frac{C_3}{L^2}\omega(\eta-\eta')} e^{(\ell\frac{C_3}{L}+hC_3)\omega(\eta')} \\ & \times \int_{\mathbb{R}^{3d}} e^{((M+1)L^3+mL^2-\mu)(\omega(x)+\omega(s))} e^{(\ell C_3+mL-h\frac{C_3}{L})\omega(\xi)} dsdxd\xi. \end{aligned} \tag{3.17}$$

Given $\ell \in \mathbb{N}$, the integrals are convergent by the same selection as before ($h > \ell L + mL^2/C_3$ and $\mu > (M+1)L^3 + mL^2$). Therefore, for every $\lambda > 0$ there exist $C_\lambda, \mu_\lambda > 0$ such that

$$|K(y',\eta',y,\eta)| \leq C_\lambda e^{-\lambda\omega(y-y')} e^{-\lambda\omega(\eta-\eta')} e^{\mu_\lambda\omega(\eta')} e^{mL^2(\omega(y')+\omega(y))}.$$

This shows formula (3.12). We observe that, given $\ell \in \mathbb{N}$, if we take $h > 0$ as before ($h > \ell L + mL^2/C_3$) and $M \geq C_4 \max\{\ell, h\}$ satisfying also $M \geq \ell + mL^3$ then

$$e^{-M\omega(\langle y-y'\rangle)} \leq e^{-\ell\omega(y-y')} e^{-mL^2\omega(y)+mL^3\omega(y')+mL^3}.$$

By setting $\mu > 0$ as before ($\mu > (M+1)L^3 + mL^2$), we get from (3.17) that for all $\lambda > 0$ there are $C_\lambda, \mu_\lambda > 0$ such that

$$|K(y',\eta',y,\eta)| \leq C_\lambda e^{-\lambda\omega(y-y')} e^{-\lambda\omega(\eta-\eta')} e^{\mu_\lambda\omega(\eta')} e^{(mL^2+mL^3)\omega(y')}. \tag{3.18}$$

For the second part, we follow closely the proof of [37, Proposition 3.7]. By (3.16), we can proceed as in (3.17) using (3.15) and taking the infimum on $k \in \mathbb{N}_0$ to find $C''_{M,\ell,h,\mu} > 0$ such that

$$|K(y',\eta',y,\eta)| \leq C''_{M,\ell,h,\mu} e^{-M\omega(\langle y-y'\rangle)} e^{mL^3\omega(y-y')} e^{(mL^2+mL^3)\omega(y')} \times \int_{\mathbb{R}^{3d}} e^{((M+1)L^3+mL^2-\mu)(\omega(x)+\omega(s))} \times e^{mL\omega(\xi)} e^{-\ell C_3\omega(\xi-\eta)} e^{-hC_3\omega(\eta'-\xi)} dsdxd\xi \tag{3.19}$$

for all $(y',\eta',y,\eta) \in \mathbb{R}^{4d}$. Now, we assume $b(z) = 0$ for $z \in \Gamma \setminus \overline{B(0,R)}$. We set

$$D_{y',y} := \Big\{(x,s,\xi) \in \mathbb{R}^{3d} : \Big(\frac{x+y'+s+y}{2}, \xi\Big) \in (\mathbb{R}^{2d}\setminus\Gamma) \cup \overline{B(0,R)}\Big\}.$$

We want to estimate $|K(y',\eta',y,\eta)|$ for $(y',\eta') \in \Gamma'$, a conic subset of Γ with $\overline{\Gamma' \cap S_{2d-1}} \subseteq \Gamma$, and $(y,\eta) \in \mathbb{R}^{2d}$. By formula [37, (3.19)], there exists $\varepsilon > 0$ such that

$$\Big|\frac{(y',\eta')}{|(y',\eta')|} - \frac{\Big(\frac{x+y'+s+y}{2},\xi\Big)}{|(y',\eta')|}\Big| \geq \varepsilon,$$

for all $(y',\eta') \in \Gamma'$, $|(y',\eta')| \geq 2R$, $(x,s,\xi) \in D_{y',y}$, $(y,\eta) \in \mathbb{R}^{2d}$. From this, it follows, as $|y'-x-s-y| \leq 2\max\{|y'-y|, |x+s|\}$,

$$e^{\omega(\varepsilon(y',\eta'))} \leq e^{\omega\left(\frac{y'-x-s-y}{2},\eta'-\xi\right)} \leq e^{L\omega(y'-y)+L^2\omega(x)+L^2\omega(s)} e^{L\omega(\eta'-\xi)} e^{L^2+L}.$$

Thus, for that $\varepsilon > 0$, there exist $C_\varepsilon, L_\varepsilon > 0$ such that

$$e^{\frac{1}{2}\omega(y')} e^{\frac{1}{2}\omega(\eta')} \leq e^{\omega(y',\eta')} \leq C_\varepsilon e^{L_\varepsilon(\omega(y'-y)+\omega(x)+\omega(s)+\omega(\eta'-\xi))}.$$

Hence there exist $C'_\varepsilon, L'_\varepsilon > 0$ such that

$$e^{-\omega(\eta'-\xi)} \leq C'_\varepsilon e^{-L'_\varepsilon\omega(y')-L'_\varepsilon\omega(\eta')} e^{\omega(y-y')+\omega(x)+\omega(s)}. \tag{3.20}$$

Set $h = Hh' = h' + (H-1)h'$ for some $H > 1$ and $h' > 0$ to be determined later. Therefore by (3.20)

$$\begin{aligned} &e^{-(H-1)h'C_3\omega(\eta'-\xi)} e^{-h'C_3\omega(\eta'-\xi)} \\ &\quad\leq (C'_\varepsilon)^{(H-1)h'C_3} e^{-(H-1)h'C_3L'_\varepsilon\omega(y')} \\ &\qquad\times e^{-(H-1)h'C_3L'_\varepsilon\omega(\eta')} e^{(H-1)h'C_3(\omega(y-y')+\omega(x)+\omega(s))} \\ &\qquad\times e^{-h'\frac{C_3}{L}\omega(\xi)} e^{h'C_3\omega(\eta')} e^{h'C_3}. \end{aligned}$$

Since

$$\begin{aligned} e^{-\ell C_3\omega(\xi-\eta)} &\leq e^{\ell C_3} e^{-\ell\frac{C_3}{L}\omega(\eta-\eta')} e^{\ell C_3\omega(\eta'-\xi)} \\ &\leq e^{\ell C_3} e^{-\ell\frac{C_3}{L}\omega(\eta-\eta')} e^{\ell C_3L\omega(\eta')} e^{\ell C_3L\omega(\xi)} e^{\ell C_3L} \end{aligned}$$

we then find, from (3.19), $C'''_{M,\ell,h,\mu} > 0$ such that

$$|K(y',\eta',y,\eta)| \leq C'''_{M,\ell,h,\mu} e^{-M\omega(\langle y-y'\rangle)}$$

$$\times\, e^{((H-1)h'C_3+mL^3)\omega(y-y')}e^{(-(H-1)h'C_3L'_\varepsilon+mL^2+mL^3)\omega(y')}$$
$$\times\, e^{-\ell\frac{C_3}{L}\omega(\eta-\eta')}e^{(-(H-1)h'C_3L'_\varepsilon+h'C_3+\ell C_3L)\omega(\eta')}$$
$$\times \int_{\mathbb{R}^{3d}} e^{((M+1)L^3+mL^2+(H-1)h'C_3-\mu)(\omega(x)+\omega(s))}$$
$$\times\, e^{\left(-h'\frac{C_3}{L}+\ell C_3L+mL\right)\omega(\xi)}dsdxd\xi.$$

Given $\ell\in\mathbb{N}$ arbitrary, we denote $\lambda=\ell\frac{C_3}{L}>0$. We put $h'>0$ such that

$$\left(-h'\frac{C_3}{L}+\ell C_3L+mL\right)\omega(\xi)\le-\omega(\xi),$$

and then $H>1$ with

$$(-(H-1)h'L'_\varepsilon C_3+mL^2+mL^3)\omega(y')\le-2\lambda\omega(y');$$
$$(-(H-1)h'L'_\varepsilon C_3+h'C_3+\ell C_3L)\omega(\eta')\le-2\lambda\omega(\eta').$$

We take $M\in\mathbb{N}$ (which satisfies $M\ge C_4\max\{\ell,Hh'\}$) such that

$$-M\omega(\langle y-y'\rangle)+((H-1)h'C_3+mL^3)\omega(y-y')\le-\lambda\omega(y-y'),$$

and finally $\mu>0$ large enough so that

$$((M+1)L^3+mL^2+(H-1)h'C_3-\mu)(\omega(x)+\omega(s))\le-(\omega(x)+\omega(s)).$$

With these choices, the convergence of the integrals is guaranteed, and also (3.13) is satisfied for all $(y',\eta')\in\Gamma'$, $|(y',\eta')|\ge 2R$ and $(y,\eta)\in\mathbb{R}^{2d}$. The proof for $|(y',\eta')|\le 2R$ follows directly from (3.18). This completes the proof. □

4. The Weyl Wave Front Set

In the present section we introduce a new global wave front set given in terms of Weyl quantizations in the ultradifferentiable setting, similarly to the one introduced by Hörmander [28, Definition 2.1] in the classical setting. We have some restrictions on the weight functions since the definition is based on the construction of parametrices given in [4, 22]. We also show that in the definition it is enough to use symbols of order zero, so we extend [37, Proposition 2.7], which is crucial for the next sections.

Definition 4.1. Given $a\in\mathrm{GS}^{m,\omega}_\rho$, we say that $z_0\in\mathbb{R}^{2d}\backslash\{0\}$ is non-characteristic for a if there exist a Gevrey weight function σ with $\omega(t^{1/\rho})=o(\sigma(t))$ as $t\to+\infty$, $C_1,C_2>0$, $n\in\mathbb{N}$, $R\ge1$, and an open conic set $\Gamma\subset\mathbb{R}^{2d}\setminus\{0\}$ with $z_0\in\Gamma$, satisfying

$$|a(z)| \geq C_1 e^{m\omega(z)}, \quad \text{and} \tag{4.1}$$

$$|D^\alpha a(z)| \leq C_2^{|\alpha|} \langle z \rangle^{-\rho|\alpha|} e^{\frac{1}{n}\varphi^*_\sigma(n|\alpha|)} |a(z)|, \tag{4.2}$$

for all $\alpha \in \mathbb{N}_0^{2d}$, $z \in \Gamma$, $|z| \geq R$.

We recall that there are non-quasianalytic weight functions in the sense of [16] that cannot be dominated by any subadditive function that satisfies property (β) (see [24]). This motivates the following definition.

Definition 4.2. Fix $0 < \rho \leq 1$. A weight function ω is called ρ-regular if for all $m \in \mathbb{R}$ there exists $a \in \mathrm{GS}^{m,\omega}_\rho$ such that for some Gevrey weight function σ with $\omega(t^{1/\rho}) = o(\sigma(t))$ as $t \to +\infty$, the inequalities (4.1) and (4.2) hold for all $z \in \mathbb{R}^{2d}$ with $|z| \geq R$, for some $R \geq 1$.

If follows from Example 3.3 that Gevrey weights $\omega(t) = t^a$, $0 < a < 1/2$, are $(1-a)$-regular weight functions.

Definition 4.3. Let ω be a weight function, $0 < \rho \leq 1$ and $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$. We say that $z \in \mathbb{R}^{2d} \setminus \{0\}$ is not in the Weyl wave front set $\mathrm{WF}^\omega_\rho(u)$ of u if there exist $m \in \mathbb{R}$ and $a \in \mathrm{GS}^{m,\omega}_\rho$ such that $a^w(x, D)u \in \mathcal{S}_\omega(\mathbb{R}^d)$ and z is non-characteristic for a.

We need to introduce the notion of conic support [37, Definition 2.1] before the next result.

Definition 4.4. Given $u \in \mathcal{S}'_\omega(\mathbb{R}^{2d})$, the conic support of u, denoted by $\mathrm{conesupp}\,(u)$, is defined as the set of all $x \in \mathbb{R}^{2d} \setminus \{0\}$ such that any conic open set $\Gamma \subseteq \mathbb{R}^{2d} \setminus \{0\}$ that contains x satisfies that $\overline{\mathrm{supp}\,(u) \cap \Gamma}$ is not a compact set in $\mathbb{R}^{2d}$.

The following lemma is [23, Lemma 4].

Lemma 4.5. *Given a weight function σ and two cones $\Gamma, \Gamma' \subseteq \mathbb{R}^{2d} \setminus \{0\}$ such that $\overline{\Gamma' \cap S_{2d-1}} \subseteq \Gamma$, there exists $\chi \in C^\infty(\mathbb{R}^{2d})$ such that $0 \leq \chi \leq 1$, $\mathrm{supp}\,(\chi) \subseteq \Gamma$, $\chi(z) = 1$ for $z \in \Gamma'$ with $|z| \geq 1$ and for every $k \in \mathbb{N}$ there is $C_k > 0$ such that*

$$|D^\alpha \chi(z)| \leq C_k \langle z \rangle^{-|\alpha|} e^{k\varphi^*_\sigma\left(\frac{|\alpha|}{k}\right)}, \qquad \alpha \in \mathbb{N}_0^{2d},\ z \in \mathbb{R}^{2d}.$$

Moreover, if ω satisfies $\omega(t^{1/\rho}) = o(\sigma(t))$ as $t \to \infty$, for some $0 < \rho \leq 1$, then $\chi \in \mathrm{GS}^{0,\omega}_\rho$.

Now we show that in Definition 4.3, similarly as in [37, Proposition 2.7], the symbol can be taken of order zero for regular weight functions.

Proposition 4.6. *Let ω be a ρ-regular weight function, for some $0 < \rho \leq 1$, $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$, and $0 \neq z_0 \notin \mathrm{WF}^\omega_\rho(u)$. There exist $b \in \mathrm{GS}^{0,\omega}_\rho$ and an open conic set $\Gamma \subset \mathbb{R}^{2d} \setminus \{0\}$ such that $z_0 \in \Gamma$, $0 \leq b \leq 1$, $b(z) = 1$ for $z \in \Gamma$ with $|z| \geq 1$ and $b^w(x, D)u \in \mathcal{S}_\omega(\mathbb{R}^d)$.*

Proof. Since $z_0 \notin \mathrm{WF}^{\omega}_{\rho}(u)$, there are $m \in \mathbb{R}$ and $a \in \mathrm{GS}^{m,\omega}_{\rho}$ such that $a^w(x,D)u \in \mathcal{S}_{\omega}(\mathbb{R}^d)$, a Gevrey weight function σ such that $\omega(t^{1/\rho}) = o(\sigma(t))$ as $t \to \infty$, $C_1, C_2 > 0$, $n \in \mathbb{N}$, $R \geq 1$, and an open conic set $\Gamma \subset \mathbb{R}^{2d} \setminus \{0\}$ such that $z_0 \in \Gamma$, and a satisfies (4.1) and (4.2) for all $z \in \Gamma$ with $|z| \geq R$. It is not restrictive to assume $C_2 \geq 1$.

Without losing generality, we can assume that Γ is connected (if Γ is not connected, then we take the connected component in which z_0 lies). Then, since (4.1) is satisfied it is not restrictive to assume [40]

$$a(z) \geq 0, \qquad z \in \Gamma,\ |z| \geq R.$$

Moreover, we have

$$a(z) \geq C_1 e^{m\omega(z)}, \qquad z \in \Gamma,\ |z| \geq R. \tag{4.3}$$

Since ω is a ρ-regular weight there is a symbol $a_0 \in \mathrm{GS}^{m,\omega}_{\rho}$ and a Gevrey weight function that without loss of generality we can assume to be σ (if not, we take the minimum of the two Gevrey weights, which is also a Gevrey weight) such that for the same $C_1, C_2 > 0$, $n \in \mathbb{N}$, $R \geq 1$, formulas (4.1) and (4.2) are satisfied for a_0, for all $z \in \mathbb{R}^{2d}$ with $|z| \geq R$. As $\mathbb{R}^{2d} \setminus B(0,R)$ is connected, $a_0(z) \geq 0$ for all $z \in \mathbb{R}^{2d}$, $|z| \geq R$ and also

$$a_0(z) \geq C_1 e^{m\omega(z)}, \qquad |z| \geq R. \tag{4.4}$$

Let $\Gamma', \Gamma'' \subset \mathbb{R}^{2d} \setminus \{0\}$ be open conic sets such that $z_0 \in \Gamma''$, $\overline{\Gamma'' \cap S_{2d-1}} \subset \Gamma'$ and $\overline{\Gamma' \cap S_{2d-1}} \subset \Gamma$. For the weight function σ, let χ be as in Lemma 4.5 for Γ and Γ'. By proceeding in a similar way for Γ' and Γ'', we can obtain $b \in \mathrm{GS}^{0,\omega}_{\rho}$ with $0 \leq b \leq 1$, $\mathrm{supp}\,(b) \subseteq \Gamma'$, $b(z) = 1$ for $z \in \Gamma''$ with $|z| \geq 1$.

Now, we set

$$b_0(z) := \chi(z)a(z) + (1 - \chi(z))a_0(z).$$

It is clear that $b_0 \in \mathrm{GS}^{m,\omega}_{\rho}$ since $a, a_0 \in \mathrm{GS}^{m,\omega}_{\rho}$, $\chi \in \mathrm{GS}^{0,\omega}_{\rho}$. For any $z \notin \Gamma$, we have that $\chi(z) = 0$ and therefore (since a_0 satisfies (4.1) for all $|z| \geq R$),

$$|b_0(z)| = |a_0(z)| \geq C_1 e^{m\omega(z)}, \qquad z \notin \Gamma,\ |z| \geq R.$$

On the other hand, as $a(z), a_0(z) \geq 0$ for all $z \in \Gamma$ with $|z| \geq R$, and $0 \leq \chi \leq 1$, it follows $b_0(z) \geq 0$. Furthermore, from (4.3) and (4.4),

$$b_0(z) = \chi(z)a(z) + (1 - \chi(z))a_0(z) \geq C_1 e^{m\omega(z)}, \qquad z \in \Gamma,\ |z| \geq R.$$

Hence, we obtain

$$|b_0(z)| \geq C_1 e^{m\omega(z)}, \qquad |z| \geq R. \tag{4.5}$$

This obviously implies condition (i) of Theorem 3.2 for b_0. Since χ is as in Lemma 4.5, there exists $C > 0$ such that, for the previous $n \in \mathbb{N}$,

$$|D^{\alpha}\chi(z)| \leq C\langle z\rangle^{-|\alpha|} e^{\varphi^*_{\sigma}(|\alpha|)} \leq C\langle z\rangle^{-\rho|\alpha|} e^{\frac{1}{n}\varphi^*_{\sigma}(n|\alpha|)},$$

for all $\alpha \in \mathbb{N}_0^{2d}$, $z \in \mathbb{R}^{2d}$. Therefore, as a, a_0 satisfy (4.2) for $z \in \Gamma$ with $|z| \geq R$, by Leibniz rule we have

$$\begin{aligned}|D^\alpha b_0(z)| &\leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}|D^\beta\chi(z)||D^{\alpha-\beta}a(z)| + \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}|D^\beta(1-\chi)(z)||D^{\alpha-\beta}a_0(z)|\\ &\leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}C\langle z\rangle^{-\rho|\beta|}e^{\frac{1}{n}\varphi^*_\sigma(n|\beta|)}\\ &\quad\times C_2^{|\alpha-\beta|}\langle z\rangle^{-\rho|\alpha-\beta|}e^{\frac{1}{n}\varphi^*_\sigma(n|\alpha-\beta|)}(|a(z)|+|a_0(z)|).\end{aligned}$$

Since $a, a_0 \in \mathrm{GS}^{m,\omega}_\rho$ there exists $C' > 0$ such that (we observe that $\sum_{\beta\leq\alpha}\binom{\alpha}{\beta} = 2^{|\alpha|}$)

$$|D^\alpha b_0(z)| \leq C(2C_2)^{|\alpha|}\langle z\rangle^{-\rho|\alpha|}e^{\frac{1}{n}\varphi^*_\sigma(n|\alpha|)}2C'e^{m\omega(z)}.$$

We consider $D = 2C_2\max\{1, \frac{2CC'}{C_1}\} > 0$. Then from (4.5) we obtain

$$\begin{aligned}|D^\alpha b_0(z)| &\leq D^{|\alpha|}\langle z\rangle^{-\rho|\alpha|}e^{\frac{1}{n}\varphi^*_\sigma(n|\alpha|)}C_1e^{m\omega(z)}\\ &\leq D^{|\alpha|}\langle z\rangle^{-\rho|\alpha|}e^{\frac{1}{n}\varphi^*_\sigma(n|\alpha|)}|b_0(z)|\end{aligned}$$

for all $\alpha \in \mathbb{N}_0^{2d}$ and $z \in \Gamma$, $|z| \geq R$. On the other hand, if $z \notin \Gamma$, then by construction $b_0 = a_0$, which satisfies (4.2). Hence b_0 satisfies condition (ii) of Theorem 3.2 for all $z \in \mathbb{R}^{2d}$ with $|z| \geq R$.

Thus, there exists $c \in \mathrm{GS}^{|m|,\omega}_\rho$ such that

$$c\#b_0 = 1 + s, \qquad \text{for some } \; s \in \mathcal{S}_\omega(\mathbb{R}^{2d}).$$

Therefore

$$\begin{aligned}b &= b\#c\#b_0 - b\#s\\ &= b\#c\#(b_0 - a) + b\#c\#a - b\#s.\end{aligned} \tag{4.6}$$

We claim that $b^w(x, D)u \in \mathcal{S}_\omega(\mathbb{R}^d)$. Since $\mathrm{supp}\,(b) \subseteq \Gamma'$ and

$$b_0 - a = \chi a + (1-\chi)a_0 - a = (1-\chi)(a_0 - a)$$

vanishes for $z \in \Gamma'$, $|z| \geq 1$ (because $\chi(z) = 1$) we deduce that

$$E := \mathrm{supp}\,(b) \cap \mathrm{supp}\,(b_0 - a)$$

is a compact set. This implies

$$b\#c\#(b_0 - a) \in \mathcal{S}_\omega(\mathbb{R}^{2d}).$$

Indeed, let $\widetilde{\chi} \in \mathcal{S}_\omega(\mathbb{R}^{2d})$ with compact support with $\widetilde{\chi} = 1$ on E. Then $b\#c\#(b_0 - a)$ has the same asymptotic expansion of $b\#c\#(\widetilde{\chi}(b_0 - a))$. By [6, Proposition 4.3] we deduce

$$b^w(x, D)c^w(x, D)(b_0 - a)^w(x, D) = b^w(x, D)c^w(x, D)(\widetilde{\chi}(b_0 - a))^w(x, D) + R, \tag{4.7}$$

for a globally ω-regularizing operator $R:\mathcal{S}'_\omega(\mathbb{R}^d)\to\mathcal{S}_\omega(\mathbb{R}^d)$. Since $\widetilde{\chi}(b_0-a)$ has compact support, we have

$$(\widetilde{\chi}(b_0-a))^w(x,D)u\in\mathcal{S}_\omega(\mathbb{R}^d),\qquad u\in\mathcal{S}'_\omega(\mathbb{R}^d).$$

The continuity of the Weyl operator yields

$$c^w(x,D)(\widetilde{\chi}(b_0-a))^w(x,D)u\in\mathcal{S}_\omega(\mathbb{R}^d),\qquad u\in\mathcal{S}'_\omega(\mathbb{R}^d),$$

and

$$b^w(x,D)c^w(x,D)(\widetilde{\chi}(b_0-a))^w(x,D)u\in\mathcal{S}_\omega(\mathbb{R}^d),\qquad u\in\mathcal{S}'_\omega(\mathbb{R}^d).$$

Hence, by (4.7),

$$b^w(x,D)c^w(x,D)(b_0-a)^w(x,D)u\in\mathcal{S}_\omega(\mathbb{R}^d),\qquad u\in\mathcal{S}'_\omega(\mathbb{R}^d).$$

Moreover, since $s\in\mathcal{S}_\omega(\mathbb{R}^{2d})$, we have

$$b^w(x,D)s^w(x,D)u\in\mathcal{S}_\omega(\mathbb{R}^d),\qquad u\in\mathcal{S}'_\omega(\mathbb{R}^d).$$

By assumption $a^w(x,D)u\in\mathcal{S}_\omega(\mathbb{R}^d)$, so

$$b^w(x,D)c^w(x,D)a^w(x,D)u\in\mathcal{S}_\omega(\mathbb{R}^d),\qquad u\in\mathcal{S}'_\omega(\mathbb{R}^d).$$

Hence, from (4.6), we finally obtain that

$$\begin{aligned}b^w(x,D)u&=b^w(x,D)c^w(x,D)(b_0-a)^w(x,D)u+\\&\quad+b^w(x,D)c^w(x,D)a^w(x,D)u-b^w(x,D)s^w(x,D)u\end{aligned}$$

belongs to $\mathcal{S}_\omega(\mathbb{R}^d)$ for any $u\in\mathcal{S}'_\omega(\mathbb{R}^d)$, and the proof is complete. □

5. A Comparison Between Different Wave Front Sets

The following definition of wave front set has been introduced and studied in [12, Definition 3.1], which extends the Gabor wave front set given in [37] for the classical setting.

Definition 5.1. Let $u\in\mathcal{S}'_\omega(\mathbb{R}^d)$ and $\psi\in\mathcal{S}_\omega(\mathbb{R}^d)\setminus\{0\}$ be a window function. We say that $z_0\in\mathbb{R}^{2d}\setminus\{0\}$ is not in the ω-wave front set $\mathrm{WF}'_\omega(u)$ of u if there exists an open conic set $\Gamma\subseteq\mathbb{R}^{2d}\setminus\{0\}$, $z_0\in\Gamma$, such that

$$\sup_{z\in\Gamma}e^{\lambda\omega(z)}|V_\psi u(z)|<+\infty,\qquad \lambda>0.$$

In [12, Theorem 4.13] an inclusion like (5.2) for linear partial differential operators with polynomial coefficients is proven. Now, we present an extension of this result for any linear partial differential operator of order m with variable coefficients of the form

$$P(x,D)=\sum_{|\gamma|\le m}a_\gamma(x)D^\gamma,\tag{5.1}$$

where $a_\gamma\in\mathcal{S}_\omega(\mathbb{R}^d)$.

We observe that, in general, a function in $\mathcal{S}_\omega(\mathbb{R}^{2d})$ is not automatically a global symbol in $\mathrm{GS}_\rho^{m,\omega}$. Hence (5.1) is not necessarily an operator with symbol in these classes. It is proven in [6, Example 3.13(b)] that in general $\mathcal{S}_\sigma(\mathbb{R}^{2d}) \subseteq \bigcap_{m\in\mathbb{R}} \mathrm{GS}_\rho^{m,\omega} \subseteq \mathcal{S}_\omega(\mathbb{R}^{2d})$ for every pair of weights ω and σ satisfying $\omega(t^{(1+\rho)/\rho}) = O(\sigma(t))$, $t \to \infty$, for some $0 < \rho \leq 1$. Also it is given there a suitable example of a weight ω for which $\mathcal{S}_\omega(\mathbb{R}^{2d}) = \bigcap_{m\in\mathbb{R}} \mathrm{GS}_\rho^{m,\omega}$.

We show that the action of the differential operator in (5.1) to an ultradistribution $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$ shrinks the ω-wave front set $\mathrm{WF}'_\omega(u)$.

Theorem 5.2. *For the differential operator defined in (5.1), we have*

$$\mathrm{WF}'_\omega(P(x,D)u) \subseteq \mathrm{WF}'_\omega(u), \qquad u \in \mathcal{S}'_\omega(\mathbb{R}^d). \tag{5.2}$$

The ω-wave front set does not depend on the choice of the window function ψ. The following lemma is an improvement of [12, Proposition 3.2].

Lemma 5.3. *Let $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$, $\psi \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$, and $z_0 \in \mathbb{R}^{2d} \setminus \{0\}$. If there exists an open conic set $\Gamma \subseteq \mathbb{R}^{2d} \setminus \{0\}$ containing z_0 such that*

$$\sup_{z\in\Gamma} e^{\lambda\omega(z)} |V_\psi u(z)| < +\infty, \qquad \lambda > 0,$$

then, for any bounded set $\mathcal{B}$ of $\mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$ and for any open conic set $\Gamma' \subseteq \mathbb{R}^{2d} \setminus \{0\}$ containing z_0 and such that $\overline{\Gamma' \cap S_{2d-1}} \subseteq \Gamma$, where S_{2d-1} is the unit sphere in $\mathbb{R}^{2d}$, we have

$$\sup_{\phi\in\mathcal{B}} \sup_{z\in\Gamma'} e^{\lambda\omega(z)} |V_\phi u(z)| < +\infty, \qquad \lambda > 0.$$

Proof. By [12, Proposition 2.12], for any $\psi, \phi \in \mathcal{S}_\omega(\mathbb{R}^d)$, $\psi \neq 0$, we have

$$|V_\phi u(z)| \leq (2\pi)^{-d} \, \|\psi\|_{L^2}^{-2} \, (|V_\psi u| * |V_\phi \psi|)(z), \qquad z \in \mathbb{R}^{2d}.$$

By Lemma 2.6,

$$|V_\phi \psi(z')| = |\overline{V_\psi \phi(-z')}| = |V_\psi \phi(-z')|, \qquad z' \in \mathbb{R}^{2d}.$$

Then,

$$\begin{aligned}(|V_\psi u| * |V_\phi \psi|)(z) &= \int_{\mathbb{R}^{2d}} |V_\psi u(z-z')||V_\phi \psi(z')| dz' \\ &= \int_{\mathbb{R}^{2d}} |V_\psi u(z-z')||V_\psi \phi(-z')| dz'.\end{aligned}$$

For $\varepsilon > 0$, we denote for all $z \in \mathbb{R}^{2d}$,

$$\begin{aligned}I_1(z) &:= \int_{\langle z'\rangle \leq \varepsilon\langle z\rangle} |V_\psi u(z-z')||V_\psi \phi(-z')| dz', \\ I_2(z) &:= \int_{\langle z'\rangle \geq \varepsilon\langle z\rangle} |V_\psi u(z-z')||V_\psi \phi(-z')| dz'.\end{aligned}$$

We choose $\varepsilon > 0$ sufficiently small so that

$$z \in \Gamma', \ |z| \geq 1, \ \langle z'\rangle \leq \varepsilon\langle z\rangle, \quad \text{then} \quad z - z' \in \Gamma.$$

Since $V_\psi : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^{2d})$ is continuous (by [12, Proposition 2.9]), the set $V_\psi(\mathcal{B})$ is bounded in $\mathcal{S}_\omega(\mathbb{R}^{2d})$. Thus for all $\mu > 0$ there exists $C_\mu > 0$ such that

$$\sup_{\phi\in\mathcal{B}} |V_\psi\phi(-z')|e^{\mu\omega(-z')} \le C_\mu, \qquad z' \in \mathbb{R}^{2d}.$$

To estimate I_1, we use the assumption made on Γ for $V_\psi u$ as follows: for all $\lambda > 0$ there exists $C_\lambda > 0$ such that

$$\begin{aligned} I_1(z) &\le C_\lambda \int_{\langle z'\rangle \le \varepsilon\langle z\rangle} e^{-\lambda L\omega(z-z')}|V_\psi\phi(-z')|dz' \\ &\le C_\lambda e^{\lambda L} e^{-\lambda\omega(z)} \int_{\mathbb{R}^{2d}} e^{\lambda L\omega(z')}|V_\psi\phi(-z')|dz' \\ &= C_\lambda e^{\lambda L} e^{-\lambda\omega(z)} \int_{\mathbb{R}^{2d}} \big(e^{(\lambda L+1)\omega(-z')}|V_\psi\phi(-z')|\big)e^{-\omega(z')}dz' \le C'_\lambda e^{-\lambda\omega(z)}, \end{aligned}$$

for some constant $C'_\lambda > 0$, for all $z \in \Gamma'$, $|z| \ge 1$, and all $\phi \in \mathcal{B}$.

On the other hand, by [26, Theorem 2.4] (see also [12, Theorem 2.5]), $V_\psi u$ is continuous and there are constants $c, \mu > 0$ such that

$$|V_\psi u(z)| \le ce^{\mu\omega(z)}, \qquad z \in \mathbb{R}^{2d}.$$

Let $q \in \mathbb{N}_0$ be such that $\varepsilon^{-1} < 2^q$. Then, for $\langle z'\rangle \ge \varepsilon\langle z\rangle$, the properties of the weight ω yield

$$\omega(z) \le \omega(\varepsilon^{-1}\langle z'\rangle) \le \omega(2^q\langle z'\rangle) \le L^{q+1}\omega(z') + L^{q+1} + L^q + \cdots + L.$$

Then, we have

$$-L^{q+1}\omega(z') \le -\omega(z) + (L^{q+1} + L^q + \cdots + L), \qquad \text{for } \langle z'\rangle \ge \varepsilon\langle z\rangle.$$

Therefore, for all $\lambda > 0$ and all $\phi \in \mathcal{B}$, we have

$$\begin{aligned} I_2(z) &\le c \int_{\langle z'\rangle \ge \varepsilon\langle z\rangle} e^{\mu\omega(z-z')}|V_\psi\phi(-z')|dz' \\ &\le ce^{\mu L}e^{\mu L\omega(z)} \int_{\langle z'\rangle \ge \varepsilon\langle z\rangle} e^{\mu L\omega(-z')}|V_\psi\phi(-z')|dz' \\ &= ce^{\mu L}e^{\mu L\omega(z)} \int_{\langle z'\rangle \ge \varepsilon\langle z\rangle} e^{-(\lambda+\mu L)L^{q+1}\omega(z')} \\ &\qquad \times \big(|V_\psi\phi(-z')|e^{((\lambda+\mu L)L^{q+1}+1+\mu L)\omega(-z')}\big)e^{-\omega(z')}dz' \\ &\le ce^{\mu L}e^{(\lambda+\mu L)(L^{q+1}+L^q+\cdots+L)}e^{-\lambda\omega(z)} \\ &\qquad \times \int_{\mathbb{R}^{2d}} \big(|V_\psi\phi(-z')|e^{((\lambda+\mu L)L^{q+1}+1+\mu L)\omega(-z')}\big)e^{-\omega(z')}dz'. \end{aligned}$$

Hence for all $\lambda > 0$ there exists $C''_\lambda > 0$ such that

$$I_2(z) \le C''_\lambda e^{-\lambda\omega(z)}, \qquad z \in \mathbb{R}^{2d}.$$

This finishes the proof. □

Proof of Theorem 5.2. We fix a window function $\psi \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$. From the linearity of the short-time Fourier transform and Lemmas 2.8 and 2.5, we have, for $z = (x, \xi) \in \mathbb{R}^{2d}$,

$$\begin{aligned} V_\psi(P(x,D)u)(x,\xi) &= \sum_{|\gamma|\le m} V_\psi(a_\gamma \cdot D^\gamma u)(x,\xi) \\ &= (2\pi)^{-d} \sum_{|\gamma|\le m} \Big(\widehat{a_\gamma \cdot D^\gamma u} * M_{-x}\widehat{\overline{\psi}}\Big)(\xi) \\ &= (2\pi)^{-2d} \sum_{|\gamma|\le m} \Big(\big(\widehat{a_\gamma} * \widehat{D^\gamma u}\big) * M_{-x}\widehat{\overline{\psi}}\Big)(\xi) \\ &= (2\pi)^{-2d} \sum_{|\gamma|\le m} \Big(\widehat{D^\gamma u} * \big(\widehat{a_\gamma} * M_{-x}\widehat{\overline{\psi}}\big)\Big)(\xi). \end{aligned} \tag{5.3}$$

On the other hand, it is easy to see that

$$\widehat{a_\gamma} * M_{-x}\widehat{\overline{\psi}} = M_{-x}\big(M_x\widehat{a_\gamma} * \widehat{\overline{\psi}}\big).$$

Now, we define $\phi_{x,\gamma} \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$ depending on $x \in \mathbb{R}^d$, $\gamma \in \mathbb{N}_0^d$ with $|\gamma| \le m$ such that

$$\widehat{\overline{\phi}}_{x,\gamma} := M_x\widehat{a_\gamma} * \widehat{\overline{\psi}}. \tag{5.4}$$

Then, by formula (5.3), Lemma 2.8, and [12, (4.31)],

$$\begin{aligned} V_\psi(P(x,D)u)(x,\xi) &= (2\pi)^{-2d} \sum_{|\gamma|\le m} \Big(\widehat{D^\gamma u} * M_{-x}\big(M_x\widehat{a_\gamma} * \widehat{\overline{\psi}}\big)\Big)(\xi) \\ &= (2\pi)^{-d} \sum_{|\gamma|\le m} V_{\phi_{x,\gamma}}(D^\gamma u)(x,\xi) \\ &= (2\pi)^{-d} \sum_{|\gamma|\le m} \sum_{\beta\le\gamma} \binom{\gamma}{\beta} \xi^{\gamma-\beta} V_{D^\beta\phi_{x,\gamma}}(u)(x,\xi). \end{aligned} \tag{5.5}$$

We show that the set

$$\mathcal{B} := \{M_x\widehat{a_\gamma} * \widehat{\overline{\psi}} \ : \ x \in \mathbb{R}^d, |\gamma| \le m\} \tag{5.6}$$

is bounded in $\mathcal{S}_\omega(\mathbb{R}^d)$. For all $\lambda > 0$, we have, by the Young inequality,

$$\begin{aligned} \left|e^{\lambda\omega(y)}\big(M_x\widehat{a_\gamma} * \widehat{\overline{\psi}}\big)(y)\right| &= \left|\int e^{\lambda\omega(y)} M_x\widehat{a_\gamma}(s)\widehat{\overline{\psi}}(y-s)ds\right| \\ &= \left|\int e^{\lambda\omega(y)} e^{ix\cdot s}\widehat{a_\gamma}(s)\widehat{\overline{\psi}}(y-s)ds\right| \\ &\le e^{\lambda L}\int e^{\lambda L\omega(s)}|\widehat{a_\gamma}(s)|e^{\lambda L\omega(y-s)}|\widehat{\overline{\psi}}(y-s)|ds \\ &\le e^{\lambda L} \max_{|\gamma|\le m} \left\|e^{\lambda L\omega(\cdot)}\widehat{a_\gamma}(\cdot)\right\|_{L^1(\mathbb{R}^d)} \left\|e^{\lambda L\omega(\cdot)}\widehat{\overline{\psi}}(\cdot)\right\|_{L^\infty(\mathbb{R}^d)}. \end{aligned} \tag{5.7}$$

On the other hand, by Lemmas 2.7 and 2.5 we have

$$M_x\widehat{a_\gamma} * \widehat{\widehat{\overline{\psi}}} = \widehat{T_{-x}a_\gamma} * \widehat{\widehat{\overline{\psi}}} = (2\pi)^d(\widehat{T_{-x}a_\gamma \cdot \overline{\psi}}),$$

so its Fourier transform satisfies

$$(\widehat{M_x\widehat{a_\gamma} * \widehat{\widehat{\overline{\psi}}}})(\eta) = (2\pi)^d(\widehat{\widehat{T_{-x}a_\gamma \cdot \overline{\psi}}})(\eta) = (2\pi)^{2d}(T_{-x}a_\gamma \cdot \overline{\psi})(-\eta).$$

Thus, for all $\lambda > 0$,

$$\begin{aligned}\left|e^{\lambda\omega(\eta)}(\widehat{M_x\widehat{a_\gamma} * \widehat{\widehat{\overline{\psi}}}})(\eta)\right| &= (2\pi)^{2d}|e^{\lambda\omega(\eta)}T_{-x}a_\gamma(-\eta)\overline{\psi}(-\eta)| \\ &= (2\pi)^{2d}|a_\gamma(x-\eta)e^{\lambda\omega(-\eta)}\overline{\psi}(-\eta)| \\ &\leq (2\pi)^{2d}\max_{|\gamma|\leq m}\|a_\gamma(\cdot)\|_{L^\infty(\mathbb{R}^d)}\left\|e^{\lambda\omega(\cdot)}\overline{\psi}(\cdot)\right\|_{L^\infty(\mathbb{R}^d)}.\end{aligned} \tag{5.8}$$

Formulas (5.7) and (5.8) show that the set given in (5.6) is bounded in $\mathcal{S}_\omega(\mathbb{R}^d)$ [26, Corollary 2.9] (we are using the seminorms given by (2.4)).

Since the Fourier transform is an isomorphism in $\mathcal{S}_\omega(\mathbb{R}^d)$, the set

$$\mathcal{F}^{-1}(\mathcal{B}) = \{\phi : \widehat{\phi} = f, \quad \text{for some } f \in \mathcal{B}\}$$

is bounded in $\mathcal{S}_\omega(\mathbb{R}^d)$, and therefore

$$\mathcal{B}' := \{\phi \ : \ \widehat{\widehat{\overline{\phi}}} = f, \quad \text{for some } f \in \mathcal{B}\}$$

is also a bounded set in $\mathcal{S}_\omega(\mathbb{R}^d)$, and the function $\phi_{x,\gamma}$ taken in (5.4) belongs to $\mathcal{B}'$. Now, we see that

$$\mathcal{B}'' := \{D^\beta\phi \ : \ \phi \in \mathcal{B}', |\beta| \leq m\}$$

is also bounded in $\mathcal{S}_\omega(\mathbb{R}^d)$. We consider the following system of seminorms in $\mathcal{S}_\omega(\mathbb{R}^d)$ (see (2.3)):

$$q_\lambda(\phi) = \sup_{\alpha,\delta\in\mathbb{N}_0^d}\sup_{x\in\mathbb{R}^d}|x^\alpha D^\delta\phi(x)|e^{-\lambda\varphi^*\left(\frac{|\alpha+\delta|}{\lambda}\right)}, \qquad \phi \in \mathcal{S}_\omega(\mathbb{R}^d), \quad \lambda > 0.$$

We fix $\lambda > 0$. From the convexity of φ^* (Lemma 2.2(*iv*)) we have for $\beta \in \mathbb{N}_0^d$, $|\beta| \leq m$,

$$\begin{aligned}q_\lambda(D^\beta\phi) &= \sup_{\alpha,\delta\in\mathbb{N}_0^d}\sup_{x\in\mathbb{R}^d}|x^\alpha D^{\delta+\beta}\phi(x)|e^{-\lambda\varphi^*\left(\frac{|\alpha+\delta|}{\lambda}\right)} \\ &\leq e^{\lambda\varphi^*\left(\frac{|\beta|}{\lambda}\right)}\sup_{\alpha,\delta\in\mathbb{N}_0^d}\sup_{x\in\mathbb{R}^d}|x^\alpha D^{\delta+\beta}\phi(x)|e^{-2\lambda\varphi^*\left(\frac{|\alpha+(\delta+\beta)|}{2\lambda}\right)} \\ &\leq e^{\lambda\varphi^*\left(\frac{|\beta|}{\lambda}\right)}\sup_{\alpha,\delta'\in\mathbb{N}_0^d}\sup_{x\in\mathbb{R}^d}|x^\alpha D^{\delta'}\phi(x)|e^{-2\lambda\varphi^*\left(\frac{|\alpha+\delta'|}{2\lambda}\right)} = e^{\lambda\varphi^*\left(\frac{|\beta|}{\lambda}\right)}q_{2\lambda}(\phi).\end{aligned}$$

Since $\phi \in \mathcal{B}'$ and $e^{\lambda\varphi^*\left(\frac{|\beta|}{\lambda}\right)} \leq e^{\lambda\varphi^*(\frac{m}{\lambda})}$, we get $q_\lambda(D^\beta\phi) < +\infty$ as we wanted.

Let us show (5.2). To this aim, we denote $z = (x, \xi) \in \mathbb{R}^{2d}$ and assume that $0 \neq z_0 \notin \mathrm{WF}'_\omega(u)$. Then, there exists $\Gamma \subseteq \mathbb{R}^{2d} \setminus \{0\}$ an open conic set containing z_0 such that

$$\sup_{z\in\Gamma} e^{\lambda\omega(z)}|V_\psi u(z)| < +\infty, \qquad \lambda > 0.$$

By Lemma 5.3 we obtain that, for any open cone Γ' containing z_0 with $\overline{\Gamma' \cap S_{2d-1}} \subseteq \Gamma$,

$$\sup_{\substack{\beta,\gamma\in\mathbb{N}_0^d:|\beta|,|\gamma|\leq m\\ x\in\mathbb{R}^d}} \sup_{z\in\Gamma'} e^{\lambda\omega(z)}|V_{D^\beta\phi_{x,\gamma}}(u)(z)| < +\infty, \qquad \lambda > 0. \tag{5.9}$$

From (5.5) we have, for all $\lambda > 0$,

$$e^{\lambda\omega(z)}|V_\psi(P(x,D)u)(z)| \leq (2\pi)^{-d} \times \sum_{|\gamma|\leq m}\sum_{\beta\leq\gamma}\binom{\gamma}{\beta}|\xi^{\gamma-\beta}|e^{-L\omega(z)}e^{(\lambda+L)\omega(z)}|V_{D^\beta\phi_{x,\gamma}}(u)(z)|. \tag{5.10}$$

Since $|\gamma-\beta| \leq |\gamma| \leq m$ we have, by Lemma 2.2(i), that

$$|\xi^{\gamma-\beta}| \leq |\xi|^{|\gamma-\beta|} \leq \langle z\rangle^m \leq e^{\varphi^*(m)}e^{\omega(\langle z\rangle)} \leq e^{\varphi^*(m)}e^{L\omega(z)+L}$$

for all $z = (x,\xi) \in \mathbb{R}^{2d}$. Therefore

$$\sup_{z\in\mathbb{R}^{2d}} |\xi^{\gamma-\beta}|e^{-L\omega(z)} < +\infty$$

for every $\beta \leq \gamma$, $|\gamma| \leq m$. When taking the supremum in (5.10) in $z \in \Gamma'$, by (5.9) we obtain

$$\sup_{z\in\Gamma'} e^{\lambda\omega(z)}|V_\psi(P(x,D)u)(z)| < +\infty, \qquad \lambda > 0.$$

Hence $z_0 \notin \mathrm{WF}'_\omega(P(x,D)u)$ and the proof is complete. □

First Inclusion

Now, we compare the Weyl wave front set defined in Sect. 4 with the ω-wave front set $\mathrm{WF}'_\omega(u)$, for certain weight functions ω and any ultradistribution $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$.

Proposition 5.4. *Let ω be a weight function, $b \in \mathrm{GS}^{m,\omega}_\rho$, and $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$. Then,*

$$\mathrm{WF}'_\omega(b^w(x,D)u) \subset \mathrm{conesupp}\,(b).$$

Proof. For a window function $\psi \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$, by formula (3.13) the kernel K in (3.4) satisfies the same estimates as in [12, Proposition 4.4]. The proof is therefore analogous to that of [12, Proposition 4.11] (with the only difference that now ω is not necessarily subadditive). □

The same result holds for the Kohn-Nirenberg quantization, and the proof is analogous. As a consequence of Proposition 5.4, we obtain as in [12, Corollary 4.12] the following

Corollary 5.5. *If $b \in \mathrm{GS}_\rho^{m,\omega}$ has compact support, then $b^w(x,D)$ is globally ω-regularizing.*

Theorem 5.6. *Let ω be a ρ-regular weight function, for some $0 < \rho \leq 1$. If $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$, then*

$$\mathrm{WF}'_\omega(u) \subset \mathrm{WF}^\omega_\rho(u).$$

Proof. Let $0 \neq z_0 \notin \mathrm{WF}^\omega_\rho(u)$. According to Proposition 4.6, there exist $b \in \mathrm{GS}_\rho^{0,\omega}$ and an open conic set $\Gamma \subset \mathbb{R}^{2d} \setminus \{0\}$ such that $z_0 \in \Gamma$, $0 \leq b \leq 1$, $b(z) = 1$ for $z \in \Gamma$ with $|z| \geq 1$ and $b^w(x,D)u \in \mathcal{S}_\omega(\mathbb{R}^d)$. We consider $\widetilde{b} = 1 - b \in \mathrm{GS}_\rho^{0,\omega}$, which satisfies $\widetilde{b}(z) = 0$ for $z \in \Gamma$ with $|z| \geq 1$, so in particular $z_0 \notin \operatorname{conesupp}(\widetilde{b})$. Since $b^w(x,D)u \in \mathcal{S}_\omega(\mathbb{R}^d)$ we obtain, by Proposition 5.4,

$$\mathrm{WF}'_\omega(u) = \mathrm{WF}'_\omega(b^w(x,D)u + \widetilde{b}^w(x,D)u) = \mathrm{WF}'_\omega(\widetilde{b}^w(x,D)u) \subset \operatorname{conesupp}(\widetilde{b}).$$

Hence, $z_0 \notin \mathrm{WF}'_\omega(u)$. □

Second Inclusion

Theorem 5.7. *Let ω be a weight function and $0 < \rho \leq 1$ such that*

$$\omega(t^{1/\rho}) = o(\sigma(t)), \qquad \sigma(t^{1+\rho/2}) = O(\gamma(t)), \tag{5.11}$$

as $t \to \infty$ for some Gevrey weight function σ and some weight function γ. If $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$, then

$$\mathrm{WF}^\omega_\rho(u) \subset \mathrm{WF}'_\omega(u).$$

Remark 5.8. The assumption in Theorem 5.7 implies

$$\omega(t^{(2+\rho)/(2\rho)}) = o(\gamma(t)), \qquad t \to \infty.$$

For $\omega(t) = t^a$, $a = 1 - \rho$, this condition holds if $(1-\rho)(\frac{2+\rho}{2\rho}) < 1$, accordingly $1 > \rho > \frac{-3+\sqrt{17}}{2} \approx 0.56155$; and $0 < a < \frac{5-\sqrt{17}}{2}$.

Proof of Theorem 5.7. First, we recall that the Wigner transform of $\psi(x) = e^{-\frac{1}{2}|x|^2}$, $x \in \mathbb{R}^d$, is (see for instance [37, Theorem 4.2])

$$\mathrm{Wig}(\psi)(z) = (4\pi)^{\frac{d}{2}} e^{-|z|^2}, \qquad z \in \mathbb{R}^{2d}.$$

Let $0 \neq z_0 \notin \mathrm{WF}'_\omega(u)$. Then, there exists an open conic set $\Gamma \subseteq \mathbb{R}^{2d} \setminus \{0\}$ such that $z_0 \in \Gamma$ and

$$\sup_{z\in\Gamma} e^{\lambda\omega(z)} |V_\psi u(z)| < +\infty, \qquad \lambda > 0. \tag{5.12}$$

We consider $\Gamma' \subseteq \mathbb{R}^{2d}\setminus\{0\}$ an open conic set such that $z_0 \in \Gamma'$ and $\overline{\Gamma' \cap S_{2d-1}} \subseteq \Gamma$. By Lemma 4.5, we can construct $b \in \mathrm{GS}_1^{0,\gamma}$ such that $0 \leq b \leq 1$, whose support is contained in Γ, and $b(z) = 1$ for $z \in \Gamma'$, $|z| \geq 1$.

We define $a := b * \mathrm{Wig}(\psi)$. To estimate its derivatives, we use $b \in \mathrm{GS}_1^{0,\gamma}$, Peetre's inequality and Lemma 2.2(*iii*) to obtain that for all $\lambda > 0$ there exists $C_\lambda > 0$ such that

$$\begin{aligned}
|D^\alpha a(z)| &\le \int_{\mathbb{R}^{2d}} |D_z^\alpha b(z-w)|\, \mathrm{Wig}(\psi)(w) dw \\
&\le \int_{\mathbb{R}^{2d}} C_\lambda \langle z-w\rangle^{-|\alpha|} e^{\lambda L \varphi_\gamma^*\left(\frac{|\alpha|}{\lambda L}\right)} (4\pi)^{\frac{d}{2}} e^{-|w|^2} dw \\
&\le \int_{\mathbb{R}^{2d}} C_\lambda \langle z-w\rangle^{-\rho|\alpha|} e^{\lambda L \varphi_\gamma^*\left(\frac{|\alpha|}{\lambda L}\right)} (4\pi)^{\frac{d}{2}} e^{-|w|^2} dw \\
&\le C_\lambda (4\pi)^{\frac{d}{2}} \langle z\rangle^{-\rho|\alpha|} \left(2^{\frac{\rho}{2}|\alpha|} e^{\lambda L \varphi_\gamma^*\left(\frac{|\alpha|}{\lambda L}\right)}\right) \int_{\mathbb{R}^{2d}} \langle w\rangle^{\rho|\alpha|} e^{-|w|^2} dw \\
&\le C_\lambda (4\pi)^{\frac{d}{2}} \langle z\rangle^{-\rho|\alpha|} e^{\lambda L} e^{\lambda \varphi_\gamma^*\left(\frac{|\alpha|}{\lambda}\right)} \int_{\mathbb{R}^{2d}} \langle w\rangle^{\rho|\alpha|} e^{-|w|^2} dw \qquad (5.13)
\end{aligned}$$

for all $\alpha \in \mathbb{N}_0^{2d}$ and $z \in \mathbb{R}^{2d}$. Since $\gamma(t) = o(t)$, for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\gamma(\langle w\rangle^2) \le \varepsilon \langle w\rangle^2 + C_\varepsilon = \varepsilon |w|^2 + (\varepsilon + C_\varepsilon), \qquad w \in \mathbb{R}^{2d}.$$

Therefore, we obtain for some $C'_\varepsilon > 0$, by Lemma 2.2(*i*),

$$\begin{aligned}
\int_{\mathbb{R}^{2d}} \langle w\rangle^{\rho|\alpha|} e^{-|w|^2} dw &\le \int_{\mathbb{R}^{2d}} \langle w\rangle^{\rho|\alpha|} e^{-\frac{1}{\varepsilon}\gamma(\langle w\rangle^2)} C'_\varepsilon dw \\
&= C'_\varepsilon \int_{\mathbb{R}^{2d}} \left(\langle w\rangle^2\right)^{\frac{\rho}{2}|\alpha|} e^{-\lambda\frac{\rho}{2}\gamma(\langle w\rangle^2)} e^{(-\frac{1}{\varepsilon}+\lambda\frac{\rho}{2})\gamma(\langle w\rangle^2)} dw \\
&= C'_\varepsilon \int_{\mathbb{R}^{2d}} \left(\left(\langle w\rangle^2\right)^{|\alpha|} e^{-\lambda\gamma(\langle w\rangle^2)}\right)^{\frac{\rho}{2}} e^{(-\frac{1}{\varepsilon}+\lambda\frac{\rho}{2})\gamma(\langle w\rangle^2)} dw \\
&\le C'_\varepsilon e^{\lambda\frac{\rho}{2}\varphi_\gamma^*\left(\frac{|\alpha|}{\lambda}\right)} \int_{\mathbb{R}^{2d}} e^{(-\frac{1}{\varepsilon}+\lambda\frac{\rho}{2})\gamma(\langle w\rangle^2)} dw. \qquad (5.14)
\end{aligned}$$

We take $\varepsilon > 0$ small enough ($\varepsilon < \frac{2}{\lambda\rho}$) so that the integral converges, and we fix it. By assumption we have $\omega(t^{(2+\rho)/(2\rho)}) = o(\gamma(t))$ as $t \to \infty$. Therefore, from (5.13), (5.14), and Lemma 2.3(*i*) we obtain that for all $\lambda > 0$ there exists $C'_\lambda > 0$ such that

$$\begin{aligned}
|D^\alpha a(z)| &\le C'_\lambda \langle z\rangle^{-\rho|\alpha|} e^{\lambda(1+\rho/2)\varphi_\gamma^*\left(\frac{|\alpha|}{\lambda}\right)} \\
&\le C''_\lambda \langle z\rangle^{-\rho|\alpha|} e^{\lambda\rho\varphi_\omega^*\left(\frac{|\alpha|}{\lambda}\right)}, \qquad (5.15)
\end{aligned}$$

for another constant $C''_\lambda > 0$ depending on $\lambda > 0$. This shows $a \in \mathrm{GS}_\rho^{0,\omega}$.

Let $\Gamma'' \subseteq \Gamma'$ be another open conic set such that $z_0 \in \Gamma''$ and $\overline{\Gamma'' \cap S_{2d-1}} \subseteq \Gamma'$. Then, there exists $\delta > 0$ such that $z - \frac{w}{t} \in \Gamma'$ for $z \in \Gamma''$ with $|z| = 1$, $|w| \le \delta$, and $t \ge 1$. Since $|z-w| \ge |z| - \delta \ge 1$ holds if $|w| \le \delta$ and $|z| \ge 1+\delta$, we have for $z \in \Gamma''$, $|z| \geq 1+\delta$,

$$\begin{aligned}|a(z)| &= \int_{\mathbb{R}^{2d}} b(z-w)\,\mathrm{Wig}(\psi)(w)dw \\ &\geq \int_{\{|w|\leq\delta\}} b\Big(|z|\Big(\frac{z}{|z|}-\frac{w}{|z|}\Big)\Big)\,\mathrm{Wig}(\psi)(w)dw \\ &= \int_{\{|w|\leq\delta\}} \mathrm{Wig}(\psi)(w)dw =: C^* > 0.\end{aligned}$$

Hence (4.1) is satisfied for $m=0$. Moreover, as $\sigma(t^{1+\rho/2}) = O(\gamma(t))$, $t\to\infty$, we use Lemma 2.3(ii) to get, by (5.15), that there exist $C'>0$ and $n\in\mathbb{N}$ such that for $z\in\Gamma''$, $|z|\geq 1+\delta$, and $\alpha\in\mathbb{N}_0^{2d}$,

$$|D^\alpha a(z)| \leq C'\langle z\rangle^{-\rho|\alpha|}e^{\frac{1}{n}\varphi_\sigma^*(n|\alpha|)} \leq \frac{C'}{C^*}\langle z\rangle^{-\rho|\alpha|}e^{\frac{1}{n}\varphi_\sigma^*(n|\alpha|)}|a(z)|,$$

and (4.2) is satisfied, too. Therefore z_0 is non-characteristic for a.

It only remains to show that $a^w(x,D)u \in \mathcal{S}_\omega(\mathbb{R}^d)$. We recall that the Weyl operator $a^w(x,D)$ coincides with the localization operator given by (see for instance [18, (6), (3)])

$$a^w(x,D)u(x) = \int_{\mathbb{R}^{2d}} b(z)V_\psi u(z)\Pi(z)\psi(x)dz. \tag{5.16}$$

Since $\mathrm{supp}\,(b) \subseteq \Gamma$ and $0\leq b\leq 1$, given $\alpha\in\mathbb{N}_0^d$ we have by (5.16), for $z=(t,\xi)\in\mathbb{R}^{2d}$,

$$\begin{aligned}|D^\alpha a^w(x,D)u(x)| &\leq \int_\Gamma |V_\psi u(t,\xi)||D_x^\alpha(e^{ix\cdot\xi}\psi(x-t))|dtd\xi \\ &\leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\int_\Gamma |V_\psi u(t,\xi)||\xi|^{|\beta|}|D_x^{\alpha-\beta}\psi(x-t)|dtd\xi.\end{aligned}$$

From (5.12), Lemma 2.2(i), and since $\psi\in\mathcal{S}_\omega(\mathbb{R}^d)$ we have that for all $\lambda>0$ there exist $C_\lambda, C'_\lambda>0$ such that

$$\begin{aligned}|D^\alpha a^w(x,D)u(x)| \leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\int_\Gamma & C_\lambda e^{-2(\lambda L+1)\omega(t,\xi)}e^{\lambda L\varphi^*\left(\frac{|\beta|}{\lambda L}\right)}e^{\lambda L\omega(\xi)} \\ &\times C'_\lambda e^{\lambda L\varphi^*\left(\frac{|\alpha-\beta|}{\lambda L}\right)}e^{-\lambda L\omega(x-t)}dtd\xi.\end{aligned}$$

As

$$\begin{aligned}-2(\lambda L+1)\omega(t,\xi) &\leq -(\lambda L+1)(\omega(t)+\omega(\xi)) \\ &\leq -(\omega(t)+\omega(\xi)) - \lambda L\omega(\xi) + \lambda L\omega(x-t) - \lambda\omega(x) + \lambda L,\end{aligned}$$

we obtain

$$-2(\lambda L+1)\omega(t,\xi) + \lambda L\omega(\xi) - \lambda L\omega(x-t) \leq -(\omega(t)+\omega(\xi)) - \lambda\omega(x) + \lambda L,$$

which shows that the integral converges. Moreover, by the convexity of φ^* (Lemma 2.2(*iv*), (*iii*)) we have $a^w(x,D)u \in \mathcal{S}_\omega(\mathbb{R}^d)$, and therefore $z_0 \notin \mathrm{WF}^\omega_\rho(u)$. □

Corollary 5.9. *Let ω be a ρ-regular weight function for some $0<\rho\le 1$ that satisfies* (5.11) *for two weight functions σ and γ as in Theorem* 5.7*. Then, for $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$, we have*

$$\mathrm{WF}^\omega_\rho(u) = \mathrm{WF}'_\omega(u).$$

Example 5.10. Let $\frac{-3+\sqrt{17}}{2} < \rho < 1$ and $\omega(t) = t^a$ with $a = 1-\rho$. Then for every $b, c > 0$ such that $\frac{1-\rho}{\rho} < b < \frac{2}{2+\rho}$ and $b(1+\rho/2) < c < 1$, the weight functions ω, $\sigma(t) = t^b$ and $\gamma(t) = t^c$ satisfy the hypotheses of Corollary 5.9 (see Remark 5.8).

6. Weyl Wave Front Set and Propagation of Singularities

In this section we study the propagation of singularities for Weyl quantizations with the Weyl wave front set with symbols in the class $\mathrm{GS}^{m,\omega}_\rho$.

Lemma 6.1. *Let ω be a ρ-regular weight function for some $0<\rho\le 1$ and $u\in\mathcal{S}'_\omega(\mathbb{R}^d)$. Then $\mathrm{WF}^\omega_\rho(u)$ is empty if and only if $u \in \mathcal{S}_\omega(\mathbb{R}^d)$.*

Proof. Let us first assume that $u \in \mathcal{S}_\omega(\mathbb{R}^d)$. Taking $a \equiv 1 \in \mathrm{GS}^{0,\omega}_\rho$ we have that z is non-characteristic for a, for every $z \in \mathbb{R}^{2d}\setminus\{0\}$, and $a^w(x,D)u = u \in \mathcal{S}_\omega(\mathbb{R}^d)$, so $\mathrm{WF}^\omega_\rho(u)$ is empty.

Assume now that $\mathrm{WF}^\omega_\rho(u)$ is empty. From Theorem 5.6 we have that $\mathrm{WF}'_\omega(u)$ is empty, and then from [12, Proposition 3.18] we obtain $u \in \mathcal{S}_\omega(\mathbb{R}^d)$. □

Proposition 6.2. *Let ω be a ρ-regular weight function for some $0<\rho\le 1$, and $m\in\mathbb{R}$; fix a symbol $a \in \mathrm{GS}^{m,\omega}_\rho$. We have*

$$\mathrm{WF}^\omega_\rho(u) \subset \mathrm{WF}^\omega_\rho(a^w(x,D)u) \cup \mathrm{char}(a),$$

for every $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$, where $\mathrm{char}(a)$ is the complement in $\mathbb{R}^{2d}$ of the set of non-characteristic points for a in the sense of Definition 4.1*.*

Proof. Let $z_0 \neq 0$ satisfying $z_0 \notin \mathrm{WF}^\omega_\rho(a^w(x,D)u) \cup \mathrm{char}(a)$. By Proposition 4.6 we have that there exist $b \in \mathrm{GS}^{0,\omega}_\rho$ and an open conic set $\Gamma \subset \mathbb{R}^{2d}\setminus\{0\}$ containing z_0 such that $0 \le b \le 1$, $b(z) = 1$ for $z \in \Gamma$, $|z| \ge 1$, and

$$b^w(x,D)a^w(x,D)u \in \mathcal{S}_\omega(\mathbb{R}^d), \tag{6.1}$$

for every $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$. We already know that the Weyl product $b\#a$ of the composition $b^w(x,D)a^w(x,D)$ has an asymptotic expansion as in (3.1), i.e.

$b\#a(x,\xi) \sim \sum_{j=0}^{\infty} c_j(x,\xi)$, where

$$c_j(x,\xi) = \sum_{|\beta+\gamma|=j} \frac{(-1)^{|\beta|}}{\gamma!\beta!} 2^{-|\beta+\gamma|} \partial_\xi^\gamma D_x^\beta b(x,\xi)\, \partial_\xi^\beta D_x^\gamma a(x,\xi), \tag{6.2}$$

for every $j \in \mathbb{N}_0$. Now, we apply [6, Theorem 4.6] to obtain a symbol $c(x,\xi) \in \mathrm{GS}_\rho^{m,\omega}$ with asymptotic expansion $\sum c_j$ and satisfying

$$c(x,\xi) = b(x,\xi)a(x,\xi) + \sum_{n=1}^{\infty} \sum_{j=j_n}^{j_{n+1}-1} \Psi_{j,n}(x,\xi) c_j(x,\xi), \tag{6.3}$$

where $(j_n)_n$ and $\Psi_{j,n}$ are defined in [6, formula (4.4)]. We observe that, from the properties of $b(x,\xi)$, we have

$$c(x,\xi) = a(x,\xi) \quad \text{for all } z = (x,\xi) \in \Gamma,\ |z| \geq 1. \tag{6.4}$$

On the other hand, since $z_0 \notin \mathrm{char}(a)$, the symbol $a(x,\xi)$ satisfies (4.1) and (4.2) (we can assume without loss of generality that the open conic set Γ appearing in (4.1) and (4.2) is the same Γ appearing in (6.4)). By (6.4), we have that z_0 is non-characteristic for $c(x,\xi)$. Finally, since for every $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$ we have

$$b^w(x,D)a^w(x,D)u = c^w(x,D)u + Ru,$$

where R is a globally ω-regularizing operator, by (6.1), it follows that $c^w(x,D)u \in \mathcal{S}_\omega(\mathbb{R}^d)$. Hence, $z_0 \notin \mathrm{WF}_\rho^\omega(u)$. □

Lemma 6.3. *Let ω be a ρ-regular weight function for some $0 < \rho \leq 1$, $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$ and $v \in \mathcal{S}_\omega(\mathbb{R}^d)$. We have*

$$\mathrm{WF}_\rho^\omega(u) = \mathrm{WF}_\rho^\omega(u+v).$$

Proof. Let $z_0 \notin \mathrm{WF}_\rho^\omega(u)$. Then there exists a symbol $a(x,\xi) \in \mathrm{GS}_\rho^{m,\omega}$ for some $m \in \mathbb{R}$ such that z_0 is non-characteristic for $a(x,\xi)$ and $a^w(x,D)u \in \mathcal{S}_\omega(\mathbb{R}^d)$. Since $v \in \mathcal{S}_\omega(\mathbb{R}^d)$, we have $a^w(x,D)(u+v) \in \mathcal{S}_\omega(\mathbb{R}^d)$ (see, for instance, [4, Lemma 3.3] and [6, Theorem 3.7]). Therefore $z_0 \notin \mathrm{WF}_\rho^\omega(u+v)$ and, so

$$\mathrm{WF}_\rho^\omega(u+v) \subset \mathrm{WF}_\rho^\omega(u).$$

By the same procedure we get

$$\mathrm{WF}_\rho^\omega(u) = \mathrm{WF}_\rho^\omega(u+v-v) \subset \mathrm{WF}_\rho^\omega(u+v),$$

so the proof is complete. □

Proposition 6.4. *Let ω be a ρ-regular weight function for some $0 < \rho \leq 1$, $m \in \mathbb{R}$ and $a(x,\xi) \in \mathrm{GS}_\rho^{m,\omega}$. Then*

$$\mathrm{WF}_\rho^\omega(a^w(x,D)u) \subset \mathrm{conesupp}(a),$$

for every $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$.

Proof. Fix $0 \neq z_0 \notin \text{conesupp}(a)$. Then there exists an open conic set Γ containing z_0 such that $a(z) = 0$ for every $z \in \Gamma$, $|z| \geq R$, for some $R > 0$. We choose Γ' and $\chi \in \text{GS}_\rho^{0,\omega}$ as in Lemma 4.5, with $z_0 \in \Gamma'$. Since $\chi(z) = 1$ for $z \in \Gamma'$, $|z| \geq 1$, we trivially have that χ satisfies (4.1) and (4.2), so z_0 is non-characteristic for χ. Now, we can argue as in the proof of Proposition 6.2 to obtain that the Weyl product of the composition $\chi^w(x, D)a^w(x, D)$ has an asymptotic expansion $\sum c_j$ where c_j, $j \in \mathbb{N}_0$, is given as in (6.2) with χ instead of b, and we can consider the symbol $c \in \text{GS}_\rho^{m,\omega}$ as in (6.3) whose asymptotic expansion is $\sum c_j$, obtaining that for every $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$,

$$\chi^w(x, D)a^w(x, D)u = c^w(x, D)u + Ru$$

for some globally ω-regularizing operator R. Since $\text{supp}(a) \cap \text{supp}(\chi)$ is compact, we have that $\text{supp}(c)$ is compact. So, from Corollary 5.5 we obtain that $c^w(x, D)$ is globally ω-regularizing. Consequently, for every $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$, we have

$$\chi^w(x, D)a^w(x, D)u \in \mathcal{S}_\omega(\mathbb{R}^d),$$

and so $z_0 \notin \text{WF}_\rho^\omega(a^w(x, D)u)$. □

Remark 6.5. We observe that Proposition 5.4 and Corollary 5.9 imply the thesis in Proposition 6.4 under the extra assumption (5.11) for two weight functions σ and γ as in Theorem 5.7, but this assumption is not necessary in the proof of Proposition 6.4.

Proposition 6.6. *Let ω be a ρ-regular weight function for some $0 < \rho \leq 1$ and $a \in \text{GS}_\rho^{m,\omega}$. Then*

$$\text{WF}_\rho^\omega(a^w(x, D)u) \subset \text{WF}_\rho^\omega(u),$$

for every $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$.

Proof. Fix $0 \neq z_0 \notin \text{WF}_\rho^\omega(u)$; by Proposition 4.6 there exist $b \in \text{GS}_\rho^{0,\omega}$ and an open conic set Γ containing z_0 with $b(z) = 1$ for $z \in \Gamma$, $|z| \geq 1$ and $b^w(x, D)u \in \mathcal{S}_\omega(\mathbb{R}^d)$. Set $\tilde{b} = 1 - b \in \text{GS}_\rho^{0,\omega}$ and observe that

$$a^w(x, D)u = a^w(x, D)\tilde{b}^w(x, D)u + a^w(x, D)b^w(x, D)u.$$

Since $a^w(x, D) : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$, we have that $a^w(x, D)b^w(x, D)u \in \mathcal{S}_\omega(\mathbb{R}^d)$, and so, by Lemma 6.3,

$$\text{WF}_\rho^\omega(a^w(x, D)u) = \text{WF}_\rho^\omega(a^w(x, D)\tilde{b}^w(x, D)u). \tag{6.5}$$

Since $\tilde{b}(z) = 0$ for every $z \in \Gamma$, $|z| \geq 1$, arguing as in the proof of Proposition 6.2, there exists a symbol c that vanishes for $z \in \Gamma$, $|z| \geq 1$ and

$$a^w(x, D)\tilde{b}^w(x, D)u = c^w(x, D)u + Ru, \quad u \in \mathcal{S}'_\omega(\mathbb{R}^d),$$

for a globally ω-regularizing operator R. Therefore from (6.5), we use Lemma 6.3 and Proposition 6.4 to obtain

$$\text{WF}_\rho^\omega(a^w(x, D)u) = \text{WF}_\rho^\omega(c^w(x, D)u) \subset \text{conesupp}(c).$$

Since $z_0 \in \Gamma$ we have that $z_0 \notin \mathrm{conesupp}(c)$ and then $z_0 \notin \mathrm{WF}_\rho^\omega(a^w(x,D)u)$. □

We have the following result as in [37, Proposition 2.11].

Corollary 6.7. *Let ω be a ρ-regular weight function for some $0 < \rho \le 1$, $a \in \mathrm{GS}_\rho^{m,\omega}$, and $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$. If*

$$\mathrm{conesupp}\,(a) \cap \mathrm{WF}_\rho^\omega(u) = \emptyset,$$

then $a^w(x,D)u \in \mathcal{S}_\omega(\mathbb{R}^d)$.

Proof. From Propositions 6.4 and 6.6 we obtain by assumption that WF_ρ^ω $(a^w(x,D)u) = \emptyset$. The result then follows by Lemma 6.1. □

From Propositions 6.2, 6.4, and 6.6 we immediately have the following result.

Theorem 6.8. *Let ω be a ρ-regular weight function for some $0 < \rho \le 1$ and $a \in \mathrm{GS}_\rho^{m,\omega}$. Then*

$$\mathrm{WF}_\rho^\omega(a^w(x,D)u) \subset \mathrm{WF}_\rho^\omega(u) \cap \mathrm{conesupp}(a) \subset \mathrm{WF}_\rho^\omega(u) \subset \mathrm{WF}_\rho^\omega(a^w(x,D)u) \cup \mathrm{char}(a).$$

Corollary 6.9. *Let ω be a ρ-regular weight function for some $0 < \rho \le 1$ which satisfies (5.11) for two weight functions σ and γ as in Theorem 5.7, and $a \in \mathrm{GS}_\rho^{m,\omega}$. Then*

$$\mathrm{WF}'_\omega(a^w(x,D)u) \subset \mathrm{WF}'_\omega(u) \cap \mathrm{conesupp}(a) \subset \mathrm{WF}'_\omega(u) \subset \mathrm{WF}'_\omega(a^w(x,D)u) \cup \mathrm{char}(a).$$

Proof. It is an immediate consequence of Theorem 6.8 and Corollary 5.9. □

Remark 6.10. If ω is ρ-regular for some $0 < \rho \le 1$, then for all $m \in \mathbb{R}$ there exists $a \in \mathrm{GS}_\rho^{m,\omega}$ such that every $z \in \mathbb{R}^{2d} \setminus \{0\}$ is non-characteristic for a in the sense of Definition 4.1 ; so $\mathrm{char}\,(a) = \emptyset$, and then by Theorem 6.8,

$$\mathrm{WF}_\rho^\omega(a^w(x,D)u) = \mathrm{WF}_\rho^\omega(u), \qquad u \in \mathcal{S}'_\omega(\mathbb{R}^d). \tag{6.6}$$

By Example 3.3, for Gevrey weights $\omega(t) = t^h$ with $0 < h < 1/2$, which are $(1-h)$-regular, we have that for every $m \in \mathbb{R}$, the Weyl operator $a^w(x,D)$ associated to the symbol

$$a(z) := e^{|m|\langle z\rangle^h}, \qquad z \in \mathbb{R}^{2d},$$

satisfies (6.6).

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References

[1] Albanese, A.A., Jornet, D., Oliaro, A.: Quasianalytic wave front sets for solutions of linear partial differential operators. Integral Equ. Oper. Theory **66**(2), 153–181 (2010)

[2] Albanese, A.A., Jornet, D., Oliaro, A.: Wave front sets for ultradistribution solutions of linear partial differential operators with coefficients in non-quasianalytic classes. Math. Nachr. **285**(4), 411–425 (2012)

[3] Annovazzi, E.: Esempi notevoli di funzioni gevrey ed ultradistribuzioni. Master's thesis, University of Torino (1999/2000) Italian

[4] Asensio, V.: Quantizations and global hypoellipticity for pseudodifferential operators of infinite order in classes of ultradifferentiable functions (2022). arXiv:2104.09198. Accepted for publication in Mediterr. J. Math

[5] Asensio, V.: Global pseudodifferential operators in spaces of ultradifferentiable functions [Tesis doctoral]. Universitat Politècnica de València. https://doi.org/10.4995/Thesis/10251/174847

[6] Asensio, V., Jornet, D.: Global pseudodifferential operators of infinite order in classes of ultradifferentiable functions. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113**(4), 3477–3512 (2019)

[7] Björck, G.: Linear partial differential operators and generalized distributions. Ark. Mat. **6**, 351–407 (1966)

[8] Boiti, C., Jornet, D.: A simple proof of Kotake–Narasimhan theorem in some classes of ultradifferentiable functions. J. Pseudo-Differ. Oper. Appl. **8**(2), 297–317 (2017)

[9] Boiti, C., Jornet, D.: A characterization of the wave front set defined by the iterates of an operator with constant coefficients. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **111**(3), 891–919 (2017)

[10] Boiti, C., Jornet, D., Juan-Huguet, J.: Wave front sets with respect to the iterates of an operator with constant coefficients. Abstr. Appl. Anal. **2014**, 438716 (2014)

[11] Boiti, C., Jornet, D., Oliaro, A.: Regularity of partial differential operators in ultradifferentiable spaces and Wigner type transforms. J. Math. Anal. Appl. **446**(1), 920–944 (2017)

[12] Boiti, C., Jornet, D., Oliaro, A.: The Gabor wave front set in spaces of ultradifferentiable functions. Monatsh. Math. **188**(2), 199–246 (2019)

[13] Boiti, C., Jornet, D., Oliaro, A.: Real Paley–Wiener theorems in spaces of ultradifferentiable functions. J. Funct. Anal. **278**(4), 108348 (2020)

[14] Boiti, C., Jornet, D., Oliaro, A., Schindl, G.: Nuclearity of rapidly decreasing ultradifferentiable functions and time-frequency analysis. Collect. Math. **72**(2), 423–442 (2021)

[15] Boiti, C., Jornet, D., Oliaro, A., Schindl, G.: Nuclear global spaces of ultradifferentiable functions in the matrix weighted setting. Banach J. Math. Anal. **15**(14), 1–39 (2021)

[16] Braun, R.W., Meise, R., Taylor, B.A.: Ultradifferentiable functions and Fourier analysis. Results Math. **17**(3–4), 206–237 (1990)

[17] Cappiello, M., Schulz, R.: Microlocal analysis of quasianalytic Gelfand–Shilov type ultradistributions. Complex Var. Elliptic Equ. **61**(4), 538–561 (2016)

[18] Cordero, E., Gröchenig, K.: Time-frequency analysis of localization operators. J. Funct. Anal. **205**(1), 107–131 (2003)

[19] Debrouwere, A., Neyt, L., Vindas, J.: Characterization of nuclearity for Beurling–Björck spaces. Proc. Am. Math. Soc. **148**(12), 5171–5180 (2020)

[20] Debrouwere, A., Neyt, L., Vindas, J.: The nuclearity of Gelfand–Shilov spaces and kernel theorems. Collect. Math. **72**(1), 203–227 (2021)

[21] Fernández, C., Galbis, A.: Superposition in classes of ultradifferentiable functions. Publ. Res. Inst. Math. Sci. **42**(2), 399–419 (2006)

[22] Fernández, C., Galbis, A., Jornet, D.: ω-Hypoelliptic differential operators of constant strength. J. Math. Anal. Appl. **297**(2), 561–576 (2004). Special issue dedicated to John Horváth

[23] Fernández, C., Galbis, A., Jornet, D.: Pseudodifferential operators of Beurling type and the wave front set. J. Math. Anal. Appl. **340**(2), 1153–1170 (2008)

[24] Franken, U.: Weight functions for classes of ultradifferentiable functions. RM **25**(1–2), 50–53 (1994)

[25] Gröchenig, K.: Foundations of Time-frequency Analysis. Applied and Numerical Harmonic Analysis. Birkhäuser Boston Inc, Boston (2001)

[26] Gröchenig, K., Zimmermann, G.: Spaces of test functions via the STFT. J. Funct. Spaces Appl. **2**(1), 25–53 (2004)

[27] Hörmander, L.: Fourier integral operators. I. Acta Math. **127**(1–2), 79–183 (1971)

[28] Hörmander, L.: Quadratic Hyperbolic Operators, Microlocal Analysis and Applications (Montecatini Terme, 1989). Lecture Notes in Mathematics, vol. 1495, pp. 118–160. Springer, Berlin (1991)

[29] Mascarello, M., Rodino, L.: Partial Differential Equations with Multiple Characteristics, Mathematical Topics, vol. 13. Akademie, Berlin (1997)

[30] Nakamura, S.: Propagation of the homogeneous wave front set for Schrödinger equations. Duke Math. J. **126**(2), 349–367 (2005)

[31] Nicola, F., Rodino, L.: Global Pseudo-Differential Calculus on Euclidean Spaces, Pseudo-Differential Operators. Theory and Applications, vol. 4. Birkhäuser, Basel (2010)

[32] Petzsche, H.-J., Vogt, D.: Almost analytic extension of ultradifferentiable functions and the boundary values of holomorphic functions. Math. Ann. **267**(1), 17–35 (1984)

[33] Pilipović, S., Prangoski, B.: Anti-Wick and Weyl quantization on ultradistribution spaces. J. Math. Pures Appl. (9) **103**(2), 472–503 (2015)

[34] Pilipović, S., Prangoski, B., Vindas., J.: Infinite order ψ DOs: Composition with entire functions, new Shubin-Sobolev spaces, and index theorem. Anal. Math. Phys. https://doi.org/10.1007/s13324-021-00545-w

[35] Prangoski, B.: Pseudodifferential operators of infinite order in spaces of tempered ultradistributions. J. Pseudo-Differ. Oper. Appl. **4**(4), 495–549 (2013)

[36] Rodino, L.: Linear Partial Differential Operators in Gevrey Spaces. World Scientific Publishing Co., River Edge (1993)

[37] Rodino, L., Wahlberg, P.: The Gabor wave front set. Monatsh. Math. **173**(4), 625–655 (2014)

[38] Schulz, R., Wahlberg, P.: Equality of the homogeneous and the Gabor wave front set. Commun. Partial Differ. Equ. **42**(5), 703–730 (2017)

[39] Shubin, M.A.: Pseudodifferential Operators and Spectral Theory, 2nd edn. Springer, Berlin (2001)

[40] Sjöstrand, J.: Singularités analytiques microlocales, Astérisque, 95, Astérisque, vol. 95, pp. 1–166. Soc. Math. France, Paris (1982)

Vicente Asensio and David Jornet
Instituto Universitario de Matemática Pura y Aplicada IUMPA
Universitat Politècnica de València
Camino de Vera, s/n
46071 Valencia
Spain
e-mail: vasensio@edem.es;
viaslo@upv.es;
djornet@mat.upv.es

Chiara Boiti
Dipartimento di Matematica e Informatica
Università di Ferrara
Via Machiavelli n. 30
44121 Ferrara
Italy
e-mail: chiara.boiti@unife.it

Alessandro Oliaro
Dipartimento di Matematica
Università di Torino
Via Carlo Alberto n. 10
10123 Torino
Italy
e-mail: alessandro.oliaro@unito.it

Vicente Asensio
Centro Universitario EDEM
Muelle de la Aduana, s/n, La Marina de Valencia
E-46024 Valencia
Spain

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