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Hybrid topologies on the real line

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Abstract

Given $A \subseteq \mathbb{R}$, the Hattori space $H(A)$ is the topological space $(\mathbb{R}, \tau_A)$ where each $a \in A$ has a τ_A-neighborhood base $\{(a-\varepsilon, a+\varepsilon) : \varepsilon > 0\}$ and each $b \in \mathbb{R} - A$ has a τ_A-neighborhood base $\{[b, b+\varepsilon) : \varepsilon > 0\}$. Thus, τ_A may be viewed as a hybrid of the Euclidean topology and the lower-limit topology. We investigate properties of Hattori spaces as well as other hybrid topologies on $\mathbb{R}$ using various combinations of the discrete, left-ray, lower-limit, upper-limit, and Euclidean topologies. Since each of these topologies is generated by a quasi-metric on $\mathbb{R}$, we investigate hybrid quasi-metrics which generate these hybrid topologies.

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1. Introduction

Among the uncountably many topologies on $\mathbb{R}$, the most familiar ones include the Euclidean, lower-limit (or Sorgenfrey), upper-limit, left-ray, and discrete topologies. The left-ray topology, having a basis $\{(-\infty, x+\varepsilon) : x \in \mathbb{R}, \varepsilon > 0\}$, is clearly not T_2 and thus not metrizable. In the lower-limit topology, each $x \in \mathbb{R}$ has a neighborhood base of sets $[x, x+\varepsilon)$ which are not symmetric around x, so one would (correctly) suspect that this topology does not arise from a metric, where distances satisfy the symmetry condition. A quasi-metric is a metric without the symmetry condition.

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Definition 1.1. A quasi-metric on X is a function $q : X \times X \to \mathbb{R}$ which satisfies, for all $x, y, z \in X$, (a) $q(x, y) \geq 0$, (b) $x = y$ if and only if $q(x, y) = 0 = q(y, x)$, and (c) $q(x, y) + q(y, z) \geq q(x, z)$. With $B(x, \varepsilon) = \{y \in X : q(x, y) < \varepsilon\}$, the collection $\{B(x, \varepsilon) : x \in X, \varepsilon > 0\}$ is a basis for the associated quasi-metric topology.

If a quasi-metric q on X also satisfies $q(x, y) = q(y, x)$ for all $x, y \in X$, then q is a metric. All of the familiar topologies on $\mathbb{R}$ mentioned above arise from quasi-metrics. Besides the well-known Euclidean (quasi-)metric $d_E(x, y) = |x - y|$ and the discrete (quasi-)metric $m(x, y) = 0$ if $x = y$ and $m(x, y) = 1$ if $x \neq y$, the lower limit topology and the left ray topology, respectively, arise from the quasi-metrics

$$q_{ll}(x, y) = \begin{cases} y - x & \text{if } y \geq x \\ 1 & \text{if } y < x \end{cases} \quad \text{and} \quad q_{lr}(x, y) = \begin{cases} y - x & \text{if } y \geq x \\ 0 & \text{if } y < x. \end{cases}$$

The upper-limit topology has basis $\{(x-\varepsilon, x] : x \in \mathbb{R}, \varepsilon > 0\}$ and arises from the quasi-metric $q_{ul}(x, y) = q_{ll}(y, x)$. If q is a quasi-metric, the associated standard bounded quasi-metric $\bar{q}(x, y) = \max\{1, q(x, y)\}$ generates the same topology as q. Further information on quasi-metrics and other topological concepts can be found in [5]. We assume that normality includes the T_1 axiom and T_4 does not.

2. Hattori Spaces.

Given $A \subseteq \mathbb{R}$, the Hattori space $H(A)$ is $\mathbb{R}$ with the topology having basis $\{(a-\varepsilon, a+\varepsilon) : a \in A, \varepsilon > 0\} \cup \{[b, b+\varepsilon) : b \notin A, \varepsilon > 0\}$. $H(A)$ may be viewed as a hybrid of the Euclidean topology and the lower-limit topology on $\mathbb{R}$. Hattori spaces were introduced in [3] and have been studied in [2, 1]. $H(\mathbb{R} - \mathbb{Z})$ models nearness for ants living on the graph of $y = \lfloor x \rfloor$ who cannot go from one level to another.

The basis for $H(A)$ consists of the d_E-balls centered at $a \in A$ and the q_{ll}-balls centered at $b \notin A$. Thus, a natural candidate for a quasi-metric for $H(A)$ is

$$q_1(x, y) = \begin{cases} d_E(x, y) & \text{if } x \in A \\ q_{ll}(x, y) & \text{if } x \notin A \end{cases} \quad = \quad \begin{cases} |y - x| & \text{if } x \in A \\ y - x & \text{if } x \notin A, y \geq x \\ 1 & \text{if } x \notin A, y < x. \end{cases}$$

However, this is not a quasi-metric unless $A = \varnothing$ or $A = \mathbb{R}$ (in which case $q_1 = d_{ll}$ or $q_1 = q_E$ isn't really a 'hybrid'). Indeed, suppose there exists $a \in A$ and $b \notin A$. If $b < a$, then $q_1(a, b) + q_1(b, b - 2) = |b - a| + 1 = a - b + 1 < a - b + 2 = |a - (b - 2)| = q_1(a, b - 2)$, so the triangle inequality fails. If $a < b$, then $q_1(a, b) + q_1(b, a - |b - a| - 2) = b - a + 1 < b - a + 2 = q_1(a, a - |b - a| - 2)$, and the triangle inequality fails.

As this example illustrates, possible issues with the triangle inequality for a hybrid quasi-metric may arises if one of the quasi-metrics is bounded in one direction while the other is unbounded. For example, when measuring distances to the left, q_{ll} is bounded while d_E is not. This suggests that we are more likely to get a quasi-metric if all distances are bounded. Thus, we consider

$$q_2(x,y) = \begin{cases} \bar{d}_E(x,y) & \text{if } x \in A \\ \bar{q}_{ll}(x,y) & \text{if } x \notin A \end{cases} = \begin{cases} |y-x| & \text{if } x \in A, y \in (x-1,x+1) \\ y-x & \text{if } x \notin A, y \in [x,x+1) \\ 1 & \text{otherwise.} \end{cases}$$

However, this fails the triangle inequality unless $A = \varnothing$ or A is a ray to the left. Otherwise, there exists $a \in A, b \notin A$ with $b < a < b + 1/4$. Then $q_2(b,a) + q_2(a, b-1/4) < 1/4 + 1/2 < 1 = q_2(b, b-1/4)$, showing that q_2 is not a quasi-metric.

Before defining a hybrid quasi-metric for less restrictive sets A, we fix some terminology.

For $x, y \in \mathbb{R}$, $[\{x,y\}]$ will denote the set of all point between x and y. Thus, $[\{x,y\}] = [x,y] \cup [y,x]$ is the convex hull of $\{x,y\}$.

For $B \subseteq \mathbb{R}$ and $b \in B$, if $\{y \in B : y > b\}$ has a least element, we call this element the successor of b, denoted b^+. Now $b \in B$ is the maximum element of B if and only if $\{y \in B : y > b\} = \varnothing$, and in this case we take $b^+ = \infty$. We say the set B *has successors* if every $b \in B$ which is not maximum in B has a successor in B. Thus, B has successors if and only if no accumulation point of B in the lower-limit topology is contained in B.

Theorem 2.1. *If $\mathbb{R} - A$ has successors, then the function q_3 given below is a quasi-metric.*

$$q_3(x,y) = \begin{cases} |y-x| & \textit{if } x \in A, y \in (x-1,x+1), [\{x,y\}] \subseteq A \\ y-x & \textit{if } x \notin A, y \in [x,x+1) \cap [x,x^+) \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. For $x, y \in \mathbb{R}$, clearly $q_3(x,y) \geq 0$, and $q_3(x,y) = 0 = q_3(y,x)$ if and only if $x = y$. It only remains to check the triangle inequality $q_3(x,y) + q_3(y,z) \geq q_3(x,z)$ for distinct $x, y, z \in \mathbb{R}$. Since $q_3(x,z) \leq 1$, we need only check the triangle inequality when $q_3(x,y) < 1$ and $q_3(y,z) < 1$. Now $q_3(x,y) < 1$ if (1) $x \in A, y \in (x-1,x+1)$, $[\{x,y\}] \subseteq A$ or (2) $x \notin A, y \in [x,x+1) \cap [x,x^+)$, and $q_3(y,z) < 1$ if (a) $y \in A, z \in (y-1,y+1)$, $[\{y,z\}] \subseteq A$ or (b) $y \notin A, z \in [y,y+1) \cap [y,y^+)$.

In case (1)(a), $[\{x,y\}] \cup [\{y,z\}] \subseteq A$ implies $[\{x,z\}] \subseteq A$, and $q_3(x,y)$, $q_3(y,z)$, $q_3(x,z)$ are each just the bounded Euclidean metric, so the triangle inequality holds in this case.

In case (1)(b), $x \in A, y \notin A$ implies $q_3(x,y) = 1$, so the triangle inequality holds.

In case (2)(a), $y \in (x,x^+)$ and $[\{y,z\}] \subseteq A$ implies $z \in (x,x^+)$. If $z \in (x,x+1)$, then $q_3(x,z) = z - x$. If $z \notin (x,x+1)$, then $z \in [x+1,x^+)$ and $q_3(x,z) = 1 \leq z - x$. Either way, $q_3(x,z) \leq z - x = |z-x| \leq |y-x| + |z-y| = q_3(x,y) + q_3(y,z)$.

In case (2)(b), $y \in [x,x^+) - A$ implies $y = x$, so the triangle inequality holds.

Thus, q_3 is a quasi-metric. □

While the quasi-metric q_3 is a hybrid of the quasi-metrics generating the basis elements of $H(A)$, q_3 will not necessarily generate the topology of $H(A)$.

Theorem 2.2. *The quasi-metric q_3 generates $H(A)$ if and only if for every $a \in A$ there exists $\varepsilon_a > 0$ such that $(a - \varepsilon_a, a + \varepsilon_a) \subseteq A$, that is, if and only if A is open in the Euclidean topology.*

Proof. If the condition fails, there exists $a \in A$ with either (a) $\sup\{b \notin A : b < a\} = a$ or (b) $\inf\{b \notin A : b > a\} = a$. From the definition of q_3, there exists $\varepsilon > 0$ with $B(a, \varepsilon) \subseteq [a, a + \varepsilon)$ in case (a) and $B(a, \varepsilon) \subseteq (a - \varepsilon, a]$ in case (b). There is no basis element for $H(A)$ containing a which is contained in such a $B(a, \varepsilon)$. Thus, the q_3 quasi-metric topology does not give $H(A)$.

Suppose the condition holds. By completeness of $\mathbb{R}$, given $a \in A$, there exist $b_a \notin A$ with $a \in (b_a, b_a^+) \subseteq A$. If $\varepsilon \leq 1$ and $(a - \varepsilon, a + \varepsilon) \subseteq (b_a, b_a^+)$, then $B(a, \varepsilon) = (a - \varepsilon, a + \varepsilon)$. If $\varepsilon \leq 1$ and $(a - \varepsilon, a] \subseteq (b_a, b_a^+)$ but $[a, a + \varepsilon) \not\subseteq (b_a, b_a^+)$, then $B(a, \varepsilon) = (a - \varepsilon, b_a^+)$. If $\varepsilon \leq 1$ and $[a, a + \varepsilon) \subseteq (b_a, b_a^+)$ but $(a - \varepsilon, a] \not\subseteq (b_a, b_a^+)$, then $B(a, \varepsilon) = (b_a, a + \varepsilon)$. In all other cases, $B(a, \varepsilon) = (-\infty, \infty)$.

Now consider $b \notin A$. If $b + \varepsilon \leq \min\{b^+, b + 1\}$, then $B(b, \varepsilon) = [b, b + \varepsilon)$. If $b^+ \leq b + \varepsilon \leq b + 1$, then $B(b, \varepsilon) = [b, b^+)$. In all other cases $B(b, \varepsilon) = (-\infty, \infty)$.

Note that all of these q_3-balls are open in $H(A)$, and the collection of q_3-balls with small radii is a basis for $H(A)$, so the q_3 quasi-metric topology is the topology of $H(A)$. □

Since q_3 is only defined if $\mathbb{R} - A$ has successors, Theorem 2.2 shows that q_3 generates $H(A)$ if and only if A is a Euclidean-open set and $\mathbb{R} - A$ has no accumulation points in the Euclidean topology.

While Theorem 2 does not answer the question of when $H(A)$ is quasi-metrizable, it essentially answers the question of when the natural approach for finding a hybrid quasi-metric for the hybrid space $H(A)$ is successful.

We mention a few other properties of $H(A)$. Clearly $\mathbb{Q}$ is dense in $H(A)$, so $H(A)$ is separable. If $A \neq \mathbb{R}$ (so $H(A)$ is not the Euclidean line), then $H(A)$ is not connected, for if $b \notin A$, then $[b, \infty)$ is closed and open. Since $H(A)$ is finer than the Euclidean topology, $H(A)$ is T_0, T_1, and T_2. In [2], it is shown that $H(A)$ is regular. We can improve on this.

Theorem 2.3. *Every Hattori space $H(A)$ is T_4.*

Proof. Our proof that $H(A)$ is T_4 is modeled on the proof that metric spaces are T_4. Where the argument for metric spaces uses the triangle inequality, we use order properties of $\mathbb{R}$. Suppose $A \subseteq \mathbb{R}$ is given. For $x \in \mathbb{R}$, define $U(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$ if $x \in A$ and $U(x, \varepsilon) = [x, x + \varepsilon)$ for $x \notin A$. Now $\{U(x, \varepsilon) : x \in \mathbb{R}, \varepsilon > 0\}$ is a basis for $H(A)$. Suppose C and D are disjoint closed sets in $\tau(A)$. For any $c \in C$, there exists $\varepsilon_c > 0$ such that $U(c, \varepsilon_c) \cap D = \varnothing$ and for any $d \in D$, there exists $\varepsilon_d > 0$ such that $U(d, \varepsilon_d) \cap C = \varnothing$. Let $U = \bigcup_{c \in C} U(c, \varepsilon_c/2)$ and $V = \bigcup_{d \in D} U(d, \varepsilon_d/2)$. Clearly U and V are open, $C \subseteq U$, and $D \subseteq V$. It remains to show $U \cap V = \varnothing$. Suppose to the contrary there exists $x \in U \cap V$. Then there exist $c^* \in C, d^* \in D$ with $x \in U(c^*, \varepsilon_{c^*}/2) \cap U(d^*, \varepsilon_{d^*}/2)$. Now $U(c^*, \varepsilon_{c^*}/2)$ is either $(c^* - \varepsilon_{c^*}/2, c^* + \varepsilon_{c^*}/2)$ or the right half of that interval, and $U(d^*, \varepsilon_{d^*}/2)$ is either $(d^* - \varepsilon_{d^*}/2, d^* + \varepsilon_{d^*}/2)$ or the right half of that interval. In any event, either x falls in the right halves of both intervals, x falls in the left halves of both intervals, or x falls in the right half of one interval and the left half of the other (and by symmetry of the argument, without loss of generality, we may say in the right half of the the c^*-neighborhood and the left half of the d^*-neighborhood).

First suppose $x \in [c^*, c^*+\varepsilon_{c^*}/2) \cap [d^*, d^*+\varepsilon_{d^*}/2)$. If $c^* < d^*$, then $c^* < d^* < x < c^* + \varepsilon_{c^*}/2$ which gives the contradiction that $d^* \in [c^*, c^* + \varepsilon_{c^*}/2) \cap D$. If $d^* < c^*$, then $d^* < c^* < x < d^*+\varepsilon_{d^*}/2$, contradicting that $[d^*, d^*+\varepsilon_{d^*}/2) \cap C = \varnothing$.

The case $x \in (c^* - \varepsilon_{c^*}/2, c^*] \cap (d^* - \varepsilon_{d^*}/2, d^*]$ is dual.

Finally, suppose $x \in [c^*, c^* + \varepsilon_{c^*}/2) \cap (d^* - \varepsilon_{d^*}/2, d^*]$. If $\varepsilon_{c^*} \leq \varepsilon_{d^*}$, then

$$x - \frac{\varepsilon_{d^*}}{2} \leq x - \frac{\varepsilon_{c^*}}{2} < c^* \leq x < c^* + \frac{\varepsilon_{c^*}}{2}$$

and since $d^* - \varepsilon_{d^*}/2 < x$,

$$d^* - \varepsilon_{d^*} < x - \frac{\varepsilon_{d^*}}{2} < c^* \leq x \leq d^*,$$

giving the contradiction that $c^* \in C \cap (d^* - \varepsilon_{d^*}, d^*] \subseteq U(d^*, \varepsilon_{d^*})$. If $\varepsilon_{d^*} < \varepsilon_{c^*}$, then

$$c^* \leq x < c^* + \frac{\varepsilon_{c^*}}{2} \leq x + \frac{\varepsilon_{c^*}}{2} < c^* + \varepsilon_{c^*}$$

and since $d^* - \varepsilon_{c^*}/2 < x$, we have

$$c^* < d^* < x + \frac{\varepsilon_{c^*}}{2} < c^* + \varepsilon_{c_*},$$

giving the contradiction that $d^* \in D \cap [c^*, c^* + \varepsilon_{c^*}) \subseteq U(c^*, \varepsilon)$.

This shows $H(A)$ is T_4. □

It follows that $H(A)$ is normal, $T_{3.5}$, completely regular, T_3, and regular. While this direct proof is instructive, we note that if $H^*(A)$ is formed from $H(A)$ by adding an extra point $b^- < b$ for every $b \in \mathbb{R} - A$ with $x < b^- < y$ if and only if $x < b < y$ and given the topology having a subbase of open rays, then $H(A)$ is a subspace of the LOTS $H^*(A)$. Thus, $H(A)$ is a *GO*-space, and *GO*-spaces are known to be normal (see [4], for example).

3. Lower-Limit Upper-Limit Hybrids.

Given $A \subseteq \mathbb{R}$, define $J(A)$ to be the topology on $\mathbb{R}$ having a basis $\{(a-\varepsilon, a] : a \in A, \varepsilon > 0\} \cup \{[b, b+\varepsilon) : b \in B = \mathbb{R} - A, \varepsilon > 0\}$.

The hybrid $J(\mathbb{R} - \mathbb{Q})$ appears as a recurring example in [5] (p. 48, 166, 244, 281).

It is easy to see that for any $A \subseteq \mathbb{R}$, $J(A)$ is separable and T_2. For any $A \subseteq \mathbb{R}$, either A or $B = \mathbb{R} - A$ contains two distinct points. If A contains distinct points $a < a'$, then $(a, a']$ and its complement are both open, so $J(A)$ is not connected. Similarly, if $b < b'$ in B, then $[b, b')$ is closed and open. Thus, $J(A)$ is never connected. Omitting some extraneous cases in the proof of Theorem 2.3 shows that $J(A)$ is T_4. With natural quasi-metrics q_{ll} and q_{ul} for the lower-limit and upper-limit topologies, we may ask if there is a hybrid quasi-metric for $J(A)$.

Theorem 3.1. *The function q_4 given below is a quasi-metric, and q_4 generates $J(A)$ if and only if for every $a \in A$ there exists $\varepsilon_a > 0$ such that $(a-\varepsilon_a, a] \subseteq A$ and for every $b \notin A$, there exists $\varepsilon_b > 0$ such that $[b, b+\varepsilon_b) \subseteq \mathbb{R} - A$.*

$$q_4(x,y) = \begin{cases} x-y & \text{if } x \in A, y \in (x-1,x], (y,x] \subseteq A \\ y-x & \text{if } x \notin A, y \in [x,x+1), [x,y) \subseteq \mathbb{R}-A \\ 1 & \text{otherwise} \end{cases}$$

Proof. To show q_4 is a quasi-metric, only the triangle inequality is non-trivial. Since q_4 is bounded by 1, if $x, y, z \in \mathbb{R}$ and $q_4(x,y) = 1$ or $q_4(y,z) = 1$, then clearly $q_4(x,y) + q_4(y,z) \geq q_4(x,z)$. Thus, suppose $q_4(x,y)$ and $q_4(y,z)$ are both less than 1. Also, since $q_4(x,x) = 0$ for all $x \in \mathbb{R}$, we may assume x, y, z are all distinct.

If $\{x,y\} \subseteq A$, then $q_4(x,y) + q_4(y,z) = x - y + y - z = x - z$ where $y - 1 < z < y < x$ and $(z,x] = (z,y] \cup (y,x] \subseteq A$. If $z \in (x-1,x]$, then $q_4(x,z) = x - z$; otherwise $q_4(x,z) = 1 < x - z = q_4(x,y) + q_4(y,z)$, so the triangle inequality holds in this case. The case of $\{x,y\} \subseteq \mathbb{R} - A$ is dual.

If $x \in A, y \notin A$, and x, y, z are distinct, then $q_4(x,y) < 1$ implies $y \leq x$ and $(y,x] \subseteq A$, and $q_4(y,z) < 1$ implies $z > y$ and $[y,z) \subseteq \mathbb{R} - A$. This gives the contradiction that $(y, \min\{x,z\})$ is contained in both A and $\mathbb{R} - A$, so this case cannot occur. Dually, the case $x \notin A, y \in A$ cannot occur.

Thus, q_4 is a quasi-metric.

To see that q_4 gives the topology we wish if the spacing condition holds, suppose the condition holds. For $a \in A$ and $\varepsilon \leq \min\{\varepsilon_a, 1\}$, $B(a,\varepsilon) = (a-\varepsilon, a]$; for $\varepsilon_a \leq \varepsilon \leq 1$, $B(a,\varepsilon) = (a - \varepsilon_a, a]$; and for $\varepsilon > 1$, $B(a,\varepsilon) = \mathbb{R}$. Dual results hold for $b \in \mathbb{R} - A$. Thus, every ball is open in $J(A)$, and already the small balls are a basis for the topology of $J(A)$. Thus, q_4 generates the topology of $J(A)$.

If the spacing condition fails, say there exists $a \in A$ such that for every $n \in \mathbb{N}$ there exists $b_n \notin A$ with $b_n \in (a - 1/n, a)$. Now $B(a, .5) = \{a\}$, which is not open in $J(A)$. Similarly, if the spacing condition fails at a point $b \notin A$, then the q_4 topology is not the topology of $J(A)$. □

Observe that Theorem 3.1 says the lower-limit upper-limit hybrid $J(A)$ arises from the quasi-metric q_4 if and only if $\mathbb{R} - A$ is open in the lower-limit topology and A is open in the upper-limit topology.

4. Lower-limit Left-ray Hybrids

Given $A \subseteq \mathbb{R}$, let $K(A)$ be the topology on $\mathbb{R}$ having a basis

$$\mathcal{B} = \{(-\infty, a+\varepsilon) : a \in A, \varepsilon > 0\} \cup \{[b, b+\varepsilon) : b \in B = \mathbb{R} - A, \varepsilon > 0\},$$

and let $K^*(A)$ to be the topology on $\mathbb{R}$ having a basis

$$\mathcal{B}^* = \{(-\infty, a+\varepsilon) : a \in A, \varepsilon \in (0,1]\} \cup \{[b, b+\varepsilon) : b \in B = \mathbb{R} - A, \varepsilon \in (0,1]\}.$$

Note that $\mathcal{B}^*$ differs from $\mathcal{B}$ only in restricting the radii ε to be small. In a metric space, the basis of balls of radius $\varepsilon \in (0,1]$ generates the same topology as the basis of balls of arbitrary positive radius. However, intuition from metric spaces is not applicable here. If $a \in A$, it is tempting to think that in $K(A)$, a will have a neighborhood base of sets of form $(-\infty, a+\varepsilon)$, which extend to $-\infty$. However, if there exists $a \in A, b \notin A$ with $b < a$, then $[b, a+\varepsilon)$ is a neighborhood of a which does not extend to $-\infty$. Although each $a \in A$ is used to index a subcollection $\mathcal{B}_a = \{(-\infty, a+\varepsilon) : \varepsilon > 0\}$ of $\mathcal{B}$, the subcollection $\mathcal{B}_a$ is not a neighborhood base for a (unless there are no points $b \in \mathbb{R} - A$ below a). For the hybrid spaces considered in previous sections, the subcollection of the basis indexed by $x \in \mathbb{R}$ was indeed a neighborhood base for x.

To see the distinction between $K(A)$ and $K^*(A)$, let $\mathbb{E} = \{2n : n \in \mathbb{N}\}$ be the set of positive even integers, and set $A = \mathbb{R} - \mathbb{E}$. Now $[2, 3.5)$ is open set in $K(A)$ (determined by $b = 2 \notin A, \varepsilon = 1.5$) but is not open in $K^*(A)$. In $K(A)$, $(2-1/n)_{n=1}^{\infty}$ does not converge and $(2+1/n)_{n=1}^{\infty} \to y$ for every $y \in [2, 4)$. Now consider $K^*(A)$. If $x < 2$, then x has a neighborhood base of form $(-\infty, x+\varepsilon)$ for $\varepsilon \in (0, 1]$. If $k \in \mathbb{N}$ and $x \in [2k, 2k+1)$, then x has a neighborhood base of form $[2k, z)$ for $z \in (x, 2k+1)$. If $k \in \mathbb{N}$ and $x \in [2k+1, 2k+2)$, then x has a neighborhood base of form $(-\infty, z)$ for $z \in (x, 2k+2)$. Thus, in $K^*(A)$, $(2-1/n)_{n=1}^{\infty} \to y$ for all $y \in [3, 4) \cup [5, 6) \cup [7, 8) \cup \cdots$ and $(2+1/n)_{n=1}^{\infty} \to y$ for all $y \in [2, 4) \cup [5, 6) \cup [7, 8) \cup \cdots$.

Recall that S is *order dense* in $\mathbb{R}$ if for every $x < y$ in $\mathbb{R}$, there exists $s \in S$ with $x < s < y$. Order dense is equivalent to dense in $(\mathbb{R}, d_E)$. Observe that if $\mathbb{R} - A$ is order dense, then for $\varepsilon \in (0, 1]$, $(a-\varepsilon, a+\varepsilon)$ is a neighborhood of $a \in A$ in $K^*(A)$, so the topology of $K^*(A)$ is finer than the topology of the Euclidean lower-limit hybrid $H(A)$.

We summarize some common properties of $K(A)$ and $K^*(A)$.

Theorem 4.1.

(a) *$K(A)$ and $K^*(A)$ are separable.*
(b) *$K(A)$ and $K^*(A)$ are T_0.*
(c) *The following are equivalent: (i) $K(A)$ is T_1, (ii) $K(A)$ is T_2, (iii) $\mathbb{R} - A$ is order dense, (iv) $K^*(A)$ is T_1, (v) $K^*(A)$ is T_2.*
(d) *D is dense in $K(A)$ iff D is dense in $K^*(A)$ iff* $\inf D = -\infty$ *and for each $b \in \mathbb{R} - A$, $b = \inf\{d \in D : d \geq b\}$.*

Proof. (a) Every basis element for $K(A)$ or $K^*(A)$ intersects $\mathbb{Q}$.

(b) Given $x < y$, with $\varepsilon \in (0, 1]$ such that $x + \varepsilon < y$, $(-\infty, x+\varepsilon)$ is a neighborhood of x which excludes y in $K(A)$ and $K^*(A)$, so the spaces are T_0.

(c) Since $\mathcal{B}^* \subseteq \mathcal{B}$, the topology of $K(A)$ is finer than that of $K^*(A)$, so (v) implies (iv), (ii), and (i) and if (i) fails, then (ii), (iv), and (v) fail. Thus, it suffices to show (iii) implies (v) and if (iii) fails, then (i) fails. Suppose (iii) and $x < y$. Pick $b \in (y-1, y] \cap (x, y] \cap (\mathbb{R} - A)$. Choose $\varepsilon \in (0, 1]$ with $x + \varepsilon < b$. Now $[b, b+1)$ is a neighborhood of y disjoint from the neighborhood $(-\infty, x+\varepsilon)$ of x, so $K^*(A)$ is T_2. Alternately, as noted above, (iii) implies the topology of $K^*(A)$ is finer than the T_2 topology of $H(A)$, so $K^*(A)$ is T_2. Now suppose (iii) fails. Then there exist $x < y$ with $[x, y] \subseteq A$. Now every neighborhood of y contains x, so $K^*(A)$ is not T_1.

(d) If $\inf D \neq -\infty$, then the closure of D is contained in $[\inf D, \infty)$, so D is not dense in $K(A)$. If there exists $b \notin A$ with $c = \inf\{d \in D : d \geq b\} > b$, then for $\varepsilon \in (0,1]$ chosen so that $b + \varepsilon < c$, $[b, b+\varepsilon)$ is a neighborhood of b disjoint from D, so D is not dense. Conversely, suppose $\inf D = \infty$ and $b = \inf\{d \in D : d \geq b\}$ for every $b \in \mathbb{R} - A$. Clearly if $b \notin A$, every neighborhood $[b, b+\varepsilon)$ intersects D and if $a \in A$, neighborhoods of A either have form $(-\infty, a+\varepsilon)$ or $[b, a+\varepsilon)$ for some $b \notin A$, and these must intersect D. This argument also holds for $K^*(A)$. □

The characterizations of connectedness of $K(A)$ and $K^*(A)$ differ.

Theorem 4.2.

(a) *$K(A)$ is connected iff $A = \mathbb{R}$.*
(b) *$K^*(A)$ is connected iff there exists an unbounded strictly increasing sequence $(a_i)_{i=1}^{\infty}$ such that $(a_i - 1, a_i] \subseteq A$ for every $i \in \mathbb{N}$.*

Proof. (a) If $A = \mathbb{R}$, $K(A)$ is the left-ray topology which has no two disjoint nonempty open sets, so is connected. If $b \notin A$, then $\bigcup_{n \in \mathbb{N}} [b, b+n) = [b, \infty)$ is open and $\{(-\infty, b), [b, \infty)\}$ is a separation of $K(A)$.

(b) To show the connectedness of $K^*(A)$ implies the existence of a strictly increasing sequence $(a_i)_{i=1}^{n}$ as described, we will show the contrapositive: if there exists $K \in \mathbb{R}$ such that every interval $(x-1, x] \subseteq [K, \infty)$ contains a point of $\mathbb{R} - A$, then $K^*(A)$ is disconnected. Suppose there exists such a K. Then there exists $b \in [K, \infty) - A$. Now $[b, x)$ is open for all $x \in (b, b+1]$. Let $m = \sup\{y : [b, x) \text{ is open for all } x \in (b, y]\}$. To see $m = \infty$, suppose to the contrary m is finite. Note that $m \geq b+1$ and $[b, m) = \bigcup\{[b, x) : x \in (b, m)\}$ is open. By the hypothesis, there exists $b' \in (m-1, m] - A$, and thus $[b', b'+1) \cup [b, m) = [b, b'+1)$ is open and $b'+1 > m$, contradicting our choice of m. Thus, $[b, \infty)$ is open. Next, we show $(-\infty, b)$ is open. If $x < b$, there exist $\varepsilon \in (0,1]$ with $x < x+\varepsilon < b$, and either $(-\infty, x+\varepsilon)$ or $[x, x+\varepsilon)$ is an open neighborhood of x contained in $(-\infty, b)$. Thus, $(-\infty, b)$ and its complement are both open, so $K^*(A)$ is disconnected.

Now suppose there exists a strictly increasing sequence $(a_i)_{i=1}^{n}$ as described. Suppose (U, V) is a separation of $K^*(A)$ and $a_1 \in U$. For $i \in \mathbb{N}$, since $(a_i - 1, a_i] \subseteq A$, the only basic open neighborhoods containing a_i also contain $(-\infty, a_i]$, so $(-\infty, a_i] \subseteq U$. Since this holds for every i and $(a_i)_{i=1}^{n}$ is unbounded, we have $\mathbb{R} \subseteq U$, contrary to (U, V) being a separation. □

Since the lower-limit left-ray hybrid $K(A)$ has a basis of q_{ll}- and q_{lr}-balls, we might hope that it arises from a hybrid of the quasi-metrics $\bar{q}_{ll}$ and $\bar{q}_{lr}$.

Theorem 4.3. *The function q_K defined below is a quasi-metric.*

$$q_K(x,y) = \begin{cases} y - x & \text{if } y \in [x, x+1) \\ 0 & \text{if } y < x, (y, x] \subseteq A \\ 1 & \text{otherwise} \end{cases}$$

Proof. The only non-trivial thing to check is the triangle inequality for distinct points x, y, z where $q_K(x,y) < 1$ and $q_K(y,z) < 1$. In such a situation, $q_K(x,y)$ may be

$$(1)\ y - x \text{ if } y \in [x, x+1) \quad \text{or} \quad (2)\ 0 \text{ if } y < x,\ (y, x] \subseteq A,$$

and $q_K(y,z)$ may be

$$(a)\ z - y \text{ if } z \in [y, y+1) \quad \text{or} \quad (b)\ 0 \text{ if } z < y,\ (z, y] \subseteq A.$$

In case (1)(a), $q_K(x,y) + q_K(y,z) = z - x \geq q_K(x,z)$ since $q_K(x,z)$ is $z - x$ if $z \in [x, x+1)$ and is 1 if $z \geq x+1$. In case (1)(b), if $x < z < y$, then $q_K(x,z) = z - x < y - x = q_K(x,y)$. If $z < x < y$, $(z, y] \subseteq A$ implies $(z, x] \subseteq A$, so $q_K(x,z) = 0$. In case (2)(a), if $y < z < x$, then $(y, x] \subseteq A$ implies $(z, x] \subseteq A$, so $q_K(x,z) = 0$. If $y < x < z$, then $q_K(x,z) = z - x < z - y = q_K(y,z)$. Case (2)(b) implies $q_K(x,z) = 0$. □

Theorem 4.4. *The quasi-metric of Theorem 4.3 generates* $K(A)$ *if and only if every strictly increasing bounded sequence* $(b_n)_{n=1}^{\infty}$ *in* $\mathbb{R} - A$ *has* $\sup\{b_n\}_{n=1}^{\infty} \in \mathbb{R} - A$.

Proof. Suppose the condition on $\mathbb{R} - A$ fails. That is, suppose there exists a strictly increasing bounded sequence $(b_n)_{n=1}^{\infty}$ in $\mathbb{R} - A$ with $\sup\{b_n\}_{n=1}^{\infty} = a \in A$. Now for $\varepsilon \in (0, 1]$, $B(a, \varepsilon) = [a, a+\varepsilon)$ is open in the quasi-metric topology. If $[a, a+\varepsilon)$ were open in $K(A)$, there must be an element B of $\mathcal{B}$ with $a \in B \subseteq [a, a+\varepsilon)$. In particular, B must be one of the bounded elements of $\mathcal{B}$, and thus must have form $[b, b+\varepsilon)$ for some $b \notin A$. Now $a \in A$ and $a \in [b, b+\varepsilon)$ implies $b < a$, which contradicts $[b, b+\varepsilon) \subseteq [a, a+1)$. Thus, $B(a, \varepsilon)$ is not open in $K(A)$, so q_K does not generate $K(A)$.

Now suppose every strictly increasing bounded sequence $(b_n)_{n=1}^{\infty}$ in $\mathbb{R} - A$ has $\sup\{b_n\}_{n=1}^{\infty} \in \mathbb{R} - A$. For $\varepsilon \leq 1$, and $b \notin A$, $B(b, \varepsilon)$ is $[b, b+\varepsilon)$. If $\varepsilon \leq 1$ and $a \in A$, $B(a, \varepsilon)$ is $(-\infty, a+\varepsilon)$ if $\{b \notin A : b < a\} = \varnothing$ and otherwise is $[b_a, a+\varepsilon)$ where $b_a = \sup\{b \notin A : b < a\}$. Since all distances are bounded by 1, if $\varepsilon > 1$, then $B(x, \varepsilon) = \mathbb{R}$. Note that all of these q_K-balls are in the basis $\mathcal{B}$ for $K(A)$, and every element of $\mathcal{B}$ contains such a q_K-ball, so the q_K-topology is that of $K(A)$. □

In the proof above, the q_K-balls of form $B(a, \varepsilon) = [b_a, a+\varepsilon)$ for $\varepsilon \leq 1$ are not in $\mathcal{B}^*$ (and may not be open in $K^*(A)$) if the length of this interval exceeds 1.

To illustrate some of the subtleties, we note that a proof similar to that of Theorem 4.3 shows that the following function is also a quasi-metric on $\mathbb{R}$.

$$q_5(x,y) = \begin{cases} y - x & \text{if } y \in [x, x+1) \\ 0 & \text{if } y < x, [y, x] \subseteq A \\ 1 & \text{otherwise} \end{cases}$$

It differs from q_K slightly in which distances are zero for $y < x$. However, with $A = \mathbb{R} - \{0\}$, we see that the q_5-ball $B(2.5, .5) = (0, 3)$ is not open in $K(A)$ nor $K^*(A)$.

5. Discrete Hybrids

We will consider hybrids of the discrete topology with the Euclidean, the lower-limit, and the left-ray topologies. Given $A \subseteq \mathbb{R}$, for $i = 1, 2, 3$, let $D_i(A)$ be the topology on $\mathbb{R}$ having a basis

$$\mathcal{B}_i = \{\{a\} : a \in A\} \cup \{U_i(b, \varepsilon) : b \in B = \mathbb{R} - A, \varepsilon > 0\}$$

where $U_1(b, \varepsilon) = (b - \varepsilon, b + \varepsilon)$, $U_2(b, \varepsilon) = [b, b + \varepsilon)$, and $U_3(b, \varepsilon) = (-\infty, b + \varepsilon)$. Thus, $D_i(A)$ is discrete on A.

With $U_i(b, \varepsilon)$ defined as above, we will first show that the basis

$$\mathcal{B}_i^* = \{\{a\} : a \in A\} \cup \{U_i(b, \varepsilon) : b \in B = \mathbb{R} - A, \varepsilon \in (0, 1]\}$$

also generates the same topology generated by $\mathcal{B}_i$. $\mathcal{B}_i^* \subseteq \mathcal{B}_i$, so $\mathcal{B}_i$ generates a finer topology than $\mathcal{B}_i^*$. For $x \in \mathbb{R}$, suppose B_i is an element of $\mathcal{B}_i$ containing x. If $x \in A$, then $B_i^* = \{x\} \subseteq B_i$, so we may assume $x \notin A$ and $B_i = U_i(b, \varepsilon)$ for some $b \notin A, \varepsilon > 0$. If $i = 2, 3$, then $x \in B_i^* = U_i(x, \min\{1, b + \varepsilon - x\}) \subseteq B$ and if $i = 1$, $x \in B_1^* = U_1(x, \min\{1, b + \varepsilon - x, x - b + \varepsilon\}) \subseteq B$. Thus, $\mathcal{B}_i^*$ generates a finer topology than $\mathcal{B}_i$.

These discrete hybrids are especially nice since their bases are closed under finite intersections.

Theorem 5.1.

(a) *For $i = 1, 2, 3$, $D_i(A)$ is separable iff A is countable.*
(b) *$D_1(A)$ and $D_2(A)$ are T_2 for all $A \subseteq \mathbb{R}$.*
$D_3(A)$ is T_1 iff $D_3(A)$ is T_2 iff $A = \mathbb{R}$ iff $D_3(A)$ is discrete.
(c) *$D_1(A)$, $D_2(A)$, and $D_3(A)$ are T_4.*
(d) *$D_1(A)$ is connected iff $A = \varnothing$.*
$D_2(A)$ is never connected.
$D_3(A)$ is connected iff there is a sequence $(b_n)_{n=1}^{\infty}$ in $\mathbb{R} - A$ which diverges to ∞.

Proof. (a) If A is countable, then every basic open set in $D_i(A)$ contains a point of $A \cup \mathbb{Q}$, so $D_i(A)$ is separable. If C is a dense set, then for any $a \in A$, the neighborhood $\{a\}$ of a must intersect C, so every dense set contains A. Thus, if A is uncountable, then $D_i(A)$ is not separable.

(b) For $i = 1, 2$, $D_i(A)$ is finer than the Euclidean topology and thus is T_2. If there exists $x \in \mathbb{R} - A$, then in $D_3(A)$, every neighborhood of x contains $(-\infty, x]$ and in particular, contains $x - 1$. Thus, $D_3(A)$ is not T_1. Thus, X is T_1 iff $A = \mathbb{R}$, and the result follows.

(c) Consider $D_1(A)$. For $x \in A$ and $\varepsilon > 0$, define $U(x, \varepsilon) = \{x\}$. For $x \notin A, \varepsilon > 0$, define $U(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$. Suppose C and D are disjoint closed sets. Construct U and V as in the proof of Theorem 2.3. Then U and V are open and $C \subseteq U$, $D \subseteq V$. Suppose $x \in U \cap V$. Then there exist $c^* \in C, d^* \in D$ with $x \in U(c^*, \varepsilon_{c^*}/2) \cap U(d^*, \varepsilon_{d^*}/2)$. If $c^* \in A$, then we have $x \in \{c^*\} \cap U(d^*, \varepsilon_{d^*}/2)$, contrary to $U(d^*, \varepsilon_{d^*}/2) \cap C = \varnothing$. Similarly, $d \notin A$, so $U(c^*, \varepsilon_{c^*}/2) \cap U(d^*, \varepsilon_{d^*}/2) = (c^* - \varepsilon_{c^*}/2, c^* + \varepsilon_{c^*}/2) \cap (d^* - \varepsilon_{d^*}/2, d^* + \varepsilon_{d^*}/2)$.

If $\varepsilon_{c^*} \leq \varepsilon_{d^*}$, then $|c^* - d^*| \leq |c^* - x| + |x - d^*| < \varepsilon_{c^*}/2 + \varepsilon_{d^*}/2 < \varepsilon_{d^*}$ gives the contradiction that $c^* \in U(d^*, \varepsilon_{d^*})$. A similar contradiction arises if $\varepsilon_{d^*} < \varepsilon_{c^*}$.

A similar argument shows $D_2(A)$ is T_4. Indeed, if C and D are disjoint closed sets in $D_2(A)$, for each $c \in C$ there exists ε_c such that $U(c, \varepsilon) \cap D = \varnothing$, where $U(c, \varepsilon)$ is $\{c\}$ if $c \in A$ and $[c, c+\varepsilon)$ if $c \notin A$. With ε_d defined similarly for each $d \in D$, we may take $U = \bigcup\{U(c, \varepsilon) : c \in C\}$ and $V = \bigcup\{U(d, \varepsilon) : d \in D\}$ (without halving the radii, here). If $x \in U \cap V$, then there exist $c^* \in C, d^* \in D$ with $x \in U(c^*, \varepsilon_{c^*}) \cap U(d^*, \varepsilon_{d^*})$. If $U(c^*, \varepsilon_{c^*})$ is a singleton, then we have the contradiction $c^* \in U(d^*, \varepsilon_{d^*})$, and similarly $U(d^*, \varepsilon_{d^*})$ is not a singleton. Now without loss of generality, say $c^* < d^*$. Now $x \in [c^*, c^* + \varepsilon_{c^*}) \cap [d^*, \varepsilon_{d^*})$ gives the contradiction that $d^* \in [c^*, c^* + \varepsilon_{c^*}) = U(c^*, \varepsilon_{c^*})$.

For $D_3(A)$, it is an easy exercise to show that there are no two disjoint nonempty closed sets, so the space is T_4.

(d) If $a \in A$, $\{a\}$ is clopen in $D_1(A)$ and $D_2(A)$, so these spaces are not connected if $A \neq \varnothing$. $D_1(\varnothing)$ is the Euclidean topology, which is connected. $D_2(\varnothing)$ is the lower limit topology, which is not connected.

For $D_3(A)$, suppose there is a sequence $(b_n)_{n=1}^{\infty}$ in $\mathbb{R} - A$ which diverges to ∞. Without loss of generality, suppose this sequence is strictly increasing. Suppose (U, V) is a separation of $D_3(A)$ with $b_1 \in U$. For $k \in \mathbb{N}$, every neighborhood of b_k contains $(-\infty, b_k]$ which intersects U at b_1, so $(-\infty, b_k] \subseteq U$ for each $k \in \mathbb{N}$, and this gives the contradiction that $U = \mathbb{R}$. Conversely, if there is no such sequence, then there exists $r \in \mathbb{R}$ with $[r, \infty) \subseteq A$. Now $[r, \infty)$ is closed and open, so $D_3(A)$ is not connected. $\square$

The natural candidate for a quasi-metric for $D_1(A)$ is given below.

$$q_6(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \in A, y \neq x \\ |y - x| & \text{if } x \notin A \end{cases}$$

However, this is a quasi-metric if and only if $A = \varnothing$ or $A = \mathbb{R}$, in which case we have the Euclidean or discrete metric (and the space is not really a hybrid). If there exist $b \in \mathbb{R} - A$, $a \in A$, then $q_6(b, a) + q_6(a, b + |b - a| + 2) = |b - a| + 1 < |b - a| + 2 = q_6(b, b + |b - a| + 2)$.

Theorem 5.2. *For $i = 1, 2, 3$, the function q_{D_i} below is a quasi-metric for $D_i(A)$.*

$$q_{D_1}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \in A, y \neq x \\ \min\{1, |y - x|\} & \text{if } x \notin A \end{cases}$$

$$q_{D_2}(x, y) = \begin{cases} 0 & \text{if } x = y \\ y - x & \text{if } x \notin A, y \in [x, x+1) \\ 1 & \text{otherwise} \end{cases}$$

$$q_{D_3}(x, y) = \begin{cases} \min\{1, y - x\} & \text{if } x \notin A, y \geq x \\ 1 & \text{if } x \in A, y \neq x \\ 0 & \text{otherwise} \end{cases}$$

Proof. Clearly $q_{D_i}(x,y)$ is nonnegative and is 0 if and only if $x = y$. Since the functions are bounded by 1, it suffices to check that $q_{D_i}(x,y)+q_{D_i}(y,z) \geq q_{D_i}(x,z)$ when x, y, z are distinct and both $q_{D_i}(x,y)$ and $q_{D_i}(y,z)$ are strictly less than 1.

For q_{D_1}, this occurs when $x, y \notin A$ and $q_{D_1}(x,y)+q_{D_1}(y,z) = |y-x|+|z-y|$. But $|y-x|+|z-y| \geq |z-x| \geq \min\{1, |z-x|\} = q_{D_1}(x,z)$. Thus, this satisfies the triangle inequality and is a quasi-metric. Furthermore, for $\varepsilon \in (0,1]$, $a \in A$, and $b \in \mathbb{R}-A$, we have $B(a,\varepsilon) = \{a\}$ and $B(b,\varepsilon) = (x-\varepsilon, x+\varepsilon)$ and for $\varepsilon > 1$, $B(x,\varepsilon) = \mathbb{R}$. Since all q_{D_1}-balls are open in $D_1(A)$ and the bounded balls are a basis for $D_1(A)$, the q_{D_1}-topology is the topology for $D_1(A)$.

The proofs for q_{D_2} and q_{D_3} are similar. □

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