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Common fixed point results for a generalized (ψ, ϕ)-rational contraction

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Abstract

In this paper, we obtain two common fixed point results for mappings satisfying the generalized (ψ, ϕ)-contractive type conditions given by a rational expression on a complete metric space. Our results generalize several well known theorems of the literature in the context of (ψ, ϕ)-rational contraction. In addition, there is an example for obtained results.

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1. Introduction

A self mapping T on metric space (X, d) is said to be contraction if there exists $0 \leq a < 1$ such that $d(Tx, Ty) \leq a\, d(x, y)$ for each $x, y \in X$. It is well known that every contraction on a complete metric space has a unique fixed point (Banach contraction principle). This result is considered as a main source of metric fixed point theory and provided a new impetus towards the existence of fixed points for the mappings not only in metric space but also for different settings of the domain of the mappings. Moreover, it plays the major role in

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solving nonlinear problems. Also, in 1969, Boyd and Wong [6] defined a new class of contractive mappings which is generally known as ϕ-contraction and generalizes the Banach contraction principle. Furthermore, in 1997, Alber and Guerre-Delabriere [1] generalized this concept in Hilbert spaces by introducing weak ϕ-contraction. However, Rhoades [18] has shown that the result of Alber et al. [1] is also valid in complete metric spaces as stated the following.

Theorem 1.1 ([18]). *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping. Assume that for every $x, y \in X$,*

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)), \tag{1.1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function with $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$. Then T has a unique fixed point.

Moreover, Dutta et al. [8] obtained the following generalization of Theorem 1.1.

Theorem 1.2 ([8]). *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping. Assume that for every $x, y \in X$,*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)), \tag{1.2}$$

where

(i) *$\psi : [0, \infty) \to [0, \infty)$ is a continuous and monotone non-decreasing function with $\psi(t) = 0$ if and only if $t = 0$.*
(ii) *$\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$.*

Then T has a unique fixed point.

Now, for self mappings T and S on a metric space (X, d), we use the following notations:

(i) $N(Tx, Sy) = \max\left\{d(x, y), d(y, Sy)\left(\frac{1+d(x,Tx)}{1+d(x,y)}\right)\right\}$.
(ii) $M(Tx, Sy) = \max\left\{d(x, y), d(x, Tx), d(y, Sy), \frac{d(y,Tx)+d(x,Sy)}{2}\right\}$.
(ii) $M_1(Tx, Sy) = \max\left\{\begin{array}{l} d(x, y), d(x, Tx), d(y, Sy), \\ \frac{d(y,Tx)+d(x,Sy)}{2}, \frac{d(x,Tx)+d(y,Sy)}{2}, \\ d(y, Sy)\left(\frac{1+d(x,Tx)}{1+d(x,y)}\right), d(x, Tx)\left(\frac{1+d(y,Sy)}{1+d(x,y)}\right) \end{array}\right\}$.

In 2009, Q. Zhang et al. [20] obtained the following generalization of Theorem 1.1.

Theorem 1.3 ([20]). *Let (X, d) be a complete metric space and let $T, S : X \to X$ be two mappings. Assume that for every $x, y \in X$,*

$$d(Tx, Sy) \leq M(Tx, Sy) - \phi(M(Tx, Sy)), \tag{1.3}$$

where ϕ is defined as in Theorem 1.2. Then there exists a unique point $z \in X$ such that $z = Tz = Sz$.

In an analogous manner, Đorić [7] has obtained the following common fixed point theorem for two mappings which also generalizes above results.

Theorem 1.4 ([7]). *Let (X,d) be a complete metric space and let $T,S : X \to X$ be two mappings. Assume that for every $x,y \in X$,*

$$\psi(d(Tx,Sy)) \le \psi(M(Tx,Sy)) - \phi(M(Tx,Sy)), \tag{1.4}$$

where ψ and ϕ are defined as in Theorem 1.2. Then there exists a unique point $z \in X$ such that $z = Tz = Sz$.

Now, we consider the following example for the comparison between Theorems 1.3 and 1.4.

Example 1.5 ([7]). Let $X = [0,1]$ be endowed with the Euclidean metric $d(x,y) = |x-y|$, $Tx = \frac{x}{3}$ and $Sx = 0$ for each $x,y \in X$. Then $d(Tx,Sy) = \frac{x}{3}$ and

$$M(Tx,Sy) = \begin{cases} x-y, & 0 \le y \le \frac{x}{3}; \\ \frac{2x}{3}, & \frac{x}{3} \le y \le \frac{2x}{3}; \\ y, & \frac{2x}{3} < y \le 1. \end{cases}$$

Moreover, for $\psi(t) = 3t$ and $\phi(t) = t$, we have $\psi(Tx,Sy) = x$ and

$$\psi(M(Tx,Sy)) - \phi(M(Tx,Sy)) = \begin{cases} 2x-2y, & 0 \le y \le \frac{x}{3}; \\ \frac{4x}{3}, & \frac{x}{3} \le y \le \frac{2x}{3}; \\ 2y, & \frac{2x}{3} < y \le 1. \end{cases}$$

Thus, the mappings T and S satisfy condition (1.4) of Theorem 1.4 with a unique common fixed point. However, they don't satisfy condition (1.3) of Theorem 1.3. Hence, Theorem 1.4 is a proper generalization of Theorem 1.3.

However, in 2017, Fei He et al. [11] proved the following common fixed point theorem for two mappings satisfying a generalized (ψ,ϕ)-Suzuki weak contractive type condition in a complete metric space.

Theorem 1.6 ([11]). *Let (X,d) be a complete metric space and let $T,S : X \to X$ be two mappings. Assume that for every $x,y \in X$,*

$$\begin{array}{l} \frac{1}{2}\min\{d(x,Tx), d(y,Sy)\} \le d(x,y) \textit{ implies} \\ \psi(d(Tx,Sy)) \le \psi(M(Tx,Sy)) - \phi(M(Tx,Sy)), \end{array} \tag{1.5}$$

where ψ and ϕ are defined as in Theorem 1.2. Then there exists a unique point $z \in X$ such that $z = Tz = Sz$.

Recently, in 2020, the authors [2] obtained the following result for the generalized (ψ,ϕ)-Suzuki weak contraction under a rational expression.

Theorem 1.7 ([2]). *Let X be a complete metric space and let $T : X \to X$ be a mapping. Assume that for every $x,y \in X$,*

$$\begin{array}{l} \frac{1}{2}d(x,Tx) \le d(x,y) \textit{ implies} \\ \psi(d(Tx,Ty)) \le \psi(N(Tx,Ty)) - \phi(N(Tx,Ty)), \end{array} \tag{1.6}$$

where ψ and ϕ are defined as in Theorem 1.2. Then T has a unique fixed point.

Besides, many researchers have studied such types of contractive conditions and have been proved some interesting fixed point results for (ψ, ϕ)-contractive mappings, see [2, 4, 5, 3, 7, 8, 10, 11, 9, 19, 12, 13, 16, 15, 17, 14, 18, 20] and references therein.

Now, we establish two common fixed theorems for two mappings satisfying a generalized (ψ, ϕ)-rational contractive condition in a complete metric space. Obtained results are also generalizations of Theorems 1.1, 1.2, 1.3, 1.4, 1.6 and 1.7 in the setting of (ψ, ϕ)-rational contraction. Moreover, an example has been presented to vindicate the results.

2. Main Results

Theorem 2.1. *Let X be a complete metric space and let $T, S : X \to X$ be two mappings. Assume that for every $x, y \in X$,*

$$\psi(d(Tx, Sy)) \leq \psi(M_1(Tx, Sy)) - \phi(M_1(Tx, Sy)), \tag{2.1}$$

where ψ and ϕ are defined as in Theorem 1.2. Then there exists a unique point $z \in X$ such that $z = Tz = Sz$.

Proof. Suppose $x_0 \in X$ is an arbitrary. Then we can choose $x_1 = Sx_0$, $x_2 = Tx_1$, $x_3 = Sx_2$ and $x_4 = Tx_3$. In general, we can construct a sequence $\{x_n\}_{n=0}^{\infty}$ in X such that $x_{2n+2} = Tx_{2n+1}$ and $x_{2n+1} = Sx_{2n}$.

Now, if n is odd, then by (2.1), we have

$$\psi(d(Tx_n, Sx_{n-1})) \leq \psi(M_1(Tx_n, Sx_{n-1})) - \phi(M_1(Tx_n, Sx_{n-1})),$$

where

$$\begin{aligned} M_1(Tx_n, Sx_{n-1}) &= \max\left\{\begin{array}{l} d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Sx_{n-1}), \\ \frac{d(x_{n-1}, Tx_n) + d(x_n, Sx_{n-1})}{2}, \\ \frac{d(x_n, Tx_n) + d(x_{n-1}, Sx_{n-1})}{2}, \\ d(x_{n-1}, Sx_{n-1})\left(\frac{1+d(x_n, Tx_n)}{1+d(x_n, x_{n-1})}\right), \\ d(x_n, Tx_n)\left(\frac{1+d(x_{n-1}, Sx_{n-1})}{1+d(x_n, x_{n-1})}\right) \end{array}\right\} \\ &= \max\left\{\begin{array}{l} d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), \\ \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2}, \frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)}{2}, \\ d(x_{n-1}, x_n)\left(\frac{1+d(x_n, x_{n+1})}{1+d(x_n, x_{n-1})}\right), \\ d(x_n, x_{n+1})\left(\frac{1+d(x_{n-1}, x_n)}{1+d(x_n, x_{n-1})}\right) \end{array}\right\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}. \end{aligned}$$

So, we obtain

$$\begin{aligned} \psi(d(x_{n+1}, x_n)) &\leq \psi(\max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}) \\ &\quad -\phi(\max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}). \end{aligned} \tag{2.2}$$

If $d(x_n, x_{n+1}) > d(x_{n-1}, x_n)$ for some n, then (2.2) gives

$$\psi(d(x_n, x_{n+1})) \leq \psi(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1})) < \psi(d(x_n, x_{n+1})),$$

which is a contradiction. Hence, for all n, we get

$$\psi(d(x_n, x_{n+1})) \leq \psi(d(x_{n-1}, x_n)) - \phi(d(x_{n-1}, x_n)).$$

Consequently, we have

$$\psi(d(x_n, x_{n+1})) \leq \psi(d(x_{n-1}, x_n)). \tag{2.3}$$

In an analogous way, we can show that condition (2.3) is true for even values of n. By the property of ψ, for all $n \in \mathbb{N}$, the positive integers, we have

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n). \tag{2.4}$$

Moreover, the sequence $\{d(x_n, x_{n+1})\}_{n=0}^{\infty}$ is non-increasing monotonic and bounded below, and so there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r = \lim_{n\to\infty} d(x_{n-1}, x_n). \tag{2.5}$$

Using the property of lower semi-continuity of ϕ, we have

$$\phi(r) \leq \liminf_{n\to\infty} \phi(d(x_{n-1}, x_n)).$$

Now, we claim that $r = 0$. In fact, taking upper limit as $n \to \infty$ on the following inequality and using (2.5), we get

$$\psi(d(x_n, x_{n+1})) \leq \psi(d(x_{n-1}, x_n)) - \phi(d(x_{n-1}, x_n)) \;\Rightarrow\; \psi(r) \leq \psi(r) - \phi(r).$$

That is, $\phi(r) \leq 0$ implies $\phi(r) = 0$, and $\phi(r) = 0$ implies $r = 0$. Hence,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.6}$$

Next, we show that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. For this, it is sufficient to prove that the subsequence $\{x_{2n}\}$ is a Cauchy sequence, but we suppose in contrary way that $\{x_{2n}\}$ is not a Cauchy sequence. Then, there is an $\epsilon > 0$ for which can find two subsequences $\{x_{2m_k}\}$ and $\{x_{2n_k}\}$ such that n_k is the smallest index for which $n_k > m_k > k$, $d(x_{2m_k}, x_{2n_k}) \geq \epsilon$ and $d(x_{2m_k}, x_{2n_k-2}) < \epsilon$. Then (2.6) and the inequality

$$\epsilon \leq d(x_{2m_k}, x_{2n_k}) \leq d(x_{2m_k}, x_{2n_k-2}) + d(x_{2n_k-1}, x_{2n_k-2}) + d(x_{2n_k-1}, x_{2n_k})$$

imply $\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k}) = \epsilon$. Also, (2.6) and the inequality

$$d(x_{2m_k}, x_{2n_k}) \leq d(x_{2m_k}, x_{2m_k+1}) + d(x_{2m_k+1}, x_{2n_k}) \text{ give } \epsilon \leq \lim_{k\to\infty} d(x_{2m_k+1}, x_{2n_k}).$$

So, (2.6) and the inequality $d(x_{2m_k+1}, x_{2n_k}) \leq d(x_{2m_k+1}, x_{2m_k}) + d(x_{2m_k}, x_{2n_k})$ yield $\lim_{k\to\infty} d(x_{2m_k+1}, x_{2n_k}) = \epsilon$. In similar way, we obtain

$$\lim_{k\to\infty} d(x_{2n_k}, x_{2m_k+1}) = \epsilon.$$

Taking $x = x_{2n_k+1}$ and $y = x_{2m_k}$ in (2.1) and (2.4), we get

$$\begin{aligned}
\psi(d(x_{2n_k+2}, x_{2m_k+1})) &= \psi(d(Tx_{2n_k+1}, Sx_{2m_k})) \\
&\leq \psi(M_1(Tx_{2n_k+1}, Sx_{2m_k})) - \phi(M_1(Tx_{2n_k+1}, Sx_{2m_k})),
\end{aligned}$$

where

$$M_1(Tx_{2n_k+1}, Sx_{2m_k}) = \max \left\{ \begin{array}{l} d(x_{2n_k+1}, x_{2m_k}), d(x_{2n_k+1}, Tx_{2n_k+1}), \\ d(x_{2m_k}, Sx_{2m_k}), \\ \frac{d(x_{2m_k}, Tx_{2n_k+1}) + d(x_{2n_k+1}, Sx_{2m_k})}{2}, \\ \frac{d(x_{2n_k+1}, Tx_{2n_k+1}) + d(x_{2m_k}, Sx_{2m_k})}{2}, \\ d(x_{2m_k}, Sx_{2m_k}) \left(\frac{1 + d(x_{2n_k+1}, Tx_{2n_k+1})}{1 + d(x_{2n_k+1}, x_{2m_k})} \right), \\ d(x_{2n_k+1}, Tx_{2n_k+1}) \left(\frac{1 + d(x_{2m_k}, Sx_{2m_k})}{1 + d(x_{2n_k+1}, x_{2m_k})} \right) \end{array} \right\},$$

for which $\lim_{k\to\infty} M_1(Tx_{2n_k+1}, Sx_{2m_k}) = \epsilon$. Hence, we have $\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon)$, which is a contradiction with $\epsilon > 0$. It follows that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X, and completeness of X ensures the convergence to a limit, say $z \in X$.

Now, we show that z is a common fixed point of T and S. For this, using (2.1), we get

$$\begin{array}{rcl} \psi(d(Tz, Sx_{2n_k})) & \leq & \psi(M_1(Tz, Sx_{2n_k})) - \psi(M_1(Tz, Sx_{2n_k})) \\ & = & \psi\left(\max \left\{ \begin{array}{l} d(z, x_{2n_k}), d(z, Tz), d(x_{2n_k}, Sx_{2n_k}), \\ \frac{d(x_{2n_k}, Tz) + d(z, Sx_{2n_k})}{2}, \\ \frac{d(z, Tz) + d(x_{2n_k}, Sx_{2n_k})}{2}, \\ d(x_{2n_k}, Sx_{2n_k}) \left(\frac{1 + d(z, Tz)}{1 + d(z, x_{2n_k})} \right), \\ d(z, Tz) \left(\frac{1 + d(x_{2n_k}, Sx_{2n_k})}{1 + d(z, x_{2n_k})} \right) \end{array} \right\} \right) \\ & & -\phi\left(\max \left\{ \begin{array}{l} d(z, x_{2n_k}), d(z, Tz), d(x_{2n_k}, Sx_{2n_k}), \\ \frac{d(x_{2n_k}, Tz) + d(z, Sx_{2n_k})}{2}, \\ \frac{d(z, Tz) + d(x_{2n_k}, Sx_{2n_k})}{2}, \\ d(x_{2n_k}, Sx_{2n_k}) \left(\frac{1 + d(z, Tz)}{1 + d(z, x_{2n_k})} \right), \\ d(z, Tz) \left(\frac{1 + d(x_{2n_k}, Sx_{2n_k})}{1 + d(z, x_{2n_k})} \right) \end{array} \right\} \right). \end{array}$$

Making $k \to \infty$, we have $\psi(d(z,Tz)) \leq \psi(d(z,Tz)) - \phi(d(z,Tz))$, which yields $z = Tz$. Further, we get

$$\begin{array}{rcl}
\psi(d(Tz,Sz)) & \leq & \psi(M_1(Tz,Sz)) - \psi(M_1(Tz,Sz)) \\
& = & \psi\left(\max\left\{\begin{array}{l} d(z,z), d(z,z), d(z,Sz), \\ \frac{d(z,z)+d(z,Sz)}{2}, \frac{d(z,z)+d(z,Sz)}{2}, \\ d(z,Sz)\left(\frac{1+d(z,z)}{1+d(z,z)}\right), \\ d(z,z)\left(\frac{1+d(z,Sz)}{1+d(z,z)}\right) \end{array}\right\}\right) \\
& & -\phi\left(\max\left\{\begin{array}{l} d(z,z), d(z,z), d(z,Sz), \\ \frac{d(z,z)+d(z,Sz)}{2}, \frac{d(z,z)+d(z,Sz)}{2}, \\ d(z,Sz)\left(\frac{1+d(z,z)}{1+d(z,z)}\right), \\ d(z,z)\left(\frac{1+d(z,Sz)}{1+d(z,z)}\right) \end{array}\right\}\right) \\
\Rightarrow \quad \psi(d(z,Sz)) & \leq & \psi(d(z,Sz)) - \phi(d(z,Sz)),
\end{array}$$

which provides $z = Sz$. Hence z is a common fixed point of T and S.

For uniqueness, we suppose that y is another fixed point of T and S, and we get

$$\begin{array}{rcl}
\psi(d(y,z)) & = & \psi(d(Ty,Sz)) \\
& \leq & \psi(M_1(Ty,Sz)) - \phi(M_1(Ty,Sz)) \\
& = & \psi(d(y,z)) - \phi(d(y,z)) \\
\Rightarrow \ \phi(d(y,z)) & = & 0.
\end{array}$$

Therefore, $y = z$. This completes the result. □

Remark 2.2. Theorem 2.1 is a proper generalization of the result of Đorić [7] (Theorem 1.4).

Now, for $\psi = I$ (identity) in Theorem 2.1, we get the following corollary.

Corollary 2.3. *Let (X,d) be a complete metric space and let $T,S : X \to X$ be two mappings. Assume that for every $x,y \in X$,*

$$d(Tx,Sy) \leq M_1(Tx,Sy) - \phi(M_1(Tx,Sy)),$$

where ϕ is defined as in Theorem 1.2. Then there exists a unique point $z \in X$ such that $z = Tz = Sz$.

Also, for $S = T$, we obtain the following corollary of Theorem 2.1.

Corollary 2.4. *Let X be a complete metric space and let $T : X \to X$ be a mapping. Assume that for every $x,y \in X$,*

$$\psi(d(Tx,Ty)) \leq \psi(M_1(Tx,Ty)) - \phi(M_1(Tx,Ty)),$$

where ψ and ϕ are defined as in Theorem 1.2. Then there exists a unique point $z \in X$ such that $z = Tz$.

Taking $M_1(Tx,Sy) = d(x,y)$ in Theorem 2.1, we get the following generalization of the results of Dutta et al. [8].

Corollary 2.5. *Let (X,d) be a complete metric space and let $T,S : X \to X$ be two mappings. Assume that for every $x,y \in X$,*

$$\psi(d(Tx,Sy)) \quad \leq \quad \psi(d(x,y)) - \phi(d(x,y)),$$

where ψ and ϕ are defined as in Theorem 1.2. Then there exists a unique point $z \in X$ such that $z = Tz = Sz$.

Further, we have the following corollary of Theorem 2.1.

Corollary 2.6. *Let (X,d) be a complete metric space and let $T,S : X \to X$ be two mappings. Assume that for every $x,y \in X$,*

$$\psi(d(Tx,Sy)) \leq \psi(N(Tx,Sy)) - \phi(N(Tx,Sy)),$$

where ψ and ϕ are defined as in Theorem 1.2. Then there exists a unique point $z \in X$ such that $z = Tz = Sz$.

Next, we discuss an example which shows that Theorem 2.1 is more general than the results in [2, 7, 8, 18, 20].

Example 2.7. Let $X = \{(0,0),(0,4),(4,0),(0,5),(5,0),(4,5),(5,4)\}$ be endowed with metric d defined by

$$d[(x_1,x_2),(y_1,y_2)] = |x_1 - y_1| + |x_2 - y_2|.$$

Suppose $T,S : X \to X$ are defined by

$$T(x_1,x_2) = \begin{cases} (x_2,0) & \text{if } x_1 \leq x_2 \\ (0,x_2) & \text{if } x_1 > x_2, \end{cases} \quad S(x_1,x_2) = \begin{cases} (x_1,0) & \text{if } x_1 \leq x_2 \\ (0,0) & \text{if } x_1 > x_2. \end{cases}$$

Choose $\psi(t) = t$ and $\phi(t) = \frac{3t}{5}$, then T and S do not satistisfy the conditions (1.1), (1.2), (1.3), and (1.4). To see this, if we take $x = (4,5)$ and $y = (5,4)$, then we have $d(x,y) = 2$, $M(Tx,Sy) = 9$, $M_1(Tx,Sy) = 21$, and we observe the followings:

(1) If $d(Tx,Ty) \leq d(x,y)) - \phi(d(x,y))$ implies $9 \leq (2 - \frac{6}{5})$, which is not possible. Hence, the condition (1.1) is not satisfied.
(2) If $\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \phi(d(x,y))$ implies $9 \leq (2 - \frac{6}{5})$, which is not possible. Hence, the condition (1.2) is not satisfied.
(3) If $(d(Tx,Sy)) \leq (M(Tx,Sy)) - \phi(M(Tx,Sy))$ implies $5 \leq (9 - \frac{27}{5})$, which is not possible. Hence, the condition (1.3) is not satisfied.
(4) If $\psi(d(Tx,Sy)) \leq \psi(M(Tx,Sy)) - \phi(M(Tx,Sy))$ implies $5 \leq (9 - \frac{27}{5})$, which is not possible. Hence, the condition (1.4) is not satisfied.

However, the condition (2.1) of our result, $\psi(d(Tx,Sy)) \leq \psi(M_1(Tx,Sy)) - \phi(M_1(Tx,Sy))$ is satisfying for all $x,y \in X$. Moreover, $(0,0) \in X$ is only common fixed point of T and S.

Next, we prove a common fixed point theorem for the (ψ,ϕ)-rational contraction in Suzuki type context.

Theorem 2.8. *Let (X,d) be a complete metric space and let $T,S:X\to X$ be two mappings. Assume that for every $x,y\in X$,*

$$\begin{array}{l}\frac{1}{2}\min\{d(x,Tx),d(y,Sy)\}\leq d(x,y) \text{ implies}\\ \psi(d(Tx,Sy))\leq\psi(M_1(Tx,Sy))-\phi(M_1(Tx,Sy)),\end{array}\tag{2.7}$$

where ψ and ϕ are defined as in Theorem 1.2. Then there exists a unique point $z\in X$ such that $z=Tz=Sz$.

Proof. Let $x_0\in X$ be an arbitrary. We construct a sequence $\{x_n\}_{n=0}^{\infty}$ in X such that $x_{2n-1}=Tx_{2n-2}$ and $x_{2n}=Sx_{2n-1}$, $n=1,2,\ldots$. The following fact will be used in the sequel.

$$\begin{array}{l}\frac{1}{2}\min\{d(x,Tx),d(y,Sy)\}\leq d(x,y) \text{ if and only if}\\ d(x,Tx)\leq d(x,y) \text{ or } d(y,Sy)\leq d(x,y).\end{array}$$

If $x_n=x_{n-1}$ for some n, then the existence of the common fixed point is obvious. Because, if we suppose $x_{2n}=x_{2n-1}$ for some $n\in\mathbb{N}$, then x_{2n-1} is a common fixed point of T and S. Indeed, using (2.16), we find

$$\frac{1}{2}d(x_{2n-1},Sx_{2n-1})=\frac{1}{2}d(x_{2n-1},x_{2n})=0\leq d(x_{2n},x_{2n-1})$$

implies

$$\psi(d(Tx_{2n},Sx_{2n-1}))\leq\psi(M_1(Tx_{2n},Sx_{2n-1}))-\phi(M_1(Tx_{2n},Sx_{2n-1})),$$

where

$$\begin{array}{rcl}M_1(Tx_{2n},Sx_{2n-1}) &=& \max\left\{\begin{array}{l} d(x_{2n},x_{2n-1}),d(x_{2n},Tx_{2n}),\\ d(x_{2n-1},Sx_{2n-1}),\\ \frac{d(x_{2n-1},Tx_{2n})+d(x_{2n},Sx_{2n-1})}{2},\\ \frac{d(x_{2n},Tx_{2n})+d(x_{2n-1},Sx_{2n-1})}{2},\\ d(x_{2n-1},Sx_{2n-1})\left(\frac{1+d(x_{2n},Tx_{2n})}{1+d(x_{2n},x_{2n-1})}\right),\\ d(x_{2n},Tx_{2n})\left(\frac{1+d(x_{2n-1},Sx_{2n-1})}{1+d(x_{2n},x_{2n-1})}\right)\end{array}\right\}\\ &=& \max\left\{\begin{array}{l} d(x_{2n},x_{2n-1}),d(x_{2n},x_{2n+1}),\\ d(x_{2n-1},x_{2n}),\\ \frac{d(x_{2n-1},x_{2n+1})+d(x_{2n},x_{2n})}{2},\\ \frac{d(x_{2n},x_{2n+1})+d(x_{2n-1},x_{2n})}{2},\\ d(x_{2n-1},x_{2n})\left(\frac{1+d(x_{2n},x_{2n+1})}{1+d(x_{2n},x_{2n-1})}\right),\\ d(x_{2n},x_{2n+1})\left(\frac{1+d(x_{2n-1},x_{2n})}{1+d(x_{2n},x_{2n-1})}\right)\end{array}\right\}\\ &=& d(x_{2n},x_{2n+1})=d(x_{2n},Tx_{2n}).\end{array}$$

Then we get

$$\psi(d(Tx_{2n},x_{2n}))\leq\psi(d(Tx_{2n},x_{2n}))-\phi(d(Tx_{2n},x_{2n})),$$

which implies $\phi(d(Tx_{2n},x_{2n}))\leq 0$. By the property of ϕ, we get $Tx_{2n}=x_{2n}$. Thus, from $Sx_{2n-1}=x_{2n}=x_{2n-1}$, it follows that $Sx_{2n-1}=x_{2n-1}=Tx_{2n-1}$, i.e., x_{2n-1} is a common fixed point of T and S. Similarly, if $x_{2n-1}=x_{2n-2}$ for some n, then x_{2n-2} is a common fixed point of T and S.

So, we always assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. Notice that, for any $n \in \mathbb{N}$, we have

$$\frac{1}{2}d(x_{2n-1}, Sx_{2n-1}) = \frac{1}{2}d(x_{2n-1}, x_{2n}) \leq d(x_{2n}, x_{2n-1}).$$

Then by (2.16), we get

$$\begin{aligned} \psi(d(Tx_{2n}, Sx_{2n-1})) \leq\ & \psi(M_1(Tx_{2n}, Sx_{2n-1})) \\ & -\phi(M_1(Tx_{2n}, Sx_{2n-1})), \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} M_1(Tx_{2n}, Sx_{2n-1}) &= \max \left\{ \begin{array}{l} d(x_{2n}, x_{2n-1}), d(x_{2n}, Tx_{2n}), \\ d(x_{2n-1}, Sx_{2n-1}), \\ \frac{d(x_{2n-1}, Tx_{2n}) + d(x_{2n}, Sx_{2n-1})}{2}, \\ \frac{d(x_{2n}, Tx_{2n}) + d(x_{2n-1}, Sx_{2n-1})}{2}, \\ d(x_{2n-1}, Sx_{2n-1}) \left(\frac{1+d(x_{2n}, Tx_{2n})}{1+d(x_{2n}, x_{2n-1})} \right), \\ d(x_{2n}, Tx_{2n}) \left(\frac{1+d(x_{2n-1}, Sx_{2n-1})}{1+d(x_{2n}, x_{2n-1})} \right) \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), \\ d(x_{2n-1}, x_{2n}), \\ \frac{d(x_{2n-1}, x_{2n+1}) + d(x_{2n}, x_{2n})}{2}, \\ \frac{d(x_{2n}, x_{2n+1}) + d(x_{2n-1}, x_{2n})}{2}, \\ d(x_{2n-1}, x_{2n}) \left(\frac{1+d(x_{2n}, x_{2n+1})}{1+d(x_{2n}, x_{2n-1})} \right), \\ d(x_{2n}, x_{2n+1}) \left(\frac{1+d(x_{2n-1}, x_{2n})}{1+d(x_{2n}, x_{2n-1})} \right) \end{array} \right\} \\ &= \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1})\}. \end{aligned}$$

If $M_1(Tx_{2n}, Sx_{2n-1}) = d(x_{2n}, x_{2n+1})$, then (2.8) becomes

$$\psi(d(x_{2n}, x_{2n+1})) \leq \psi(d(x_{2n}, x_{2n+1})) - \phi(d(x_{2n}, x_{2n+1})),$$

which implies $\phi(d(x_{2n}, x_{2n+1})) \leq 0$ and so $d(x_{2n}, x_{2n+1}) = 0$. This is contrary to the assumption $x_n \neq x_{n-1}$. Consequently, (2.8) becomes that

$$\psi(d(x_{2n}, x_{2n+1})) \leq \psi(d(x_{2n}, x_{2n-1})) - \phi(d(x_{2n}, x_{2n-1})). \tag{2.9}$$

Similarly, we can find that

$$\psi(d(x_{2n+1}, x_{2n+2})) \leq \psi(d(x_{2n}, x_{2n+1})) - \phi(d(x_{2n}, x_{2n+1})). \tag{2.10}$$

Combining (2.9) and (1.5), we get

$$\psi(d(x_n, x_{n+1})) \leq \psi(d(x_n, x_{n-1})) - \phi(d(x_n, x_{n-1})), \tag{2.11}$$

for all $n \in \mathbb{N}$. Since $\phi(d(x_n, x_{n-1})) > 0$, we have

$$\psi(d(x_{n+1}, x_n)) < \psi(d(x_n, x_{n-1})).$$

By the property of ψ, for all $n \in \mathbb{N}$, we have

$$d(x_{n+1}, x_n) < d(x_n, x_{n-1}).$$

Moreover, the sequence $\{d(x_{n+1}, x_n)\}_{n=0}^{\infty}$ is non-increasing monotonic and bounded below, and so there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r = \lim_{n\to\infty} d(x_n, x_{n-1}).$$

Now, we claim that $r = 0$. In fact, taking upper limit as $n \to \infty$ on each side of (2.11), we get

$$\psi(r) \leq \psi(r) - \phi(r).$$

That is, $\phi(r) \leq 0$ implies $\phi(r) = 0$, and $\phi(r) = 0$ implies $r = 0$. Hence, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.12}$$

Moreover, we get

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{2.13}$$

Now, we claim that for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ with $m - n \equiv 1 \pmod 2$, then $d(x_n, x_{n+1}) < \epsilon$. Suppose to the contrary that there exists $\epsilon_0 > 0$ such that for any $N \in \mathbb{N}$, we can find $m > n > N$ with $m - n \equiv 1 \pmod 2$ satisfying $d(x_m, x_n) > \epsilon_0$. Using (2.12) and (2.13), for this ϵ_0, we find N_0 such that $n > N_0$ implies

$$d(x_n, x_{n+1}) < \epsilon_0 \text{ and } d(x_n, x_{n+2}) < \epsilon_0. \tag{2.14}$$

Following proof lines of Fei He et al. [11], we can find two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\begin{aligned} &d(x_{m_k}, x_{n_k})) \geq \epsilon_0, d(x_{m_k-2}, x_{n_k})) < \epsilon_0 \\ &\text{and } m - n \equiv 1 \pmod 2. \end{aligned} \tag{2.15}$$

Observe that, (2.13) and the inequality

$$\epsilon_0 \leq d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k-2}) + d(x_{m_k-2}, x_{n_k}) + d(x_{2n_k-1}, x_{n_k})$$

imply $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon_0$. Also, it implies that

$$\lim_{k\to\infty} d(x_{2m_k+1}, x_{2n_k}) = \epsilon_0.$$

In similar way, we obtain

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k+1}) = \epsilon_0,$$

and

$$\lim_{k\to\infty} d(x_{n_k+1}, x_{m_k+1}) = \epsilon_0.$$

Now, using $m_k - n_k \equiv 1 \pmod 2$, we consider the following cases:

Case 1. Let $m_k = 2p_k - 1$ and $m_k = 2q_k$ for some p_k, q_k. Then, from (2.14) and (2.15), we get

$$\frac{1}{2} d(x_{m_k}, Sx_{m_k}) = \frac{1}{2} d(x_{m_k}, x_{m_k+1}) < \epsilon_0 \leq d(x_{n_k}, x_{m_k}).$$

So, (2.16) implies that

$$\begin{array}{rcl}\psi(d(x_{n_k+1},x_{m_k+1})) &=& \psi(d(Tx_{n_k},Sx_{m_k}))\\ &\leq& \psi(M_1(Tx_{n_k},Sx_{m_k}))-\phi(M_1(Tx_{n_k},Sx_{m_k})),\end{array}$$

where

$$M_1(Tx_{n_k},Sx_{m_k})=\max\left\{\begin{array}{l} d(x_{n_k},x_{m_k}),d(x_{n_k},Tx_{n_k}),\\ d(x_{m_k},Sx_{m_k}),\\ \frac{d(x_{m_k},Tx_{n_k})+d(x_{n_k},Sx_{m_k})}{2},\\ \frac{d(x_{n_k},Tx_{n_k})+d(x_{m_k},Sx_{m_k})}{2},\\ d(x_{m_k},Sx_{m_k})\left(\frac{1+d(x_{n_k},Tx_{n_k})}{1+d(x_{n_k},x_{m_k})}\right),\\ d(x_{n_k},Tx_{n_k})\left(\frac{1+d(x_{m_k},Sx_{m_k})}{1+d(x_{n_k},x_{m_k})}\right)\end{array}\right\},$$

for which $\lim_{k\to\infty}M_1(Tx_{n_k},Sx_{m_k})=\epsilon_0$. Hence, we have $\psi(\epsilon_0)\leq\psi(\epsilon_0)-\phi(\epsilon_0)$, which is a contradiction with $\epsilon_0>0$.

Case 2. Let Let $m_k=2p_k$ and $m_k=2q_k-1$ for some p_k,q_k. Then, from (2.14) and (2.15), we get

$$\frac{1}{2}d(x_{m_k},Tx_{m_k})=\frac{1}{2}d(x_{m_k},x_{m_k+1})<\epsilon_0\leq d(x_{m_k},x_{n_k}).$$

So, (2.16) implies that

$$\begin{array}{rcl}\psi(d(x_{m_k+1},x_{n_k+1})) &=& \psi(d(Tx_{m_k},Sx_{n_k}))\\ &\leq& \psi(M_1(Tx_{m_k},Sx_{n_k}))-\phi(M_1(Tx_{m_k},Sx_{n_k})),\end{array}$$

where

$$M_1(Tx_{m_k},Sx_{n_k})=\max\left\{\begin{array}{l} d(x_{m_k},x_{n_k}),d(x_{m_k},Tx_{m_k}),\\ d(x_{n_k},Sx_{n_k}),\\ \frac{d(x_{n_k},Tx_{m_k})+d(x_{m_k},Sx_{n_k})}{2},\\ \frac{d(x_{m_k},Tx_{m_k})+d(x_{n_k},Sx_{n_k})}{2},\\ d(x_{n_k},Sx_{n_k})\left(\frac{1+d(x_{m_k},Tx_{m_k})}{1+d(x_{m_k},x_{n_k})}\right),\\ d(x_{m_k},Tx_{m_k})\left(\frac{1+d(x_{n_k},Sx_{n_k})}{1+d(x_{m_k},x_{m_k})}\right)\end{array}\right\},$$

for which $\lim_{k\to\infty}M_1(Tx_{m_k},Sx_{n_k})=\epsilon_0$. Hence, we have $\psi(\epsilon_0)\leq\psi(\epsilon_0)-\phi(\epsilon_0)$, which is again a contradiction with $\epsilon_0>0$.

Next, we show that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Using the claim, we find $N_1\in\mathbb{N}$ such that if $m>n>N_1$ with $m-n\equiv 1 \pmod 2$, then $d(x_n,x_m)<\frac{\epsilon}{2}$. Also, using (2.12), we can find $N_2\in\mathbb{N}$ such that $n>N_2$, we get $d(x_n,x_{n+1})<\frac{\epsilon}{2}$. Suppose $m,n>N=\max\{N_1,N_2\}$ with $m>n$, then we get following two cases:

(i) If $m-n\equiv 1 \pmod 2$, then $d(x_m,x_n)<\epsilon$

(ii) If $m-n\equiv 0 \pmod 2$, then $d(x_m,x_n)\leq d(x_m,x_{n+1})+d(x_{n+1},x_n)<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$

Hence, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X, and completeness of X ensures the convergence to a limit, say $z \in X$.

Further, to show z is a fixed point of T, we claim that either $\frac{1}{2}d(x_{2n}, Tx_{2n}) \leq d(x_{2n}, z)$ or $\frac{1}{2}d(x_{2n+1}, Sx_{2n+1}) \leq d(x_{2n+1}, z)$. Otherwise, as $\{d(x_n, x_{n+1})\}_{n=0}^{\infty}$ is non-increasing, we have

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &\leq d(x_{2n}, z) + p(z, x_{2n+1}) \\ &< \frac{1}{2}d(x_{2n}, Tx_{2n}) + \frac{1}{2}d(x_{2n+1}, Sx_{2n+1}) \\ &= \frac{1}{2}[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] \\ &\leq d(x_{2n}, x_{2n+1}), \end{aligned}$$

which is a contradiction. Hence, there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that

$$\frac{1}{2}d(x_{2n_k}, Tx_{2n_k}) \leq d(x_{2n_k}, z) \text{ or } \frac{1}{2}d(x_{2n_k+1}, Sx_{2n_k+1}) \leq d(x_{2n_k+1}, z).$$

We consider the following two case:

Case I. If $\frac{1}{2}d(x_{2n_k}, Tx_{2n_k}) \leq d(x_{2n_k}, z)$, then by (2.16), we get

$$\begin{aligned} \psi(d(Tx_{2n_k}, Sz)) &\leq \psi(M_1(Tx_{2n_k}, Sz)) - \psi(M_1(Tx_{2n_k}, Sz)) \\ &= \psi\left(\max\left\{\begin{array}{l} d(x_{2n_k}, z), d(x_{2n_k}, Tx_{2n_k}), \\ d(z, Sz), \frac{d(z, Tx_{2n_k}) + d(x_{2n_k}, Sz)}{2}, \\ \frac{d(x_{2n_k}, Tx_{2n_k}) + d(z, Sz)}{2}, \\ d(z, Sz)\left(\frac{1 + d(x_{2n_k}, Tx_{2n_k})}{1 + d(x_{2n_k}, z)}\right), \\ d(x_{2n_k}, Tx_{2n_k})\left(\frac{1 + d(z, Sz)}{1 + d(x_{2n_k}, z)}\right) \end{array}\right\}\right) \\ &\quad -\phi\left(\max\left\{\begin{array}{l} d(x_{2n_k}, z), d(x_{2n_k}, Tx_{2n_k}), \\ d(z, Sz), \frac{d(z, Tx_{2n_k}) + d(x_{2n_k}, Sz)}{2}, \\ \frac{d(x_{2n_k}, Tx_{2n_k}) + d(z, Sz)}{2}, \\ d(z, Sz)\left(\frac{1 + d(x_{2n_k}, Tx_{2n_k})}{1 + d(x_{2n_k}, z)}\right), \\ d(x_{2n_k}, Tx_{2n_k})\left(\frac{1 + d(z, Sz)}{1 + d(x_{2n_k}, z)}\right) \end{array}\right\}\right). \end{aligned}$$

Making $n \to \infty$, we get $\psi(d(z, Sz)) \leq \psi(d(z, Sz)) - \phi(d(z, Sz))$, which yields $z = Sz$. Thus, we obtain $\frac{1}{2}d(z, Sz) = 0 \leq d(z, z)$. Using (2.16), we get

$$\begin{aligned} \psi(d(Tz, Sz)) &\leq \psi(M_1(Tz, Sz)) - \psi(M_1(Tz, Sz)) \\ &= \psi(d(Tz, z)) - \psi(d(Tz, z)), \end{aligned}$$

which implies $Tz = z$. Thus z is a common fixed point of T and S.

Case II. If $\frac{1}{2}d(x_{2n_k+1}, Sx_{2n_k+1}) \leq d(x_{2n_k+1}, z)$, then by (2.16), we get

$$\begin{aligned}
\psi(d(Tz, Sx_{2n_k+1})) &\leq \psi(M_1(Tz, Sx_{2n_k+1})) - \psi(M_1(Tz, Sx_{2n_k+1})) \\
&= \psi\left(\max\left\{\begin{array}{l} d(z, x_{2n_k+1}), d(x_{2n_k+1}, Sx_{2n_k+1}), \\ d(z, Tz), \frac{d(x_{2n_k+1}, Tz)+d(z, Sx_{2n_k+1})}{2}, \\ \frac{d(x_{2n_k+1}, Sx_{2n_k+1})+d(z, Tz)}{2}, \\ d(z, Tz)\left(\frac{1+d(x_{2n_k+1}, Sx_{2n_k+1})}{1+d(z, x_{2n_k+1})}\right), \\ d(x_{2n_k+1}, Sx_{2n_k+1})\left(\frac{1+d(z, Tz)}{1+d(z, x_{2n_k+1})}\right) \end{array}\right\}\right) \\
&\quad -\phi\left(\max\left\{\begin{array}{l} d(z, x_{2n_k+1}), d(x_{2n_k+1}, Sx_{2n_k+1}), \\ d(z, Tz), \frac{d(x_{2n_k+1}, Tz)+d(z, Sx_{2n_k+1})}{2}, \\ \frac{d(x_{2n_k+1}, Sx_{2n_k+1})+d(z, Tz)}{2}, \\ d(z, Tz)\left(\frac{1+d(x_{2n_k+1}, Sx_{2n_k+1})}{1+d(z, x_{2n_k+1})}\right), \\ d(x_{2n_k+1}, Sx_{2n_k+1})\left(\frac{1+d(z, Tz)}{1+d(z, x_{2n_k+1})}\right) \end{array}\right\}\right).
\end{aligned}$$

Making $n \to \infty$, we get $\psi(d(z, Tz)) \leq \psi(d(z, Tz)) - \phi(d(z, Tz))$, which yields $z = Tz$. Thus, as of case I, we obtain that z is a common fixed point of T and S.

In order to prove uniqueness, suppose that y is another common fixed point of T and S. Then, $\frac{1}{2}d(z, Tz) = 0 \leq d(y, z)$ implies

$$\begin{aligned}
\psi(d(y, z)) = \psi(d(Ty, Sz)) &\leq \psi(M_1(Ty, Sz)) - \phi(M_1(Ty, Sz)) \\
&= \psi(d(y, z)) - \phi(d(y, z)),
\end{aligned}$$

which leads to $y = z$. This completes the proof. □

Remark 2.9. Theorem 2.8 is a proper generalization of results due to Fei He et al. [11] (Theorem 1.6) and the authors [2] (Theorem 1.7).

However, for $S = T$ in Theorem 2.8, we obtain the following corollary.

Corollary 2.10. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping. Assume that for every $x, y \in X$,*

$$\begin{array}{l} \frac{1}{2}d(x, Tx) \leq d(x, y) \text{ implies} \\ \psi(d(Tx, Ty)) \leq \psi(M_1(Tx, Ty)) - \phi(M_1(Tx, Ty)), \end{array}$$

where ψ and ϕ are defined as in Theorem 1.2. Then there exists a unique point $z \in X$ such that $z = Tz$.

Furthermore, for $\psi = I$ (identity) in Theorem 2.8, we get the following result.

Corollary 2.11. *Let (X, d) be a complete metric space and let $T, S : X \to X$ be two mappings. Assume that for every $x, y \in X$,*

$$\begin{array}{l} \frac{1}{2}\min\{d(x, Tx), d(y, Sy)\} \leq d(x, y) \text{ implies} \\ d(Tx, Sy)) \leq M_1(Tx, Sy) - \phi(M_1(Tx, Sy)), \end{array} \tag{2.16}$$

where ϕ is defined as in Theorem 1.2. Then there exists a unique point $z \in X$ such that $z = Tz = Sz$.

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