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Uniformly refinable maps

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Abstract

We introduce the notion of uniformly refinable map for compact, Hausdorff spaces, as a generalization of refinable maps originally defined for metric continua by Jo Ford (Heath) and Jack W. Rogers, Jr., Refinable maps, Colloq. Math., 39 (1978), 263-269.

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1. Introduction

Refinable maps are introduced by Jo Ford (Heath) and Jack W. Rogers, Jr. in [5] for compact metric spaces. These maps have been studied extensively. We extend this concept to compact Hasudorff spaces as *uniformly refinable maps* using uniformities. Results known for metric compact spaces are stated and proved for compact Hausdorff spaces, we put the reference of the results for the metric case. Recently, interest for compact Hausdorff spaces has been increasing.

The paper has six sections. After this Introduction 1 and the section of Definitions 2, in Section 3, we introduce uniformly refinable maps for compact Hausdorff spaces and show the equivalence of both definitions for compact metric spaces (Theorem 3.1). We prove that the composition of uniformly refinable maps is uniformly refinable (Theorem 3.4). We show that uniformly refinable

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maps are weakly confluent (Corollary 3.7) and they are monotone when the range space is locally connected (Corollary 3.9). Given Hausdorff continua X and Y, we prove the equivalence of decomposabilty (Corollary 3.11) and atriodicity (Theorem 3.13) of X and Y in the presence of uniformly refinable maps. In Section 4, we show that aposyndesis (Corollary 4.2), semi-aposyndesis (Theorem 4.7) and mutual aposyndesis (Theorem 4.8) of Hausdorff continua are preserved by uniformy refinable maps. Also, we prove results involving F. Burton Jones' set funcitons $\mathcal{T}$ (Theorem 4.1) and $\mathcal{K}$ (Theorem 4.5). In Section 5, we consider irreducible Hausdorff continua. Given Hausdorff continua X and Y, we show the equivalence of property (B_2) (Theorem 5.3) and hereditary indecomposability (Theorem 5.7) of X and Y. Property (B_2) is used in [11] to prove the equivalence of irreducibility for metric continua, we were not able to show the equivalence of property (B_2) and irreducibility for Hausdorff continua. We prove that if the range of a uniformly refinable map between Hausdorff continua is irreducible, then the domain is irreducible (Theorem 5.6). In Section 6, we show that the induced maps of a uniformly refinable map on hyperspaces are uniformly refinable maps (Theorem 6.7) and that the induced maps of a monotonly uniformly refinable map on hyperspaces are monotonly uniformly refinable maps (Theorem 6.8).

2. Definitions

A topological space Z is a *Hausdorff space*, if for each pair of points z_1 and z_2 of Z, there exist two disjoint open subsets W_1 and W_2 of Z such that z_1 is in W_1 and z_2 belongs to W_2. The topological space Z is a *compact space* provided that for each family $\{W_\gamma\}_{\gamma\in\Gamma}$ of open subsets of Z satisfyaing that $Z \subset \bigcup_{\gamma\in\Gamma} W_\gamma$, there exists a finite subfamily $\{W_{\gamma_1}, \dots, W_{\gamma_n}\}$ of $\{W_\gamma\}_{\gamma\in\Gamma}$ such that $Z \subset \bigcup_{j=1}^{n} W_{\gamma_j}$.

If Z is a Hausdorff topological space, given a subset A of Z, the interior of A is denoted by $Int_Z(A)$, the boundary of A by $Bd_Z(A)$, and the closure of A by $Cl_Z(A)$. Let X and Z be topological spaces if f is a surjective function from X onto Z, we write $f\colon X \twoheadrightarrow Z$; if the function is not necessarily surjective, we write $f\colon X \to Z$.

A *map* is a continuous function. A surjective map $f\colon X \twoheadrightarrow Y$ between topological spaces is *monotone* provided that $f^{-1}(C)$ is connected for every connected subset C of Y. The map f is *weakly confluent* if for each compact connected subset Q of Y, there exists a component K of $f^{-1}(Q)$ such that $f(K) = Q$.

A *compactum* is a compact Hausdorff space. A *subcompactum* of a space Z is a compactum contained in Z. A compactum X is *connected im kleinen at a point* p provided that for each open subset A of X, with $p \in A$, there exists a connected subcompactum K of X such that $p \in Int_X(K) \subset K \subset A$. A *continuum* is a connected compactum. A *subcontinuum* of a space Z is a continuum contained in Z. A continuum is *decomposable* if it is the union of two of its proper subcontinua. A continuum is *indecomposable* if it is not decomposable. A point x of a continuum X is a *weak cut point* provided that there exists two points z_1 and z_2 of X such that for each subcontinuum M of X with $\{z_1, z_2\} \subset M$, we have that $x \in M$. A point x of a continuum X is a *cut point* if $X \setminus \{x\}$ is not connected.

Let Z be a Hausdorff space. If V and W are subsets of $Z \times Z$, then

$$-V = \{(z', z) \in Z \times Z \mid (z, z') \in V\}$$

and

$$V + W = \{(z, z'') \mid \text{there exists } z' \in Z \text{ such that } (z, z') \in V \text{ and } (z', z'') \in W\}.$$

We write $1V = V$ and for each positive integer n, $(n+1)V = nV + 1V$.

The diagonal of Z is the set $\Delta_Z = \{(z, z) \mid z \in Z\}$. An *entourage* of Δ_Z is a subset V of $Z \times Z$ such that $\Delta_Z \subset V$ and $V = -V$. The family of all entourages of the diagonal of Z is denoted by $\mathfrak{D}_Z$. If $V \in \mathfrak{D}_Z$ and $z \in Z$, then $B_Z(z, V) = \{z' \in Z \mid (z, z') \in V\}$. If A is a nonempty subset of Z and $V \in \mathfrak{D}_Z$, then $B_Z(A, V) = \bigcup\{B_Z(a, V) \mid a \in A\}$. If $V \in \mathfrak{D}_Z$ and $(z, z') \in V$, then we write $\rho_Z(z, z') < V$. If $(z, z') \notin V$, then we write $\rho_Z(z, z') \geq V$. If A is a nonempty subset of Z and $\rho_Z(a, a') < V$ for each pair of points of A, we write $\delta_Z(A) < V$. If there exist two points a_1 and a_2 in A such that $\rho_Z(a_1, a_2) \geq V$, then we write $\delta_Z(A) \geq V$. We have that if z, z' and z'' are points of Z, and V and W belong to $\mathfrak{D}_Z$ then the following hold [4, p. 426]:

(*i*) $\rho_Z(z, z) < V$.

(*ii*) $\rho_Z(z, z') < V$ if and only if $\rho_Z(z', z) < V$.

(*iii*) If $\rho_Z(z, z') < V$ and $\rho_Z(z', z'') < W$, then $\rho_Z(z, z'') < V + W$.

Let Z be a Tychonoff space. A *uniformity* on Z is a subfamily $\mathfrak{U}$ of $\mathfrak{D}_Z \setminus \{\Delta_Z\}$ such that:

(1) If $V \in \mathfrak{U}$, $W \in \mathfrak{D}_Z$ and $V \subset W$, then $W \in \mathfrak{U}$.

(2) If V and W belong to $\mathfrak{U}$, then $V \cap W \in \mathfrak{U}$.

(3) For every $V \in \mathfrak{U}$, there exists $W \in \mathfrak{U}$ such that $2W \subset V$.

(4) $\bigcap\{V \mid V \in \mathfrak{U}\} = \Delta_Z$.

A *uniform space* is a pair $(Z, \mathfrak{U})$ consisting of a nonempty set Z and a uniformity on the set Z. For any uniformity $\mathfrak{U}$ on a set Z, the family $\mathfrak{O} = \{G \subset Z \mid \text{for every } z \in G, \text{ there exists } V \in \mathfrak{U} \text{ such that } B(z, V) \subset G\}$ is a topology on the set Z [4, 8.1.1]. The topology $\mathfrak{O}$ is called the *topology induced by the uniformity* $\mathfrak{U}$. Observe that, if the topology of Z is induced by a uniformity $\mathfrak{U}$ and $V \in \mathfrak{U}$, then, by [4, 8.1.3], $Int_Z(B(z, V))$ is a neighbourhood of z.

Notation 2.1. Let Z be a nonempty set Z and let $\mathfrak{U}$ be a uniformity on Z. If $V \in \mathfrak{U}$, then $\mathfrak{C}_Z(V) = \{B_Z(z, V) \mid z \in Z\}$.

Remark 2.2. Note that by [4, 8.3.13], for every compactum Z, there exists a unique uniformity $\mathfrak{U}_Z$ on Z that induces the original topology of Z.

Notation 2.3. Let X be a compact Hausdorff space and let $\mathfrak{U}_X$ be the unique uniformity of X that induces its topology. If Y is a subspace of X and $U \in \mathfrak{U}_X$, then $U|_Y = U \cap (Y \times Y)$.

Notation 2.4. Let X and Y be compacta and let $\mathfrak{U}_X$ and $\mathfrak{U}_Y$ be the unique uniformities that induce the topology of X and Y, respectively (Remark 2.2). Let

$$\Lambda_{(X,Y)} = \{(V,U) \mid V \in \mathfrak{U}_X \text{ and } U \in \mathfrak{U}_Y\}.$$

If (V,U) and (V',U') are elements of $\Lambda_{(X,Y)}$, we say that

$$(V,U) \geq (V',U') \text{ if and only if } V \subset V' \text{ and } U \subset U'.$$

Then $(\Lambda_{(X,Y)}, \geq)$ is a directed set.

3. Uniformly Refinable Maps

Refinable maps are introduced in [5] for metric compact spaces. We extend this concept to compacta. We recall the appropriate defintions:

Let X and Y be compact metric spaces, let $f\colon X \twoheadrightarrow Y$ be a surjective map and let $\varepsilon > 0$. Then f is an *ε-map* if for every $y \in Y$ and each pair points x_1 and x_2 of $f^{-1}(y)$, $d_X(x_1,x_2) < \varepsilon$. If $g\colon X \twoheadrightarrow Y$ is another map, then g is *ε-near* f provided that for each $x \in X$, $d_Y(f(x),g(x)) < \varepsilon$. Now, f is a *refinable map* provided that for each $\varepsilon > 0$, there exists an ε-map $g\colon X \twoheadrightarrow Y$ that is ε-near f.

Let X and Y be compacta, let $f\colon X \twoheadrightarrow Y$ be a surjective map and let $V \in \mathfrak{U}_X$. Then f is a *V-map* if for every $y \in Y$ and each pair of points x_1 and x_2 of $f^{-1}(y)$, $\rho_X(x_1,x_2) < V$. If $g\colon X \twoheadrightarrow Y$ is another map and $U \in \mathfrak{U}_Y$, then g is *U-near* f provided that for each $x \in X$, $\rho_Y(f(x),g(x)) < U$. Now, f is a *uniformly refinable map* if for each $V \in \mathfrak{U}_X$ and every $U \in \mathfrak{U}_Y$, there exists a V-map $g\colon X \twoheadrightarrow Y$ that is U-near f. The map g is called a *uniform (V,U)-refinament of f*.

Our first result shows that both definitions coincide for compact metric spaces.

Theorem 3.1. *Let X and Y be compact metric spaces and let $f\colon X \twoheadrightarrow Y$ be a surjective map. Then f is refinable if and only if f is uniformly refinable.*

Proof. Suppose f is refinable, let $V \in \mathfrak{U}_X$ and let $U \in \mathfrak{U}_Y$. Let $V' \in \mathfrak{U}_X$ and $U' \in \mathfrak{U}_Y$ be such that $2V' \subset V$ and $2U' \subset U$. Note that $\mathcal{W}_c = \{Int_X(B_X(x,V')) \mid x \in X\}$ and $\mathcal{U}_c = \{Int_Y(B_Y(y,U')) \mid y \in Y\}$ are open covers of X and Y, respectively. Let λ_X and λ_Y be Lebesgue numbers for $\mathcal{W}_c$ and $\mathcal{U}_c$, respectively, [13, Theorem 1.6.6]. Let $\lambda = \min\{\lambda_X,\lambda_Y\}$. Since f is refinable, there exists a λ-map $g\colon X \twoheadrightarrow Y$ that is λ-near f. Let $y \in Y$ and let x_1 and x_2 be points of $g^{-1}(y)$. Since g is a λ-map, $d_X(x_1,x_2) < \lambda$. Hence, since λ is a Lebesgue number for $\mathcal{W}_c$, there exists $x \in X$ such that $\{x_1,x_2\} \subset B_X(x,V')$. Thus, $\rho_X(x_1,x) < V'$ and $\rho_X(x,x_2) < V'$. Since $2V' \subset V$, $\rho_X(x_1,x_2) < V$. Therefore, g is a V-map. Now, let $x \in X$. Since g is λ-near f, $d_Y(f(x),g(x)) < \lambda$. Hence, since λ is a Lebesgue number for $\mathcal{U}_c$, there exists $y \in Y$ such that $\{f(x), g(x)\} \subset B_Y(y, U')$. Thus, $\rho_Y(f(x), y) < U'$ and $\rho_Y(y, g(y)) < U'$. Since $2U' \subset U$, $\rho_Y(f(x), g(x)) < U$. Hence, g is U-near to f. Therefore, f is a uniformly refinable map.

Next, assume f is uniformly refinable, and let $\varepsilon > 0$. Let $\mathcal{W}_c = \{\mathcal{V}_{\frac{\varepsilon}{2}}(x) \mid x \in X\}$ and let $\mathcal{U}_c = \{\mathcal{V}_{\frac{\varepsilon}{2}}(y) \mid y \in Y\}$. By [14, Theorem 1.3.6], there exist $V \in \mathfrak{U}_X$ and $U \in \mathfrak{U}_Y$ such that $\mathfrak{C}_X(V)$ refines $\mathcal{W}_c$ and $\mathfrak{C}_Y(U)$ refines $\mathcal{U}_c$. Since f is a uniformly refinable map, there exists a V-map $g\colon X \twoheadrightarrow Y$ that is U-near f. Let $y \in Y$ and let x_1 and x_2 be points of $g^{-1}(y)$. Since g is a V-map, $\rho_X(x_1, x_2) < V$. Thus, $x_2 \in B_X(x_1, V)$. Since $\mathfrak{C}_X(V)$ refines $\mathcal{W}_c$, there exists $x \in X$ such that $B_X(x_1, V) \subset \mathcal{V}_{\frac{\varepsilon}{2}}(x)$. Hence, $d_X(x_1, x_2) \leq d_X(x_1, x) + d_X(x, x_2) < \varepsilon$. Therefore, g is an ε-map. Now, let $x \in X$. Since g is U-near f, $\rho_Y(f(x), g(x)) < U$. Thus, $g(x) \in B_Y(f(x), U)$. Since $\mathfrak{C}_Y(U)$ refines $\mathcal{U}_c$, there exists $y \in Y$ such that $B_Y(f(x), U) \subset \mathcal{V}_{\frac{\varepsilon}{2}}(y)$. Hence, $d_Y(f(x), g(x)) \leq d_Y(f(x), y) + d_Y(y, g(x)) < \varepsilon$. Thus, g is ε-near to f. Therefore, f is a refinable map. □

Theorem 3.2. *Let X and Y be compacta and let $f\colon X \twoheadrightarrow Y$ be a surjective map. Then f is a uniformly refinable map if and only if for each $V \in \mathfrak{U}_X$ and every $U \in \mathfrak{U}_Y$, there exists an $Int_{X\times X}(V)$-map $g\colon X \twoheadrightarrow Y$ that is U-near f.*

Proof. Suppose f is uniformly refinable. Let $V \in \mathfrak{U}_X$ and let $U \in \mathfrak{U}_Y$. Let $V' \in \mathfrak{U}_X$ be such that $2V' \subset V$. Since f is uniformly refinable, there exists a V'-map $g\colon X \twoheadrightarrow Y$ that is U-near f. By [14, Lemma 1.3.10], $V' \subset Int_{X\times X}(2V')$. Hence, g is an $Int_{X\times X}(V)$-map that is U-near f. The converse implication is clear. □

The following lemma is an extension of [13, Lemma 2.4.20] to compacta.

Lemma 3.3. *Let X and Y be compacta, let $V \in \mathfrak{U}_X$ and let $f\colon X \twoheadrightarrow Y$ be a surjective $Int_{X\times X}(V)$-map. Then there exists $U \in \mathfrak{U}_Y$ such that, for each subset B of Y with $\delta_Y(B) < U$, we have that $\delta_X(f^{-1}(B)) < Int_{X\times X}(V)$.*

Proof. Suppose the result is not true. Hence, for each $U \in \mathfrak{U}_Y$, there exists a subset B_U of Y with $\delta_Y(B_U) < U$ such that $\delta_X(f^{-1}(B_U)) \geq Int_{X\times X}(V)$. Hence, for each $U \in \mathfrak{U}_Y$, there exist $x_U, x'_U \in f^{-1}(B_U)$ with $\rho_X(x_U, x'_U) \geq Int_{X\times X}(V)$. Then we have two nets $\{x_U\}_{U\in\mathfrak{U}_Y}$ and $\{x'_U\}_{U\in\mathfrak{U}_Y}$ in X. Since X is a compactum, without loss of generality, we assume that these nets converge to the points x and x' of X, respectively [4, 3.1.23 and 1.6.1]. Note that $\rho_X(x, x') \geq Int_{X\times X}(V)$. Since, by construction, $\rho_Y(f(x_U), f(x'_U)) < U$, for each $U \in \mathfrak{U}_Y$, we have, by continuity, that $f(x) = f(x')$ [16, 3.38]. A contradiction to the fact that f is $Int_{X\times X}(V)$-map. □

Theorem 3.4 ([3, Lemma 3.2])**.** *Let X, Y and Z be compacta and let $f\colon X \twoheadrightarrow Y$ and $h\colon Y \twoheadrightarrow Z$ be maps. If f and h are uniformly refinable, then $h \circ f$ is uniformly refinable.*

Proof. Let $V_X \in \mathfrak{U}_X$, and $W_Z \in \mathfrak{U}_Z$. Let $W'_Z \in \mathfrak{U}_Z$ be such that $2W'_Z \subset W_Z$. Since h is uniformly continuous [14, Theorem 1.3.15], there exists $U_Y \in \mathfrak{U}_Y$ such that if y_1 and y_2 are points of Y with $\rho_Y(y_1, y_2) < U_Y$, then $\rho_Z(h(y_1), h(y_2)) < W'_Z$. Let $g_{XY} \colon X \twoheadrightarrow Y$ be an $Int_{X\times X}(V_X)$-map that is U_Y-near f. By Lemma 3.3, there exists $U'_Y \in \mathfrak{U}_Y$ such that if B is a nonempty subset of Y with $\delta_Y(B) < U'_Y$, then $\delta_X(g_{XY}^{-1}(B)) < Int_{X\times X}(V_X)$. Without loss of generality, we assume that $U'_Y \subset U_Y$. Let $g_{YZ} \colon Y \twoheadrightarrow Z$ be a U'_Y-map that is W'_Z-near h.

Let x be a point of X. Then $\rho_Y(f(x), g_{XY}(x)) < U_Y$. Also, we have that $\rho_Z(h(f(x)), h(g_{XY}(x)) < W'_Z$ and $\rho_Z(h(g_{XY}(x)), g_{YZ}(g_{XY}(x)) < W'_Z$. Hence, $\rho_Z(h(f(x)), g_{YZ}(g_{XY}(x))) < 2W'_Z$. Thus, $\rho_Z(h(f(x)), g_{YZ}(g_{XY}(x))) < W_Z$. Therefore, $g_{YZ} \circ g_{XY}$ is W_Z-near $h \circ f$. Let z be a point of Z. Since g_{YZ} is a U'_Y-map, we have that $\delta_Y(g_{YZ}^{-1}(z)) < U'_Y$. Hence, by the choice of U'_Y, we have that $\delta_X(g_{XY}^{-1}(g_{YZ}^{-1}(z))) < Int_{X\times X}(V)$. Therefore, by Theorem 3.2, $h \circ f$ is uniformly refinable. □

Theorem 3.5 ([2, Theorem *A*])**.** *Let X_1, X_2, Z_1 and Z_2 be compacta. Suppose X_1 is homeomorphic to X_2 and Z_1 is homeomorphic to Z_2. If $V \in \mathfrak{U}_{X_1}$, then there exists $U \in \mathfrak{U}_{X_2}$ such that for any U-map from X_2 onto Z_2, there exists a V-map $g_1 \colon X_1 \twoheadrightarrow Z_1$. Moreover, if Z_2 is a monotone (weakly confluent, respectively) image of X_2, then there exists a monotone (weakly confluent, respectively) map from X_1 onto Z_1.*

Proof. Let $h_1 \colon X_1 \twoheadrightarrow X_2$ and $h_2 \colon Z_1 \twoheadrightarrow Z_2$ be homeomorphisms and let $V \in \mathfrak{U}_{X_1}$. Since h_1^{-1} is uniformly continuous [14, Theorem 1.3.15], there exists $U \in \mathfrak{U}_{X_2}$ such that for each pair of points x_2 and x'_2 of X_2 with $\rho_{X_2}(x_2, x'_2) < U$, $\rho_{X_1}(h_1^{-1}(x_2), h_1^{-1}(x_2)) < V$. Let $g_2 \colon X_2 \twoheadrightarrow Z_2$ be a U-map and let $g_1 \colon X_1 \twoheadrightarrow Z_1$ be given by $g_1 = h_2^{-1} \circ g_2 \circ h_1$. Then g_1 is a map. We show that g_1 is a V-map. Let $z \in Z_1$ and let x_1 and x'_1 be points of $g_1^{-1}(z)$. This implies that $g_1(x_1) = g_1(x'_1)$. Hence, $h_2^{-1} \circ g_2 \circ h_1(x_1) = h_2^{-1} \circ g_2 \circ h_1(x'_1)$. Since h_2 is a homeomorphism, we obtain that $g_2 \circ h_1(x_1) = g_2 \circ h_1(x'_1)$. Now, since g_2 is a U-map, we have that $\rho_{X_2}(h_1(x_1), h_1(x'_1)) < U$. Thus, by the construction of U, $\rho_{X_1}(x_1, x'_1) < V$. Therefore, g_1 is a V-map.

The second part of the theorem follows from the definition of g_1 and the fact that h_1 and h_2 are homeomorphisms. □

Theorem 3.6 ([5, Theorem 1])**.** *Let X and Y be compacta and let $f \colon X \twoheadrightarrow Y$ be a uniformly refinable map. Then for every subcontinuum Q of Y, there exists a subcontinuum K_Q of X such that $f(K_Q) = Q$ and K_Q contains $f^{-1}(Int_Y(Q))$.*

Proof. Since f is uniformly refinable, for each $V \in \mathfrak{U}_X$ and every $U \in \mathfrak{U}_Y$, there exists a V-map $g_{(V,U)} \colon X \twoheadrightarrow Y$ that is U-near f. By Notation 2.4, $(\Lambda_{(X,Y)}, \geq)$ is a directed set.

Let Q be a subcontinuum of Y. Then $\{g_{(V,U)}^{-1}(Q)\}_{(V,U)\in\Lambda_{(X,Y)}}$ is a net of closed subsets of X. Since 2^X is a compactum [14, Theorem 1.6.7], by [4, 3.1.23 and 1.6.1], there exists a cofinal subset $\Lambda'_{(X,Y)}$ of $\Lambda_{(X,Y)}$ such that $\{g_{(V,U)}^{-1}(Q)\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges to an element K_Q of 2^X. We prove that

K_Q is connected. Suppose this is not true. Then there exist two nonempty proper disjoint closed subsets K_1 and K_2 of X such that $K_Q = K_1 \cup K_2$. Since X is a normal space, there exist disjoint open subsets A_1 and A_2 of X such that $K_1 \subset A_1$ and $K_2 \subset A_2$. By [4, Lemma 8.2.5], there exist $V_{A_1}, V_{A_2} \in \mathfrak{U}_X$ such that $B_X(K_1, V_{A_1}) \subset A_1$ and $B_X(K_2, V_{A_2}) \subset A_2$. Let $V_K = V_{A_1} \cap V_{A_2}$. Then $B_X(K_1, V_K) \subset B_X(K_1, V_{A_1})$ and $B_X(K_2, V_K) \subset B_X(K_2, V_{A_2})$. Since $\{g^{-1}_{(V,U)}(Q)\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges to K_Q, there exists $(V_0, U_0) \in \Lambda'_{(X,Y)}$ such that $g^{-1}_{(V,U)}(Q) \subset Int_X(B_X(K_1, V_K) \cup Int_X(B_X(K_2, V_K))$, for every $(V,U) \geq (V_0, U_0)$. Observe that $g^{-1}_{(V,U)}(Q) \cap Int_X(B_X(K_1, V_K)) \neq \varnothing$ and $g^{-1}_{(V,U)}(Q) \cap Int_X(B_X(K_2, V_K)) \neq \varnothing$, for each $(V,U) \geq (V_0, U_0)$. Let $(V, U_0) \in \Lambda'_{(X,Y)}$ be such that $(V, U_0) \geq (V_0, U_0)$ and $(V, U_0) \geq (V_K, U_0)$. Then $g^{-1}_{(V,U_0)}(Q) \subset Int_X(B_X(K_1, V_K) \cup Int_X(B_X(K_2, V_K))$, $g^{-1}_{(V,U_0)}(Q) \cap Int_X(B_X(K_1, V_K)) \neq \varnothing$ and $g^{-1}_{(V,U_0)}(Q) \cap Int_X(B_X(K_2, V_K)) \neq \varnothing$. Since Q is connected, there exists $y \in Q$ such that $g^{-1}_{(V,U_0)}(y) \cap Int_X(B_X(K_1, V_K)) \neq \varnothing$ and $g^{-1}_{(V,U_0)}(y) \cap Int_X(B_X(K_2, V_K)) \neq \varnothing$. This implies that $g_{(V,U_0)}$ is not a V-map, a contradiction. Therefore, K_Q is connected.

Next, we show that $f(K_Q) = Q$. Let $x \in K_Q$. Then, for each $(V,U) \in \Lambda'_{(X,Y)}$, there exists $x_{(V,U)} \in g^{-1}_{(V,U)}(Q)$ such that the net $\{x_{(V,U)}\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges to x. By continuity, $\{f(x_{(V,U)})\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges to $f(x)$ [16, 3.38]. Since each $g_{(V,U)}$ is U-near f, $\rho_Y(f(x_{(V,U)}), g_{(V,U)}(x_{(V,U)})) < U$. Since $\Lambda'_{(X,Y)}$ is a cofinal subset of $\Lambda_{(X,Y)}$, we have that $\bigcap\{U \mid (V,U) \in \Lambda'_{(X,Y)}\} = \Delta_Y$. Since $g_{(V,U)}(x_{(V,U)}) \in Q$, for all $(V,U) \in \Lambda'_{(X,Y)}$, and Q is a closed subset of X, we obtain that $f(x) \in Q$. Therefore, $f(K_Q) \subset Q$. Now, let $y \in Q$. For each $(V,U) \in \Lambda'_{(X,Y)}$, let $x_{(V,U)} \in g^{-1}_{(V,U)}(Q)$ be such that $g_{(V,U)}(x_{(V,U)}) = y$. Without loss of generality, we assume that the net $\{x_{(V,U)}\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges to a point x of X [4, 3.1.23 and 1.6.1]. Note that, in fact, $x \in K_Q$. Since each $g_{(V,U)}$ is U-near f, $\rho_Y(f(x_{(V,U)}), g_{(V,U)}(x_{(V,U)})) = \rho_Y(f(x_{(V,U)}), y) < U$. Hence, the net $\{f(x_{(V,U)}\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges to y. By continuity, $\{f(x_{(V,U)}\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges to $f(x)$. Hence, $f(x) = y$, and $Q \subset f(K_Q)$. Therefore, $f(K_Q) = Q$.

Suppose x is a point of X such that $f(x) \in Int_Y(Q)$. Then there exists $(V_0, U_0) \in \Lambda(X,Y)'$ such that $g_{(V,U)}(x) \in Int_Y(Q)$, for every $(V,U) \geq (V_0, U_0)$. Since $\{g_{(V,U)}(x)\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges to $f(x)$, we have that $x \in g^{-1}_{(V,U)}(Q)$, for each $(V,U) \geq (V_0, U_0)$. Hence, $x \in K_Q$, and $f^{-1}(Int_Y(Q)) \subset K_Q$. $\square$

Corollary 3.7 ([5, Corollary 1.1])**.** *Let X and Y be compacta and let $f\colon X \twoheadrightarrow Y$ be a uniformly refinable map. Then f is weakly confluent.*

Theorem 3.8 ([8, Lemma 1])**.** *Let X and Y be compacta, let $f\colon X \twoheadrightarrow Y$ be a uniformly refinable map, let Q be a subcontinuum of Y, and let K_Q be the subcontinuum given by Theorem 3.6. If L is a subcontinuum of Q, then there exists a subcontinuum M of K_Q such that $f(M) = L$ and K contains $(f|_{K_Q})^{-1}(Int_Q(L))$. In particular, $f|_{K_Q}\colon K_Q \twoheadrightarrow Q$ is weakly confluent.*

Proof. By the proof of Theorem 3.6, there exists a cofinal subset $\Lambda'_{(X,Y)}$ of $\Lambda_{(X,Y)}$ (Notation 2.4) such that $\{g^{-1}_{(V,U)}(Q)\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges to K_Q. Let L be a subcontinuum of Q. Since 2^X is a compactum [14, Theorem 1.6.7], without loss of generality, we assume that $\{g^{-1}_{(V,U)}(L)\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges to a subcontinuum M of X ([4, 3.1.23 and 1.6.1] and [19, Theorem 4]). In fact, M is a subcontinuum of K_Q. By the proof of Theorem 3.6, $f(M) = L$ and $f^{-1}(Int_Y(L)) \subset M$.

The proof of the fact that $(f|_{K_Q})^{-1}(Int_Q(L)) \subset M$ is similar to the one given in Theorem 3.6 taking the appropriate intersections. □

Corollary 3.9 ([5, Corollary 1.2])**.** *Let X and Y be compacta and let $f\colon X \twoheadrightarrow Y$ be a uniformly refinable map. If Y is connected im kleinen at a point p, then $f^{-1}(p)$ is connected. Hence, f is monotone if Y is locally connected.*

Proof. Suppose Y is connected im kleinen at p, but $f^{-1}(p)$ is not connected. Then there exist two disjoint open subsets A and B of X such that $f^{-1}(p) \subset A \cup B$, $f^{-1}(p) \cap A \neq \varnothing$ and $f^{-1}(p) \cap B \neq \varnothing$. The compactness of X implies that $O = Y \setminus f(X \setminus (A \cup B))$ is an open subset of Y and $p \in O$. Since Y is connected im kleinen at p, there exists a subcontinuum Q of Y such that $p \in Int_Y(Q) \subset O$. By Theorem 3.6, there exists a subcontinuum K of X such that $f^{-1}(p) \subset f^{-1}(Int_Y(Q)) \subset K$ and $f(K) = Q$. Thus, $K \subset f^{-1}(Q) \subset f^{-1}(O) \subset A \cup B$, a contradiction to the connectedness of K. Therefore, $f^{-1}(p)$ is connected.

The second part follows form the well-known fact that a compactum is locally connected if and only if it is connected im kleinen at each of its points and [14, Lemma 1.4.46]. □

From a similar proof to the one given for Corollary 3.9, we obtain:

Corollary 3.10 ([8, Corollary, p. 2])**.** *Let X and Y be compacta, let $f\colon X \twoheadrightarrow Y$ be a uniformly refinable map, let Q be a subcontinuum of X of Y, and let K_Q be the subcontinuum given by Theorem 3.6. If Q is connected im kleinen at a point p, then $(f|_{K_Q})^{-1}(p)$ is connected. Hence, $f|_{K_Q}$ is monotone if Q is locally connected.*

Corollary 3.11 ([5, Corollary 1.3])**.** *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be a uniformly refinable map. Then X is decomposable if and only if Y is decomposable.*

Proof. Suppose Y is decomposable. By [14, Lemma 1.4.34], Y contains a proper subcontinuum Q with nonempty interior. By Theorem 3.6, there exists a subcontinuum K of X such that $f(K) = Q$ and $f^{-1}(Int_Y(Q)) \subset K$. Hence, $Int_X(K) \neq \varnothing$. Therefore, X is decomposable [14, Lemma 1.4.34].

Suppose X decomposable there exist two proper subcontinua K and L of X such that $X = K \cup L$. Let $x_K \in K \setminus L$ and let $x_L \in L \setminus K$. Thus, there exists $V \in \mathfrak{U}_X$ such that $B_X(x_K, V) \subset K \setminus L$ and $B_X(x_L, V) \subset L \setminus K$. Since f is uniformly refinable, there exists a V-map $g\colon X \twoheadrightarrow Y$. Note that

$g^{-1}(g(x_K)) \subset B_X(x_K, V)$ and $g^{-1}(g(x_L)) \subset B_X(x_L, V)$. Hence, $g(K)$ is a proper subcontinuum of Y such that $g(x_K) \in Int_Y(g(K))$. Therefore, Y is a decomposable continuum [14, Lemma 1.4.34]. □

Corollary 3.12 ([11, Proposition (3.4)-(1)])**.** *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be a uniformly refinable map. If X is hereditarily decomposable, then Y is hereditarily decomposable.*

Proof. Let L be a nondegenerate subcontinuum of Y. By Corollary 3.7, f is weakly confluent. Hence, there exists a subcontinuum K' of X such that $f(K') = L$. By [12, Theorem 7, p. 171], there exists a subcontinuum K of K' such that $f(K) = L$ and $f(W) \neq L$, for each proper subcontinuum W of K. Since K is decomposable, there exist two proper subcontinua K_1 and K_2 of K such that $K = K_1 \cup K_2$. Hence, $L = f(K_1) \cup f(K_2)$. Since $f(K_1) \neq L$ and $f(K_2) \neq L$, we have that L is decomposable. Therefore, Y is hereditarily decomposable. □

A *triod* is a continuum X for which there exist three subcontinua K, L and M of X such that $X = K \cup L \cup M$, $K \cap L \cap M \neq \varnothing$, $K \setminus (L \cup M) \neq \varnothing$, $L \setminus (K \cup M) \neq \varnothing$ and $M \setminus (K \cup L) \neq \varnothing$. A continuum is *atriodic* if no subcontinuum of it is a triod.

Theorem 3.13 ([11, Proposition (3.4)-(3)])**.** *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be a uniformly refinable map. Then X is atriodic if and only if Y is atriodic.*

Proof. Suppose Y is atriodic and X contains a triod. Let K, L and M be subcontinua of X such that $K \cap L \cap M \neq \varnothing$, $K \setminus (L \cup M) \neq \varnothing$, $L \setminus (K \cup M) \neq \varnothing$ and $M \setminus (K \cup L) \neq \varnothing$. Consider $f(K)$, (L) and $f(M)$. Since Y is atriodic, without loss of generality, we assume that $f(M) \subset f(K) \cup f(L)$. Let $x_M \in M \setminus (K \cup L)$. Then there exists $V \in \mathfrak{U}_X$ such that $B_X(x_M, V) \cap (K \cup L) = \varnothing$. Since f is uniformly refinable, there exists a V-map $g\colon X \twoheadrightarrow Y$. Note that, on one hand, $g^{-1}(g(x_M)) \subset B_X(x_M, V)$ and, on the other hand, there exists a point $x \in (K \cup L)$ such that $g(x) = g(x_M)$, a contradiction to the fact that g is a V-map. Therefore, X is atriodic.

Now, assume that Y contains three subcontinua L_1, L_2 and L_3 such that $L_1 \cap L_2 \cap L_3 \neq \varnothing$, $L_1 \setminus (L_2 \cup L_3) \neq \varnothing$, $L_2 \setminus (L_1 \cup L_3) \neq \varnothing$ and $L_3 \setminus (L_1 \cup L_2) \neq \varnothing$. Let $y_1 \in L_1 \setminus (L_2 \cup L_3)$, $y_2 \in L_2 \setminus (L_1 \cup L_3)$, $y_3 \in L_3 \setminus (L_1 \cup L_2)$, and $y \in L_1 \cap L_2 \cap L_3$. Consider the directed set $\Lambda_{(X,Y)}$ (Notaion 2.4). Since f is uniformly refinable, for each $V \in \mathfrak{U}_X$ and every $U \in \mathfrak{U}_Y$, there exists a V-map $g_{(V,U)}\colon X \twoheadrightarrow Y$ that is U-near f. Then we have the nets $\{g^{-1}_{(V,U)}(L_j)\}_{(V,U)\in\Lambda_{(X,Y)}}$, $j \in \{1, 2, 3\}$, $\{g^{-1}_{(V,U)}(y_j)\}_{(V,U)\in\Lambda_{(X,Y)}}$ and $\{g^{-1}_{(V,U)}(y)\}_{(V,U)\in\Lambda_{(X,Y)}}$ of nonempty closed subsets of X. As in the proof of Theorem 3.6, there exists a cofinal subset $\Lambda'_{(X,Y)}$ of $\Lambda_{(X,Y)}$ such that each of the nets $\{g^{-1}_{(V,U)}(L_j)\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges to a subcontinuum K_j, $j \in \{1, 2, 3\}$, of X, with the property that $f(K_j) = L_j$, $j \in \{1, 2, 3\}$. Also, each of the nets $\{g^{-1}_{(V,U)}(y_j)\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges to points

x_j, $j \in \{1,2,3\}$ of X and $\{g^{-1}_{(V,U)}(y)\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges to a point x of X. Observe that $x \in K_1 \cap K_2 \cap K_3$ and $x_j \in K_j \setminus (K_l \cup K_m)$, with $j,l,m \in \{1,2,3\}$, $j \neq l$, $j \neq m$ and $l \neq m$. Therefore, $K_1 \cup K_2 \cup K_3$ is a triod in X. □

Theorem 3.14 ([5, Theorem 2]). *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be a uniformly refinable map. If y is a point of Y that cuts Y, then some point of $f^{-1}(y)$ is a weak cut point of X.*

Proof. Suppose that $Y \setminus \{y\} = A \cup B$, where A and B are disjoint open subsets of Y. Let $x_A \in f^{-1}(A)$ and let $x_B \in f^{-1}(B)$. Consider the directed set $(\Lambda_{(X,Y)}, \geq)$ (Notation 2.4). Then for each $(V,U) \in \Lambda_{(X,Y)}$, there exists a V-map $g_{(V,U)}\colon X \twoheadrightarrow Y$ that is U-near f. We have the net $\{g^{-1}_{(V,U)}(y)\}_{(V,U)\in\Lambda_{(X,Y)}}$. Since X is compact, without loss of generality, we assume that the net converges to a point x of X [4, 3.1.23 and 1.6.1]. Observe that $x \in f^{-1}(y)$.

Now, suppose Z is a subcontinuum of X such that $\{x_A, x_B\} \subset Z$ and $x \in X \setminus Z$. Let $V \in \mathfrak{U}_X$ be such that $B_X(x,V) \cap Z = \varnothing$, and let $U \in \mathfrak{U}_Y$ be such that $B_Y(f(x_A),U) \subset A$ and $B_Y(f(x_B),U) \subset B$. Let $g_{(V,U)}\colon X \twoheadrightarrow Y$ be a V-map that is U-near f. Note that for each $x' \in g^{-1}_{(V,U)}(y)$, we have that $\rho_X(x',x) < V$. Thus, $g^{-1}_{(V,U)}(y) \subset B_X(x,V)$ and $g^{-1}_{(V,U)}(y) \cap Z = \varnothing$. Since $g_{(V,U)}$ is U-near f, $g_{(V,U)}(x_A) \in B_Y(f(x_A),U) \subset A$ and $g_{(V,U)}(x_B) \in B_Y(f(x_B),U) \subset B$. Therefore, Z is a subcontinuum of X that intersects both disjoint open subsets $g^{-1}_{(V,U)}(A)$ and $g^{-1}_{(V,U)}(B)$ and $Z \subset g^{-1}_{(V,U)}(A) \cup g^{-1}_{(V,U)}(B)$, a contradiction. Therefore, x is weak cut point of X. □

Corollary 3.15 ([5, Corollary 2.1]). *Let X and Y be metric continua, where X is locally connected and has no cut points, and let $f\colon X \twoheadrightarrow Y$ be a uniformly refinable map. Then for each $y \in Y$, $X \setminus f^{-1}(y)$ is connected. Hence, Y has no cut points.*

Proof. Suppose there exists a point y of Y such that $X \setminus f^{-1}(y)$ is not connected. Since f is monotone (Corollary 3.9), $Y \setminus \{y\}$ is not connected. By Theorem 3.14, some point x of $f^{-1}(y)$ is a weak cut point of X. Hence, x is a cut point [10, p. 143], since X locally connected (and locally connected continua are aposyndetic). This is a contradiction to our hypothesis. □

Let X be a continuum. A subcontinuum K of X is *terminal* provided that for each subcontinuum L of X such that $K \cap L \neq \varnothing$, we have that either $K \subset L$ or $L \subset K$.

Theorem 3.16 ([8, Theorem 4]). *Let X and Y be compacta, let $f\colon X \twoheadrightarrow Y$ be a uniformly refinable map, let Q be a subcontinuum of Y with $Int_Y(Q) \neq \varnothing$, and let K_Q be the subcontinuum given by Theorem 3.6. If K_Q is a terminal subcontinuum of X, then Q is a terminal subcontinuum of Y.*

Proof. Assume that Q is not a terminal subcontinuum of Y. Then there exists a subcontinuum L of Y such that $L \cap Q \neq \varnothing$, $L \setminus Q \neq \varnothing$ and $Q \setminus L \neq \varnothing$. By Corollary 3.7, f is weakly confluent. Hence, there exists a component R of

$f^{-1}(L \cup Q)$ such that $f(R) = L \cup Q$. Since $f^{-1}(Int_Y(Q)) \cap R \neq \varnothing$, we have that $R \cap K_Q \neq \varnothing$ (Theroem 3.6). Hence, $K_Q \subset R$. Let C be a component of $Cl_X(R \setminus K_Q)$. Since $Int_X(K_Q) \neq \varnothing$ (Theorem 3.6), we have that $K_Q \setminus C \neq \varnothing$. By [14, Theorem 1.4.36], $C \cap K_Q \neq \varnothing$. Also, by construction $C \setminus K_Q \neq \varnothing$. Thus, K_Q is not a terminal subcontinuum of X. □

Let X and Y be compacta and let $f\colon X \twoheadrightarrow Y$ be a surjective map. Then f *is uniformly monotonely refinable* provided that for each $V \in \mathfrak{U}_X$ and every $U \in \mathfrak{U}_Y$ there exists a monotone uniform (V, U)-refinament of f.

Theorem 3.17 ([6, Lemma 2])**.** *Let X and Y be compacta and let $f\colon X \twoheadrightarrow Y$ be a uniformly monotonely refinable map. If Y is locally conncted, then X is locally connected.*

Proof. Let x be a point of X and let A be an open subset of X with $x \in A$. Then there exists $V \in \mathfrak{U}_X$ such that $B_X(x, V) \subset A$. Since f is uniformly monotonely refinable, by Theorem 3.2, there exists a monotone $Int_{X\times X}(V)$-map $g\colon X \twoheadrightarrow Y$. By Lemma 3.3, there exists $U \in \mathfrak{U}_Y$ such that if B is a nonempty subset of Y and $\delta_Y(B) < U$, then $\delta_X(g^{-1}(B)) < Int_{X\times X}(V)$. Since Y is locally connected, there exists an open connected subset B of Y sucht that $g(x) \in B$ and $\delta_Y(B) < U$. Hence, $g^{-1}(B)$ is an open connected subset of X such that $x \in g^{-1}(B)$ and $\delta_X(g^{-1}(B)) < Int_{X\times X}(V)$. Thus, $g^{-1}(B) \subset B_X(x, V) \subset A$. Therefore, X is locally connected. □

4. Aposyndesis

Let X be a continuum and let p and q be two distinct points of X. Then X is *aposyndetic at p with respecto to q* provided that there exists a subcontinuum K of X such that $p \in Int_X(K) \subset K \subset X \setminus \{q\}$. X is *aposyndetic at p* if X is aposyndetic at p with respect to each point $q \in X \setminus \{p\}$. Finally, X is *aposyndetic* if it is aposyndetic at each of its points.

F. Burton Jones defined what are now known as the set functions $\mathcal{T}$ and $\mathcal{K}$. He used these set functions to study aposyndesis on continua. Both functions have been used to study continua, mainly $\mathcal{T}$ [14]. These set functions are considered "dual" functions. Next, we define these set functions.

Given a continuum X, we define the *set function* $\mathcal{T}_X$ as follows: if A is a subset of X, then

$$\mathcal{T}_X(A) = X \setminus \{x \in X \mid \text{there exists a subcontinuum } K \text{ of } X \\ \text{such that } x \in Int_X(K) \subset K \subset X \setminus A\}.$$

Let us observe that for any subset A of X, $\mathcal{T}_X(A)$ is a closed subset of X and $A \subset \mathcal{T}_X(A)$. Also, if K is a subcontinuum of X, then $\mathcal{T}_X(K)$ is a subcontinuum of X [14, Theorem 2.1.27].

For the continuum X, the *set function* $\mathcal{K}_X$ is defined as follows: if A is a subset of X, then

$$\mathcal{K}_X(A) = \bigcap\{K \mid K \text{ is a subcontinuum of } X \text{ such that } A \subset Int_X(K)\}.$$

We also have that for any subset A of X, $\mathcal{K}_X(A)$ is a closed subset of X and $A \subset \mathcal{K}_X(A)$. It is not necessarily true that if L is a subcontinuum of X, then $\mathcal{K}_X(L)$ is connected.

Compare the following result with [14, Theorem 2.1.51, (b) and (c)].

Theorem 4.1 ([7, Theorem 1′]). *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be a uniformly refinable map. Then the following hold:*

(1) *For each closed subset W of X, $f\mathcal{T}_X(W) \subset \mathcal{T}_Y f(W)$.*

(2) *For each closed subset Z of Y, there exists a closed subset W of X such that $f\mathcal{T}_X(W) = \mathcal{T}_Y(Z)$.*

Proof. We show (1). Suppose there exists a point $x \in \mathcal{T}_X(W)$ such that $f(x) \in Y \setminus \mathcal{T}_Y f(W)$. Then there exists a subcontinuum L of Y such that $f(x) \in Int_Y(L) \subset L \subset Y \setminus f(W)$. By Theorem 3.6, there exists a subcontinuum K of X such that $f(K) = L$ and $f^{-1}(Int_Y(L)) \subset K$. Note that $x \in Int_X(K) \subset K \subset X \setminus W$. Thus, $x \in X \setminus \mathcal{T}_X(W)$, a contradiction. Therefore, $f\mathcal{T}_X(W) \subset \mathcal{T}_Y f(W)$.

To prove (2), consider the directed set $\Lambda_{(X,Y)}$ (Notaion 2.4). Since f is uniformly refinable, for each $V \in \mathfrak{U}_X$ and every $U \in \mathfrak{U}_Y$, there exists a V-map $g_{(V,U)}\colon X \twoheadrightarrow Y$ that is U-near f. Then we have the net $\{g^{-1}_{(V,U)}(Z)\}_{(V,U)\in\Lambda_{(X,Y)}}$ of nonemtpy closed subsets of X. As in the proof of Theorem 3.6, there exists a cofinal subset $\Lambda'_{(X,Y)}$ of $\Lambda_{(X,Y)}$ such that the net $\{g^{-1}_{(V,U)}(Z)\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges to a nonempty closed subset W of X, with the property that $f(W) = Z$. By part (1), we have that $f\mathcal{T}_X(W) \subset \mathcal{T}_Y f(W) = \mathcal{T}_Y(Z)$. We show the reverse inclusion. If $\mathcal{T}_Y(Z) = Z$, then $f(W) = Z = \mathcal{T}_Y(Z) \subset f\mathcal{T}_X(W)$ (since $W \subset \mathcal{T}_X(W)$, $f(W) \subset f\mathcal{T}_X(W)$). Now, suppose there exists a point $z \in \mathcal{T}_Y(Z) \setminus Z$. We have the net $\{g^{-1}_{(V,U)}(z)\}_{(V,U)\in\Lambda'_{(X,Y)}}$. Since X is compact, without loss of generality, we assume that this net converges to a point x of X [4, 3.1.23 and 1.6.1]. Observe that $x \in f^{-1}(z)$. Suppose $x \in X \setminus \mathcal{T}_X(W)$. Then there exists a subcontinuum K of X such that $x \in Int_X(K) \subset K \subset X \setminus W$. This implies that $W \subset X \setminus K$. Since $\{g^{-1}_{(V,U)}(Z)\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges to W and $\{g^{-1}_{(V,U)}(z)\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges to x, there exists $(V,U) \in \Lambda_{(X,Y)}$ such that $g^{-1}_{(V,U)}(Z) \subset X \setminus K$ and $g^{-1}_{(V,U)}(z) \subset Int_X(K)$. Then $g_{(V,U)}(K)$ is a subcontinuum of Y such that $z \in Int_Y(g_{(V,U)}(K)) \subset g_{(V,U)}(K) \subset Y \setminus Z$. Hence, $z \in Y \setminus \mathcal{T}_Y(Z)$, a contradiction. Thus, $x \in \mathcal{T}_X(W)$ and $z \in f\mathcal{T}_X(W)$. Therefore, $\mathcal{T}_Y(Z) \subset f\mathcal{T}_X(W)$, and $f\mathcal{T}_X(W) = \mathcal{T}_Y(Z)$. □

Corollary 4.2 ([7, Corollary, p. 368]). *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be a uniformly refinable map. If X is aposyndetic, then Y is aposyndetic.*

Proof. Let $y \in Y$. By the proof of part (2) of Theorem 4.1, we may find a point $x \in X$ such that $f(x) = y$ and $f\mathcal{T}_X(\{x\}) = \mathcal{T}_Y(\{y\})$. Since X is aposyndetic, $\mathcal{T}_X(\{x\}) = \{x\}$ [14, Theorem 2.1.34]. Thus, $f\mathcal{T}_X(\{x\}) = \{f(x)\} = \{y\}$. Hence, $\mathcal{T}_Y(\{y\}) = \{y\}$. Therefore, Y is aposyndetic [14, Theorem 2.1.34]. □

Let X be a compactum. A nonempty closed subset K of X is a $\mathcal{T}_X$-closed set if $\mathcal{T}_X(K) = K$. The family of $\mathcal{T}_X$-closed sets of X is denoted by $\mathfrak{T}(X)$. If A is a set, then $|A|$ denotes the cardinality of A.

The next result extends [14, Corollary 4.1.26] to uniformly monotonely refinable maps.

Theorem 4.3. *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be a uniformly monotonely refinable map. Then $|\mathfrak{T}(Y)| \leq |\mathfrak{T}(X)|$.*

Proof. Since f is a monotonely refinable map, there exists a monotone map $g\colon X \twoheadrightarrow Y$. Hence, by [14, Corollary 4.1.26], we have that $|\mathfrak{T}(Y)| \leq |\mathfrak{T}(X)|$. □

Let X and Y be compacta and let $f\colon X \twoheadrightarrow Y$ be a surjective map. Then f is *atomic* if for each subcontinuum K of X such that $f(K)$ is nondegenerate, we have that $K = f^{-1}f(K)$.

Next, we introduce the family of uniformly atomically refinable maps. Let X and Y be compacta and let $f\colon X \twoheadrightarrow Y$ be a surjective map. Then f *is atomically refinable* provided that for each $V \in \mathfrak{U}_X$ and every $U \in \mathfrak{U}_Y$ there exists an atomic uniform (V, U)-refinament of f.

The following theorem extends [14, Theorem 4.1.32] to uniformly atomically refinable maps and gives a partial positive answer to [14, Question 8.1.18].

Theorem 4.4. *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be an atomically refinable map. Then $|\mathfrak{T}(Y)| = |\mathfrak{T}(X)|$.*

Proof. Since f is a atomically refinable map, there exists an atomic map $g\colon X \twoheadrightarrow Y$. Hence, by [14, Theorem 4.1.32], we have that $|\mathfrak{T}(Y)| = |\mathfrak{T}(X)|$. □

The following theorem is an extension to closed sets of [7, Theorem 1″]. Its proof is similar to the one given for Theorem 4.1; we include the details to see the "duality" of $\mathcal{T}$ and $\mathcal{K}$.

Theorem 4.5. *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be a uniformly refinable map. Then the following hold:*

(1) *For each closed subset W of X, $f\mathcal{K}_X(W) \subset \mathcal{K}_Y f(W)$.*

(2) *For each closed subset Z of Y, there exists a closed subset W of X such that $f\mathcal{K}_X(W) = \mathcal{K}_Y(Z)$.*

Proof. We show (1). Suppose there exists a point $x \in \mathcal{K}_X(W)$ such that $f(x) \in Y \setminus \mathcal{K}_Y f(W)$. Then there exists a subcontinuum L of Y such that $f(x) \in Y \setminus L$ and $f(W) \subset Int_Y(L)$. By Theorem 3.6, there exists a subcontinuum K of X such that $f(K) = L$ and $f^{-1}(Int_Y(L)) \subset K$. Note that $W \subset Int_X(K)$ and $x \in X \setminus K$, a contradiction to the fact that $x \in \mathcal{K}_X(W)$. Therefore, $f\mathcal{K}_X(W) \subset \mathcal{K}_Y f(W)$.

To prove (2), consider the directed set $\Lambda_{(X,Y)}$ (Notaion 2.4). Since f is uniformly refinable, for each $V \in \mathfrak{U}_X$ and every $U \in \mathfrak{U}_Y$, there exists a V-map $g_{(V,U)}\colon X \twoheadrightarrow Y$ that is U-near f. Then we have the net $\{g^{-1}_{(V,U)}(Z)\}_{(V,U)\in\Lambda_{(X,Y)}}$ of nonemtpy closed subsets of X. As in the proof of Theorem 3.6, there exists a cofinal subset $\Lambda'_{(X,Y)}$ of $\Lambda_{(X,Y)}$ such that the net $\{g^{-1}_{(V,U)}(Z)\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges to a nonempty closed subset W of X, with the property that $f(W) = Z$. By part (1), we have that $f\mathcal{K}_X(W) \subset \mathcal{K}_Y f(W) = \mathcal{K}_Y(Z)$. We show the reverse inclusion. If $\mathcal{K}_Y(Z) = Z$, then $f(W) = Z = \mathcal{K}_Y(Z) \subset f\mathcal{K}_X(W)$ (since $W \subset \mathcal{K}_X(W)$, $f(W) \subset f\mathcal{K}_X(W)$). Now, suppose there exists a point $z \in \mathcal{K}_Y(Z) \setminus Z$. We have the net $\{g^{-1}_{(V,U)}(z)\}_{(V,U)\in\Lambda'_{(X,Y)}}$. Since X is compact, without loss of generality, we assume that this net converges to a point x of X [4, 3.1.23 and 1.6.1]. Observe that $x \in f^{-1}(z)$. Suppose $x \in X\setminus\mathcal{K}_X(W)$. Then there exists a subcontinuum K of X such that $x \in X \setminus K$ and $W \subset Int_X(K)$. Since $\{g^{-1}_{(V,U)}(Z)\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges to W and $\{g^{-1}_{(V,U)}(z)\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges to x, there exists $(V,U) \in \Lambda_{(X,Y)}$ such that $g^{-1}_{(V,U)}(Z) \subset Int_X(K)$ and $g^{-1}_{(V,U)}(z) \subset X \setminus K$. Then $g_{(V,U)}(K)$ is a subcontinuum of Y such that $Z \subset Int_Y(g_{(V,U)}(K))$ and $z \in Y \setminus g_{(V,U)}(K)$. Hence, $z \in Y \setminus \mathcal{K}_Y(Z)$, a contradiction. Thus, $x \in \mathcal{K}_X(W)$ and $z \in f\mathcal{K}_X(W)$. Therefore, $\mathcal{K}_Y(Z) \subset f\mathcal{K}_X(W)$, and $f\mathcal{K}_X(W) = \mathcal{K}_Y(Z)$. □

Remark 4.6. A proof of Corollary 4.2 may be given using Theorem 4.5 and [14, Corollary 7.7.4] which says that, in particular, if X is an aposyndetic continuum, then $\mathcal{T}_X(\{x\}) = \mathcal{K}_X(\{x\})$, for each $x \in X$.

A continuum X is *semi-aposyndetic* provided that for each pair of points p and q of X, there exists a subcontinuum K of X such that $\{p,q\}\cap Int_X(K) \neq \varnothing$ and $\{p,q\} \setminus K \neq \varnothing$.

Theorem 4.7 ([7, Theorem 2]). *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be a uniformly refinable map. If X is semi-aposyndetic, then Y is semi-aposyndetic.*

Proof. Let y_1 and y_2 be two distinct points of Y. Consider the directed set $\Lambda_{(X,Y)}$ (Notaion 2.4). Since f is uniformly refinable, for each $V \in \mathfrak{U}_X$ and every $U \in \mathfrak{U}_Y$, there exists a V-map $g_{(V,U)}\colon X \twoheadrightarrow Y$ that is U-near f. Then we have the nets $\{g^{-1}_{(V,U)}(y_1)\}_{(V,U)\in\Lambda_{(X,Y)}}$ and $\{g^{-1}_{(V,U)}(y_2)\}_{(V,U)\in\Lambda_{(X,Y)}}$. Since X is compact, without loss of generality, we assume that these nets converge to the points x_1 and x_2, respectively [4, 3.1.23 and 1.6.1]. Note that $x_1 \neq x_2$. Since X is semi-aposyndetic, either $x_1 \in X \setminus \mathcal{T}_X(\{x_2\})$ or $x_2 \in X \setminus \mathcal{T}_X(\{x_2\})$ [14, Theorem 2.1.32]. Suppose that $x_1 \in X \setminus \mathcal{T}_X(\{x_2\})$. Then there exists a subcontinuum K of X such that $x_1 \in Int_X(K) \subset K \subset X \setminus \{x_2\}$. Since $\{g^{-1}_{(V,U)}(y_1)\}_{(V,U)\in\Lambda_{(X,Y)}}$ and $\{g^{-1}_{(V,U)}(y_2)\}_{(V,U)\in\Lambda_{(X,Y)}}$ converge to x_1 and x_2, respectively, there exists $(V,U) \in \Lambda_{(X,Y)}$ such that $g^{-1}_{(V,U)}(y_1) \subset Int_X(K)$ and $g^{-1}_{(V,U)}(y_2) \subset X \setminus K$. Therefore, $g_{(V,U)}(K)$ is a subcontinuum of Y such that $y_1 \in Int_Y(g_{(V,U)}(K)) \subset g_{(V,U)}(K) \subset Y \setminus \{y_2\}$. Therefore, Y is semi-aposyndetic. □

A continuum X is *mutually aposyndetic*, if for each pair of points p and q, there exist two disjoint subcontinua K and L of X such that $p \in Int_X(K)$ and $q \in Int_X(L)$.

Let X be a continuum and let $n \in \mathbb{N}$. Then X is *n-aposyndetic* provided that for each subset F of X with at most n points and every point $x \in X \setminus F$, there exists a subcontinuum K of X such that $x \in Int_X(K) \subset K \subset X \setminus F$. Observe that this is equivalent to have $\mathcal{T}_X(F) = F$.

A similar argument to the one given for Theorem 4.7 shows:

Theorem 4.8 ([7, Theorem 3]). *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be a uniformly refinable map. Then the following hold:*

(1) *If X is mutually aposyndetic, then Y is mutually aposyndetic.*

(2) *If X is n-aposyndetic, for some $n \in \mathbb{N}$, then Y is n-aposyndetic.*

5. Irreducible Continua

Let X be a decomposable continuum. We say X has *property* (B_2) if every time $X = K \cup L \cup M$, where K, L and M are proper subcontinua such that $K \setminus (L \cup M) \neq \varnothing$, $L \setminus (K \cup M) \neq \varnothing$ and $M \setminus (K \cup L) \neq \varnothing$, we have that either $K \cap L = \varnothing$ or $K \cap M = \varnothing$ or $L \cap M = \varnothing$.

This definition is due to R. H. Sorgenfrey [20]. We present a couple of consequnces of property (B_2), which are used in the metric case to prove the converse implication of Theorem 5.4, [20, Theorem]. Lemmas 5.1 and 5.2 might be useful to prove the converse of Theorem 5.4.

Lemma 5.1 ([20, Lemma 1 (a), case $n = 2$]). *Let X be a decomposable continuum with property (B_2). If L and M are proper subcontinua of X such that $X = L \cup M$ and then $X \setminus L$ and $X \setminus M$ are connected.*

Proof. Assume $X = L \cup M$, where L and M are proper subcontinua. We show that $X \setminus L$ is connected. Suppose that this is not true. Then $X \setminus L = A \cup B$, where A and B are disjoint open subsets of X. By the Boundary Bumping Theorem [14, Theorem 1.4.36], if R is a component of $Cl_X(A)$, then $R \cap L \neq \varnothing$ and if S is a component of $Cl_X(B)$, then $S \cap L \neq \varnothing$. Let $T = \bigcup\{R \mid R \text{ is a component of } Cl_X(A)\}$ and let

$$W = \bigcup\{S \mid S \text{ is a component of } Cl_X(B)\}.$$

Thus, L, $L\cup T$ and $L\cup W$ are proper subcontinua of X such that $X = L\cup(L\cup T)\cup(L\cup W)$, $L\setminus[(L\cup T)\cup(L\cup W)] \neq \varnothing$, $(L\cup T)\setminus(L\cup W) \neq \varnothing$, $(L\cup W)\setminus(L\cup T) \neq \varnothing$, but $L \cap (L \cup T) \neq \varnothing$ $L \cap (L \cup W) \neq \varnothing$ and $(L \cup T) \cap (L \cup W) \neq \varnothing$, a contradiction. Therefore, $X\setminus L$ is connected. Similarly, $X\setminus M$ is connected. □

Lemma 5.2 ([20, Lemma 2]). *Let X be a decomposable continuum with property (B_2). Let L and M be proper subcontinua such that $X \setminus L$ and $X \setminus M$ are connected and $L \cap (X \setminus M)$ and $M \cap (X \setminus L)$ contain open subsets of X. If H is a subcontinuum of X such that $H \cap L \neq \varnothing$ and $H \cap M \neq \varnothing$, then $X \setminus (L \cup M) \subset H$.*

Proof. Let H be a subcontinuum of X such that $H \cap L \neq \varnothing$ and $H \cap M \neq \varnothing$. Assume that $X \setminus (L \cup M) \not\subset H$. Then $H \cup L \cup M$ is a proper subcontinuum of X. By an argument similar to the one given in the proof of Lemma 5.1, we have that $C = X \setminus (H \cup L \cup M)$ is connected. Note that, in this case, either $Cl_X(C) \cap (H \cup L) = \varnothing$ or $Cl_X(C) \cap (H \cup M) = \varnothing$. If $Cl_X(C) \cap (H \cup L) \neq \varnothing$ and $Cl_X(C) \cap (H \cup M) \neq \varnothing$, then $Cl_X(C)$, $H \cup L$ and $H \cup M$ would be three proper subcontinua of X such that $X = Cl_X(C) \cup (H \cup L) \cup (H \cup M)$, $Cl_X(C) \setminus [(H \cup L) \cup (H \cup M)] \neq \varnothing$, $(H \cup L) \setminus [Cl_X(C) \cup (H \cup M)] \neq \varnothing$, $(H \cup M) \setminus [Cl_X(C) \cup (H \cup L)] \neq \varnothing$, $(H \cup L) \cap (H \cup M) \neq \varnothing$, $Cl_X(C) \cap (H \cup L) \neq \varnothing$ and $Cl_X(C) \cap (H \cup M) \neq \varnothing$, a contradiction. Hence, either $Cl_X(C) \cap (H \cup L) = \varnothing$ or $Cl_X(C) \cap (H \cup M) = \varnothing$. Without loss of generality, assume that $Cl_X(C) \cap (H \cup L) = \varnothing$. Then $(X \setminus M) \cap C \neq \varnothing$ and $(X \setminus M) \cap (H \cup L) \neq \varnothing$. Thus, $X \setminus M$ is not connected, a contradiction. Therefore, $X \setminus (L \cup M) \subset H$. $\square$

A continuum X is *irreducible* provided that there exist two points x_1 and x_2 of X such that if K is a subcontinuum of X and $\{x_1, x_2\} \subset K$, then $K = X$.

The following result is, essentially, [11, Theorem (3.1)]. In [11], the author uses case $n = 2$ of [20, Theorem, p. 667] to change property (B_2) to irreducible continuum. We have one implication of [11, Theorem (3.1)] in Theorem 5.6.

Theorem 5.3. *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be a uniformly refinable map. Then X has property (B_2) if and only if Y has property (B_2).*

Proof. Suppose X does not have property (B_2). Then there exist three proper subcontinua K, L and M of X such that $X = K \cup L \cup M$, $K \setminus (L \cup M) \neq \varnothing$, $L \setminus (K \cup M) \neq \varnothing$ and $M \setminus (K \cup L) \neq \varnothing$, $K \cap L \neq \varnothing$, $K \cap M \neq \varnothing$ and $L \cap M \neq \varnothing$. Let $x_K \in K \setminus (L \cup M)$, $x_L \in L \setminus (K \cup M)$ and $x_M \in M \setminus (K \cup L)$. Let $V \in \mathfrak{U}_X$ be such that $B_X(x_K, V) \cap (L \cup M) = \varnothing$, $B_X(x_L, V) \cap (K \cup M) = \varnothing$ and $B_X(x_M, V) \cap (K \cup L) = \varnothing$. Since f is a uniformly refinable map, there exists a V-map $g\colon X \twoheadrightarrow Y$. Then $g(K)$, $g(L)$ and $g(M)$ are three subcontinua of Y such that $Y = g(K) \cup g(L) \cup g(M)$, $g(x_K) \in g(K) \setminus [g(L) \cup g(M)]$, $g(x_L) \in g(L) \setminus [g(K) \cup g(M)]$, $g(x_m) \in g(M) \setminus (g(K) \cup g(L))$, $g(K) \cap g(L) \neq \varnothing$, $g(K) \cap g(M) \neq \varnothing$ and $g(L) \cap g(M) \neq \varnothing$. Therefore, Y does not have property (B_2).

Assume Y does not have property (B_2). Then there exist three proper subcontinua K, L and M of Y such that $Y = K \cup L \cup M$, $K \setminus (L \cup M) \neq \varnothing$, $L \setminus (K \cup M) \neq \varnothing$ and $M \setminus (K \cup L) \neq \varnothing$, $K \cap L \neq \varnothing$, $K \cap M \neq \varnothing$ and $L \cap M \neq \varnothing$. Let $y_K \in K \setminus (L \cup M)$, $y_L \in L \setminus (K \cup M)$ and $y_M \in K \setminus (K \cup L)$.

Consider the directed set $\Lambda_{(X,Y)}$ (Notaion 2.4). Since f is uniformly refinable, for each $(V, U) \in \Lambda_{(X,Y)}$, there exists a V-map $g_{(V,U)}\colon X \twoheadrightarrow Y$ that is U-near f. Then we have the nets $\{g^{-1}_{(V,U)}(K)\}_{(V,U)\in\Lambda_{(X,Y)}}$, $\{g^{-1}_{(V,U)}(L)\}_{(V,U)\in\Lambda_{(X,Y)}}$, $\{g^{-1}_{(V,U)}(M)\}_{(V,U)\in\Lambda_{(X,Y)}}$, $\{g^{-1}_{(V,U)}(y_K)\}_{(V,U)\in\Lambda_{(X,Y)}}$, $\{g^{-1}_{(V,U)}(y_L)\}_{(V,U)\in\Lambda_{(X,Y)}}$, and $\{g^{-1}_{(V,U)}(y_M)\}_{(V,U)\in\Lambda_{(X,Y)}}$. As in the proof of Theorem 3.6, there exists a cofinal subset $\Lambda'_{(X,Y)}$ of $\Lambda_{(X,Y)}$ such that these nets converge to K', L' and M', x'_K, x'_L and x'_M, respectively, such that K', L' and M' are subcontinua of X with the property that $f(K') = K$, $f(L') = L$ and $f(M') = M$. Since for each $(V, U) \in \Lambda'_{(X,Y)}$, $X = g^{-1}_{(V,U)}(K) \cup g^{-1}_{(V,U)}(L) \cup g^{-1}_{(V,U)}(M)$, $g^{-1}_{(V,U)}(K) \cap g^{-1}_{(V,U)}(L) \neq \varnothing$, $g^{-1}_{(V,U)}(K) \cap g^{-1}_{(V,U)}(M) \neq \varnothing$ and $g^{-1}_{(V,U)}(L) \cap g^{-1}_{(V,U)}(M) \neq \varnothing$, also,

$g^{-1}_{(V,U)}(y_K)\setminus[g^{-1}_{(V,U)}(L)\cup g^{-1}_{(V,U)}(M)]\neq\varnothing$, $g^{-1}_{(V,U)}(y_L)\setminus[g^{-1}_{(V,U)}(K)\cup g^{-1}_{(V,U)}(M)]\neq\varnothing$, and $g^{-1}_{(V,U)}(y_M)\setminus[g^{-1}_{(V,U)}(K)\cup g^{-1}_{(V,U)}(L)]\neq\varnothing$, we have that $X=K'\cup L'\cup M'$, $K'\cap L'\neq\varnothing$, $K'\cap M'\neq\varnothing$, $L'\cap M'\neq\varnothing$, $x'_K\in[X\setminus(L\cup M)]$, $x'_L\in[X\setminus(K\cup M)]$, and $x'_M\in[X\setminus(K\cup L)]$. Therefore, X does not have property (B_2). □

Theorem 5.4. *If X is a decomposable irreducible continuum X, then X has property (B_2).*

Proof. Suppose X is a decomposable continuum without property (B_2). Then there exist three proper subcontinua K, L and M of X such that $X=K\cup L\cup M$, $K\setminus(L\cup M)\neq\varnothing$, $L\setminus(K\cup M)\neq\varnothing$, $M\setminus(K\cup L)\neq\varnothing$, $K\cap L\neq\varnothing$, $K\cap M\neq\varnothing$ and $L\cap M\neq\varnothing$. Let x_1 and x_2 be two points of X. Without loss of generality, assume that $x_1\in K$. Then $x_2\in K\cup L\cup M$. Suppose $x_2\in L$. Then $K\cup L$ is a proper subcontinuum of X such that $\{x_1,x_2\}\subset K\cup L$. Therefore, X is not irreducible. □

Question 5.5. Is the converse of Theorem 5.4 true?

Theorem 5.6. *Let X and Y be continua and let $f\colon X\twoheadrightarrow Y$ be a uniformly refinable map. If Y is irreducible, then X is irreducible.*

Proof. Let y_1 and y_2 be two points of Y such that Y is irreducible about them. Let $x_1\in f^{-1}(y_1)$ and let $x_2\in f^{-1}(y_2)$. Suppose X is not irreducible about x_1 and x_2. Then there exists a proper subcontinuum K of X such that $\{x_1,x_2\}\subset K$. Let $x\in X\setminus K$ and let $V\in\mathfrak{U}_X$ be such that $B_X(x,V)\cap K=\varnothing$. Since f is uniformly refinable, there exists a V-map $g\colon X\twoheadrightarrow Y$. Then $g(K)$ is a subcontinuum of Y and $\{y_1,y_2\}\subset g(K)$. Since Y is irreducible about y_1 and y_2, $g(K)=Y$. This contradicts the fact that g is a V-map. Therefore, X is irreducible. □

Theorem 5.7 ([11, Proposition (3.4)-(2)])**.** *Let X and Y be continua and let $f\colon X\twoheadrightarrow Y$ be a uniformly refinable map. Then X is hereditarily indecomposable if and only if Y is hereditarily indecomposable.*

Proof. Suppose X is hereditarily indecomposable and let L be a decomposable subcontinuum of Y. Then there exist two proper subcontinua L_1 and L_2 of L such that $L=L_1\cup L_2$. Consider the directed set $\Lambda_{(X,Y)}$ (Notaion 2.4). Since f is uniformly refinable, for each $V\in\mathfrak{U}_X$ and every $U\in\mathfrak{U}_Y$, there exists a V-map $g_{(V,U)}\colon X\twoheadrightarrow Y$ that is U-near f. Then we have the nets $\{g^{-1}_{(V,U)}(L_j)\}_{(V,U)\in\Lambda_{(X,Y)}}$, $j\in\{1,2\}$, of nonemtpy closed subsets of X. As in the proof of Theorem 3.6, there exists a cofinal subset $\Lambda'_{(X,Y)}$ of $\Lambda_{(X,Y)}$ such that each of the nets $\{g^{-1}_{(V,U)}(L_j)\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges to a subcontinuum K_j, $j\in\{1,2\}$, of X, with the property that $f(K_j)=L_j$, $j\in\{1,2\}$. Since X is hereditarily indecomposable, we have that $K_1\cap K_2=\varnothing$. Let A_1 and A_2 be disjoint open subsets of X such that $K_j\subset A_j$, $j\in\{1,2\}$. By [4, Lemma 8.2.5], there exist V_{A_1} and V_{A_2} in $\mathfrak{U}_X$ such that $B(K_j,V_{A_j})\subset A_j$, $j\in\{1,2\}$. Let $V'=V_{A_1}\cap V_{A_2}$. Since $\{g^{-1}_{(V,U)}(L_j)\}_{(V,U)\in\Lambda'_{(X,Y)}}$ converges K_j, $j\in\{1,2\}$, there exists $(V,U) \in \Lambda'_{(X,Y)}$ such that $g^{-1}_{(V,U)}(L_j) \subset Int_X(B_X(K_j,V'))$. Without loss of generality, we assume that $V \subset V'$. Let $y \in L_1 \cap L_2$. Then $g^{-1}_{(V,U)}(y) \cap Int_X(B_X(K_j,V')) \neq \varnothing$, $j \in \{1,2\}$, a contradition to the fact that g is a V-map. Therefore, L is indecomposable and Y is hereditarily indecomposable.

Suppose that Y is hereditarily indecomposable. By Corollary 3.11, X is indecomposable. Suppose X contains two subcontinua K and L such that $K \cap L \neq \varnothing$, $K \setminus L \neq \varnothing$ and $L \setminus K \neq \varnothing$. Since Y is hereditarily indecomposable, we have that either $f(L) \subset f(K)$ or $f(K) \subset f(L)$. Without loss of generality, we assume that $f(L) \subset f(K)$. Let $x_L \in L \setminus K$. Thus, there exists $V \in \mathfrak{U}_X$ such that $B_X(x_L,V) \cap K = \varnothing$. Since f is uniformly refinable, there exists a V-map $g\colon X \twoheadrightarrow Y$. Note that, on one hand, $g^{-1}(g(x_L)) \subset B_X(x_L,V)$ and, on the other hand, there exists a point $x_K \in K$ such that $g(x_K) = g(x_L)$, a contradiction to the fact that g is a V-map. Therefore, X is hereditarily indecomposable. □

6. Induced Maps on Hyperspaces

Let Z be a compactum. We consider the following *hyperspaces* of Z:

$$2^Z = \{A \subset Z \mid A \text{ is closed and nonempty}\};$$

$$\mathcal{C}_n(Z) = \{A \in 2^Z \mid A \text{ has at most } n \text{ components}\}, n \in \mathbb{N};$$

$$\mathcal{F}_n(Z) = \{A \in 2^Z \mid A \text{ has at most } n \text{ points}\}, n \in \mathbb{N}.$$

We define a uniformity on 2^Z as follows: If $U \in \mathfrak{U}_Z$, then let $2^U = \{(A,A') \in 2^Z \times 2^Z \mid A \subset B(A',U) \text{ and } A' \subset B(A,U)\}$. Let $\mathfrak{B}_Z = \{2^U \mid U \in \mathfrak{U}_Z\}$. Then $\mathfrak{B}_Z$ is a base for a uniformity, denoted by $2^{\mathfrak{U}_Z}$ [4, 8.5.16]. Observe that the topology generated by $2^{\mathfrak{U}_Z}$ coincides with the Vietoris topology [17, 3.3]. Hence, 2^Z is compact and Hausdorff [17, 4.9]. Thus, $2^{\mathfrak{U}_Z}$ is unique (Remark 2.2), and

$$2^{\mathfrak{U}_Z} = \{\mathcal{U} \subset 2^Z \times 2^Z \mid \mathcal{U} = -\mathcal{U}, \Delta_{2^Z} \subset \mathcal{U} \text{ and there exists } U \in \mathfrak{U}_Z \text{ such that } 2^U \subset \mathcal{U}\}.$$

For the other hyperspaces, we use the restriction of $2^{\mathfrak{U}_Z}$ to the corresponding hyperspace and we denote such restriction by: $\mathcal{C}_n(\mathfrak{U}_Z)$, and $\mathcal{F}_n(\mathfrak{U}_Z)$, respectively. In order to avoid confusion, we put a subindex to the expressions: $\rho_Z(z,z') < U$, $\rho_{2^Z}(A,A') < \mathcal{U}$, $\rho_{\mathcal{C}_n(Z)}(A,A') < \mathcal{U}$, and $\rho_{\mathcal{F}_n(Z)}(A,A') < \mathcal{U}$, respectively.

Let X and Z be compacta and let $f\colon X \to Z$ be a map. Then the functions: 2^f, $\mathcal{C}_n(f)$, and $\mathcal{F}_n(f)$, given by $2^f(A) = f(A)$ for all $A \in 2^X$, $\mathcal{C}_n(f) = 2^f|_{\mathcal{C}_n(X)}$,and $\mathcal{F}_n(f) = 2^f|_{\mathcal{F}_n(X)}$ are the *induced maps* of f. By [17, 5.10], all these induced maps are continuous.

The following result follows from the definition of weakly confluent map.

Theorem 6.1. *Let X and Z be compacta and let $n \in \mathbb{N}$. If $f\colon X \twoheadrightarrow Z$ is a weakly confluent map, then $\mathcal{C}_n(f)$ is surjective.*

Theorem 6.2. *Let X and Z be continua, let $n \in \mathbb{N}$ and let $f\colon X \twoheadrightarrow Z$ be a map. Then the following are equivalnet:*

(1) *f is a homeomorphism.*
(2) *2^f is a homeomorphism.*
(3) *$\mathcal{F}_n(f)$ is a homeomorphism.*
(4) *$\mathcal{C}_n(f)$ is a homeomorphism.*

Proof. Suppose f is a homeomorphism. We see that 2^f is a homeomorphism. Let B be an element of 2^Z. Note that, since f continuous and surjective, $f^{-1}(B)$ is an element of 2^X and $f(f^{-1}(B)) = B$. Thus, $2^f(f^{-1}(B)) = B$, and 2^f is surjective. Now, let A_1 and A_2 be elements of 2^X such that $2^f(A_1) = 2^f(A_2)$. Then $f(A_1) = f(A_2)$. Since f is one-to-one, we have that $A_1 = A_2$. Hence, 2^f is one-to-one. Therefore, 2^f is a homeomorphism. Next, assume that 2^f is a homeomorphism. Let z be an element of Z. Thus, $\{z\}$ belongs to 2^Z. Since 2^f is surjective, there exists an element A of 2^X such that $2^f(A) = \{z\}$. Hence, if a belongs to A, we have that $f(a) = z$. Thus, f is surjective. Now, let x_1 and x_2 be two distinct elements of X. Note that $\{x_1\}$ and $\{x_2\}$ belong to 2^X and, since 2^f is one-to-one, $2^f(\{x_1\}) \neq 2^f(\{x_2\})$. Hence, $f(x_1) \neq f(x_2)$, and f is one-to-one. Therefore, f is a homeomorphism.

Note that (1) is equivalent to (3) by [1, Theorem 3.3], noting that this result does not use the metric hypothesis.

The proof of the equivalence between (1) and (4) is similar to the one given for the equivalence between (1) and (2). We need to use the fact that a homeomorphism is a weakly confluent map, this implies that $\mathcal{C}_n(f)$ is surjective (Theorem 6.1). $\square$

Theorem 6.3. *Let X and Z be continua, let $n \in \mathbb{N}$ and let $f\colon X \twoheadrightarrow Z$ be a map. Then the following are equivalnet:*

(1) *f is monotone.*
(2) *2^f is monotone.*
(3) *$\mathcal{F}_n(f)$ is monotone.*
(4) *$\mathcal{C}_n(f)$ is monotone.*

Proof. Suppose f is monotone. We show that 2^f is monotone. By [14, Lemma 1.4.46], we only need to prove that if $B \in 2^Z$, then $(2^f)^{-1}(B)$ is connected. Let $B \in 2^B$. Since f is monotone, the cardinality of the family of the components of B and $f^{-1}(B)$ is the same. Let $A \in (2^f)^{-1}(B)$. Then $2^f(A) = A$ and $A \subset f^{-1}(B)$. Also, each component of $f^{-1}(B)$ intersects A. Hence, by the proof of [18, Corollary 2.7], there exists an order arc α from A to $f^{-1}(B)$. Note that $2^f(\alpha) = \{B\}$. Thus, $(2^f)^{-1}(B)$ is arcwise connected. Therefore, 2^f is monotone. Now, assume that 2^f is monotone. We prove that f is monotone. Let z be an element of Z. Then $\{z\}$ belongs to 2^Z. Since 2^f is monotone, $(2^f)^{-1}(\{z\})$ is a subcontinuum of 2^X. Note that $(2^f)^{-1}(\{z\}) \cap \mathcal{C}_1(X) \neq \varnothing$. Hence, by [14, Lemma 1.6.8], $\bigcup (2^f)^{-1}(\{z\}) = f^{-1}(z)$ is a subcontinuum of X. Thus, f is monotone.

Observe that (1) is equivalent to (3) by [1, Theorem 4.1], noting that this result and [15, Lemma 1] do not use the metric hypothesis.

The proof of the equivalence between (1) and(4) is similar to the one given for the equivalence between (1) and (2). □

Lemma 6.4. *Let X and Z be compacta, let $n \in \mathbb{N}$, let $V \in \mathfrak{U}_X$ and let $f: X \twoheadrightarrow Z$ be a surjective map. Consider the following statements:*

(1) *f is a V-map.*

(2) *2^f is a 2^V-map.*

(3) *If f is a weakly confluent, then $\mathcal{C}_n(f)$ is a $2^V|_{\mathcal{C}_n(X)}$-map (Notation 2.3).*

(4) *$\mathcal{F}_n(f)$ is a $2^V|_{\mathcal{F}_n(X)}$-map (Notation 2.3).*

Then (1) *implies* (2), (3) *and* (4). *Also* (2) *implies* (3) *and* (4).

Proof. Assume (1), we prove (2). Since f is surjective, 2^f is surjective. Let $B \in 2^Z$ and let A_1 and A_2 be two elements of $(2^f)^{-1}(B)$. Hence, $2^f(A_1) = 2^f(A_2) = B$. Thus, for each $a_1 \in A_1$, there exists $a_2 \in A_2$ such that $f(a_1) = f(a_2)$. Since f is a V-map, we have that $\rho_X(a_1, a_2) < V$. This implies that $A_1 \subset B(A_2, V)$. Similarly, $A_2 \subset B(A_1, V)$. Hence, $\rho_{2^X}(A_1, A_2) < 2^V$. Therefore, 2^f is a 2^V-map.

The proofs of the facts that (1) implies (2) and (3) are done in a similar way, we need to use Theorem 6.1 to ensure that $\mathcal{C}_n(f)$ is surjective.

The proofs of (2) implies (3) and (4) follow from the fact that we only intersect 2^U with the appropriate hyperspace. □

Lemma 6.5. *Let X and Z be compacta, let $n \in \mathbb{N}$, let $V \in \mathfrak{U}_X$ and let $f: X \twoheadrightarrow Z$ be a surjective map. If $\mathcal{C}_n(f)$ is a $2^V|_{\mathcal{C}_n(X)}$-map, then $\mathcal{F}_n(f)$ is a $2^V|_{\mathcal{F}_n(X)}$-map*

Proof. The lemma follows from the fact that we are only intersecting 2^U with the appropriate hyperspace. □

Lemma 6.6. *Let X and Z be compacta, let $V \in \mathfrak{U}_X$, let $n \in \mathbb{N}$, and let $g, f: X \twoheadrightarrow Z$ be surjective maps. Consider the following statements:*

(1) *$\rho_X(f(x), g(x)) < U$, for every $x \in X$.*

(2) *$\rho_{2^Z}(2^f(A), 2^g(A)) < 2^U$, for each $A \in 2^X$.*

(3) *$\rho_{\mathcal{C}_n(Z)}(\mathcal{C}_n(f(A)), \mathcal{C}_n(g(A))) < 2^U|_{\mathcal{C}_n(Z)}$, for every $A \in \mathcal{C}_n(X)$ (Notation 2.3).*

(4) *$\rho_{\mathcal{F}_n(Z)}(\mathcal{F}_n(f(A)), \mathcal{F}_n(g(A))) < 2^U|_{\mathcal{F}_n(Z)}$, for all $A \in \mathcal{F}_n(X)$ (Notation 2.3).*

Then (1) *implies* (2), (3) *and* (4). *Also* (2) *implies* (3) *and* (4); *and* (3) *implies* (4).

Proof. Assume (1), we show (2). Let $A \in 2^X$ and let $a \in A$. By hypothesis, we have that $\rho_Z(f(a), g(a)) < U$. Hence, $f(A) \subset B(g(A), U)$. Similarly, $g(A) \subset B(f(A), U)$. Hence, $\rho_{2^Z}(2^f(A), 2^g(A)) < 2^U$. The proofs of (1) implies (3) and (4) are done in a similar way.

The proofs of the other implications follow from the fact that we intersect 2^U with the appropriate hyperspace. $\square$

A continuum Z is in $Class(W)$ provided that for each continuum X, every surjective map $f\colon X \twoheadrightarrow Z$ is weakly confluent.

Theorem 6.7. *Let X and Z be continua, let $n \in \mathbb{N}$ and let $f\colon X \twoheadrightarrow Z$ be a map. Consider the following statements:*

(1) *$f\colon X \twoheadrightarrow Z$ is a uniformly refinable map.*

(2) *2^f is a uniformly refinable map.*

(3) *If Z is in $Class(W)$, then $\mathcal{C}_n(f)$ is a uniformly refinable map.*

(4) *$\mathcal{F}_n(f)$ is a uniformly refinable map.*

Then (1) *implies* (2), (3) *and* (4).

Proof. Assume (1), we prove (2). Since f is surjective, 2^f is surjective. Let $\mathcal{V} \in 2^{\mathfrak{U}_X}$ and let $\mathcal{U} \in 2^{\mathfrak{U}_Z}$. Then there exist $V \in \mathfrak{U}_X$ and $U \in \mathfrak{U}_Z$ such that $2^V \subset \mathcal{V}$ and $2^U \subset \mathcal{U}$. Since f a uniformly refinable map, there exists a V-map $g\colon X \twoheadrightarrow Z$ that is U-near f. By Lemma 6.4, 2^g is a 2^V-map. In particular, 2^g is a $\mathcal{V}$-map. By Lemma 6.6, 2^g is 2^U-near 2^f. In particular, 2^g is $\mathcal{U}$-near 2^f. Therefore, 2^f is a uniformly refinable map.

The proofs of the facts that (1) implies (2) and (3) are done in a similar way. We need to use the fact that Z is in $Class(W)$ and Theorem 6.1 to ensure that $\mathcal{C}_n(f)$ and $\mathcal{C}_n(g)$ are surjective. $\square$

As a consequence of Theorems 6.7 and 6.3, we have that:

Theorem 6.8. *Let X and Z be continua, let $n \in \mathbb{N}$ and let $f\colon X \twoheadrightarrow Z$ be a map. Consider the following statements:*

(1) *f is a monotonly uniformly refinable map.*

(2) *2^f is a monotonly uniformly refinable map.*

(3) *If Z is in $Class(W)$, then $\mathcal{C}_n(f)$ is a monotonly uniformly refinable map.*

(4) *$\mathcal{F}_n(f)$ is a monotonly uniformly refinable map.*

Then (1) *implies* (2), (3) *and* (4).

Let X and Z be compacta and let $f\colon X \twoheadrightarrow Z$ be a map. Then f is a *near-homeomorphism* provided that for each $U \in \mathfrak{U}_Z$, there exists a homeomorphism $g\colon X \twoheadrightarrow Z$ such that $\rho_Z(f(x), g(x)) < U$ for all elements x of X.

Theorem 6.9. *Let X and Z be continua, let $n \in \mathbb{N}$ and let $f\colon X \twoheadrightarrow Z$ be a map. Consider the following statements:*

(1) *f is a near-homeomorphism.*

(2) *2^f is a near-homeomorphism.*

(3) *If Z is in $Class(W)$, then $\mathcal{C}_n(f)$ is a near-homeomorphism.*

(4) *$\mathcal{F}_n(f)$ is a near-homeomorphism.*

Then (1) *implies* (2), (3) *and* (4).

Proof. Assume (1), we prove (2). Since f is a near-homeomorphism, f is a uniformly refinable map. Let $\mathcal{U} \in 2^{\mathfrak{U}_Z}$ and let $U \in \mathfrak{U}_Z$ be such that $2^U \subset \mathcal{U}$. Since f is a near-homeomorphism, there exists a homeomorphism $g\colon X \twoheadrightarrow Z$ such that $\rho_Z(f(x), g(x)) < U$ for all points x of X. By Theorem 6.2, 2^g is a homeomorphism, and by Lemma 6.7, $\rho_{2^Z}(2^f(A), 2^g(A)) < 2^U$, for each $A \in 2^X$. Therefore, 2^f is a near-homeomorphism.

The proofs of (1) implies (3) and (4) are done in a similar way. We need to use the fact that Z is in $Class(W)$ and Theorem 6.1 to ensure that $\mathcal{C}_n(f)$ is surjective. □

Lemma 6.10. *Let X and Z be continua. If X is arcwise connected and $f\colon X \twoheadrightarrow Z$ is an atomic map, then f is a homeomorphism.*

Proof. Note that the proof of [13, Corollary 8.1.26] may be done using the fact that continua are normal spaces instead of the metric hypothesis. □

Theorem 6.11. *Let X and Z be continua, let $n \in \mathbb{N}$ and let $f\colon X \twoheadrightarrow Z$ be a map.*

(1) *If 2^f is an atomically refinable map, then 2^f is a near-homeomorphism.*

(2) *If $\mathcal{C}_n(f)$ is an atomically refinable map, then $\mathcal{C}_n(f)$ is a near-homeomorphism.*

Proof. We show (1). Suppose 2^f is an atomically refinable map. Then for each $\mathcal{V} \in 2^{\mathfrak{U}_X}$ and $\mathcal{U} \in 2^{\mathfrak{U}_Z}$, there exists an atomic $\mathcal{V}$-map $G\colon 2^X \twoheadrightarrow 2^Z$ that is $\mathcal{U}$-near 2^f. By [18, Corollary 2.7], we have that 2^X is arcwise connected. Hence, by Lemma 6.10, G is a homeomorphism. Therefore, 2^f is a near-homeomorphism. Since $\mathcal{C}_n(X)$ is arcwise connected, [18, Corollary 2.7], the proof of (2) is done in a similar way. □

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