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Basic Contractions of Suzuki-Type on Quasi-Metric Spaces and Fixed Point Results

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Abstract: This paper deals with the question of achieving a suitable extension of the notion of Suzuki-type contraction to the framework of quasi-metric spaces, which allows us to obtain reasonable fixed point theorems in the quasi-metric context. This question has no an easy answer; in fact, we here present an example of a self map of Smyth complete quasi-metric space (a very strong kind of quasi-metric completeness) that fulfills a simple and natural contraction of Suzuki-type but does not have fixed points. Despite it, we implement an approach to obtain two fixed point results, whose validity is supported with several examples. Finally, we present a general method to construct non-T_1 quasi-metric spaces in such a way that it is possible to systematically generate non-Banach contractions which are of Suzuki-type. Thus, we can apply our results to deduce the existence and uniqueness of solution for some kinds of functional equations which is exemplified with a distinguished case.

Keywords: contraction of Suzuki-type; fixed point; quasi-metric; complete

MSC: 54H25; 54E50; 47H10

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1. Introduction

In the late 1980s and during the 1990s, several researchers conducted a line of work consisting of establishing connections between quasi-metric spaces and domain theory with application to the mathematical foundations of computer science, where the construction of iterations and the obtaining of fixed points constituted essential instruments [1–6] (at this point, it is interesting to emphasize that most of the quasi-metric spaces used to mathematically model the corresponding computational processes are non-T_1). This fruitful approach has continued to progress during this century (cf. [7–14]).

Partly stimulated by these developments, the research about the fixed point theory on quasi-metric spaces has received a powerful boost in the last 12 years, during which many papers have been published in this area, so we will limit ourselves here to citing some of the most recent ones [15–22] with the references therein.

On the other hand, Suzuki published in 2008 his renowned article [23] in which he presented a necessary and sufficient condition for the metric completeness by utilizing an appealing generalization of the Banach contraction principle. This new and compelling approach was successfully continued by him in [24], and by other authors who generalized and extended the type of contractions proposed by Suzuki to obtain new fixed point theorems both in metric spaces and in b-metric spaces, partial metric spaces, G-metric spaces, quasi-metric spaces, fuzzy metric spaces, and others (see [19,25–35]) and the references therein).

Encouraged by the interesting facts set forth in the two previous paragraphs, we here focus our attention in exploring basic contractions of Suzuki-type in the realm of quasi-metric spaces. Our starting point is the following visual and direct consequence of [23] (Theorem 2).

Theorem 1 (Suzuki)**.** *Let $\mathscr{F}$ be a self map of a complete metric space $(\mathscr{X}, \varrho)$, and let $c \in (0,1)$ be a constant, such that for every $x, y \in \mathscr{X}$, the following contraction condition holds:*

$$\varrho(x, \mathscr{F}x) \leq 2\varrho(x,y) \implies \varrho(\mathscr{F}x, \mathscr{F}y) \leq c\varrho(x,y).$$

Then, $\mathscr{F}$ has a unique fixed point $\xi \in \mathcal{X}$. Furthermore, $\varrho(\xi, \mathscr{F}^n x_0) \to 0$ as $n \to \infty$ for all $x_0 \in \mathcal{X}$.

The above theorem suggests the following natural question (see Section 2 for notation and concepts).

Question. Let $\mathscr{F}$ be a self map of a bicomplete (or at least, Smyth complete) quasi-metric space $(\mathscr{X}, \varrho)$ and let $c \in (0,1)$ be a constant, such that for every $x, y \in \mathscr{X}$, the following contraction condition holds:

$$\varrho(x, \mathscr{F}x) \leq 2\varrho(x,y) \implies \varrho(\mathscr{F}x, \mathscr{F}y) \leq c\varrho(x,y). \quad (1)$$

Under the above assumptions, does $\mathscr{F}$ admit a fixed point?

In Section 3, we will give an example showing that this question has a negative answer in the general quasi-metric context. Nevertheless, and based on an interesting contraction condition introduced by Fulga, Karapinar, and Petrusel in [19], we are able to obtain a couple of fixed point theorems whose validities are supported with some enlightening examples. Finally, we present a methodology to construct non-T_1 quasi-metric spaces in such a way that it is possible to systematically generate non-Banach contractions that are of Suzuki-type. Thus, we can apply our fixed point results to deduce the existence and uniqueness of solution for some kinds of functional equations that are exemplified with a case from which we derive the existence and uniqueness of a solution for an outstanding kind of difference equations.

2. Preliminaries

In the sequel, we will use the following notation: $\mathbb{N}$ and $\mathbb{R}$ denote the sets of positive integers, respectively, the set of real numbers, while $\mathbb{N}_0$ and $\mathbb{R}^+$ denote the sets of non-negative integers, respectively, the set of non-negative real numbers. For the notions and properties of general topology employed here, we refer to the reader to [36].

The concept of quasi-metric space has its origin in the articles of Niemytzki [37] and Wilson [38], in which these authors worked with asymmetric distances for purely topological reasons. For instance, Wilson proved that every T_1 topological space with a countable base is quasi-metrizable. Later on, many authors contributed to the progress of the theory of quasi-metric spaces in the field of general topology. An excellent compilation of articles on quasi-metric spaces published up to 1982 can be found in the monograph by Fletcher and Lindgren [39] where the authors provided a detailed and systematized study of these structures and other related ones (for subsequent updates, see the survey article from Künzi [40], and the book from Cobzaş [41], and the references therein).

Let $\mathscr{X}$ be a set. A function $\varrho : \mathscr{X} \times \mathscr{X} \to \mathbb{R}^+$ is a quasi-metric on $\mathscr{X}$, provided that it verifies the following two conditions for every $x, y, z \in \mathscr{X}$:

(i_1) $\varrho(x,y) = \varrho(y,x) = 0$ if and only if $x = y$;

(i_2) $\varrho(x,z) \leq \varrho(x,y) + \varrho(y,z)$.

In that case, we say that $(\mathscr{X}, \varrho)$ is a quasi-metric space.

If ϱ fulfills condition (i_2) and the next strengthening of condition (i_1): $\varrho(x,y) = 0$, if and only if $x = y$, we will refer to ϱ as a T_1 quasi-metric on $\mathscr{X}$.

In that case, we say that $(\mathscr{X}, \varrho)$ is a T_1 quasi-metric space.

Let ϱ be a quasi-metric on a set $\mathscr{X}$. Then, we have the following notions and fundamental properties, denoted by (N) and (P), respectively, which will be utilized in the rest of the paper:

(N1) The function $\varrho^{\mathbf{r}} : \mathscr{X} \times \mathscr{X} \to \mathbb{R}^+$, defined as $\varrho^{\mathbf{r}} x, y) = \varrho(y,x)$, is also a quasi-metric on $\mathscr{X}$ called the reverse (or the conjugate) quasi-metric of ϱ, and the function

$\varrho^{\max} : \mathscr{X} \times \mathscr{X} \to \mathbb{R}^+$ defined as $\varrho^{\max}(x,y) = \max\{\varrho(x,y), \varrho^{\mathbf{r}}(x,y)\}$, is a metric on $\mathscr{X}$. Notice that if ϱ is T_1, then $\varrho^{\mathbf{r}}$ is also a T_1 quasi-metric.

(P1) As in the metric case, for each $x \in \mathscr{X}$ and $\rho > 0$, we refer to the set $\mathcal{B}(x,\rho) := \{y \in \mathscr{X} : \varrho(x,y) < \rho\}$ as the ϱ-ball of center x and radius ρ. It is well known that the family $\{\mathcal{B}(x,\rho) : x \in \mathscr{X}, \rho > 0\}$ is a base of open sets for a T_0 topology $\mathfrak{T}_\varrho$ on $\mathscr{X}$, called the topology induced by ϱ. If ϱ is a T_1 quasi-metric, the topology $\mathfrak{T}_\varrho$ is a T_1 topology on $\mathscr{X}$. If $\mathfrak{T}_\varrho$ is a Hausdorff (or T_2) topology on $\mathcal{X}$, we say that $(\mathcal{X}, \varrho)$ is a Hausdorff quasi-metric space.

(P2) A sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathscr{X}$ is $\mathfrak{T}_\varrho$-convergent to a point $\xi \in \mathscr{X}$ if and only if $\varrho(\xi, x_n) \to 0$ as $n \to \infty$.

(N2) A sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathscr{X}$ is called left Cauchy in $(\mathscr{X}, \varrho)$ if for each $\rho > 0$ there is an $n_\rho \in \mathbb{N}$, such that $\varrho(x_n, x_m) < \rho$ whenever $n_\rho \leq n \leq m$; it is called right Cauchy in $(\mathscr{X}, \varrho)$ if it is left Cauchy in $(\mathscr{X}, \varrho^{\mathbf{r}})$, and it is called Cauchy in $(\mathscr{X}, \varrho)$ if it is left and right Cauchy in $(\mathscr{X}, \varrho)$.

(P3) A sequence in $\mathscr{X}$ is Cauchy in $(\mathscr{X}, \varrho)$ if and only if it is a Cauchy sequence in the metric space $(\mathscr{X}, \varrho^{\max})$.

(N4) $(\mathscr{X}, \varrho)$ is said to be bicomplete if the metric space $(\mathscr{X}, \varrho^{\max})$ is complete, and it is said to be Smyth complete if every left Cauchy sequence in $(\mathscr{X}, \varrho)$ is $\mathfrak{T}_{\varrho^{\max}}$-convergent.

(P4) Smyth completeness implies bicompleteness, but the converse does not hold in general (see Example 1 below).

To the end, in this section, we remind of two basic examples of non-T_1 quasi-metrics that correspond to the asymmetric counterparts of the usual metric on $\mathbb{R}$, and we also recall a well-known full quasi-metric version of the Banach contraction principle.

Example 1. *Denote by $\mathbf{u}$ the non-T_1 quasi-metric on $\mathbb{R}$ given by $\mathbf{u}(x,y) = \max\{y - x, 0\}$ for all $x, y \in \mathbb{R}$. Then, $\mathbf{u}^{\mathbf{r}}(x,y) = \max\{x - y, 0\}$ for all $x, y \in \mathbb{R}$. Since $\mathbf{u}^{\max}(x,y) = |x - y|$ for all $x, y \in \mathbb{R}$, we infer that $(\mathbb{R}, \mathbf{u})$ is bicomplete. However, it is not Smyth complete because $(-n)_{n\in\mathbb{N}}$ is a left Cauchy sequence in $(\mathbb{R}, \mathbf{u})$ that is not $\mathfrak{T}_{\mathbf{u}^{\max}}$-convergent.*

Example 2. *It is well known that the non T_1 quasi-metric space $(\mathbb{R}^+, \mathbf{u})$ is Smtyh complete, where we have also denoted by $\mathbf{u}$ the restriction of the quasi-metric $\mathbf{u}$ to $\mathbb{R}^+$. However $(\mathbb{R}^+, \mathbf{u}^{\mathbf{r}})$ is not Smyth complete because $(n)_{n\in\mathbb{N}}$ is a left Cauchy sequence in $(\mathbb{R}^+, \mathbf{u}^{\mathbf{r}})$ which is not $\mathfrak{T}_{\mathbf{u}^{\max}}$-convergent.*

Theorem 2. *Let $\mathscr{F}$ be a self map of a bicomplete quasi-metric space $(\mathscr{X}, \varrho)$, and let $c \in (0,1)$ be a constant such that, for every $x, y \in \mathscr{X}$, we have $\varrho(\mathscr{F}x, \mathscr{F}y) \leq c\varrho(x,y)$. Then, $\mathscr{F}$ has a unique fixed point $\xi \in \mathcal{X}$. Furthermore, $\varrho^{\max}(\xi, \mathscr{F}^n x_0) \to 0$ as $n \to \infty$ for all $x_0 \in \mathcal{X}$.*

Let $(\mathscr{X}, \varrho)$ be a quasi-metric space. As in the metric case, a self map $\mathscr{F}$ of $\mathscr{X}$ that satisfies the contraction condition of Theorem 2 will be called a Banach contraction (on $(\mathscr{X}, \varrho)$).

Remark 1. *Note that Theorem 2 can be obtained as a consequence of the classical Banach contraction principle because, clearly, every Banach contraction on $(\mathcal{X}, \varrho)$ is a Banach contraction on the metric space $(\mathcal{X}, \varrho^{\max})$, and the bicompleteness of $(\mathcal{X}, \varrho)$ coincides, by definition, with the completeness of $(\mathcal{X}, \varrho^{\max})$ (see (N4)).*

3. Contractions of Suzuki-Type and Fixed Point Results

We begin this section by presenting an example of a self map $\mathscr{F}$ of a Smyth complete quasi-metric space that satisfies the contraction condition (1), but it is free of fixed points. Actually, our self map verifies the following contraction condition apparently stronger than (1): There is a constant $c \in (0,1)$ such that, for every $x, y \in \mathscr{X}$,

$$\min\{\varrho(x, \mathscr{F}x), \varrho(y, \mathscr{F}y)\} \leq 2\varrho(x,y) \implies \varrho(\mathscr{F}x, \mathscr{F}y) \leq c\varrho(x,y). \quad (2)$$

Example 3. *Let $\mathscr{X} = \mathbb{N} \cup \{\infty\}$ and let ϱ be the quasi-metric on $\mathscr{X}$ given by*

$$\varrho(x,x) = 0 \quad \text{for all } x \in \mathscr{X},$$

$\varrho(n,\infty)=0$ *for all* $n\in\mathbb{N}$,
$\varrho(\infty,n)=1/n$ *for all* $n\in\mathbb{N}$,

and

$\varrho(n,m)=1/m$ *for all* $n,m\in\mathbb{N}$ *with* $n\neq m$.

Since every non-eventually left Cauchy sequence $\mathfrak{T}_{\varrho^{\max}}$*-converges to* ∞*, we deduce that* $(\mathscr{X},\varrho)$ *is Smyth complete.*

Now let $\mathscr{F}$ *be the self map of* $\mathscr{X}$ *defined as* $\mathscr{F}\infty=1$*, and* $\mathscr{F}n=2n$ *for all* $n\in\mathbb{N}$*. Obviously* $\mathscr{F}$ *has no fixed points. We prove that, nevertheless, it fulfills the condition (2) for* $c=1/2$*.*

- *If* $x=n$ *and* $y=m$*, with* $n\neq m$*, we obtain*
$$\varrho(\mathscr{F}x,\mathscr{F}y)=\varrho(2n,2m)=\frac{1}{2m}=\frac{1}{2}\varrho(n,m)=\frac{1}{2}\varrho(x,y).$$
- *If* $x=\infty$ *and* $y=n$*,* $n\in\mathbb{N}$*, we obtain*
$$\varrho(\mathscr{F}x,\mathscr{F}y)=\varrho(1,2n)=\frac{1}{2n}=\frac{1}{2}\varrho(\infty,n)=\frac{1}{2}\varrho(x,y).$$
- *If* $x=n$*,* $n\in\mathbb{N}$*, and* $y=\infty$*, we obtain*
$$\min\{\varrho(x,\mathscr{F}x),\varrho(y,\mathscr{F}y)\}=\min\{\frac{1}{2n},1\}>0=2\varrho(x,y).$$

We have shown that $\mathscr{F}$ *fulfills the contraction condition (2), and hence the contraction condition (1).*

Remark 2. *It is interesting to emphasize that both the quasi-metric* ϱ *of the preceding example and other of their variants (see [3]) can be used in modeling increasing sequences* $(x_n)_{n\in\mathbb{N}}$ *of information where, roughly speaking, the element* x_{n+1} *contains more information than the element* x_n *and the supremum element* ∞ *(also denoted by* $\top$*) is an "ideal" element that captures the information of all of the elements of the sequence. Thus, in Example 3, one has that* $((\varrho(n,n+1))_{n\in\mathbb{N}}$ *is a strictly decreasing sequence, which can be interpreted to mean that the element represented by* $n+1$ *contains more information than the one represented by* n*. Furthermore,* $\varrho(n,n+1)\to 0$ *and* $\varrho^{\max}(n,\infty)\to 0$ *as* $n\to\infty$*, as we could expect in a reasonable model.*

Definition 1. *Let* $(\mathscr{X},\varrho)$ *be a quasi-metric space. A self map* $\mathscr{F}$ *of* $\mathscr{X}$ *that satisfies the contraction condition (1), will be called a basic contraction of Suzuki-type (on* $(\mathscr{X},\varrho)$*), while a self map* $\mathscr{F}$ *of* $\mathscr{X}$ *that satisfies the contraction condition (2), will be called a 2-basic contraction of Suzuki-type (on* $(\mathscr{X},\varrho)$*).*

Evidently, every 2-basic contraction of Suzuki-type on a quasi-metric space $(\mathscr{X},\varrho)$ is a basic contraction of Suzuki-type. Moreover, it is clear that if $(\mathcal{X},\varrho)$ is a metric space, both concepts coincide via the symmetry of ϱ. The following is an example of a basic contraction of Suzuki-type on a Hausdorff quasi-metric space that is not a 2-basic contraction.

Example 4. *Let* $\mathscr{X}=\mathbb{N}\cup\{\infty\}$ *and* $\varrho:\mathscr{X}\times\mathscr{X}\to\mathbb{R}^+$ *defined as:*

$\varrho(x,x)=0$ *for all* $x\in\mathscr{X}$,
$\varrho(n,m)=(2^{|n-m|}-1)/2^{\max\{n,m\}-1}$ *if* $n,m\in\mathbb{N}$ *with* $n\neq m$,
$\varrho(n,\infty)=(2^n-1)/2^{n-1}$ *for all* $n\in\mathbb{N}$,
$\varrho(\infty,1)=2/3$,

and

$\varrho(1,n)=1/6$ *for all* $n\in\mathbb{N}\backslash\{1\}$.

We want to show that ϱ *is a* T_1 *quasi-metric on* $\mathscr{X}$*. Since* $\varrho(x,y)>0$ *whenever* $x\neq y$*, we focus our attention in checking that the triangle inequality* (i_2) *is fulfilled for all* $x,y,z\in\mathscr{X}$*.*

If $x=\infty$*,* $y=1$ *and* $z=m>1$*, we obtain*

$$\varrho(x,y)=\frac{2}{3}=\frac{1}{6}+\frac{1}{2}\leq\frac{1}{6}+\frac{2^{m-1}-1}{2^{m-1}}=\varrho(x,z)+\varrho(z,y).$$

If $x = \infty$, $y = n > 1$ and $z = m$, $z \neq y$, we obtain

$$\varrho(x, y) = \frac{1}{6} \leq \varrho(x, z).$$

The rest of cases are obtained as a direct consequence of the following easy inequality:

$$\frac{2^{|n-m|} - 1}{2^{\max\{n,m\}-1}} \leq \frac{2^n - 1}{2^{n-1}},$$

for all $n, m \in \mathbb{N}$.

Therefore, $(\mathscr{X}, \varrho)$ is a T_1 quasi-metric space. Actually, it is Hausdorff because every point of $\mathscr{X}$ is isolated, i.e., $\{x\}$ is $\mathfrak{T}_\varrho$-open for all $x \in \mathscr{X}$, and thus, $\mathfrak{T}_\varrho$ agrees with the discrete topology on $\mathscr{X}$, so $(\mathscr{X}, \mathfrak{T}_\varrho)$ is a metrizable topological space.

Now define a self map $\mathscr{F}$ of $\mathscr{X}$ as $\mathscr{F}n = n + 1$ for all $n \in \mathbb{N}$ and $\mathscr{F}\infty = 1$.

We are going to show that $\mathscr{F}$ is a basic contraction of Suzuki-type on $(\mathscr{X}, \varrho)$.

Indeed, for every $n, m \in \mathbb{N}$, we obtain

$$\varrho(\mathscr{F}n, \mathscr{F}m) = \frac{2^{|n-m|} - 1}{2^{\max\{n+1,m+1\}-1}} = \frac{1}{2}\frac{2^{|n-m|} - 1}{2^{\max\{n,m\}-1}} = \frac{1}{2}\varrho(n, m).$$

Furthermore, for every $n \in \mathbb{N}$, we obtain

$$\varrho(\mathscr{F}n, \mathscr{F}\infty) = \varrho(n + 1, 1) = \frac{2^n - 1}{2^n} = \frac{1}{2}\varrho(n, \infty).$$

We also have

$$\varrho(\mathscr{F}\infty, \mathscr{F}1) = \varrho(1, 2) = \frac{1}{2} = \frac{3}{4}\varrho(\infty, 1),$$

and, for $n > 1$,

$$\varrho(\infty, \mathscr{F}\infty) = \varrho(\infty, 1) = \frac{2}{3} > \frac{1}{3} = 2\varrho(\infty, n).$$

We conclude that $\mathscr{F}$ is a basic contraction of Suzuki-type with constant $c = 3/4$.

Finally, note that for $n \geq 2$, we obtain

$$\min\{\varrho(\infty, \mathscr{F}\infty), \varrho(n, \mathscr{F}n)\} = \varrho(n, \mathscr{F}n) = \frac{1}{2^n} < \frac{1}{3} = 2\varrho(\infty, n),$$

and, nevertheless,

$$\varrho(\mathscr{F}\infty, \mathscr{F}n) = \frac{2^n - 1}{2^n} > \frac{1}{6} = \varrho(\infty, n),$$

which implies that $\mathscr{F}$ is not a 2-basic contraction of Suzuki-type on $(\mathscr{X}, \varrho)$.

Remark 3. *In Remark 1, we have underlined that Theorem 2 can be obtained as a consequence of the classical Banach contraction principle. In order to guarantee that this situation does not occur in our context is crucial to obtain an example of a basic contraction of Suzuki-type on a bicomplete quasi-metric space $(\mathscr{X}, \varrho)$, which is not a basic contraction of Suzuki-type on the complete metric space $(\mathscr{X}, \varrho^{\max})$. Fortunately, the 2-basic contraction of Example 3 fulfills this requirement via Theorem 1 (see also Example 7 below).*

In what follows, we will present positive results that are partially inspired in the recent article [19] by Fulga, Karapinar, and Petrusel, where the authors obtained, among other results, two terrific and very general fixed point theorems for bicomplete T_1 quasi-metric spaces, by combining conditions of Suzuki-type, contraction conditions of $\alpha - \psi$-type in the style of Samet, Vetro, and Vetro [42] and interpolation conditions. For our goals here, it will be enough to consider the following consequence of [19] (Theorem 2).

Theorem 3. *Let $\mathscr{F}$ be a self map of a bicomplete T_1 quasi-metric space $(\mathscr{X}, \varrho)$, and let $c \in (0,1)$ be a constant such that, for every $x, y \in \mathscr{X}$, the following contraction condition holds:*

$$\min\{\varrho(x, \mathscr{F}x), \varrho(y, \mathscr{F}y), \varrho(\mathscr{F}y, y)\} \leq 2\varrho(x, y) \implies \varrho(\mathscr{F}x, \mathscr{F}y) \leq c\varrho(x, y). \tag{3}$$

Then, $\mathscr{F}$ has a unique fixed point.

Definition 2. *Let $(\mathscr{X}, \varrho)$ be a quasi-metric space. A self map $\mathscr{F}$ of $\mathscr{X}$ that satisfies the contraction condition (3), will be called an FKP-contraction (on $(\mathscr{X}, \varrho)$).*

Evidently, every FKP-contraction on a quasi-metric space $(\mathscr{X}, \varrho)$ is a 2-basic contraction of Suzuki-type. It is clear that both concepts coincide when $(\mathcal{X}, \varrho)$ is a metric space. However, it follows from Theorem 5 below that the 2-basic contraction of Suzuki-type of Example 3 is not an FKP-contraction.

The proofs of the following auxiliary lemmas use standard methods. Notwithstanding, in order to help the readers and for the sake of completeness, we give outlined demos of them.

Lemma 1. *Let $\mathscr{F}$ be a self map of a quasi-metric space $(\mathcal{X}, \varrho)$. Then, for every $x, y \in \mathscr{X}$,*

$$\varrho(x, \mathscr{F}x) \leq 2\max\{\varrho(x, y), \varrho(y, \mathscr{F}x)\}.$$

Proof. Assume the contrary. Then, there exist $x, y \in \mathscr{X}$, such that

$$\varrho(x, \mathscr{F}x) > 2\varrho(x, y) \quad \text{and} \quad \varrho(x, \mathscr{F}x) > 2\varrho(y, \mathscr{F}x).$$

Therefore,

$$\varrho(x, \mathscr{F}x) \leq \varrho(x, y) + \varrho(y, \mathscr{F}x) < \frac{1}{2}\varrho(x, \mathscr{F}x) + \frac{1}{2}\varrho(x, \mathscr{F}x),$$

a contradiction. □

Lemma 2. *Let $\mathscr{F}$ be a basic contraction of Suzuki-type on a quasi-metric space $(\mathscr{X}, \varrho)$. Then, for each $x_0 \in \mathscr{X}$, the sequence $(\mathscr{F}^n x_0)_{n\in\mathbb{N}_0}$ is left Cauchy in $(\mathscr{X}, \varrho)$, and $(\varrho(\mathscr{F}^n x_0, \mathscr{F}^{n+1} x_0))_{n\in\mathbb{N}_0}$ is a non-increasing sequence in $\mathbb{R}^+$.*

Proof. Let $c \in (0,1)$, for which the contraction condition (1) holds. Fix $x_0 \in \mathscr{X}$. Since $\varrho(x_0, \mathscr{F}x_0) \leq 2\varrho(x_0, \mathscr{F}x_0)$, we deduce from (1) that $\varrho(\mathscr{F}x_0, \mathscr{F}^2 x_0) \leq c\varrho(x_0, \mathscr{F}x_0)$. Following this process, we obtain

$$\varrho(\mathscr{F}^n x_0, \mathscr{F}^{n+1} x_0) \leq c\varrho(\mathscr{F}^{n-1} x_0, \mathscr{F}^n x_0),$$

for all $n \in \mathbb{N}$, so $\varrho(\mathscr{F}^n x_0, \mathscr{F}^{n+1} x_0) \leq \varrho(\mathscr{F}^{n-1} x_0, \mathscr{F}^n x_0)$, and $\varrho(\mathscr{F}^n x_0, \mathscr{F}^{n+1} x_0) \leq c^n \varrho(x_0, \mathscr{F}x_0)$, for all $n \in \mathbb{N}$. By standard arguments, we deduce that $(\mathscr{F}^n x_0)_{n\in\mathbb{N}_0}$ is a left Cauchy sequence in $(\mathscr{X}, \varrho)$ and $(\varrho(\mathscr{F}^n x_0, \mathscr{F}^{n+1} x_0))_{n\in\mathbb{N}_0}$ is a non-increasing sequence in $\mathbb{R}^+$. □

Lemma 3. *Let $\mathscr{F}$ be a self map of a quasi-metric space $(\mathscr{X}, \varrho)$, and let $c \in (0,1)$ be a constant, such that, for every $x, y \in \mathscr{X}$, the following contraction condition holds:*

$$\min\{\varrho(x, \mathscr{F}x), \varrho(\mathscr{F}y, y)\} \leq 2\varrho(x, y) \implies \varrho(\mathscr{F}x, \mathscr{F}y) \leq c\varrho(x, y). \tag{4}$$

Then, for each $x_0 \in \mathcal{X}$, the sequence $(\mathscr{F}^n x_0)_{n\in\mathbb{N}_0}$ is Cauchy in $(\mathcal{X}, \varrho)$.

Proof. Fix $x_0 \in \mathscr{X}$. It is clear that $\mathscr{F}$ is a basic contraction of Suzuki-type, so, by Lemma 2, the sequence $(\mathscr{F}^n x_0)_{n\in\mathbb{N}_0}$ is left Cauchy in $(\mathscr{X}, \varrho)$.

We are going to show that it is also a right Cauchy sequence in $(\mathscr{X}, \varrho)$. We have

$$\min\{\varrho(\mathscr{F}x_0, \mathscr{F}^2 x_0), \varrho(\mathscr{F}x_0, x_0)\} \leq \varrho(\mathscr{F}x_0, x_0).$$

Hence, $\varrho(\mathscr{F}^2x_0, \mathscr{F}x_0) \leq c\varrho(\mathscr{F}x_0, x_0)$, by the contraction condition (4). By repeating the process, we infer that

$$\varrho(\mathscr{F}^{n+1}x_0, \mathscr{F}^n x_0) \leq c^n \varrho(\mathscr{F}x_0, x_0)$$

for all $n \in \mathbb{N}$. Consequently, $(\mathscr{F}^n x_0)_{n\in\mathbb{N}_0}$ is a right Cauchy sequence in $(\mathscr{X}, \varrho)$. Hence, it is a Cauchy sequence in $(\mathscr{X}, \varrho)$ (see (N2)). □

Theorem 4. *Let $\mathscr{F}$ be a 2-basic contraction of Suzuki-type on a Smyth complete quasi-metric space $(\mathscr{X}, \varrho)$. Then, for each $x_0 \in \mathscr{X}$ there exists $\xi \in \mathscr{X}$, such that $\varrho(\mathscr{F}\xi, \xi) = 0$ and $\varrho^{\max}(\xi, \mathscr{F}^n x_0) \to 0$ as $n \to \infty$.*

Proof. Fix $x_0 \in \mathscr{X}$. By Lemma 2, $(x_n)_{n\in\mathbb{N}_0}$ is a left Cauchy sequence in $(\mathscr{X}, \varrho)$, where $x_n := \mathscr{F}^n x_0$ for all $n \in \mathbb{N}_0$. Since $(\mathscr{X}, \varrho)$ is Smyth complete, there exists $\xi \in \mathscr{X}$, such that $\varrho^{\max}(\xi, x_n) \to 0$ as $n \to \infty$.

By Lemma 1, for each $n \in \mathbb{N}$, we have

$$\varrho(x_n, x_{n+1}) \leq 2\max\{\varrho(x_n, \xi), \varrho(\xi, x_{n+1})\},$$

so

$$\min\{\varrho(x_n, x_{n+1}), \varrho(\xi, \mathscr{F}\xi)\} \leq 2\varrho(\xi, x_{n+1}),$$

or

$$\min\{\varrho(x_n, x_{n+1}), \varrho(\xi, \mathscr{F}\xi)\} \leq 2\varrho(x_n, \xi).$$

Consequently, we can find a subsequence $(x_{n(k)})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, such that

$$\min\{\varrho(x_{n(k)}, x_{n(k)+1}), \varrho(\xi, \mathscr{F}\xi)\} \leq 2\varrho(\xi, x_{n(k)+1}), \tag{5}$$

or

$$\min\{\varrho(x_{n(k)}, x_{n(k)+1}), \varrho(\xi, \mathscr{F}\xi)\} \leq 2\varrho(x_{n(k)}, \xi), \tag{6}$$

for all $k \in \mathbb{N}$.

If (5) is met, since, by Lemma 2, $\varrho(x_{n+1}, x_{n+2}) \leq \varrho(x_n, x_{n+1})$, we obtain

$$\min\{\varrho(x_{n(k)+1}, x_{n(k)+2}), \varrho(\xi, \mathscr{F}\xi)\} \leq 2\varrho(\xi, x_{n(k)+1}),$$

for all $k \in \mathbb{N}$. Hence, $\varrho(\mathscr{F}\xi, \mathscr{F}x_{n(k)+1}) \leq c\varrho(\xi, x_{n(k)+1})$ for all $k \in \mathbb{N}$, where c is the contraction constant. Therefore,

$$\varrho(\mathscr{F}\xi, \xi) \leq \varrho(\mathscr{F}\xi, x_{n(k)+2}) + \varrho(x_{n(k)+2}, \xi) \leq c\varrho(\xi, x_{n(k)+1}) + \varrho(x_{n(k)+2}, \xi),$$

for all $k \in \mathbb{N}$.

Since $\varrho^{\max}(\xi, x_n) \to 0$ as $n \to \infty$, we deduce that $\varrho(\mathscr{F}\xi, \xi) = 0$, with $\varrho^{\max}(\xi, \mathscr{F}^n x_0) \to 0$ as $n \to \infty$.

If (6) is met, from the contraction condition, we infer that $\varrho(x_{n(k)+1}, \mathscr{F}\xi) \leq c\varrho(x_{n(k)}, \xi)$ for all $k \in \mathbb{N}$.

Since $\varrho(x_{n(k)}, \xi) \to 0$ as $k \to \infty$, we obtain $\varrho(x_{n(k)+1}, \mathscr{F}\xi) \to 0$ as $k \to \infty$. We also have that $\varrho(\xi, x_{n(k)}) \to 0$ and $\varrho(x_{n(k)}, x_{n(k)+1}) \to 0$ as $k \to \infty$, so by applying the triangle inequality, we obtain $\varrho(\xi, \mathscr{F}\xi) = 0$. Hence, $\min\{\varrho(\xi, \mathscr{F}\xi), \varrho(x_{n(k)}, x_{n(k)+1})\} \leq 2\varrho(\xi, x_{n(k)})$ for all $k \in \mathbb{N}$, so $\varrho(\mathscr{F}\xi, x_{n(k)+1}) \leq c\varrho(\xi, x_{n(k)})$ for all $k \in \mathbb{N}$. From the triangle inequality, it follows that $\varrho(\mathscr{F}\xi, \xi) = 0$, which concludes the proof. □

Corollary 1. *Let $\mathscr{F}$ be a 2-basic contraction of Suzuki-type on a Smyth complete T_1 quasi-metric space $(\mathscr{X}, \varrho)$. Then, $\mathscr{F}$ has a unique fixed point $\xi \in \mathscr{X}$. Furthermore, $\varrho^{\max}(\xi, \mathscr{F}^n x_0) \to 0$ as $n \to \infty$ for all $x_0 \in X$.*

Proof. Fix $x \in \mathscr{X}$. By Theorem 5, there exists $\xi \in \mathscr{X}$, such that $\varrho(\mathscr{F}\xi, \xi) = 0$ and

$\varrho^{\max}(\xi, \mathscr{F}^n x) \to 0$ as $n \to \infty$. Since $(\mathscr{X}, \varrho)$ is T_1, we have that $\mathscr{F}\xi = \xi$. Suppose that $\zeta \in \mathscr{X}$ is another fixed point of $\mathscr{F}$. Then

$$\min\{\varrho(\xi, \mathscr{F}\xi), \varrho(\zeta, \mathscr{F}\zeta)\} = 0.$$

By the contraction condition, $\varrho(\mathscr{F}\xi, \mathscr{F}\zeta) \le c\varrho(\xi, \zeta)$, so $\varrho(\xi, \zeta) = 0$, i.e., $\xi = \zeta$.

Finally, given any $x_0 \in \mathscr{X}$, by Theorem 4, there exists $\xi_{x_0} \in \mathscr{X}$ such that $\mathscr{F}\xi_{x_0} = \xi_{x_0}$ and $\varrho^{\max}(\xi_{x_0}, \mathscr{F}^n x_0) \to 0$ as $n \to \infty$. Since ξ is the unique fixed point of $\mathscr{F}$, we conclude that $\varrho^{\max}(\xi, \mathscr{F}^n x_0) \to 0$ as $n \to \infty$. □

The next result generalizes Theorem 3 to non-T_1 quasi-metric spaces. In this way, we can recover the main part of the quasi-metric spaces that appear in the modeling of several processes in the theory of computation, which, as we pointed out in Section 1 are non-T_1.

Theorem 5. *Let $\mathscr{F}$ be a FKP-contraction on a bicomplete quasi-metric space $(\mathscr{X}, \varrho)$. Then, $\mathscr{F}$ has a unique fixed point $\xi \in \mathscr{X}$. Furthermore, $\varrho^{\max}(\xi, \mathscr{F}^n x_0) \to 0$ as $n \to \infty$, for all $x_0 \in \mathscr{X}$.*

Proof. Fix $x_0 \in \mathscr{X}$. Since $\mathscr{F}$ is an FKP-contraction, it satisfies the contraction condition (4). So, by Lemma 3, $(x_n)_{n \in \mathbb{N}_0}$ is a Cauchy sequence in $(\mathscr{X}, \varrho)$, where $x_n := \mathscr{F}^n x_0$ for all $n \in \mathbb{N}_0$. Since $(\mathscr{X}, \varrho)$ is bicomplete, there exists $\xi \in \mathscr{X}$, such that $\varrho^{\max}(\xi, x_n) \to 0$ as $n \to \infty$.

We shall prove that ξ is a fixed point of $\mathscr{F}$.

As in the proof of Theorem 4, it follows from Lemma 1 that

$$\min\{\varrho(\xi, \mathscr{F}\xi), \varrho(x_n, x_{n+1}), \varrho(x_{n+1}, x_n)\} \le 2\varrho(\xi, x_{n+1}),$$

or

$$\min\{\varrho(x_n, x_{n+1}), \varrho(\xi, \mathscr{F}\xi), \varrho(\mathscr{F}\xi, \xi)\} \le 2\varrho(x_n, \xi),$$

for all $n \in \mathbb{N}$.

Consequently, we can find a subsequence $(x_{n(k)})_{k \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$, such that

$$\min\{\varrho(\xi, \mathscr{F}\xi), \varrho(x_{n(k)}, x_{n(k)+1}), \varrho(x_{n(k)+1}, x_{n(k)})\} \le 2\varrho(\xi, x_{n(k)+1}), \tag{7}$$

or

$$\min\{\varrho(x_{n(k)}, x_{n(k)+1}), \varrho(\xi, \mathscr{F}\xi), \varrho(\mathscr{F}\xi, \xi)\} \le 2\varrho(x_{n(k)}, \xi), \tag{8}$$

for all $k \in \mathbb{N}$.

If (7) is met, since $\mathscr{F}$ is an FKP-contraction, we deduce that $\varrho(\mathscr{F}\xi, \mathscr{F}x_{n(k)+1}) \le c\varrho(\xi, x_{n(k)+1})$ for all $k \in \mathbb{N}$, where c is the contraction constant.

Exactly as in the proof of Theorem 4, we obtain $\varrho(\mathscr{F}\xi, \xi) = 0$ and $\varrho^{\max}(\xi, \mathscr{F}^n x_0) \to 0$ as $n \to \infty$. Since $\mathscr{F}$ is an FKP-contraction, the equality $\varrho(\mathscr{F}\xi, \xi) = 0$ implies that $\varrho(\mathscr{F}x_{n(k)+1}, \mathscr{F}\xi) \le c\varrho(x_{n(k)+1}, \xi)$ for all $k \in \mathbb{N}$. Therefore, $\varrho(\mathscr{F}x_{n(k)+1}, \mathscr{F}\xi) \to 0$ as $n \to \infty$, so $\varrho(\xi, \mathscr{F}\xi) = 0$. We conclude that ξ is a fixed point of $\mathscr{F}$.

If (8) is met, reasoning as in the preceding case, we obtain $\varrho(\xi, \mathscr{F}\xi) = 0$ with $\varrho^{\max}(\xi, \mathscr{F}^n x_0) \to 0$ as $n \to \infty$. Since $\mathscr{F}$ is an FKP-contraction, from the equality $\varrho(\xi, \mathscr{F}\xi) = 0$ we deduce, as in the preceding case, that $\varrho(\mathscr{F}\xi, \xi) = 0$. Hence, ξ is a fixed point of $\mathscr{F}$.

Finally, suppose that ζ is another fixed point of $\mathscr{F}$. Then,

$$\min\{\varrho(\xi, \mathscr{F}\xi), \varrho(\zeta, \mathscr{F}\zeta), \varrho(\mathscr{F}\zeta, \zeta)\} = 0.$$

Since $\mathscr{F}$ is an FKP-contraction, we deduce that $\varrho(\xi, \zeta) = 0$. Similarly, we show that $\varrho(\zeta, \xi) = 0$. Hence, $\zeta = \xi$, and thus, ξ is the unique fixed point of $\mathscr{F}$. □

The following is an example where we can apply Theorem 4 but not Theorem 5.

Example 5. *Let $(\mathbb{R}^+, \mathbf{u})$ be the Smyth complete non-T_1 quasi-metric space of Example 2. Let $\mathscr{F}$ be the self map of $\mathbb{R}^+$ defined as $\mathscr{F}0 = 3/2$ and $\mathscr{F}x = 2$ for all $x > 0$. We first show that $\mathscr{F}$*

is not a Banach contraction on $(\mathbb{R}^+, \mathbf{u})$. *Indeed, choose an arbitrary* $c \in (0,1)$. *Let* $x = 0$ *and* $y \in (0, 1/2c)$. *Then,* $\mathbf{u}(\mathscr{F}x, \mathscr{F}y) = \mathbf{u}(3/2, 2) = 1/2 > cy = c\mathbf{u}(0, y) = c\mathbf{u}(x, y)$.

Next we show that, however, $\mathscr{F}$ *is a 2-basic contraction on* $(\mathbb{R}^+, \mathbf{u})$ *with constant* $c = 3/4$. *Indeed, since for every* $x > 0$, $\mathbf{u}(\mathscr{F}x, \mathscr{F}0) = \mathbf{u}(2, 3/2) = 0$, *we only need to analyze the case where* $x = 0$ *and* $y > 0$. *To reach this, suppose that* $\min\{\mathbf{u}(x, \mathscr{F}x), \mathbf{u}(y, \mathscr{F}y)\} \leq 2\mathbf{u}(x, y)$, *i.e.,* $\min\{3/2, \mathbf{u}(y, 2)\} \leq 2y$. *If* $\min\{3/2, \mathbf{u}(y, 2)\} = 3/2$, *we obtain* $3/4 \leq y$ *and* $3/2 \leq \max\{2 - y, 0\}$. *Thus, we come to a contradiction. Hence,* $\min\{3/2, \mathbf{u}(y, 2)\} = \mathbf{u}(y, 2)$, *so* $\mathbf{u}(y, 2) \leq 2y$ *and* $\mathbf{u}(y, 2) \leq 3/2$, *which implies* $y \geq 2/3$. *Therefore,* $\mathbf{u}(\mathscr{F}x, \mathscr{F}y) = \mathbf{u}(3/2, 2) = 1/2 \leq 3y/4 = 3\mathbf{u}(x, y)/4$.

We conclude that $\mathscr{F}$ *is a 2-basic contraction on* $(\mathbb{R}^+, \mathbf{u})$. *By Theorem 4, there is* $\xi \in \mathbb{R}^+$ *such that* $\varrho(\mathscr{F}\xi, \xi) = 0$ *(note that in this example that condition is satisfied by all points of* $\mathbb{R}^+$*).*

Finally, we check that $\mathscr{F}$ *is not an FKP-contraction on* $(\mathbb{R}^+, \mathbf{u})$. *To this end, it suffices to note that for an arbitrary* $c \in (0,1)$ *that we obtain, taking* $x = 0$ *and* $y \in (0, 1/2]$, *that* $\min\{\mathbf{u}(x, \mathscr{F}x), \mathbf{u}(y, \mathscr{F}y), \mathbf{u}(\mathscr{F}y, y)\} = 0$, *but* $\mathbf{u}(\mathscr{F}x, \mathscr{F}y) = 1/2 > cy = c\mathbf{u}(x, y)$. *Therefore, we cannot apply Theorem 5.*

Next we give an example where we can apply Theorem 5, but not Theorem 3.

Example 6. *Let* $\mathscr{X} = \{0, 1, 2, 3\}$, *and let* ϱ *be the non-*T_1 *quasi-metric on* $\mathscr{X}$ *given by* $\varrho(x, x) = 0$ *for all* $x \in \mathscr{X}$, $\varrho(0, 1) = 0$, $\varrho(1, 0) = 2$, $\varrho(x, y) = 1$ *if* $x, y \in \{2, 3\}$ *with* $x \neq y$, $\varrho(x, y) = 2$, *if* $x \in \{0, 1\}$ *and* $y \in \{2, 3\}$, *and* $\varrho(x, y) = 2$, *if* $x \in \{2, 3\}$ *and* $y \in \{0, 1\}$.

$(\mathscr{X}, \varrho)$ *is Smyth complete, and hence, bicomplete because the left Cauchy sequences in* $(\mathscr{X}, \varrho)$ *are eventually constant.*

Now, define a self map $\mathscr{F}$ *of* $\mathscr{X}$ *by* $\mathscr{F}0 = 2$, *and* $\mathscr{F}1 = \mathscr{F}2 = \mathscr{F}3 = 3$.

We first note that $\mathscr{F}$ *is not a Banach contraction on* $(\mathscr{X}, \varrho)$ *because* $\varrho(\mathscr{F}0, \mathscr{F}1) = \varrho(2, 3) = 1$ *but* $\varrho(0, 1) = 0$.

We assert that $\mathscr{F}$ *is an FKP-contraction on* $(\mathscr{X}, \varrho)$ *with constant* $c = 1/2$. *By the construction of* $\mathscr{F}$, *we only need to focus our attention in the next cases:*

Case (a) $x = 0, y = 1$. *Then,* $\varrho(\mathscr{F}x, \mathscr{F}y) = \varrho(2, 3) = 1 > \varrho(x, y)$, *but in this case, we have:*

$$\min\{\varrho(x, \mathscr{F}x), \varrho(y, \mathscr{F}y), \varrho(\mathscr{F}y, y)\} = 2 > 2\varrho(0, 1).$$

Case (b) $x = 1, y = 0$. *Then,* $\varrho(\mathscr{F}x, \mathscr{F}y) = \varrho(3, 2) = 1 = \varrho(x, y)/2$.

Case (c) $x = 0$ *and* $y \in \{2, 3\}$. *Then,* $\varrho(\mathscr{F}x, \mathscr{F}y) = 1 = \varrho(x, y)/2$.

Case (d) $x \in \{2, 3\}$ *and* $y = 0$ *Then,* $\varrho(\mathscr{F}x, \mathscr{F}y) = 1 = \varrho(x, y)/2$.

We have checked that $\mathscr{F}$ *is an FKP-contraction on* $(\mathscr{X}, \varrho)$, *so we can apply Theorem 5. In fact* $\mathscr{F}$ *has a unique fixed point, namely* $\xi = 3$. *Finally, we cannot apply Theorem 3 because* $(\mathscr{X}, \varrho)$ *is not a* T_1 *quasi-metric space.*

Related to Remark 3, we give an example where we can apply Theorem 3 (and also Corollary 1) to a Smyth complete Hausdorff qusi-metric space $(\mathscr{X}, \varrho)$ but not Theorem 1 to the complete metric space $(\mathscr{X}, \varrho^{\max})$.

Example 7. *Let* $\mathscr{X} = \{0, 1, 2, 3, 4\}$ *and let* $\varrho : \mathscr{X} \times \mathscr{X} \to \mathbb{R}^+$ *defined as* $\varrho(x, x) = 0$ *for all* $x \in \mathscr{X}$, $\varrho(0, 1) = 1/2$, $\varrho(1, 0) = 2$, $\varrho(2, 3) = 2$, $\varrho(x, y) = 2$ *if* $x \in \{0, 1\}$ *and* $y \in \{2, 3, 4\}$, $\varrho(x, y) = 2$ *if* $x \in \{2, 3, 4\}$ *and* $y \in \{0, 1\}$, *and* $\varrho(x, y) = 1$ *otherwise.*

It is easily checked that ρ *is a quasi-metric on* $\mathscr{X}$, *and that* $\mathfrak{T}_\varrho$ *is a Hausdorff topology on* $\mathscr{X}$ *(in fact,* $\mathfrak{T}_\varrho$ *is a compact topology and* $\mathfrak{T}_\varrho = \mathfrak{T}_{\varrho^r} = \mathfrak{T}_{\varrho^{\max}}$*). Clearly* $(\mathscr{X}, \varrho)$ *is Smyth complete.*

Now define a self map $\mathscr{F}$ *of* $\mathscr{X}$ *as* $\mathscr{F}0 = 2$, $\mathscr{F}1 = 3$, *and* $\mathscr{F}2 = \mathscr{F}3 = \mathscr{F}4 = 4$.

We first note that $\mathscr{F}$ *is not a basic contraction of Suzuki-type on the complete metric space* $(\mathscr{X}, \varrho^{\max})$, *so we cannot apply Theorem 1. Indeed, we have* $\varrho^{\max}(0, \mathscr{F}0) = 2 < 2\varrho^{\max}(0, 1)$, *but* $\varrho^{\max}(\mathscr{F}0, \mathscr{F}1) = \varrho^{\max}(2, 3) = 2 = \varrho^{\max}(0, 1)$.

We shall prove that, however, $\mathscr{F}$ *is an FKP-contraction on* $(\mathscr{X}, \varrho)$, *with constant* $c = 1/2$.

- *If* $x = 0, y = 1$, *we obtain*

$$\min\{\varrho(x, \mathscr{F}x), \varrho(y, \mathscr{F}y), \varrho(\mathscr{F}y, y)\} = 2 > 2\varrho(0, 1).$$

- *If $x, y \in \{2, 3, 4\}$ we obtain $\varrho(\mathscr{F}x, \mathscr{F}y) = 0$.*
- *In the rest of the cases, it is routine to check that $\varrho(\mathscr{F}x, \mathscr{F}y) = \varrho(x, y)/2$.*

Therefore, we can apply Theorem 5 (note that we can also apply Corollary 1). In fact, $\mathscr{F}$ has a unique fixed point, namely $\xi = 4$.

As we pointed out in Section 1, the last part of the paper is devoted to developing a general method that allows us to construct non-T_1 quasi-metric spaces on which it is possible to systematically generate contractions of Suzuki-type that are not Banach contractions. We illustrate this approach by applying our fixed point results to show the existence and uniqueness of solution for some kinds of functional equations from which we derive the existence and uniqueness of a solution for a very well-known difference equation. To this purpose, the following result will be crucial.

Proposition 1. *Let $(\mathscr{X}, \sigma)$ be a quasi-metric space, such that $|\mathscr{X}| \geq 2$ and $\sigma \leq 1$, let $\mathscr{Y}$ be a strict non-empty subset of $\mathscr{X}$, and $\perp$ be an element, such that $\perp \notin \mathscr{X}$. Put $\mathscr{X}' = \mathscr{X} \cup \{\perp\}$ and $\mathscr{Y}' = \mathscr{Y} \cup \{\perp\}$. Define a function $\varrho : \mathscr{X}' \times \mathscr{X}' \to [0, 1]$ as:*

$$\varrho(x, x) = 0 \quad \text{for all } x \in \mathscr{X},$$
$$\varrho(\perp, y) = 0 \quad \text{for all } y \in \mathscr{Y},$$
$$\varrho(x, y) = \sigma(x, y) \quad \text{for all } x, y \in \mathscr{X} \backslash \mathscr{Y}',$$

and

$$\varrho(x, y) = 2, \quad \text{otherwise}.$$

Then, we have:

(A) $(\mathscr{X}', \varrho)$ is a non-T_1 quasi-metric space. Moreover, $(\mathscr{X}', \varrho)$ is bicomplete if $(\mathscr{X} \backslash \mathscr{Y}, \sigma)$ is.
(B) If $\mathscr{F}$ is a self map of $\mathscr{X}'$, such that $\mathscr{F}y \in \mathscr{X} \backslash \mathscr{Y}'$ for all $y \in \mathscr{Y}'$, and the set $\{y \in \mathscr{Y} : \sigma(\mathscr{F}\perp, \mathscr{F}y) > 0\}$ is non-empty, then $\mathscr{F}$ is not a Banach contraction on $(\mathscr{X}', \varrho)$.

Proof. (A) We omit the easy proof that $(\mathscr{X}', \varrho)$ is a non-T_1 quasi-metric space. Now, let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $(\mathscr{X}', \varrho)$. Via the definition of ϱ, there is $n_0 \in \mathbb{N}$ such that $x_n \in \mathscr{X} \backslash \mathscr{Y}'$ for all $n \geq n_0$, and $\varrho(x_n, x_m) = \sigma(x_n, x_m)$ for all $n, m \geq n_0$. Hence, $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $(\mathscr{X} \backslash \mathscr{Y}, \sigma)$. Let $p \in \mathscr{X} \backslash \mathscr{Y}$, such that $\sigma^{\max}(p, x_n) \to 0$ as $n \to \infty$. Since $\varrho = \sigma$ on $\mathscr{X} \backslash \mathscr{Y}$, we infer that $\varrho^{\max} = \sigma^{\max}$ on $\mathscr{X} \backslash \mathscr{Y}$. So $\varrho^{\max}(p, x_n) \to 0$ as $n \to \infty$. We conclude that $(\mathscr{X}', \varrho)$ is bicomplete.

(B) Take any $y \in \mathscr{Y}$, such that $\sigma(\mathscr{F}\perp, \mathscr{F}y) > 0$. Since $\mathscr{F}y, \mathscr{F}\perp \in \mathscr{X} \backslash \mathscr{Y}'$, we obtain

$$\varrho(\mathscr{F}\perp, \mathscr{F}y) = \sigma(\mathscr{F}\perp, \mathscr{F}y) > 0 = \varrho(\perp, y).$$

Therefore, $\mathscr{F}$ is not a Banach contraction on $(\mathscr{X}, \varrho)$. □

In the sequel, we shall denote by $f_{\mathbf{0}}$ the zero function on $\mathbb{R}$, i.e., $f_{\mathbf{0}}(x) = 0$ for all $x \in \mathbb{R}$. Adopting the notation of Proposition 1, let $(\mathscr{X}, \sigma)$ be the quasi-metric space, such that $\mathscr{X} = [0, 1]^{\mathbb{R}} \backslash \{f_{\mathbf{0}}\}$ and σ is the supremum quasi-metric on $\mathscr{X}$, i.e.,

$$\sigma(f, g) = \sup_{x \in \mathbb{R}} \max\{f(x) - g(x), 0\},$$

for all $f, g \in [0, 1]^{\mathbb{R}} \backslash \{f_{\mathbf{0}}\}$. Of course, $\sigma \leq 1$ on X.

Let $\mathscr{Y} := \{f \in \mathscr{X} : \sup_{x \in \mathbb{R}} f(x) < 1\}$. Since,

$$\sigma^{\max}(f, g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)|,$$

for all $f, g \in \mathscr{X}$, we infer that $(\mathscr{X} \backslash \mathscr{Y}, \sigma^{\max})$ is a complete metric space, i.e., $(\mathscr{X} \backslash \mathscr{Y}, \sigma)$ is a bicomplete quasi-metric space.

Put $\perp = f_{\mathbf{0}}$, $\mathscr{X}' = \mathscr{X} \cup \{f_{\mathbf{0}}\}$, i.e., $\mathscr{X}' = [0, 1]^{\mathbb{R}}$, and let $\mathscr{Y}' = \mathscr{Y} \cup \{f_{\mathbf{0}}\}$.

Denote by ϱ the quasi-metric on $\mathscr{X}'$ constructed in Proposition 1. Thus, $(\mathscr{X}', \varrho)$ is bicomplete by Proposition 1(A).

Now, let $F : \mathbb{R} \to [0,1]$ be the functional equation with initial value $F(0) = a \in [0,1]$, and such that

$$F(x) = \frac{\alpha F(x-1)}{\beta + \gamma F(x-1)},$$

for all $x \in \mathbb{R}$, with $0 < \alpha < \beta$ and $\gamma > 0$.

Define a self map $\mathscr{F}$ of $\mathscr{X}'$ as follows:

$\mathscr{F} f_0(0) = a$, and $\mathscr{F} f_0(x) = 1$ for all $x \in \mathbb{R}\backslash\{0\}$.

If $f \in \mathscr{Y}$:

$\mathscr{F} f(0) = a$, $\mathscr{F} f(1) = 1$, and $\mathscr{F} f(x) = 0$ otherwise.

If $f \in \mathscr{X}\backslash\mathscr{Y}$:

$\mathscr{F} f(0) = a$, and

$$\mathscr{F} f(x) = \frac{\alpha f(x-1)}{\beta + \gamma f(x-1)},$$

for all $x \in \mathbb{R}\backslash\{0\}$.

Observe that

$$\mathscr{X}\backslash\mathscr{Y}' = \{f \in [0,1]^{\mathbb{R}} : \sup_{x \in \mathbb{R}} f(x) = 1\},$$

and thus, $\mathscr{F} f \in \mathscr{X}\backslash\mathscr{Y}'$ for all $f \in \mathscr{Y}'$. Moreover, $\sigma(\mathscr{F} f_0, \mathscr{F} f) = 1$ for all $f \in \mathscr{Y}$. Hence, $\mathscr{F}$ is not a Banach contraction on $(\mathscr{X}', \varrho)$ by Proposition 1(B).

We are going to check that $\mathscr{F}$ is an FKP-contraction on $(\mathscr{X}', \varrho)$. To this end, it is appropriate to note that $\varrho(f,g) \leq 1$ for all $f, g \in \mathscr{X}\backslash\mathscr{Y}'$.

- For f_0 and $f \in \mathscr{Y}$, we have

$$\min\{\varrho(f_0, \mathscr{F} f_0), \varrho(f, \mathscr{F} f), \varrho(\mathscr{F} f, f)\} = 2 > 0 = 2\varrho(f_0, f),$$

and

$$\varrho(\mathscr{F} f, \mathscr{F} f_0) \leq 1 = \frac{1}{2}\varrho(f, f_0).$$

- For $f, g \in \mathscr{Y}$, we have $\varrho(\mathscr{F} f, \mathscr{F} g) = 0$.
- For $f \in \mathscr{Y}'$ and $g \in \mathscr{X}\backslash\mathscr{Y}'$, or $f \in \mathscr{X}\backslash\mathscr{Y}'$ and $g \in \mathscr{Y}'$, we have

$$\varrho(\mathscr{F} f, \mathscr{F} g) \leq 1 = \frac{1}{2}\varrho(f,g).$$

- For $f, g \in \mathscr{X}\backslash\mathscr{Y}'$, we have

$$\begin{aligned} \varrho(\mathscr{F} f, \mathscr{F} g) &= \sup_{x \in \mathbb{R}} \max\{\frac{\alpha\beta(f(x-1) - g(x-1))}{(\beta + \gamma f(x-1))(\beta + \gamma g(x-1))}, 0\} \\ &\leq \sup_{x \in \mathbb{R}} \max\{\frac{\alpha(f(x-1) - g(x-1))}{\beta}, 0\} = \frac{\alpha}{\beta}\varrho(f,g). \end{aligned}$$

Therefore, $\mathscr{F}$ is an FKP-contraction with contraction constant $c = \max\{1/2, \alpha/\beta\}$.

We have proven that all conditions of Theorem 5 are fulfilled, so that $\mathscr{F}$ has a unique fixed point $h \in \mathscr{X}'$, which is obviously the unique solution of the functional equation F.

Remark 4. *This approach has the advantage that we only need pay attention to calculating* $\varrho(\mathscr{F} f, \mathscr{F} g)$ *when* $f, g \in \mathscr{X}\backslash\mathscr{Y}$. *Moreover, we also deduce that* $h \in \mathscr{X}\backslash\mathscr{Y}'$, *i.e.,* $\sup_{x \in \mathbb{R}} h(x) = 1$, *because* $\mathscr{F} f \neq f$ *for all* $f \in \mathscr{Y}'$.

As a consequence, we obtain that the difference equation, with initial value $x_0 = a$, *and*

$$x_n = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-1}},$$

for all $n \in \mathbb{N}$, *with* $0 < \alpha < \beta$ *and* $\gamma > 0$, *has a unique solution which consists of the restriction of* h *to* $\mathbb{N}_0$. *This type of difference equations is well-known, and belongs to the family of difference* *equations for population growth (also called, sometimes, logistic difference equations); see e.g., [43] (p. 515).*

4. Conclusions

We have presented an example of a basic contraction of Suzuki-type on a Smyth complete quasi-metric space, which has no fixed points. This shows the great difficulty in obtaining a full natural quasi-metric generalization of Suzuki's fixed point theorem. However, and inspired in a type of contraction stated by Fulga, Karapinar, and Petrusel [19], we are able to obtain fixed point theorems for contractions of Suzuki-type, both for Smyth complete and bicomplete quasi-metric spaces. Our results were accompanied with some key examples. In particular, Example 7 shows that such results provide real generalizations of the corresponding ones for metric spaces. Finally, we have implemented a method to construct basic contractions of Suzuki-type that are not Banach contractions, illustrating this approach with an application to a featured class of difference equations.

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