mathematics

Article

On Four Classical Measure Theorems

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Abstract: A subset $\mathcal{B}$ of an algebra $\mathcal{A}$ of subsets of a set Ω has property (N) if each $\mathcal{B}$-pointwise bounded sequence of the Banach space $ba(\mathcal{A})$ is bounded in $ba(\mathcal{A})$, where $ba(\mathcal{A})$ is the Banach space of real or complex bounded finitely additive measures defined on $\mathcal{A}$ endowed with the variation norm. $\mathcal{B}$ has property (G) $[(VHS)]$ if for each bounded sequence [if for each sequence] in $ba(\mathcal{A})$ the $\mathcal{B}$-pointwise convergence implies its weak convergence. $\mathcal{B}$ has property (sN) $[(sG)$ or $(sVHS)]$ if every increasing covering $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of $\mathcal{B}$ contains a set $\mathcal{B}_p$ with property (N) $[(G)$ or $(VHS)]$, and $\mathcal{B}$ has property (wN) $[(wG)$ or $(wVHS)]$ if every increasing web $\{\mathcal{B}_{n_1 n_2 \cdots n_m} : n_i \in \mathbb{N}, 1 \leq i \leq m, m \in \mathbb{N}\}$ of $\mathcal{B}$ contains a strand $\{\mathcal{B}_{p_1 p_2 \cdots p_m} : m \in \mathbb{N}\}$ formed by elements $\mathcal{B}_{p_1 p_2 \cdots p_m}$ with property (N) $[(G)$ or $(VHS)]$ for every $m \in \mathbb{N}$. The classical theorems of Nikodým–Grothendieck, Valdivia, Grothendieck and Vitali–Hahn–Saks say, respectively, that every σ-algebra has properties (N), (sN), (G) and (VHS). Valdivia's theorem was obtained through theorems of barrelled spaces. Recently, it has been proved that every σ-algebra has property (wN) and several applications of this strong Nikodým type property have been provided. In this survey paper we obtain a proof of the property (wN) of a σ-algebra independent of the theory of locally convex barrelled spaces which depends on elementary basic results of Measure theory and Banach space theory. Moreover we prove that a subset $\mathcal{B}$ of an algebra $\mathcal{A}$ has property $(wWHS)$ if and only if $\mathcal{B}$ has property (wN) and $\mathcal{A}$ has property (G).

Keywords: algebra and σ-algebra of subsets; bounded finitely additive scalar measure; Nikodým; strong and web Nikodým properties; Grothendieck; strong and web Grothendieck properties; Vitali–Hahn–Saks; strong and web Vitali–Hahn–Saks properties

MSC: 28A60; 46G10

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Citation: López-Alfonso, S.; López-Pellicer, M.; Moll-López, S. On Four Classical Measure Theorems. *Mathematics* **2021**, *9*, 526. https://doi.org/10.3390/math9050526

Academic Editor: Rekha Srivastava

Received: 7 February 2021
Accepted: 26 February 2021
Published: 3 March 2021

1. Introduction

In this paper $\mathcal{A}$ and $\mathcal{S}$ denote, respectively, an algebra and a σ-algebra of subsets of a set Ω. We will refer to an algebra and a σ-algebra of subsets of Ω simply as an algebra and a σ-algebra, respectively. The real or complex linear hull $L(\mathcal{A})$ of the set

$$\{e(B) : B \in \mathcal{A}\}$$

of characteristics functions of the elements of $\mathcal{A}$ with the norm

$$\| f \|_\infty := \sup\{|f(x)| : x \in \Omega\}, \quad f \in L(\mathcal{A})$$

is a normed space and its completion is the Banach space $L_\infty(\mathcal{A})$ of all $\mathcal{A}$-measurable real or complex bounded functions defined on Ω. By [1], Theorem 1.13, its dual endowed with the polar norm is the Banach space $ba(\mathcal{A})$ of scalar bounded finitely additive measures defined on $\mathcal{A}$, and the polar norm of every $\mu \in ba(\mathcal{A})$ is the variation of μ, given by

$$|\mu| := \sup\{\Sigma_{i=1}^{n}\{|\mu(A_i)|\} : \{A_i : 1 \le i \le n\} \in \mathcal{F}\},$$

where $\mathcal{F}$ is the family of finite partitions of Ω by elements of $\mathcal{A}$ and $\mu(e(C)) := \mu(C)$. For an element B of $\mathcal{A}$ the variation of μ on B

$$|\mu|(B) := \sup\{\Sigma_{i=1}^{n}\{|\mu(A_i)|\} : \{A_i : 1 \le i \le n\} \in \mathcal{F}_B\},$$

defines a seminorm on $ba(\mathcal{A})$ and for each finite partition $\{B_i : B_i \in \mathcal{A}, 1 \leqslant i \leqslant n\}$ of B we have $|\mu|(B) = \Sigma\{|\mu|(B_i) : 1 \leqslant i \leqslant n\}$.

The polar set [2], §20, 8 (named absolute polar set), of a subset M of $L(\mathcal{A})$ or $ba(\mathcal{A})$ is the subset M° defined by

$$M^\circ := \{\mu \in ba(\mathcal{A}) : |\mu(f)| \le 1, \text{ for every } f \in M\}, \text{ if } M \subset L(\mathcal{A})$$

or

$$M^\circ := \{f \in L(\mathcal{A}) : |\mu(f)| \le 1, \text{ for every } \mu \in M\}, \text{ if } M \subset L(\mathcal{A}).$$

The topology in $ba(\mathcal{A})$ of pointwise convergence on a subset $\mathcal{B}$ of $\mathcal{A}$ is denoted by $\tau_s(\mathcal{B})$ in $ba(\mathcal{A})$. Clearly a subset M of $ba(\mathcal{A})$ is $\tau_s(\mathcal{B})$-bounded if and only if

$$\sup\{|\mu(C)| : \mu \in M\} < \infty, \text{ for every } C \in \mathcal{B}.$$

In particular, $\tau_s(\mathcal{A})$ is the weak* topology in $ba(\mathcal{A})$.

By absco$H$ is denoted the absolutely convex hull of H and the gauge or Minkowski functional of the subset $G := \text{absco}(\{e(C) : C \in \mathcal{A}\})$ of $L(\mathcal{A})$ is a norm in $L(\mathcal{A})$ defined by

$$\|f\|_G := \inf\{|\lambda| : f \in \lambda G\}, \quad f \in L(\mathcal{A}),$$

which is equivalent to the supremum norm ([3], Propositions 1 and 2). Its polar norm in $ba(\mathcal{A})$ is the supremum of the modulus, i.e., for every $\mu \in ba(\mathcal{A})$ of μ in $\mathcal{A}$,

$$|\mu|_\infty := \sup\{|\mu(C)| : C \in \mathcal{A}\}, \quad \mu \in ba(\mathcal{A}),$$

hence in $ba(\mathcal{A})$ the norms variation and supremum are equivalent. For each $B \in \mathcal{A}$ the seminorms defined by the variation on B, $|\mu|(B)$, and the supremum of the modulus on $\{C \in \mathcal{A} : C \subset B\}$, $|\mu|_{\infty,B} := \sup\{|\mu(C)| : C \in \mathcal{A}, C \subset B\}$, are equivalent seminorms in $ba(\mathcal{A})$.

A subset $\mathcal{B}$ of an algebra $\mathcal{A}$ is a Nikodým set for $ba(\mathcal{A})$, or $\mathcal{B}$ has property (N), if the $\tau_s(\mathcal{B})$-boundedness of a subset M of $ba(\mathcal{A})$ implies

$$\sup\{|\mu| : \mu \in M\} < \infty,$$

or, equivalently

$$\sup\{|\mu(C)| : \mu \in M,\ C \in \mathcal{A}\} < \infty,$$

i.e., M is uniformly bounded in $\mathcal{A}$. Note that $\mathcal{B}$ has property (N) if and only if $\{e(C) : C \in \mathcal{B}\}$ is a uniform bounded deciding subset of $L(\mathcal{A})$ ([4], Example 2). We may suppose that the subset M above is absolutely convex and weak*-closed. Clearly $\mathcal{B}$ has property (N) if each $\mathcal{B}$-pointwise bounded sequence of $ba(\mathcal{A})$ is uniformly bounded in $\mathcal{A}$. The above set $\mathcal{B}$ is a strong Nikodým set for $ba(\mathcal{A})$, or $\mathcal{B}$ has property (sN), if each increasing countable covering of $\mathcal{B}$ contains an element that has property (N), and $\mathcal{B}$ is a web Nikodým set for $ba(\mathcal{A})$, or $\mathcal{B}$ has property (wN), if each increasing web $\{\mathcal{B}_{n_1 n_2 \cdots n_m} : n_i \in \mathbb{N}, 1 \le i \le m, m \in \mathbb{N}\}$ of $\mathcal{B}$ contains a decreasing sequence $\{\mathcal{B}_{p_1 p_2 \cdots p_m} : m \in \mathbb{N}\}$ formed by subsets with property (N). Let us recall that by definition $\{\mathcal{B}_{n_1 n_2 \cdots n_m} : n_i \in \mathbb{N}, 1 \le i \le m, m \in \mathbb{N}\}$ is an increasing web of $\mathcal{B}$ if the sequence $\{\mathcal{B}_{n_1} : n_1 \in \mathbb{N}\}$ is an increasing covering of $\mathcal{B}$ and for each $m \in \mathbb{N}$ and each $(n_1 n_2 \cdots n_m) \in \mathbb{N}^m$ the countable family $\{\mathcal{B}_{n_1 n_2 \cdots n_m n_{m+1}} : n_{m+1} \in \mathbb{N}\}$ is an increasing covering of $\mathcal{B}_{n_1 n_2 \cdots n_m}$. It is obvious that $(wN) \Rightarrow (sN) \Rightarrow (N)$, and that $\mathcal{B}$ has property (wN) if and only if each increasing web of $\mathcal{B}$ contains an increasing subweb formed by sets that have property (wN). It is straightforward to prove that properties (wN), $(s(wN))$, $(w(sN))$ and $(w(wN))$ are equivalent ([5], Proposition 1).

A subset $\mathcal{B}$ of an algebra $\mathcal{A}$ is a Grothendieck set for $ba(\mathcal{A})$ if each bounded sequence in $ba(\mathcal{A})$ that $\mathcal{B}$-pointwise converges to the null measure converges weakly, and $\mathcal{B}$ is a Vitaly-Hahn-Saks set for $ba(\mathcal{A})$ if in $ba(\mathcal{A})$ each sequence that $\mathcal{B}$-pointwise converges to the null measure converges weakly. In brief we will say that $\mathcal{B}$ has property (G) or property (VHS), respectively. The above subset $\mathcal{B}$ has property (VHS) if and only if $\mathcal{B}$ has properties (N) and (G) (see Proposition 6). Properties (sG), (wG), $(sVHS)$ and $(wVHS)$ are defined as in the case of properties (sN) and (wN), changing N into G or VHS. For instance, $\mathcal{B}$ has property (sG) if for each increasing covering $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of $\mathcal{B}$ there exists $p \in \mathbb{N}$ such that $\mathcal{B}_p$ has property (G). Clearly $(wG) \Rightarrow (sG) \Rightarrow (G)$, and $(wVHS) \Rightarrow (sVHS) \Rightarrow (VHS)$. Let us recall (see [6,7]) that a subset C of the closed dual unit ball B_{E^*} of a Banach space E is a Rainwater set for E if for every bounded sequence $\{x_n : n \in \mathbb{N}\}$ the conditions

$$\lim_{n\to\infty} f(x_n) = 0, \text{ for every } f \in C$$

imply

$$\lim_{n\to\infty} f(x_n) = 0, \text{ for every } f \in E^*.$$

Hence for a subset $\mathcal{B}$ of an algebra $\mathcal{A}$ the conditions $\mathcal{B}$ has property (G) and $C = \{e(B) : B \in \mathcal{B}\}$ is a Rainwater set for $ba(\mathcal{A})$ are equivalent ([8], Proposition 4.1) , where $e(B)$ is the element of the closed dual unit ball of $ba(\mathcal{A})^*$ such that $e(B)(\mu) := \mu(B)$, for each measure $\mu \in ba(\mathcal{A})$.

The four classical theorems of Nikodým–Grothendieck, Valdivia, Grothendieck and Vitali–Hahn–Saks say, respectively, that each σ-algebra $\mathcal{S}$ has properties (N), (sN), (G) and (VHS) (see [1,3,9–15]). Equivalent definitions of properties (G) and (VHS) are given in [14]. We may find in [11] that for each sequence in ℓ_∞^* the weak* convergence implies the weak convergence. Because of this deep property, a Banach space E is called a Grothendieck space if for each sequence in its dual E^* the weak* convergence implies its weak convergence, so ℓ_∞ is a Grothendieck space. Notice that by the Banach–Steinhaus theorem every weak* convergent sequence of the dual E^* of a Banach space E is bounded, hence an algebra $\mathcal{A}$ has property (G) if and only if $L_\infty(\mathcal{A})^*$ is a Grothendieck space. Recently it has been proved that every σ-algebra $\mathcal{S}$ has properties (wN) (see [16,17]) and (wG) (see [18] and ([19], Theorem 1)). It has also property $(wVHS)$, because a set $\mathcal{B}$ has property $(wVHS)$ if and only if $\mathcal{B}$ has properties (wN) and (G) (Corollary 2).

The situation with algebras is different. There are many examples of algebras that do not have property (N) ([1], Chapter I, Example 5). Schachermayer [14] proved that the algebra $\mathcal{J}([0,1])$ of all Jordan measurable subsets of $[0,1]$ has property (N) but fails property (G). In 2013, Valdivia proved that the algebra $\mathcal{J}([0,1]^p)$ has property (sN) [15]. This result motivated paper [20] where if was proved that $\mathcal{J}([0,1]^p)$ has property (wN). It has been found recently in [8] that there exists a class of rings of sets with property (wN).

Valdivia improved some results concerning the range localization of vector measures defined in a σ-algebra by showing that each σ-algebra has property (sN) [3]. The extension of these new range localization results to vector measures defined on an algebra motivates the following open problem proposed by Valdivia in 2013 [15]:

Is it true that in an algebra $\mathcal{A}$ that property (N) implies property (sN)?

Valdivia's original proof that every σ-algebra has property (sN) depends on properties of locally convex barrelled spaces (contained among others in the books [21,22], and also in the papers [23–25]). As a help to solve the mentioned open problem proposed by Valdivia, in [19], Section 3, was given a new proof independent from barrelledness properties. In [19], Problem 2, it was proposed to prove that every σ-algebra has property (wN) using basic results of Measure theory and Banach space theory. We give such proof in Section 3 of this paper. For the sake of completeness we include several proofs of previous known results with the corresponding references.

The last section is motivated by [26,27]. We prove that for a subset $\mathcal{B}$ of an algebra $\mathcal{A}$ properties (sN) and (G) imply property $(sVHS)$ and properties (wN) and (G) imply property $(wVHS)$ (Corollaries 1 and 2). Therefore in a class of algebras where property (N) implies property (sN) we will have also that property (VHS) imply property $(sVHS)$.

2. Preliminary Results

The next well known proposition characterizes when a subset $\mathcal{B}$ of an algebra $\mathcal{A}$ has property (N) (see [19], Proposition 1). We give a reduced proof for the sake of completeness.

Proposition 1. *$\mathcal{B}$ has property (N) if and only if for each increasing covering $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of $\mathcal{B}$ there exists $\mathcal{B}_p$ such that*

$$\overline{\operatorname{absco}\{e(A) : A \in \mathcal{B}_p\}}^{L(\mathcal{A})}$$

is a neighborhood of zero in $L(\mathcal{A})$.

Proof. If $\mathcal{B}$ does not have property (N) there exists a subset M in $ba(\mathcal{A})$ such that

$$\sup\{|\mu| : \mu \in M\} = \infty$$

and

$$\sup\{|\mu(C)| : \mu \in M\} < \infty, \quad \text{for each } C \in \mathcal{B}.$$

The first equality implies that M° is not a neighborhood of zero in $L(\mathcal{A})$. The above inequalities imply that the sets

$$\mathcal{B}_n = \{A \in \mathcal{B} : \sup_{\mu \in M} |\mu(A)| \leq n\}, \quad n \in \mathbb{N},$$

are increasing, cover $\mathcal{B}$ and

$$\{e(A) : A \in \mathcal{B}_n\} \subset nM^\circ, \quad \text{for each } n \in \mathbb{N}.$$

Hence the inclusions $D_n := \overline{\operatorname{absco}\{e(A) : A \in \mathcal{B}_n\}}^{L(\mathcal{A})} \subset nM^\circ$, for each $n \in \mathbb{N}$, imply that for each natural number n the set D_n is not a neighborhood of zero in $L(\mathcal{A})$.

Conversely, if there exists an increasing covering $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of $\mathcal{B}$ such that

$$\overline{\operatorname{absco}\{e(A) : A \in \mathcal{B}_n\}}^{L(\mathcal{A})}$$

is not a neighborhood of zero in $L(\mathcal{A})$ for every $n \in \mathbb{N}$, then there exists $\mu_n \in \{e(A) : A \in \mathcal{B}_n\}^\circ$ such that $|\mu_n| \geq n$, for each $n \in \mathbb{N}$, and, by definition of polar set,

$$\sup\{|\mu_n(A)| : A \in \mathcal{B}_n\} \leq 1.$$

Hence $M = \{\mu_n : n \in \mathbb{N}\}$ is an unbounded subset of $ba(\mathcal{A})$ and if $C \in \mathcal{B}$ there exists $q_C \in \mathbb{N}$ such that $C \in \mathcal{B}_n$ for each $n \geq q_C$, hence

$$\sup\{|\mu(C)| : \mu \in M\} \leq 1 + \Sigma\{|\mu_n(C)| : n < q_C\} < \infty,$$

so $\mathcal{B}$ does not have property (N). □

In particular, if $\mathcal{B}$ is a Nikodým set for $ba(\mathcal{A})$ then $\overline{\operatorname{absco}\{e(A) : A \in \mathcal{B}\}}^{L(\mathcal{A})}$ is a neighborhood of zero in $L(\mathcal{A})$ and $\overline{\operatorname{span}\{e(A) : A \in \mathcal{B}\}}^{L(\mathcal{A})} = L(\mathcal{A})$.

We need to complement Proposition 1 with Proposition 2, which provides a property of a subset $\mathcal{B}$ of an algebra $\mathcal{A}$ that has property (N) but fails property (sN).

Let B and C be two subsets of a vector space E. Note that if the sum span $B +$ span C is not direct there exists $x \in C\backslash\{0\}$ such that

$$x = \Sigma_{i=1}^{p}\beta_i y_i + \Sigma_{j=1}^{q}\gamma_j z_j,$$

with $y_i \in B$, $1 \leq i \leq p$, and $z_j \in C\backslash\{x\}$, $1 \leq j \leq q$. Then each $w \in$ absco$(B \cup C)$ may be represented as

$$w = \Sigma_{i=1}^{s}\delta_i s_i + \epsilon x + \Sigma_{j=1}^{t}\epsilon_j t_j,$$

with $\Sigma_{i=1}^{s}|\delta_i| + |\epsilon| + \Sigma_{j=1}^{t}|\epsilon_j| \leq 1$ and $(s_i, t_j) \in B \times (C\backslash\{x\})$. The two above equalities imply that

$$x = \Sigma_{i=1}^{s}\delta_i s_i + \Sigma_{i=1}^{p}\epsilon\beta_i y_i + \Sigma_{j=1}^{q}\epsilon\gamma_j z_j + \Sigma_{j=1}^{t}\epsilon_j t_j,$$

hence

$$x \in (1+h)absco(B \cup (C\backslash\{x\})),$$

with $h = \Sigma_{i=1}^{p}|\beta_i| + \Sigma_{j=1}^{q}|\gamma_j|$. This relation proves the non trivial inclusion in

$$absco(B \cup (C\backslash\{q_1\})) \subset absco(B \cup C) \subset (1+h)absco(B \cup (C\backslash\{x\})). \tag{1}$$

From (1) it follows that the gauges defined by absco$(B \cup C)$ and absco$(B \cup C\backslash\{x\})$ are equivalent and

$$\text{span}B + \text{span}C = \text{span}B + \text{span}(C\backslash\{x\}).$$

By a direct finite induction we deduce the well known property ([19], Claim 2) that each finite subset C of E contains a subset D such that span$(B \cup C)$ is equal to the direct sum span $B \oplus$ span D and the gauges of absco$(B \cup C)$ and absco$(B \cup D)$ are equivalent. This property is used in the following Remark 1 ([19], Claim 3), that implies Proposition 2 obtained in [19], Proposition 3. To help the reader we present simplified proofs.

Remark 1. *Let E be a normed space and let B be a closed absolutely convex subset of E which is not a zero neighborhood in E and such that its linear hull is dense in E. Then for each finite subset C of E the absolutely convex hull of $B \cup C$ is not a zero neighborhood in E.*

Proof. If absco$(B \cup C)$ is a neighborhood of 0 in E then C contains a subset D such that absco$(B \cup D)$ is a neighborhood of 0 in E and $E =$ span$B \oplus$ span$D$. Then $(\text{absco}(B \cup D)) \cap (\text{span}B) = B$ is a zero neighborhood in span$B$, implying that $B = \overline{B}^E$ is a neighborhood of zero in $\overline{\text{span}B}^E$, since for each $x \in E$ with $0 < \|x\| < r$ there exists a sequence $(x_n)_{n=1}^{\infty}$ in span$B$ with $\|x_n\| < r$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = x$. We get the contradiction that B is neighborhood of zero in E. □

Proposition 2. *Let $\mathcal{A}$ be an algebra that has a subset $\mathcal{B}$ enjoying property (N) with an increasing covering $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of $\mathcal{B}$ such that each $\mathcal{B}_n$ does not have property (N). Then there exists $p \in \mathbb{N}$ such that for every $n \geq p$ the space $ba(\mathcal{A})$ contains a weak*-closed, absolutely convex, $\mathcal{B}_n$-pointwise bounded subset M_n such that for each finite subset Q of $\mathcal{A}$ we have that $M_n \cap \{e(A) : A \in Q\}^\circ$ is unbounded in $ba(\mathcal{A})$, i.e.,*

$$\sup_{\mu \in M_n \cap \{e(A):A\in Q\}^\circ} \{|\mu|(\Omega)\} = \infty. \tag{2}$$

Proof. Since $\mathcal{B}_n$ does not have property (N) there exists an unbounded, weak* closed, absolutely convex subset M_n of $ba(\mathcal{A})$ which is $\mathcal{B}_n$-pointwise bounded. Hence the polar set M_n° is a closed absolutely convex subset of $L(\mathcal{A})$ which is not neighborhood of zero and span$\{e(A) : A \in \mathcal{B}_n\} \subset$ spanM_n°. By Proposition 1 there exists p such that for each $n \geq p$

$$\overline{\text{span}\{e(A) : A \in \mathcal{B}_n\}}^{L(\mathcal{A})} = L(\mathcal{A}), \tag{3}$$

hence, for each $n \geq p$, the relation (3) implies that $\overline{\text{span} M_n^\circ}^{L(\mathcal{A})} = L(\mathcal{A})$, so the set $\text{absco}(M_n^\circ \cup \{e(A) : A \in Q\})$ is not a zero neighborhood in $L(\mathcal{A})$, implying the unboundedness of its polar set

$$\{\text{absco}(M_n^\circ \cup \{e(A) : A \in Q\})\}^\circ = M^{\circ\circ} \cap \{e(A) : A \in Q\}^\circ,$$

and the proof follows from the equality $M = M^{\circ\circ}$. □

Assume that an element B of an algebra $\mathcal{A}$ and an absolutely convex a subset M of $ba(\mathcal{A})$ verify that for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M \cap \{e(A):A \in Q\}^\circ} \{|\mu|(B)\} = \infty \tag{4}$$

then for each finite partition $\{B_1, B_2, \ldots, B_q\}$ of B by elements of $\mathcal{A}$ the equality

$$|\mu|(B) = |\mu|(B_1) + |\mu|(B_2) + \cdots + |\mu|(B_q)$$

implies that there exists B_j, with $1 \leq j \leq q$, such that for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M \cap \{e(A):A \in Q\}^\circ} \{|\mu|(B_j)\} = \infty.$$

This observation implies that for each $\alpha \in \mathbb{R}^+$ and each finite subset $\{P_i : 1 \leq i \leq n\}$ of $\mathcal{A}$ there exists $(\mu_1, B_1) \in M \times \mathcal{A}$, $B_1 \subset B$ such that

$$|\mu_1(e(B_1))| > \alpha, \quad |\mu_1(e(B \backslash B_1))| > \alpha, \quad \Sigma_{j=1}^n \big|\mu_1(e(P_j))\big| \leq 1$$

and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M \cap \{e(A):A \in Q\}^\circ} |\mu|(B \backslash B_1) = \infty,$$

because by (4) with $Q = \{B, P_1, \cdots, P_n\}$ there exists

$$(\nu_1, P_{11}) \in (M \cap \{e(D) : D \in Q\}^\circ) \times \mathcal{A},$$

with $P_{11} \subset B$ such

$$|\nu_1(P_{11})| > n(\alpha + 1), \quad |\nu_1(B)| \leq 1 \quad \text{and} \quad \big|\nu_1(P_j)\big| \leq 1, \text{ for } 1 \leq j \leq n.$$

Let $P_{12} := B \backslash P_{11}$ and $\mu_1 = n^{-1}\nu_1$. The measure $\mu_1 \in M$ and verifies that

$$|\mu_1(P_{11})| > \alpha + 1, \ |\mu_1(B)| \leq 1, \ \Sigma_{j=1}^n \big|\mu_1(e(P_j))\big| \leq 1,$$

hence

$$|\mu_1(P_{12})| = |\mu_1(B) - \mu_1(P_{11})| \geq |\mu_1(P_{11})| - |\mu_1(B)| > \alpha.$$

Moreover, it holds at least one of the equalities

$$\sup_{\mu \in M \cap \{e(D):D \in Q\}^\circ} \{|\mu|(P_{11})\} = \infty, \text{ for each finite subset } Q \in \mathcal{A}$$

or

$$\sup_{\mu \in M \cap \{e(D):D \in Q\}^\circ} \{|\mu|(P_{12})\} = \infty, \text{ for each finite subset } Q \in \mathcal{A}$$

If the first equality holds we define $B_1 := P_{12}$ and if this is not the case we take $B_1 := P_{11}$ to get this Claim. Proposition 3 follows from this observation.

Proposition 3. *Let $\mathcal{A}$ be an algebra and let M be a weak*-closed and absolutely convex subset of $ba(\mathcal{A})$. If there exists $B \in \mathcal{A}$ such that for each finite subset* Q *of* $\mathcal{A}$

$$\sup_{\mu \in M \cap \{e(A): A \in Q\}^{\circ}} |\mu|(B) = \infty,$$

then for each natural number $p > 1$, each $\alpha \in \mathbb{R}^+$ and each finite subset $\{P_i : 1 \leq i \leq n\}$ of $\mathcal{A}$ there exists a partition $\{B_i : B_i \in \mathcal{A}, 1 \leq i \leq p\}$ of B and a subset $\{\mu_i : 1 \leq i \leq p\}$ of M such that

$$|\mu_i(e(B_i))| > \alpha \quad \text{and} \quad \Sigma_{j=1}^{n} \big|\mu_i(e(P_j))\big| \leq 1, \;\; \text{for } 1 \leq i \leq p \tag{5}$$

Proof. We have seen that there exists in B a partition $\{B_1, B \backslash B_1\} \in \mathcal{A} \times \mathcal{A}$ and a measure $\mu_1 \in M$ such that

$$|\mu_1(e(B_1))| > \alpha, \;\; |\mu_1(e(B \backslash B_1))| > \alpha, \;\; \Sigma_{j=1}^{n} \big|\mu_1(e(P_j))\big| \leq 1$$

and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M \cap \{e(A): A \in Q\}^{\circ}} |\mu|(B \backslash B_1) = \infty.$$

This equality implies that there exists in $B \backslash B_1$ a subset $B_2 \in \mathcal{A}$ and a measure $\mu_2 \in M$ such that

$$|\mu_2(e(B_2))| > \alpha, \;\; |\mu_2(e(B \backslash (B_1 \cup B_2)))| > \alpha, \;\; \Sigma_{j=1}^{n} \big|\mu_2(e(P_j))\big| \leq 1$$

and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M \cap \{e(D): D \in Q\}^{\circ}} |\mu|(B \backslash (B_1 \cup B_2)) = \infty.$$

Repeating this method we get in $B \backslash (B_1 \cup B_2 \cup \cdots \cup B_{p-2})$ a partition $\{B_{p-1}, B \backslash (B_1 \cup B_2 \cup \cdots \cup B_{p-2} \cup B_{p-1})\} \in \mathcal{A} \times \mathcal{A}$ and a measure $\mu_{p-1} \in M$ such that

$$\big|\mu_{p-1}(e(B_{p-1}))\big| > \alpha, \;\; \big|\mu_{p-1}(e(B \backslash (B_1 \cup \cdots \cup B_{p-1})))\big| > \alpha, \;\; \Sigma_{j=1}^{n} \big|\mu_{p-1}(e(P_j))\big| \leq 1.$$

To finish the proof we define $B_p := B \backslash (B_1 \cup B_2 \cup \cdots \cup B_{p-2} \cup B_{p-1})$ and $\mu_p := \mu_{p-1}$. □

3. A Proof of the Web Nikodým Property of σ-Algebras

Let $\mathbb{N}^{<\infty} := \cup \{\mathbb{N}^s : s \in \mathbb{N}\}$ be the set of finite sequences of natural numbers, let $t = (t_1, t_2, \ldots, t_p)$ and $s = (s_1, s_2, \ldots, s_q)$ be two elements of $\mathbb{N}^{<\infty}$ and let T and U be two subsets of $\mathbb{N}^{<\infty}$. Then the element

$$(t, s) := (t_1, t_2, \ldots, t_p, s_1, s_2, \ldots, s_q)$$

is a proper continuation of t and the sets $(t, U) := \{(t, u) : u \in U\}$ and $(T, U) := \{(t, u) : t \in T, u \in U\}$ are named the concatenations of t and U, and T and U, respectively. The element $t(i) := (t_1, t_2, \ldots, t_i)$, if $1 \leqslant i \leqslant p$ and $t(i) := \emptyset$ if $i > p$, and the set $T(i) := \{t(i) : t \in T\}$ are named the sections of length i of t and T. A sequence $(t^n : n \in \mathbb{N})$ formed by elements of $\mathbb{N}^{<\infty}$ is a strand if $t^{n+1}(n) = t^n(n)$, for each $n \in \mathbb{N}$. For simplicity (t_1) will be represented by t_1 and when $U = \emptyset$ then, by agreement, $(t, U) = t$ and $(T, U) = T$.

A non-void subset U of $\mathbb{N}^{<\infty}$ is increasing at $t = (t_1, t_2, \ldots, t_p) \in \mathbb{N}^{<\infty}$ if there exists p scalars t_i^i verifying $t_i < t_i^i$, for $1 \leqslant i \leqslant p$, and $p - 1$ elements v^i of $\mathbb{N}^{<\infty}$, $1 \leq i < p - 1$, such that $(t_1^1, v^1) \in U$, $((t_1, t_2, \ldots, t_{i-1}, t_i^i), v^i) \in U$, $1 < i < p$, and $(t_1, t_2, \ldots, t_{p-1}, t_p^p) \in U(p)$. U is increasing (increasing respect to a subset V of $\mathbb{N}^{<\infty}$) if U is increasing at each $t \in U$ (at each $t \in V$). Clearly U is increasing if and only if for each $t = (t_1, t_2, \ldots, t_p) \in U$ the sets $U(1)$ and $\{n \in \mathbb{N} : (t(i), n) \in U(i+1)\}$, $1 \leqslant i < p$, are infinite.

The next definition provides a particular type of increasing subsets U of $\mathbb{N}^{<\infty}$ considered in [16], Definition 2, and [5], Definition 1, and named NV-trees, reminding O.M. Nikodým and M. Valdivia.

Definition 1. *An NV-tree is a non-void increasing subset T of $\mathbb{N}^{<\infty}$ without strands and such that every $t = (t_1, t_2, \ldots, t_p) \in T$ has no proper continuation in T.*

An NV-tree T is an infinite subset of $\mathbb{N}$ if and only if $T = T(1)$. Then it is said that T is an NV-tree trivial. The sets $\mathbb{N}^i$, $i \in \mathbb{N} \backslash \{1\}$, and the set $\cup\{(i, \mathbb{N}^i) : i \in \mathbb{N}\}$ are non trivial NV-trees.

If T is an increasing subset of $\mathbb{N}^{<\infty}$ and $\{B_u : u \in \mathbb{N}^{<\infty}\}$ is an increasing web in B then $(B_{u(1)})_{u \in T}$ is an increasing covering of B and for each $u = (u_1, u_2, \ldots, u_p) \in T$ and each $i < p$ the sequence $(B_{u(i)\times n})_{u(i)\times n \in T(i+1)}$ is an increasing covering of $B_{u(i)}$. In particular if T is an NV-tree then $B = \cup\{B_t : t \in T\}$ because T does not contain strands.

As every increasing subset S of an NV-tree T is an NV-tree, then we have that if $(S_n)_n$ is a sequence of non-void subsets of an NV-tree T such that each S_{n+1} is increasing with respect to S_n then $\cup_n S_n$ is an NV-tree. The following Proposition 4 may be found in ([5], Proposition 2) with a long detailed inductive proof. For the sake of completeness we present here only a sketch of its proof.

Proposition 4. *Let U be a subset of an NV-tree T. If U does not contain an NV-tree then $T \backslash U$ contains an NV-tree.*

Proof. This proposition is obvious if T is a trivial NV-tree, so we suppose that T is a non-trivial NV-tree.

By hypothesis on U there exists $m_1' \in T(1)$ such that for each $n_1 \geqslant m_1'$ the set $\{v \in \mathbb{N}^{<\infty} : (n_1, v) \in U\}$ does not contain an NV-tree. We define $Q_1 := \emptyset$ and $Q_1' := \{n_1 \in T(1) \backslash T : n_1 \geqslant m_1\}$.

Fix $n_1 \in Q_1'$ and then we have one of the following two possible cases:

- There exists $p_2 \in \mathbb{N}$ such that $(n_1, p_2) \in T$. As

$$\{v \in \mathbb{N}^{<\infty} : (n_1, v) \in U\} = \{v : (n_1, v) \in U\}$$

does not contain an NV-tree, there exists $m_2(n_1) \in \mathbb{N}$ such that

$$Q_2(n_1) = \{(n_1, n_2) \in T : n_2 \geqslant m_2(n_1)\} \subset T \backslash U.$$

In this case we define $Q_2'(n_1) = \emptyset$.
- Or for each $p_2 \in \mathbb{N}$ we have that $(n_1, p_2) \notin T$. Then there exists $m_2'(n_1) \in \mathbb{N}$ such that for each $n_2 \geqslant m_2'(n_1)$ the set $\{v \in \cup_s \mathbb{N}^s : ((n_1, n_2), v) \in U\}$ does not contain an NV-tree. In this case we define

$$Q_2'(n_1) = \{(n_1, n_2) \in T(2) : n_2 \geqslant m_2'(n_1)\}$$

and $Q_2(n_1) = \emptyset$.

We finish this second step of the inductive process defining

$$Q_2 = \cup\{Q_2(n_1) : n_1 \in Q_1'\} \text{ and } Q_2' = \cup\{Q_2'(n_1) : n_1 \in Q_1'\}.$$

The induction continues in an obvious way.

By construction $Q = \cup\{Q_i : i \in \mathbb{N}\} \subset T \backslash U$. As T does not have strands we have that the set Q is non-void. Moreover Q is increasing because $t = (t_1, t_2, \ldots, t_p) \in Q$ if and only if $t(i) \in Q'(i)$ for $1 \leq i < p$ and $t \in Q(p)$. Hence Q is an NV-tree contained in $T \backslash U$. □

Proposition 5 is a simplified version of Proposition 9 and 10 of [16].

Proposition 5. *Let $\{B, P_1, \ldots, P_r\}$ be a subset of an algebra $\mathcal{A}$ of subsets of Ω and $\{M_t : t \in T\}$ a family of absolutely convex subsets of $ba(\mathcal{A})$, indexed by a NV-tree T such that for each $t \in T$ and for each finite subset Q of $\mathcal{A}$*

$$\sup_{\mu \in M_t \cap \{e(A):A \in Q\}^\circ} \{|\mu|(B)\} = \infty. \tag{6}$$

Then for each positive real number α and each finite subset $\{t^j : 1 \leqslant j \leqslant k\}$ of T there exist k pairwise disjoint sets $B_j \in \mathcal{A}$ that are subsets of B, k measures $\mu_j \in M_{t^j}$, $1 \leqslant j \leqslant k$, and a NV-tree T^* such that:

$$\{t^j : 1 \leqslant j \leqslant k\} \subset T^* \subset T$$

and for each $t \in T^*$ and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M_t \cap \{e(A):A \in Q\}^\circ} \left\{ |\mu|(B \backslash \cup_{1 \leqslant j \leqslant k} B_j) \right\} = \infty,$$

$$\left|\mu_j(B_j)\right| > \alpha \text{ and } \Sigma\{\left|\mu_j(P_i)\right| : 1 \leqslant i \leqslant r\} \leqslant 1, \text{ for } j = 1, 2, \ldots, k.$$

Proof. Let $t^j := (t^j_1, t^j_2, \ldots, t^j_{p_j})$, for $1 \leqslant j \leqslant k$. By Proposition 3 applied to

$$B, \ \alpha, \ q = 2 + \Sigma_{1 \leqslant j \leqslant k} p_j \ \text{ and } \ M_{t^1}$$

there exist a partition $\{C_1, C_2, \cdots, C_q\}$ of B by elements of $\mathcal{A}$ and $\{\lambda_1, \lambda_2, \cdots, \lambda_q\} \subset M_{t^1}$ such that:

$$|\lambda_k(C_k)| > \alpha \quad \text{and} \quad \Sigma_{1 \leqslant i \leqslant r} |\lambda_k(P_i)| \leqslant 1, \quad \text{for} \quad k = 1, 2, \ldots, q. \tag{7}$$

Let t be an element of T. Equality (6) enables us to fix one element

$$C_{i_t} \in \{C_1, C_2, \cdots, C_q\}$$

such that for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M_t \cap \{e(A):A \in Q\}^\circ} \{|\mu|(C_{i_t})\} = \infty,$$

and then there exists a map $\varphi : T \to \{1, 2, \cdots, q\}$ defined by $\varphi(t) = i_t$ with the following properties:

1. $T = \cup\{\varphi^{-1}(i) : 1 \leqslant i \leqslant q\}$. Hence there exists $i_0 \in \{1, 2, \ldots, q\}$ and an NV-tree T_{i_0} contained in $\varphi^{-1}(i_0)$. Then for each finite subset Q of $\mathcal{A}$ and each $t \in T_{i_0}$

$$\sup_{\mu \in M_t \cap \{e(A):A \in Q\}^\circ} \left\{ |\mu|(C_{i_0}) \right\} = \infty.$$

2. Let $S := \{j : 1 \leqslant j \leqslant k, t^j = (t^j_1, t^j_2, \ldots, t^j_{p_j}) \notin T_{i_0}\}$. For each $j \in S$, the element $\varphi(t^j) = i^j \in \{1, 2, \ldots, q\}$. Hence for each finite subset Q of $\mathcal{A}$ we have

$$\sup_{\mu \in M_{t^j} \cap \{e(A):A \in Q\}^\circ} \{|\mu|(C_{i^j})\} = \infty.$$

3. For each $j \in S$ and for each m, $2 \leqslant m \leqslant p_j$, $1 \leqslant j \leqslant k$, the set $W^j_m := \{w \in \mathbb{N}^{<\infty} : t^j(m-1) \times w \in T\}$ is an NV-tree. The map $\varphi_{jm} : W^j_m \to \{1, 2, \cdots, q\}$ defined by

$$\varphi_{jm}(w) = \varphi(t^j(m-1) \times w)$$

verifies that $W_m^j = \cup\{\varphi_{jm}^{-1}(i) : 1 \leqslant i \leqslant q\}$; hence, there exists $i_m^j \in \{1, 2, \ldots, q\}$ and an NV-tree V_m^j contained in $\varphi_{jm}^{-1}(i_m^j)$ such that for each finite subset Q of $\mathcal{A}$ and each $v \in V_m^j$ we have

$$\sup_{\mu \in M_{t^j(m-1)\times v} \cap \{e(A):A\in Q\}^\circ} \left\{|\mu|(C_{i_m^j})\right\} = \infty.$$

The number of sets defining

$$D := C_{i_0} \cup (\cup\{C_{ij} \cup C_{i_m^j} : j \in S,\ 2 \leqslant m \leqslant p_j\})$$

is less or equal than $2 + \Sigma_{1 \leqslant j \leqslant k} p_j = q - 1$, hence there exists $C_h \in \{C_1, C_2, \cdots, C_q\}$ such that $D \subset B \backslash C_h$. By construction

$$T_1 := T_{i_0} \cup \{t^j : j \in S\} \cup \{t^j(m-1) \times V_m^j : j \in S, 2 \leqslant m \leqslant p_j\}$$

is an increasing subset of the NV-tree T, therefore T_1 is also an NV-tree in T. By preceding points 1, 2 and 3 for each $t \in T_1$ and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M_t \cap \{e(A):A\in Q\}^\circ} \{|\mu|(D)\} = \infty,$$

hence

$$\sup_{\mu \in M_t \cap \{e(A):A\in Q\}^\circ} \{|\mu|(B \backslash C_h)\} = \infty.$$

Inequalities (7) enable us to define $B_1 := C_h^1$ and $\mu_1 := \lambda_h$. With a repetition, changing B by $B \backslash B_1$, we get B_2, μ_2 and T_2. After k repetitions we get the proof with $T^* := T_k$. □

Lemma 1. *Let $\{\mathcal{B}_{n_1 n_2 \cdots n_m} : n_i \in \mathbb{N}, 1 \leq i \leq m, m \in \mathbb{N}\}$ be an increasing web of the algebra $\mathcal{A}$ such that for each sequence $(p_m)_{m=1}^\infty$ there exists $r \in \mathbb{N}$ such that $\mathcal{B}_{p_1 p_2 \cdots p_r}$ does not have property (N). If $\mathcal{A}$ has property (N) then there exists an NV-tree T such that for each $t = (t_1, t_2, \cdots, t_p) \in T$ there exists a $\tau_s(\mathcal{A})$-closed absolutely convex subset M_t of $ba(\mathcal{A})$ which is $\mathcal{B}_t$-pointwise bounded and for each finite subset Q of $\mathcal{A}$*

$$\sup_{\mu \in M_t \cap \{e(A):A\in Q\}^\circ} \{|\mu|(\Omega)\} = \infty.$$

Proof. The increasing sequence $\{\mathcal{B}_{n_1} : n_1 \in \mathbb{N}\}$ verifies one of the following two properties:

- Each $\mathcal{B}_{n_1}$, $n_1 \in \mathbb{N}$, does not have property (N). Then by Proposition 2 there exists a natural number p such that for each $n_1 \in \mathbb{N}$ with $p < n_1$ there exists a $\tau_s(\mathcal{A})$-closed absolutely convex subset M_{n_1} of $ba(\mathcal{A})$ which is $\mathcal{B}_{n_1}$-pointwise bounded and such that for each finite subset Q of $\mathcal{A}$ we have

$$\sup_{\mu \in M_{n_1} \cap \{e(A):A\in Q\}^\circ} \{|\mu|(\Omega)\} = \infty.$$

  Then let $T_1 := \{n_1 \in \mathbb{N} : n_1 > p\}$ and $T_1' := \emptyset$.
- Or there exists $m_1 \in \mathbb{N}$ such that $\mathcal{B}_{n_1}$ has property (N) for each $n_1 > m_1$.
  In this case we write $T_1 := \emptyset$ and $T_1' := \{n_1 \in \mathbb{N} : n_1 > m_1\}$ and we finish the first step of the proof.

  If $T_1' := \emptyset$ then the trivial NV-tree $T = T_1$ verifies this lemma.
  If $T_1' \neq \emptyset$ and $n_1 \in T_1'$ then the increasing sequence $\{\mathcal{B}_{n_1 n_2} : n_2 \in \mathbb{N}\}$ may have one of the two following two properties:

- Each $\mathcal{B}_{n_1 n_2}$, $n_2 \in \mathbb{N}$, does not have property (N). Again Proposition 2 implies that there exists $p(n_1) \in \mathbb{N}$ such that for each $n_2 \in \mathbb{N}$ with $p(n_1) < n_2$ there exists a $\tau_s(\mathcal{A})$-

closed absolutely convex subset $M_{n_1n_2}$ of $ba(\mathcal{A})$ which is $\mathcal{B}_{n_1n_2}$-pointwise bounded and such that for each finite subset Q of $\mathcal{A}$ we have

$$\sup_{\mu\in M_{(n_1,n_2)}\cap\{e(A):A\in Q\}^\circ}\{|\mu|(\Omega)\}=\infty.$$

Then we define $T_{n_1}:=\{(n_1,n_2):n_2>p(n_1)\}$ and $T'_{n_1}:=\emptyset$.

- Or there exists $m_2(n_1)\in\mathbb{N}$ such that for each natural number $n_2>m_2(n_1)$ the set $\mathcal{B}_{n_1n_2}$ has property (N).
  In this case let $T_{n_1}:=\emptyset$ and $T'_{n_1}:=\{(n_1,n_2):n_2>m_2(n_1)\}$.

  We finish this second step writing

$$T_2=\cup\{T_{n_1}:n_1\geqslant m_1\}\ \text{ and }\ T'_2=\cup\{T'_{n_1}:n_1\geqslant m_1\}.$$

Notice that if $(n_1,n_2)\in T_2$ we have determined a $\tau_s(\mathcal{A})$-closed absolutely convex subset $M_{n_1n_2}$ of $ba(\mathcal{A})$ which is $\mathcal{B}_{n_1n_2}$-pointwise bounded and such that for each finite subset Q of $\mathcal{A}$ we have that

$$\sup_{\mu\in M_{(n_1,n_2)}\cap\{e(A):A\in Q\}^\circ}\{|\mu|(\Omega)\}=\infty.$$

If $T'_2\neq\emptyset$ we continue the induction in a natural way because if $(n_1,n_2)\in T'_2$ considering as before the two cases in the increasing covering $\{\mathcal{B}_{n_1n_2n_3}:n_3\in\mathbb{N}\}$ of $\mathcal{B}_{n_1n_2}$ we get $T_{n_1,n_2}:=\{(n_1,n_2,n_3):n_3>p(n_1,n_2)\}$ and $T'_{n_1,n_2}:=\emptyset$ in one case and $T_{n_1,n_2}:=\emptyset$ and $T'_{n_1,n_2}:=\{(n_1,n_2,n_3):n_3>m_3(n_1,n_2)\}$ in the other case. For each $(n_1,n_2,n_3)\in T_3:=\cup\{T_{n_1,n_2}:(n_1,n_2)\in T'_2\}$ we have determined a $\tau_s(\mathcal{A})$-closed absolutely convex subset $M_{n_1n_2n_3}$ of $ba(\mathcal{A})$ which is $\mathcal{B}_{n_1n_2n_3}$-pointwise bounded and such that for each finite subset Q of $\mathcal{A}$ we have

$$\sup_{\mu\in M_{(n_1,n_2)}\cap\{e(A):A\in Q\}^\circ}\{|\mu|(\Omega)\}=\infty.$$

This third step finish writing $T'_3:=\cup\{T_{n_1,n_2}:(n_1,n_2)\in T'_2\}$. For brevity, we omit the clear and easy formalism of the induction.

Let $T:=\cup\{T_j:j\in\mathbb{N}\}$. If $T=T_1$ then T is a trivial NV-tree that verifies this lemma. If $T\neq T_1$, i.e., $T_1=\emptyset$, then according to the hypothesis of the increasing web

$$\{\mathcal{B}_{n_1n_2\cdots n_m}:n_i\in\mathbb{N},1\leq i\leq m,m\in\mathbb{N}\}$$

for each sequence $(p_m)_{m=1}^\infty$ there exists $r\in\mathbb{N}$ such that $\mathcal{B}_{p_1p_2\cdots p_r}$ does not have property (N), hence $T:=\cup\{T_j:j\in\mathbb{N}\}$ is a non-void subset of $\mathbb{N}^{<\infty}$ without strands. Moreover, by construction T is increasing, because if $t=(t_1,t_2,\cdots,t_h)\in T$ then $t(i)\in T'_i$, for $1\leq i<h$, and $t\in T_h$. Therefore T is an NV-tree that verifies the lemma, since, by construction, for each $t\in T$ there exists a $\tau_s(\mathcal{A})$-closed absolutely convex subset M_t of $ba(\mathcal{A})$ which is $\mathcal{B}_t$-pointwise bounded and such that for each finite subset Q of $\mathcal{A}$ we have obtained that

$$\sup_{\mu\in M_t\cap\{e(A):A\in Q\}^\circ}\{|\mu|(\Omega)\}=\infty.$$

□

Theorem 1. *A σ-algebra $\mathcal{S}$ of subsets of a set Ω has property wN.*

Proof. If the σ-algebra $\mathcal{S}$ does not have property wN then there exists in $\mathcal{S}$ an increasing web $\{\mathcal{B}_{m_1m_2\ldots m_p}:p,m_1,m_2,\cdots,m_p\in\mathbb{N}\}$ such that for each sequence $(m_p:p\in\mathbb{N})$ there exists q such that $\mathcal{B}_{m_1m_2\ldots m_q}$ does not have property (N). By Nikodým theorem $\mathcal{S}$ has property (N), hence by Lemma 1 there exists an NV-tree T such that for each

$t = (m_1, m_2, \cdots, m_p) \in T$ there exists a $\tau_s(\mathcal{A})$-closed absolutely convex subset M_t of $ba(\mathcal{S})$ which is $\mathcal{B}_t$-pointwise bounded and such that for each finite subset Q of $\mathcal{S}$

$$\sup_{\mu \in M_t \cap \{e(A):A \in Q\}^\circ} \{|\mu|(\Omega)\} = \infty.$$

By induction on j we will determine a strictly increasing sequence of natural numbers $(k_j)_j$, with $k_1 = 1$, and a countable NV-tree

$$T^* = \{t^1, t^2, \cdots, t^{k_2}, t^{k_2+1}, \cdots, t^{k_3}, \cdots\}$$

contained in T such that for each $(i, j) \in \mathbb{N}^2$ with $i \leqslant k_j$ there exists a set $B_{ij} \in \mathcal{S}$ and $\mu_{ij} \in M_{t^i}$ that verify

$$\Sigma_{s,v}\{|\mu_{ij}(B_{sv})| : s \leqslant k_v,\ 1 \leqslant v < j\}) < 1, \tag{8}$$

$$|\mu_{ij}(B_{ij})| > j, \tag{9}$$

and $B_{ij} \cap B_{i'j'} = \emptyset$ if $(i, j) \neq (i', j')$.

In fact, select the number $k_1 = 1$ and an element $t^1 \in T$. Proposition 5 with $B := \Omega$ and $\alpha = 1$ provides $B_{11} \in \mathcal{S}$, $\mu_{11} \in M_{t^1}$ and an NV-tree T_1 such that $|\mu_{11}(B_{11})| > 1$, $t^1 \in T_1 \subset T$ and for each finite subset Q of $\mathcal{A}$ and each $t \in T_1$

$$\sup_{\mu \in M_t \cap \{e(A):A \in Q\}^\circ} \{|\mu|(\Omega)\} = \infty.$$

The first induction step finishes writing $k_1 := 1$, $S^1 := \{t^1\}$ and $B^1 := B_{11}$.

Let us suppose that for $j = 1, 2, \cdots, n$ we have obtained the natural numbers $k_1 < k_2 < k_3 < \ldots < k_n$, the NV-trees $T_1 \supset T_2 \supset T_3 \supset \ldots \supset T_n$, the elements $\{t^1, t^2, \ldots, t^{k_n}\}$, the measures $\mu_{ij} \in M_{t^i}$ and the pairwise disjoint elements $B_{ij} \in \mathcal{S}$, $i \leqslant k_j$ and $j \leqslant n$, such that

1. $S^j := \{t^i : i \leqslant k_j\} \subset T_j$ and $S_j := \{t^{k_{j-1}+1}, \ldots, t^{k_j}\}$ has the increasing property respect to S^{j-1}, for each $1 < j \leqslant n$,
2. $|\mu_{ij}(B_{ij})| > j$ and $\Sigma_{s,v}\{|\mu_{ij}(B_{sv})| : s \leqslant k_v, 1 \leqslant v < j\} < 1$, if $i \leqslant k_j$ and $j \leqslant n$,
3. and the union $B^j := \cup\{B_{sv} : s \leqslant k_v, 1 \leqslant v \leqslant j\}$ verifies that for each finite subset Q of $\mathcal{A}$ and for each t belonging to the NV-tree T_j, for each $j < n$.

$$\sup_{\mu \in M_t \cap \{e(A):A \in Q\}^\circ} \left\{|\mu|(\Omega \backslash B^j)\right\} = \infty,$$

To finish the induction procedure select in $T_n \backslash \{t^i : i \leqslant k_n\}$ a subset $S_{n+1} := \{t^{k_n+1}, \ldots, t^{k_{n+1}}\}$ that has the increasing property respect to S^n. Then Proposition 5 with

1. $B = \Omega \backslash B^n$,
2. $\{P_1, \cdots, P_r\} = \{B_{sv} : s \leqslant k_v, 1 \leqslant v \leqslant n\}$,
3. the NV-tree T_n,
4. $\alpha = n + 1$
5. and the finite subset $S^{n+1} := \{t^i : i \leqslant k_{n+1}\}$ of T_n,

provides k_{n+1} pairwise disjoint sets $B_{in+1} \in \mathcal{A}$ that are subsets of $\Omega \backslash B^n$, k_{n+1} measures $\mu_{in+1} \in M_{t^i}$, $1 \leq i \leqslant k_{n+1}$, and an NV-tree T_{n+1} such that

1. $S^{n+1} \subset T_{n+1} \subset T_n$ and for each $t \in T_{n+1}$ and for each finite subset Q of $\mathcal{A}$ the set M_t verifies that

$$\sup_{\mu \in M_t \cap \{e(A):A \in Q\}^\circ} \left\{|\mu|(\Omega \backslash B^{n+1})\right\} = \infty, \quad \text{with } B^{n+1} = B^n \cup \{B_{in+1} : i \leqslant k_{n+1}\},$$

2. $|\mu_{in+1}(B_{in+1})| > n+1$ and $\Sigma_{s,v}\{|\mu_{in+1}(B_{sv})| : s \leqslant k_v, 1 \leqslant v \leqslant n\} < 1$, for $1 \leq i \leqslant k_{n+1}$.

From the increasing property of $S_{j+1} := \{t^{k_j+1}, \ldots, t^{k_{j+1}}\}$ with respect to

$$S^j := \{t^i : i \leqslant k_j\},$$

for each $j \in \mathbb{N}$, we get that $T^* := \{t^i : i \in \mathbb{N}\}$ is an increasing subset of the NV-tree T. Therefore T^* is an NV-tree.

We claim that there exists a sequence $(i_n, j_n)_{n\in\mathbb{N}}$ such that $(i_n)_{n\in\mathbb{N}}$ is the sequence of first components of the sequence obtained ordering the elements of $\mathbb{N}^2$ by the diagonal order, i.e.,

$$(i_1, i_2, i_3, i_4, i_5, i_6, i_7, \cdots) = (1, 1, 2, 1, 2, 3, 1, \cdots),$$

and $(j_n)_{n\in\mathbb{N}}$ is a strict increasing sequence such that for each $n \in \mathbb{N}$

$$|\mu_{i_n,j_n}|(\cup\{B_{i_m,j_m} : m > n\}) \leq 1.$$

As the sequence $(k_n : n \in \mathbb{N})$ is increasing then for each increasing sequence $(j_n : n \in \mathbb{N})$ we have $i_n \leq j_n \leq k_{j_n}$, $n \in \mathbb{N}$. To get the increasing sequence $(j_n)_{n\in\mathbb{N}}$ we fix $(i_1, j_1) := (1, 1)$ and if $|\mu_{i_1,j_1}| \leq h_1 \in \mathbb{N}$ then we split the set $\{j \in \mathbb{N} : j > 1\}$ in h_1 infinite subsets $N_{11}, \cdots, N_{1h_1}$. At least one of this subsets, named N_1, verifies that

$$|\mu_{i_1,j_1}|(\cup\{B_{i,j} : i \leq k_j, j \in N_1\}) \leq 1,$$

because

$$h_1 \geq |\mu_{i_1,j_1}| \geq \Sigma_{1\leq r\leq h_1}|\mu_{i_1,j_1}|(\cup\{B_{i,j} : i \leq k_j, j \in N_{1r}\}).$$

Then we define $j_2 := \inf\{n : n \in N_1\}$.

Suppose that we have obtained the natural number j_n and the infinite subset N_n of the set $\{j \in \mathbb{N} : j > j_n\}$ such that

$$|\mu_{i_n,j_n}|(\cup\{B_{i,j} : i \leq k_j, j \in N_n\}) \leq 1.$$

Then we define $j_{n+1} = \inf\{n : n \in N_n\}$ and if $|\mu_{i_{n+1},j_{n+1}}| \leq h_{n+1}$ we split the set $\{j \in N_n : j > j_{n+1}\}$ in h_{n+1} infinite subsets $N_{n+1,1}, \cdots, N_{n+1,h_{n+1}}$. At least one of this subsets, named N_{n+1} verifies that

$$|\mu_{i_{n+1},j_{n+1}}|(\cup\{B_{i,j} : i \leq k_j, j \in N_{n+1}\}) \leq 1$$

because

$$h_{n+1} \geq |\mu_{i_{n+1},j_{n+1}}| \geq \Sigma_{1\leq r\leq h_{n+1}}|\mu_{i_{n+1},j_{n+1}}|(\cup\{B_{i,j} : i \leq k_j, j \in N_{n+1,r}\}).$$

The relation

$$B = \cup\{B_{i_m,j_m} : m \in \mathbb{N}\} \in \mathcal{S} \tag{10}$$

and the property that $T^* := \{t^i : i \in \mathbb{N}\}$ is an NV-tree imply that there exists $r \in \mathbb{N}$ such that $B \in \mathcal{B}_r$. By construction there exists an increasing sequence $(m_s : s \in \mathbb{N})$ such that each $i_{m_s} = r$, $s \in \mathbb{N}$. Therefore the set of measures $\{\mu_{i_{m_s},j_{m_s}} : s \in \mathbb{N}\} = \{\mu_{r,j_{m_s}} : s \in \mathbb{N}\}$ is a subset of M_r, that it is pointwise bounded in $\mathcal{B}_r$. In particular

$$\sup\{|\mu_{i_{m_s},j_{m_s}}(B)| : s \in \mathbb{N}\} = \sup\{|\mu_{r,j_{m_s}}(B)| : s \in \mathbb{N}\} < \infty. \tag{11}$$

But from

$$|\mu_{i_{m_s},j_{m_s}}(B)| = \left|\mu_{i_{m_s},j_{m_s}}(\bigcup_{m\in\mathbb{N}} B_{i_m,j_m})\right| \geq$$

$$\geq \left|\mu_{i_{m_s},j_{m_s}}(B_{i_{m_s},j_{m_s}})\right| - \sum_{1\leq k\leq m<j_{m_s}} \left|\mu_{i_{m_s},j_{m_s}}(B_{km})\right| - \left|\mu_{i_{m_s},j_{m_s}}\right|\left(\bigcup_{m>j_{m_s}} B_{i_m,j_m}\right) > j_{m_s} - 2$$

we get that $\lim_{s\to\infty}\left|\mu_{i_{m_s},j_{m_s}}(B)\right| = \infty$, in contradiction with (11). □

Remark 2. *In the preceding proof it has been used the fact that the* $B = \cup\{B_{i_m,j_m} : m \in \mathbb{N}\} \in \mathcal{S}$ *in* (10) *to get the final contradiction. In [20], Theorem 1, it is proved that the algebra* $\mathcal{J}([0,1]^p)$ *of Jordan measurable subsets of* $[0,1]^p$ *has the property* (wN) *and the construction is made selecting a sequence of sets* $(B_{i_m,j_m} : m \in \mathbb{N})$ *of* $\mathcal{J}([0,1]^p)$ *such that the* $\bigcup_{m\in\mathbb{N}} B_{i_m,j_m} \in \mathcal{J}([0,1]^p$.

Remark 3. *In [8], Theorem 3.3, it is given a class of ring of subsets that have the property* (N) *if they have the property* (wN)*, i.e., for this class of rings the Valdivia problem in [15]* ($(N) \Rightarrow (sN)$?) *has a positive answer.*

Recall that a family $\mathcal{R}$ of subsets of a set Ω is a ring if $\emptyset \in \mathcal{R}$ and for every $(A,B) \in \mathcal{R}^2$ we have that $A\backslash B \in \mathcal{R}$ and $A \cup B \in \mathcal{R}$. Hence a ring $\mathcal{A}$ of subsets of a set Ω is an algebra if and only if $\Omega \in \mathcal{A}$.

The fact that the known algebras with property (sN) have the property (wN) suggests the following problem as a natural complement to before mentioned Valdivia problem [15].

Problem 1. *Is it true that in an algebra* $\mathcal{A}$ *property* (sN) *implies property* (wN)*?*

4. Sets with (wG) and $(wVHS)$ Properties

Let F be a subset of a Banach space E and let $(\mu_n)_{n=1}^{\infty}$ be a bounded sequence in its dual E^*. Then the F-pointwise convergence of the sequence to μ implies the $\overline{F}$-pointwise convergence of the sequence to μ. In fact, fix $\epsilon > 0$ and $v \in \overline{F}$, then by hypothesis there exists $f \in F$ such that $\|v - f\| < \epsilon(2(1 + |\mu| + \sup_n|\mu_n|)^{-1}$, and, again by hypothesis, for this f there exists n_ϵ such that $|(\mu_n - \mu)(f)| < 2^{-1}\epsilon$, for every $n > n_\epsilon$. Hence for $n > n_\epsilon$ we have that

$$|(\mu_n - \mu)(v - f)| + |(\mu_n - \mu)(f)| < \frac{\epsilon(|\mu| + \sup_n|\mu_n|)}{2(1 + |\mu| + \sup_n|\mu_n|} + \frac{\epsilon}{2} \leq \epsilon,$$

hence

$$|(\mu_n - \mu)(v)| \leq |(\mu_n - \mu)(v - f)| + |(\mu_n - \mu)(f)| \leq \epsilon,$$

so $(\mu_n)_{n=1}^{\infty}$ converges pointwise to μ in $\overline{F}$.

In particular, if an algebra $\mathcal{A}$ contains a subset $\mathcal{B}$ with property (N) then each sequence $(\mu_n)_{n=1}^{\infty}$ of $ba(\mathcal{A})$ that converges $\mathcal{B}$-pointwise to μ is bounded and for each $f \in L_\infty(\mathcal{A})$ we have that

$$\lim_{n\to\infty} \mu_n(f) = \mu(f) \tag{12}$$

In fact, as $\mathcal{B}$ has property (N) and the sequence $(\mu_n)_{n=1}^{\infty}$ is $\mathcal{B}$-pointwise bounded, the sequence $(\mu_n)_{n=1}^{\infty}$ is bounded in $ba(\mathcal{A})$. The norm boundedness of the sequence $(\mu_n)_{n=1}^{\infty}$, the equality $\overline{\text{span}\{e(A) : A \in \mathcal{B}\}}^{L_\infty(\mathcal{A})} = L_\infty(\mathcal{A})$ deduced from Proposition 1 and the hypothesis that $\lim_{n\to\infty} \mu_n(C) = \mu(C)$, for each $C \in \mathcal{B}$, that it is equivalent to $\lim_{n\to\infty} \mu_n(f) = \mu(f)$, for each $f \in \text{span}\{e(A) : A \in \mathcal{B}\}$, imply (12) for each $f \in L_\infty(\mathcal{A})$.

This result implies easily the following Proposition 6.

Proposition 6. *A subset* $\mathcal{B}$ *of an algebra* $\mathcal{A}$ *has property* (VHS) *if and only if* $\mathcal{B}$ *has property* (N) *and* $\mathcal{A}$ *has property* (G).

Proof. We have seen that if $\mathcal{B}$ has property (N) and $(\mu_n)_{n=1}^{\infty}$ is a sequence of $ba(\mathcal{A})$ that converges $\mathcal{B}$-pointwise to μ then the sequence $(\mu_n)_{n=1}^{\infty}$ is bounded and $(\mu_n(B))_{n=1}^{\infty}$ converges to $\mu(B)$ for each $B \in \mathcal{A}$. If additionally $\mathcal{A}$ has property (G) then $(\mu_n)_{n=1}^{\infty}$ converges weakly to μ, i.e., $(\psi(\mu_n))_{n=1}^{\infty}$ converges weakly to $\psi(\mu)$, for each $\psi \in ba(\mathcal{A})^*$. Therefore $\mathcal{B}$ has property (VHS).

To prove the converse, let's suppose that a subset $\mathcal{B}$ of an algebra $\mathcal{A}$ has property (VHS). It is obvious that $\mathcal{B}$ has property (G) and then $\mathcal{A}$ has property (G). Moreover, if $(\mu_n)_{n=1}^{\infty}$ is a sequence of $ba(\mathcal{A})$ that is $\mathcal{B}$-pointwise bounded then for every scalar sequence $(\epsilon_n)_{n=1}^{\infty}$ that converges to 0 we have that

$$\lim_{n\to\infty} \epsilon_n \mu_n(B) = 0,$$

for each $B \in \mathcal{B}$, hence as $\mathcal{B}$ has property (VHS) we have that the sequence $(\epsilon_n \mu_n)_{n=1}^{\infty}$ converges weakly to the null measure, implying that $\{\mu_n : n \in \mathbb{N}\}$ is a bounded subset of $ba(\mathcal{A})$. Therefore $\mathcal{B}$ has the property (N). □

Proposition 6 for $\mathcal{B} = \mathcal{A}$ says that an algebra $\mathcal{A}$ has property (VHS) if and only if $\mathcal{A}$ has the properties (N) and (G). In [14] (page 6, lines 23 and 24) it says that Diestel, Faires and Huff obtained this equivalence in his 1976 preprint paper Convergence and boundedness of measures on non-sigma complete algebras. It seems that this preprint has never been published, but it is cited in many other papers, for instance in [28] (reference 9 in page 113).

Corollary 1. *A subset $\mathcal{B}$ of an algebra $\mathcal{A}$ has property $(sVHS)$ if and only if $\mathcal{B}$ has property (sN) and $\mathcal{A}$ has property (G).*

Proof. Let $(\mathcal{B}_n : n \in \mathbb{N})$ be an increasing covering of $\mathcal{B}$.

If $\mathcal{B}$ has property $(sVHS)$ there exists $p \in \mathbb{N}$ such that $\mathcal{B}_p$ has property (VHS). By Proposition 6 $\mathcal{B}_p$ has property (N) and $\mathcal{A}$ has property (G). Hence $\mathcal{B}$ has property (sN).

The converse follows from the observation that if $(\mathcal{B}_n : n \in \mathbb{N})$ contains a set $\mathcal{B}_q$ with property (N) and $\mathcal{A}$ has property (G), then, by Proposition 6, $\mathcal{B}_q$ has property (VHS), so $\mathcal{B}$ has property $(sVHS)$. □

Corollary 2. *A subset $\mathcal{B}$ of an algebra $\mathcal{A}$ has property $(wVHS)$ if and only if $\mathcal{B}$ has property (wN) and $\mathcal{A}$ has property (G).*

Proof. Let $\{\mathcal{B}_{n_1 n_2 \cdots n_m} : n_i \in \mathbb{N}, 1 \le i \le m, m \in \mathbb{N}\}$ be an increasing web of $\mathcal{B}$.

If $\mathcal{B}$ has property $(wVHS)$ then there exists a sequence $(p_i : i \in \mathbb{N})$ such that $\mathcal{B}_{p_1 p_2 \cdots p_m}$ has property (VHS) for every $m \in \mathbb{N}$. By Proposition 6 each $\mathcal{B}_{p_1 p_2 \cdots p_m}$, $m \in \mathbb{N}$, has properties (N) and (G). Hence $\mathcal{B}$ has property (wN) and $\mathcal{A}$ has property (G).

The converse follows from the observation that if $\mathcal{A}$ has property (G) and for the increasing web $\{\mathcal{B}_{n_1 n_2 \cdots n_m} : n_i \in \mathbb{N}, 1 \le i \le m, m \in \mathbb{N}\}$ there exists a sequence $(q_i : i \in \mathbb{N})$ such that each $\mathcal{B}_{q_1 q_2 \cdots q_m}$, $m \in \mathbb{N}$, has property (N), then Proposition 6 implies that each $\mathcal{B}_{q_1 q_2 \cdots q_m}$, $m \in \mathbb{N}$ has property (VHS) property, hence $\mathcal{B}$ has property $(wVHS)$. □

5. Conclusions

Let $\mathcal{A}$ be an algebra and let M be a subset of $ba(\mathcal{A})$. By the Banach–Steinhaus theorem the inequalities

$$\sup\{\mu(f) : \mu \in M\} < \infty, \text{ for each } f \in L_\infty(\mathcal{A}),$$

imply that M is a bounded subset of $ba(\mathcal{A})$. According to the Nikodým–Grothendieck theorem each σ-algebra $\mathcal{S}$ has property (N), i.e., if a subset M of $ba(\mathcal{S})$ verifies the inequalities

$$\sup\{\mu(C) : \mu \in M\} < \infty, \text{ for each } C \in \mathcal{S},$$

then M is a bounded subset of $ba(\mathcal{S})$. This theorem is considered in ([29], Page 309) as a "striking improvement of the Banach–Steinhaus theorem of uniform boundedness". In the frame of locally convex barrelled spaces Nikodým-Grothendieck theorem has been improved in [3,17], obtaining that every σ-algebra $\mathcal{S}$ has properties (sN) and (wN), respectively, and both properties enable us to get new results in Functional Analysis and Measure theory.

There exists algebras with property (N) and algebras without property (N). It is unknown if property (N) in an algebra implies property (sN) ([15], Problem 1) and it is also unknown if property (sN) in an algebra implies property (wN) (see Problem 1).

As a step to solve these two open problems we have provide in Section 3 a proof of the web Nikodým property of σ-algebras which only depends on elementary basic results of Measure theory and Banach space theory. Positive solutions of this two open questions would provide new progress in Functional Analysis and Measure theory and will allow to extend results for σ-algebra $\mathcal{S}$ to results for algebras.

Moreover the results in Section 4 imply that if $\mathcal{A}$ is an algebra with property (VHS) then $\mathcal{A}$ has property $(sVHS)$ $[(wVHS)]$ if and only if $\mathcal{A}$ has property (sN) $[(wN)]$. Therefore the two above-mentioned open problems have an equivalent formulation for algebras with property (VHS).

Author Contributions: The authors (S.L.-A., M.L.-P., S.M.-L.) contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded for the second named author by grant PGC2018-094431-B-I00 of Ministry of Science, Innovation and Universities of Spain.

Acknowledgments: To Professor Manuel Valdivia (1928–2014), whose 2013 paper On Nikodým boundedness property, (see [15]) give us many suggestions for this work, and to Professor Juan Carlos Ferrando for his many interesting comments and for the detailed revision of the final version of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

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