MODELLING FOR ENGINEERING & HUMAN BEHAVIOUR 2022

PROCEEDINGS

Edited by

Juan Ramón Torregrosa
Juan Carlos Cortés
Antonio Hervás
Antoni Vidal
Elena López-Navarro

im²
Instituto Universitario
de Matemática Multidisciplinar

UNIVERSITAT
POLITÈCNICA
DE VALÈNCIA

Modelling for Engineering & Human Behaviour 2022

València, July 14th-16th, 2022

This book includes the extended abstracts of papers presented at XXIV Edition of the Mathematical Modelling Conference Series at the Institute for Multidisciplinary Mathematics *Mathematical Modelling in Engineering & Human Behaviour*.

I.S.B.N.: 978-84-09-47037-2

November 30th, 2022
Report any problems with this document to imm@imm.upv.es.

Edited by: I.U. de Matemàtica Multidisciplinar, Universitat Politècnica de València.
J.R. Torregrosa, J-C. Cortés, A. Hervás, A. Vidal-Ferràndiz and E. López-Navarro

im²
Instituto Universitario
de Matemática Multidisciplinar

Contents

5

Probabilistic analysis of scalar random differential equations with state-dependent impulsive terms via probability density functions

V. Bevia ♭,[1] J. C. Cortés ♭ M. Jornet # and R.J. Villanueva♭

(♭) Instituto de Matemática Multidisciplinar (Imm),
Universitat Politècnica de València (UPV)
Camí de Vera s/n, València, Spain.

(#) Departament Matemàtiques,
Universitat de València (UV),
Burjassot (València), Spain.

1 Introduction

Most phenomena observed in nature can be described as a function changing smoothly over time. However, sometimes the system suddenly changes its state, requiring special mathematical tools to model its dynamics correctly. This is common in biology, medicine, or engineering when determining the effectiveness of specific impulsive-type control strategies. In ecology, these are known as harvesting models [5, 9, 10].

In this contribution, we will study, from a probabilistic standpoint, the following random Initial Value Problem (IVP):

$$\begin{cases} \dfrac{\mathrm{d}X(t,\omega)}{\mathrm{d}t} &= g(X(t,\omega),t,\mathbf{A}(\omega)) - \displaystyle\sum_{k=1}^{N}\Gamma_k(\omega)\delta(t-t_k)X(t,\omega), \quad t>t_0, \\ X(t_0,\omega) &= X_0(\omega). \end{cases} \tag{1}$$

Here t_0 denotes a real number; $X_0(\omega)$, $\mathbf{A}(\omega) := (A_1(\omega),\ldots,A_m(\omega))$ and $\{\Gamma_k(\omega)\}_{k=1}^N$ are assumed to be independent absolutely continuous random variables defined on the Hilbert space $\mathrm{L}^2(\Omega,\mathbb{R})$, whose elements are real-valued random variables with finite variance and $(\Omega,\mathcal{F},\mathbb{P})$ denotes a complete probability space [6]; $\delta(t-t_k)$ stands for the Dirac delta function [4] acting at the prefixed time instants $t=t_k$, $k=1,\ldots,N$ and g is known as the *(scalar) field function* satisfying certain conditions that will be specified later. Finally, $X(t,\omega)$ denotes the solution of the random IVP (1)

[1]vibees@doctor.upv.es

2 Methods and Results

2.1 Pathwise solution

The deterministic Laplace transform [2] and the usual conditions required for the existence and uniqueness of ODEs have allowed us to construct a right-continuous pathwise solution of the random IVP (1), given by

$$X(t,\omega) = X_0(\omega) + \int_{t_0}^{t} g(X(s,\omega), s, \mathbf{A}(\omega))\mathrm{d}s - \sum_{k=1}^{N} \Gamma_k(\omega) X(t_k,\omega)\mathrm{H}(t-t_k), \quad t \geq t_0, \tag{2}$$

$$X(t_k,\omega) = \frac{X(t_k^-,\omega)}{1+\Gamma_k(\omega)} = X(t_k^+,\omega), \quad \omega \in \tilde{\Omega}. \tag{3}$$

2.2 Probability Density Function evolution

RDEs verifies a *probability conservation property*; that is, the total probability in the phase space is conserved through time. This fact gives an evolution PDE which is verified by its 1-PDF. The theorem can be stated as follows

Theorem 1. *[1] Let $b(\cdot,t) : \mathbb{R} \longrightarrow \mathbb{R}$ be a Lipschitz-continuous function for all $t \in (t_0,\infty)$, and continuous in t. Let $X(t,\omega)$, $t \geq t_0$, $\omega \in \Omega$ be the stochastic process verifying the following RDE in the almost-surely or mean square sense:*

$$\begin{cases} \dfrac{\mathrm{d}X(t)}{\mathrm{d}t} &= b(X(t),t), \quad t > t_0, \\ X(t_0) &= X_0 \in \mathrm{L}^2(\Omega,\mathbb{R}). \end{cases} \tag{4}$$

Let $\mathcal{D}$ be a set such that $\{X([t_0,\infty),\omega)\}_{\omega\in\Omega} \subset \mathcal{D}$. Then, the 1-PDF of the stochastic process $X(t)$, denoted by $f = f_{X(t)}$, verifies the Liouville PDE:

$$\begin{cases} \partial_t f(x,t) + \partial_x[b\, f](x,t) = 0, \quad x \in \mathcal{D}, \quad t > t_0, \\ f(x,t_0) = f_0(x), \quad x \in \mathcal{D}, \\ \partial_x f(x,t) = 0, \quad x \in \partial\mathcal{D}, \quad t \geq t_0, \end{cases} \tag{5}$$

where f_0 is the PDF of $X_0 = X_0(\omega)$.

When the RDE has random parameters, IVP (5) becomes a family of deterministic PDE problems indexed in the realizations, $\mathbf{a}$, of the random parameter vector, $\mathbf{A} = \mathbf{A}(\omega)$,

$$\begin{cases} \partial_t f(x,t\,|\,\mathbf{a}) + \partial_x[b(x,t,\mathbf{a})f(x,t\,|\,\mathbf{a})] = 0, \quad x \in \mathcal{D} \subseteq \mathbb{R}, \quad t > t_0, \\ f(x,t_0\,|\,\mathbf{a}) = f_0(x), \quad x \in \mathcal{D}. \end{cases} \tag{6}$$

The PDF of the RDE solution (independent of parameter realizations) is obtained by marginalizing the joint PDF of both the solution and the parameter vector $\mathbf{A}$, which, using the conditional PDF can be written as:

$$f(x,t) = \int_{\mathbb{R}^m} f(x,t\,|\,\mathbf{a}) f_{\mathbf{A}}(\mathbf{a})\mathrm{d}\mathbf{a} = \mathbb{E}_{\mathbf{A}}[f(x,t\,|\,\mathbf{A})], \tag{7}$$

where $f_{\mathbf{A}}$ is the parameters' joint PDF and $\mathbb{E}$ denotes the expectation operator. This shows that the PDF can be obtained by solving (6) for all realizations $\mathbf{a}$ of $\mathbf{A}$ and then computing its mean.

However, a *priori*, global-in-time existence of a solution to the Liouville equation can only be assured when the field function $b(\cdot, t)$ is Lipschitz continuous, uniformly in t. The field of the RDE class under study, $b(x,t) = g(x,t) - \sum_k \gamma_k \delta(t-t_k)x$, does not verify this hypothesis at the impulse times $\{t_k\}_{k=1}^N$. We want to obtain a condition such as (3), but for the PDF. This will allow the computation of the evolution of f_0, accurately capturing the discontinuities at the impulse times.

Let us turn back to the set of conditions in (3), which are identities between random variables. We are going to make use of the RVT theorem, which can be written as follows:

Theorem 2. *[2, 8] Let $\mathbf{X}, \mathbf{Y} : \Omega \to \mathbb{R}^M$ be two random vectors with PDFs $f_{\mathbf{X}}$ and $f_{\mathbf{Y}}$, respectively. Assume that there is a one-to-one, C^1 function $\mathbf{h}$ such that $\mathbf{X} = \mathbf{h}(\mathbf{Y})$. Then, denoting $\mathbf{h}^{-1}$ as the inverse mapping of $\mathbf{h}$,*

$$f_{\mathbf{X}}(\mathbf{x}) = f_{\mathbf{Y}}(\mathbf{h}^{-1}(\mathbf{x})) \left| \frac{\partial \mathbf{h}^{-1}(\mathbf{x})}{\partial \mathbf{x}} \right|, \tag{8}$$

where $\left| \frac{\partial \mathbf{h}^{-1}(\mathbf{x})}{\partial \mathbf{x}} \right|$ denotes the absolute value of the determinant of the Jacobian matrix.

Now, the Liouville equation describes the evolution of the PDF between impulse times, whereas at any impulse time, applying the RVT theorem, we obtain:

$$f(x, t_k) = \mathbb{E}_{\Gamma_k}[f\left(x\,(1+\Gamma_k), t_k^-\right)|1+\Gamma_k|], \quad \forall x > 0, \quad k = 1, \ldots, N, \tag{9}$$

because of the relations at (3).

3 Results

In this contribution, we are going to deal with the impulse-harvest generalized logistic model with a finite number of captures, say N,

$$\begin{aligned} X'(t,\omega) &= \alpha(t)\, r(\omega) X(t,\omega) \left(1 - \left(\frac{X(t,\omega)}{K(\omega)}\right)^{\nu(\omega)}\right) - \sum_{n=1}^{N} \Gamma_n(\omega)\delta(t-t_n)X(t,\omega), \\ X(t_0,\omega) &= X_0(\omega), \end{aligned} \tag{10}$$

where $t \geq t_0$ and $\omega \in \tilde{\Omega}$. As usual, t is interpreted as the time, the parameter r is the growth ($r > 0$) or decay ($r < 0$) rate, and K is the carrying capacity. The differential equation is generalized by adding two terms: a positive, monotonically growing function $\alpha(\cdot)$ and a constant positive term ν. The first term, $\alpha(\cdot)$, allows controlling the so-called *lag phase*, which is the growth phase in which the population under study has not yet achieved a fully exponential growth. In particular, we have chosen [7]:

$$\alpha(t) := \frac{q(\omega)}{q(\omega) + \mathrm{e}^{-m(\omega)\,t}}, \quad q, m > 0 \text{ a.s.}$$

The latter, ν, is a power that controls how fast the carrying capacity K is approached and is known as *deceleration* term. When $\nu = 1$, the classical logistic differential equation is obtained. And when ν tends to 0, the Gompertz equation is given. The incorporation of both the function $\alpha(\cdot)$ and the power ν allows for more flexible S-shaped curves to model growth phenomena over time.

3.1 Tumor removal

Radiotherapy, chemotherapy, and direct retrieval of a fraction of a tumor mass are some of the main techniques used to treat cancer. The first two treatments have a prolonged effect of tumor destruction, whereas the latter can be modeled via a delta-type impulse function because of the sudden extraction of the tumor mass with respect to the total treatment. Let us model this problem as (10), where the parameter vectors are chosen as follows:

- The initial tumor size $X_0 \sim \mathrm{N}|_{(0,1)}(0.15, 0.01)$, where $\mathrm{N}|_{(0,1)}$ is a normal distribution truncated on the interval $(0, 1)$.

- Variables q and m will be given the same deterministic values as in the previous example: $q = 1$ and $m = 4$.

- We consider $r \sim \mathrm{N}|_{(0,1)}(0.15, 0.0075)$, $\nu \sim \mathrm{Unif}(1, 1.25)$ and $K \sim \mathrm{Unif}(0.9, 1)$.

- We are going to consider 5 removals with equally distributed intensity given by $\Gamma \sim \mathrm{N}|_{\mathbb{R}^+}(2, 0.01)$, at times $T_{\mathrm{Tumor}} = \{t_1 = 15, t_2 = 25, t_3 = 35, t_4 = 45, t_5 = 55\}$.

Figure 1, shows the mean and 95%-confidence intervals according to the prefixed parameters and removal times. It is seen how, after each removal, the tumor size grows according to the un-removed size of the tumor. Interestingly, the confidence interval amplitude before each removal is higher than the uncertainty after the removal. Indeed, since all removals are distributed as $\Gamma \sim \mathrm{N}|_{\mathbb{R}^+}(2, 0.01)$, it is easily seen that each removal takes away half of the tumor (in average), thus reducing the uncertainty after each removal time. This case allows having a long-time prediction with a reduced level of uncertainty while still considering random impulses. This is further seen in Figure 2, where the PDF given as the solution of the Liouville equation in this particular problem setting is shown in every simulated time in T_{Tumor}.

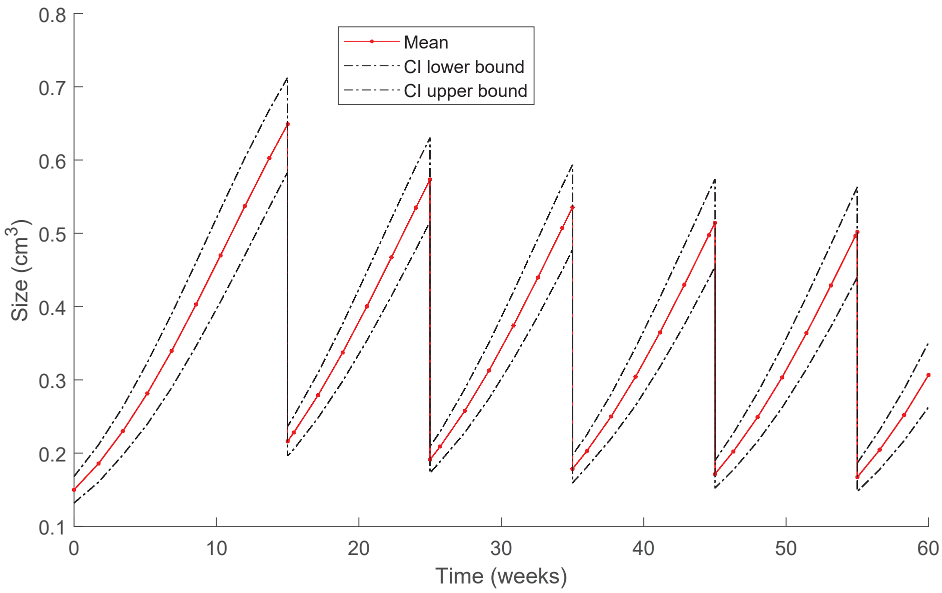

Figure 1: Time evolution of the mean tumor size and a 95% confidence interval with several extractions.

4 Conclusions

In this contribution, we have obtained a pathwise solution to a general random differential equation with a finite number of random-intensity, state-dependent, impulsive terms, with the usual assumptions on the regularity of the field function. Furthermore, we have determined the evolution

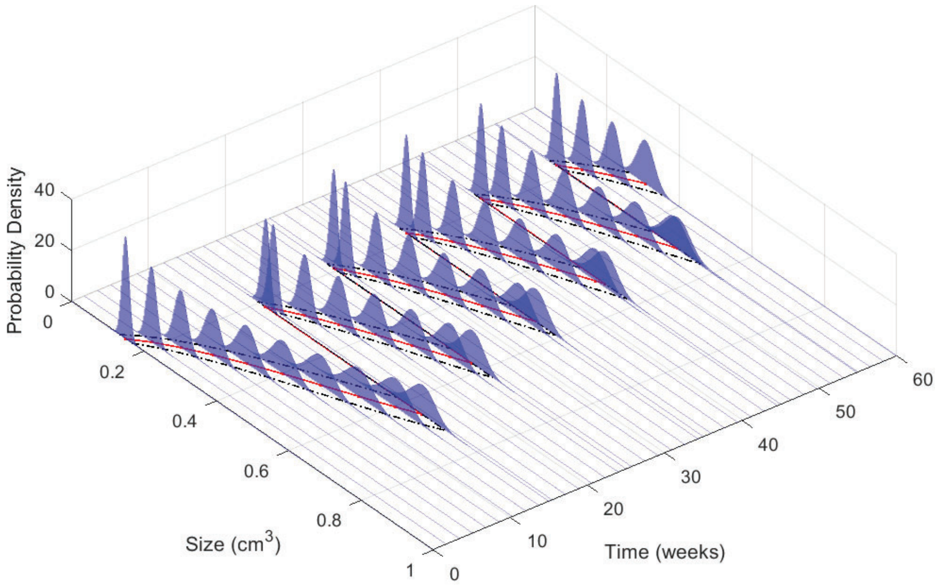

Figure 2: Full view of the PDF evolution simulations at the corresponding time values in T_{Tumor}, together with the mean (red) and 95% confidence intervals (dashed, black). Compare with Figure 1.

of the first probability density function of the solution stochastic process by combining the Liouville equation and the Random Variable Transformation theorem. We have applied our general theoretical findings to a mathematical model emerging from the generalized logistic model with natural growth altered by impulsive terms acting contrarily to its natural dynamics.

As a final note, the work on which this presentation was based has been submitted to a scientific journal as a research article.

Acknowledgments

This work has been partially supported by the Spanish Ministerio de Economía, Industria y Competitividad (MINECO), the Agencia Estatal de Investigación (AEI) and Fondo Europeo de Desarrollo Regional (FEDER UE) grant PID2020–115270GB–I00, the Generalitat Valenciana (grant AICO/2021/302) and by el Fondo Social Europeo y la Iniciativa de Empleo Juvenil EDGJID/2021/185. Vicente Bevia acknowledges the doctorate scholarship granted by Programa de Ayudas de Investigación y Desarrollo (PAID), Universitat Politècnica de València.

References

[1] V. Bevia, C. Burgos, J.C. Cortés, A. Navarro, and R.J. Villanueva. Analysing Differential Equations with Uncertainties via the Liouville-Gibbs Theorem: Theory and Applications, pages 1–23. Springer Singapore, Singapore, 2020. ISBN 978-981-15-8498-5. doi: 10.1007/978-981-15-8498-5 1. URL https://doi.org/10.1007/978-981-15-8498-5_1.

[2] V. Bevia, C. Burgos, J.C. Cortés, A. Navarro-Quiles, and R.-J. Villanueva. Uncertainty quantification analysis of the biological Gompertz model subject to random fluctuations in all its parameters. Chaos, Solitons & Fractals, 138:109908, 2020. doi: https://doi.org/10.1016/j.chaos.2020.109908.

[3] P. Dyke. An Introduction to Laplace Transforms and Fourier Series. Springer London, 2014. ISBN 978-1-4471-6394-7.

[4] C. Gasquet and P. Witomski. Fourier Analysis and Applications. Filtering, Numerical Computation, Wavelets. Springer, 1998.

[5] N. Hritonenko and Yu. Yatsenko. Bang-bang, impulse, and sustainable harvesting in age-structured populations. Journal of Biological Systems, 20(02):133–153, 2012. doi: 10.1142/S0218339012500088.

[6] M. Loève. Probability Theory I. Springer New York, NY, New York, 1977. ISBN 978-0-387-90210-4.

[7] Y. Ram, E. Dellus-Gur, M. Bibi, K. Karkare, U. Obolski, M.W. Feldman, T.F. Cooper, J. Berman, and L. Hadany. Predicting microbial growth in a mixed culture from growth curve data. Proceedings of the National Academy of Sciences, 116(29):14698–14707, 2019. ISSN 0027-8424. doi: 10.1073/pnas.1902217116. URL https://www.pnas.org/content/116/29/14698.

[8] T. Soong. Random Differential Equations in Science and Engineering. Academic Press, New York, 1973. ISBN 9780126548501.

[9] X.Huang and B. Yang. Improving energy harvesting from impulsive excitations by a nonlinear tunable bistable energy harvester. Mechanical Systems and Signal Processing, 158:107797, 2021. ISSN 0888-3270. doi: 10.1016/j.ymssp.2021.107797.

[10] X. Zhang, Z. Shuai, and K. Wang. Optimal impulsive harvesting policy for single population. Nonlinear Analysis: Real World Applications, 4(4):639–651, 2003. ISSN 1468-1218. doi: https://doi.org/10.1016/S1468-1218(02)00084-6.