# MODELLING FOR ENGINEERING & HUMAN BEHAVIOUR 2022

# PROCEEDINGS


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# Dynamical analysis of a family of Traub-type iterative methods for solving nonlinear problems

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## 1 Introduction

The problem of calculating the roots of nonlinear functions arises in any scientific and technological application. Due to the increasing volume of data available, the problems modeled by these applications are of larger dimensions and therefore more difficult to solve.

Most processes require the resolution of a system of nonlinear equations of the form $F(X) = 0$, where $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a sufficiently Fréchet differentiable function in $D$, being $D$ an open convex set.

Very often the complexity of the system $F(X) = 0$ prevent its analytical resolution, and as a consequence its solution is approximated numerically using iterative methods. There is a large classical literature regarding this issue, especially dedicated to the scalar case of solving nonlinear equations [1, 2]. For this case, a lot of iterative methods have been designed with very high orders of convergence that allow obtaining quality approximations to the solutions of the problems.

However, for multidimensional nonlinear problems it is important to take into account that the dimensions of the problem difficult the efficiency of iterative schemes to approximate their roots, so that it is more frequent to use methods that, although they have lower orders of convergence, require a lower computational cost and therefore are usually more efficient. In this sense, we recall Traub's iterative scheme [3] with cubic order of convergence and the following iterative expression:

$$\begin{aligned} y^{(k)} &= x^{(k)} - [F'(x^{(k)})]^{-1} F(x^{(k)}), \\ x^{(k+1)} &= y^{(k)} - [F'(x^{(k)})]^{-1} F(y^{(k)}), \end{aligned} \qquad k = 0, 1, 2, \ldots, \tag{1}$$

where $F'(x^{(k)})$ is the Jacobian matrix of $F$ in the $k$-th iteration.

In addition to the order of convergence, the methods can be compared in terms of their efficiency and computational cost. In this sense, Ostrowski introduced in [4] the efficiency index, defined by $EI = p^{1/d}$, where $p$ is the order of convergence of the iterative method and $d$ is the number of different functional evaluations performed on each iteration of the algorithm. Traub's method computes two evaluations of $F$ at the points $x^{(k)}$ and $y^{(k)}$, that is $2n$ functional evaluations, and a Jacobian matrix, with $n^2$ functional evaluations. Therefore, its efficiency index is

$$EI = 3^{1/(n^2+2n)}.$$

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Furthermore, Cordero et al. in [5] introduced the computational efficiency index defined by $CI = p^{1/(d+op)}$, where $op$ is the number of products-quotients required per iteration. In Traub's method we must take into account that each iteration requires solving two different linear systems with the same matrix of coefficients, $F'(x^{(k)})$. These kind of systems are solved by Gaussian elimination and need $\frac{1}{3}n^3 + n^2 - \frac{1}{3}n$ products and quotients. As in Traub's method both systems have the same matrix of coefficients, the total number of products-quotients is only $\frac{1}{3}n^3 + 2n^2 - \frac{1}{3}n$. Then, we have

$$CI = 3^{1/(\frac{1}{3}n^3+3n^2+\frac{5}{3}n)}.$$

In this work, we propose to generalise Traub's method to a two-step family of iterative schemes for solving nonlinear problems with a real parameter in its iterative structure. To this end, in Section 2 we present the iterative structure of the family and we analyze its order of convergence. We devote Section 3 to check numerically the performance of some members of the family to solve two bidimensional systems of nonlinear equations. Finally, we end this paper with some conclusions.

## 2 Design and convergence of $TM\alpha$ family

Based on Traub's iterative method, in this section we propose a generalization of the scheme by including a real parameter $\alpha \neq 0$. Our aim is to keep the order of convergence and efficiency of Traub's scheme, but at the same time to design a family of methods from which to select the most stable ones for each problem. That is, to select from among the different methods of the family obtained when selecting different values of $\alpha$ those that best approximate the solution and even improve the performance of Traub's scheme.

From Traub's iterative expression in (1), we include the sequence of points

$$z^{(k)} = x^{(k)} + \alpha(y^{(k)} - x^{(k)}), \qquad k = 0, 1, 2, \ldots$$

with a parameter $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and then by making some changes in the second step of (1), we obtain the following family of methods:

$$\begin{aligned} y^{(k)} &= x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \\ z^{(k)} &= x^{(k)} + \alpha(y^{(k)} - x^{(k)}), \\ x^{(k+1)} &= y^{(k)} - \frac{1}{\alpha^2}[F'(x^{(k)})]^{-1}\left((\alpha-1)F(x^{(k)}) + F(z^{(k)})\right), \end{aligned} \qquad k = 0, 1, 2, \ldots \tag{2}$$

Let us note that for each value of $\alpha \in \mathbb{R}$ in (2) we obtain a different method belonging to the iterative family that is denoted by $TM\alpha$. In addition, Traub's method is obtained for $\alpha = 1$.

We can observe that each iteration of $TM\alpha$ family requires three different functional evaluations: two of the vectorial functions $F(x^{(k)})$ and $F(z^{(k)})$, and one of the Jacobian matrix $F'(x^{(k)})$. The following result shows the error equation of $TM\alpha$ family and the sufficient conditions to prove that it has cubic order of convergence for any value of the parameter.

**Theorem 11.** *Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently differentiable function in an open convex set $D$ and let us denote by $x^* \in D$ a solution of $F(x) = 0$, such that $F'$ is continuous and nonsingular in $x^*$. Then, if the initial estimation $x^{(0)}$ is close enough to $x^*$, family $TM\alpha$ converges to $x^*$ with order of convergence three for any value of $\alpha \neq 0$, being its error equation*

$$e^{(k+1)} = \left(2C_2^2 + (\alpha-1)C_3\right)e^{(k)^3} + \mathcal{O}(e^{(k)^4}),$$

*where $e^{(k)} = x^{(k)} - x^*$ is the error in each iteration and $C_j = \frac{1}{j!}[F'(x^*)]^{-1}F^{(j)}(x^*)$, $j \geq 2$.*

All the methods of $TM\alpha$ family converge cubically, so all of them have the same efficiency index and computational efficiency index than Traub's method.

After generalizing Traub's iterative scheme, next section is devoted to check numerically the performance of some members of the family. The results are also compared with Traub's method in order to verify that the proposed family includes methods that even improve the performance of Traub's one.

# 3 Numerical experiments

Next we consider two polynomial systems to test the performance of $TM\alpha$ family. For the numerical implementation we have selected three members of the family, corresponding to $\alpha = 1$ (Traub's method), $\alpha = 10$ and $\alpha = -10$.

As mentioned in Theorem 11, the initial estimate $x^{(0)}$ needs to be close to the solution of the problem to guarantee convergence. Since all the iterative methods require an initial estimate $x^{(0)}$, we have previously performed a study of the basins of attraction of the selected methods for each numerical example. This dynamical study is based on the multidimensional real dynamical tools described in [6, 7]. The basins of attraction are the initial estimates that converge to the roots of the nonlinear function. The study of these initial estimates allows us to compare the performance of the different methods of the family and to determine those that have a greater number of initial estimates converging to the solution we are trying to approximate. All numerical developments in this section have been performed in two dimensions, that is, two nonlinear equations with two variables ($x_1$ and $x_2$) in order to be able to represent the basins of attraction of the methods in the dynamical planes.

In the dynamical planes we represent each variable on the coordinate axes. We define a grid of points in the plane, so that each point represents an initial estimate to start calculating iterations of each method. When there is convergence to the root on the nonlinear function, the point is represented in a colour. In other case, the point is represented in black. Therefore, with the dynamical planes we can see the set of points that if taken as initial estimation will converge to the root and we can compare these sets for different methods.

### Example 1

Let us consider the following polynomial system

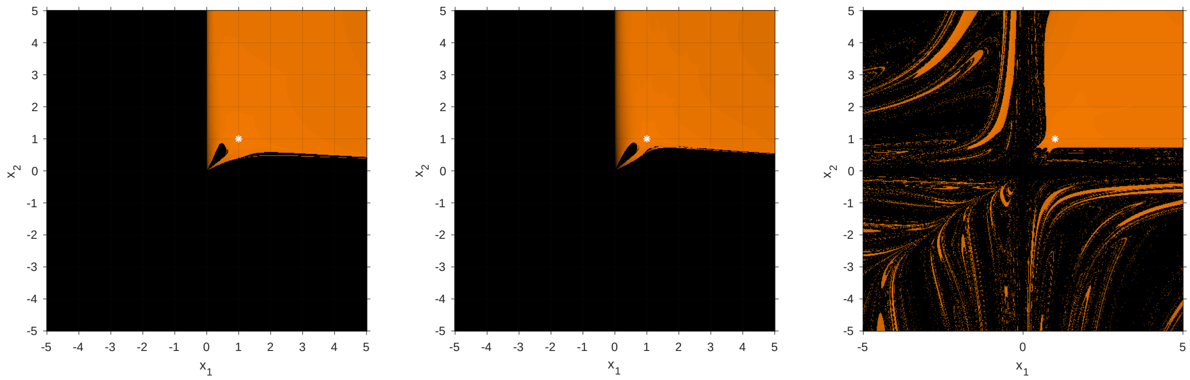
$$\begin{cases} x_1^2 x_2 &=& 1, \\ x_2^2 x_1 &=& 1, \end{cases}$$

whose only real root is $x^* = (1, 1)$. We are going to approximate $x^*$ using methods of $TM\alpha$ family.

Figure 1 represents the dynamical planes of Example 1, corresponding to Traub's method, and the iterative schemes obtained for $\alpha = 10$ and $\alpha = -10$. These plots have been generated taking a mesh of $500 \times 500$ points. The convergence is set when $||x^{(k)} - x^*|| < 10^{-10}$ or the method reach a maximum of 50 iterations. When there is convergence to $x^*$, the point is depicted in orange. We also represent the root of the nonlinear system with a white star.

We can see in Figure 1 that the method for $\alpha = -10$ improves the stability of Traub's scheme and the corresponding to $\alpha = -10$ as the number of initial estimations converging to $x^*$ (orange colour) is greater in Figure 1a than in Figures 1b and 1c.

Taking into account the basins of attraction of Figure 1, we have selected to test the methods some initial estimations taken from different regions in the plane. Table 1 shows the numerical results obtained for solving Example 1. The table includes the number of iterations until the




(a) $\alpha = 1$ (Traub) (b) $\alpha = 10$ (c) $\alpha = -10$

Figure 1: Dynamical planes for Example 1.

stopping criteria is reached, the value of $||F(x^{(k+1)})||$ in the last iteration, the difference $||x^{(k+1)} - x^{(k)}||$ between the two last iterates and an approximation to the theoretical order of convergence of the methods by means of the ACOC, defined in [8] by

$$p \approx \frac{\ln\left(||x^{(k+1)} - x^{(k)}||/||x^{(k)} - x^{(k-1)}||\right)}{\ln\left(||x^{(k)} - x^{(k-1)}||/||x^{(k-1)} - x^{(k-2)}||\right)}, \qquad k = 2, 3, \ldots$$

The convergence is set when $||x^{(k+1)} - x^{(k)}|| < 10^{-10}$ or $||F(x^{(k+1)})|| < 10^{-10}$, and the iterations stop when there is no convergence to the roots after 50 iterations. We write nc to indicate this case.

| $x^{(0)}$ | $\alpha$ | iter | $\lVert F(x^{(k+1)})\rVert$ | $\lVert x^{(k+1)} - x^{(k)}\rVert$ | ACOC |
|---|---|---|---|---|---|
| $\begin{pmatrix}1.5\\1.5\end{pmatrix}$ | 1 | 4 | 6.38446e-28 | 5.97037e-10 | 2.95792 |
| | 10 | 4 | 5.83313e-17 | 1.98129e-6 | 2.888 |
| | -10 | 3 | 6.55344e-26 | 2.97059e-9 | 2.1447 |
| $\begin{pmatrix}4\\3\end{pmatrix}$ | 1 | 5 | 3.8879e-5 | 2.10075e-14 | 2.88947 |
| | 10 | 6 | 1.27539e-13 | 2.94634e-5 | 3.71231 |
| | -10 | 4 | 5.01225e-12 | 9.9243e-5 | 2.91059 |
| $\begin{pmatrix}-0.5\\2\end{pmatrix}$ | 1 | nc | - | - | - |
| | 10 | nc | - | - | - |
| | -10 | 9 | 7.50067e-17 | 2.44021e-6 | 2.99672 |

Table 1: Numerical results for Example 1.

We can observe in Table 1 that the results agree with the dynamical planes (Figure 1). The best results approximating the solution of the problem are obtained for $\alpha = -10$, since it requires fewer iterations and in some cases it is the only method of the three considered that is convergent. In general, when the methods are convergent, we obtain approximations very close to the solution and with an ACOC close to the cubic order of convergence obtained in Section 2.

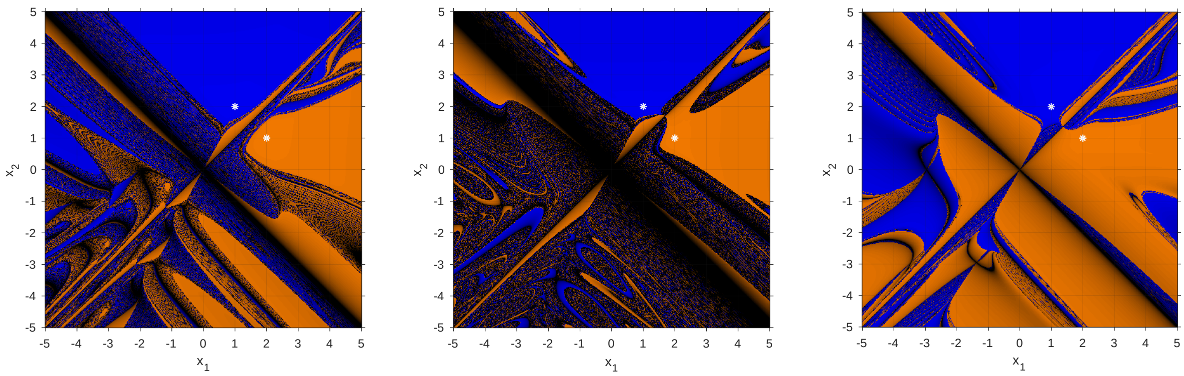
### Example 2

Now we consider a polynomial system with two real roots:

$$\left\{\begin{array}{rcl} x_1^3+x_2^3 &=& 9, \\ x_1^2x_2+x_1x_2^2 &=& 6. \end{array}\right.$$

The real solutions of the system are denoted by $x_1^*=(1,2)$ and $x_2^*=(2,1)$.

For the implementation of the dynamical planes we follow the same criteria as in Figure 1, but for Example 2 we have denoted in blue or orange the convergence to $x_1^*$ or $x_2^*$, respectively. Figure 2 shows the dynamical planes of Example 2 and the methods of $TM\alpha$ for $\alpha=\{1,10,-10\}$. We can see again that the basins of attraction of Traub and $\alpha=10$ are smaller since there are more points in the plane represented in black.



(a) $\alpha=1$ (Traub) (b) $\alpha=10$ (c) $\alpha=-10$

Figure 2: Dynamical planes for Example 2.

As the considered polynomial system has two real roots, we show in different tables the numerical results obtained when we are trying to approximate each solution. In this sense, we have select from Figure 2 initial points represented in blue or orange for Tables 3 and 4 in order to analyze the convergence to $x_1^*$ or $x_2^*$, respectively. Similar results to those in Table 1 are obtained, since the method obtained for $\alpha=-10$ is always convergent to the corresponding root even in cases where the other methods are not convergent.

| $x^{(0)}$ | $\alpha$ | iter | $\|\|F(x^{(k+1)})\|\|$ | $\|\|x^{(k+1)}-x^{(k)}\|\|$ | ACOC |
|---|---|---|---|---|---|
| $\begin{pmatrix}1.5\\3\end{pmatrix}$ | 1 | 4 | 4.88318e-27 | 9.43998e-10 | 2.95792 |
| | 10 | 4 | 4.46148e-16 | 3.13269e-6 | 2.888 |
| | -10 | 3 | 5.01241e-25 | 4.69692e-9 | 2.1447 |
| $\begin{pmatrix}-4.5\\1\end{pmatrix}$ | 1 | nc | - | - | - |
| | 10 | nc | - | - | - |
| | -10 | 10 | 9.60177e-18 | 1.33838e-6 | 2.1109 |
| $\begin{pmatrix}4.5\\-1\end{pmatrix}$ | 1 | nc | - | - | - |
| | 10 | nc | - | - | - |
| | -10 | 10 | 1.87412e-21 | 6.53728e-8 | 2.88154 |

Table 2: Numerical results for Example 2 approximating $x_1^*=(1,2)$.

| $x^{(0)}$ | $\alpha$ | iter | $\|\|F(x^{(k+1)})\|\|$ | $\|\|x^{(k+1)} - x^{(k)}\|\|$ | ACOC |
|---|---|---|---|---|---|
| $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ | 1 | 4 | 6.06358e-11 | 2.08324e-4 | 2.68829 |
| | 10 | 5 | 3.8871e-17 | 1.79614e-6 | 3.02901 |
| | -10 | 4 | 1.89885e-22 | 3.04763e-8 | 2.71486 |
| $\begin{pmatrix} 2 \\ -3.5 \end{pmatrix}$ | 1 | 7 | 3.50169e-11 | 1.73502e-4 | 2.59589 |
| | 10 | nc | - | - | - |
| | -10 | 10 | 5.54428e-22 | 4.35592e-8 | 2.88959 |
| $\begin{pmatrix} -1.5 \\ -4 \end{pmatrix}$ | 1 | nc | - | - | - |
| | 10 | nc | - | - | - |
| | -10 | 10 | 4.33411e-28 | 4.43146e-10 | 2.88404 |

Table 3: Numerical results for Example 2 approximating $x_2^* = (2, 1)$.

# 4 Conclusions

In this paper, we propose a family of iterative methods with a real parameter $\alpha$. This family is a generalization of Traub's iterative scheme, obtained for $\alpha = 1$. After determining the sufficient conditions to obtain a cubic order of convergence for any value of the parameter, a numerical analysis of some schemes of the family when applied to approximate the solution of systems of nonlinear equations is carried out. This study allows verifying that the theoretical developments are correct and that the family includes methods that improve the stability of the classical Traub's iterative scheme.

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