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ON SOLID CORES AND HULLS OF WEIGHTED BERGMAN SPACES A^1_μ

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Abstract
We consider weighted Bergman spaces A^1_μ on the unit disc as well as the corresponding spaces of entire functions, defined using non-atomic Borel measures with radial symmetry. By extending the techniques from the case of reflexive Bergman spaces, we characterize the solid core of A^1_μ. Also, as a consequence of a characterization of solid A^1_μ-spaces, we show that, in the case of entire functions, there indeed exist solid A^1_μ -spaces. The second part of the article is restricted to the case of the unit disc and it contains a characterization of the solid hull of A^1_μ, when μ equals the weighted Lebesgue measure with the weight v. The results are based on the duality relation of the weighted A^1- and H^∞-spaces, the validity of which requires the assumption that $-\log v$ belongs to the class $\mathcal{W}_0$, studied in a number of publications; moreover, v has to satisfy the condition (b), introduced by the authors. The exponentially decreasing weight $v(z) = \exp(-1/(1-|z|)$ provides an example satisfying both assumptions.

Keywords Bergman space · Weighted L^1-norm · Unit disc · Solid hull · Solid core

Introduction and preliminaries

The solid hulls and cores of spaces of analytic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ or the entire plane $\mathbb{C}$ have been investigated by many authors. We refer the reader to the recent books [12] and [16] and the many references therein. In the series of articles [2–5], the authors have presented the solid hulls and cores of the weighted H^∞-spaces H^∞_v on $\mathbb{D}$ or $\mathbb{C}$ for a large class of radial weights v as well as their Bergman space analogues A^p_μ for $1 < p < \infty$. Earlier, the cases of standard weights and $d\mu(r) = (1-r)^\alpha dr$, $\alpha > 0$, were considered in [1] and [12].
In this part, we want to extend the results of [5] to weighted Bergman spaces A^p_μ for $p = 1$. The spaces are defined on the unit disc $\mathbb{D}$ or on the entire plane. (*Fock spaces* are usually considered the Bergman space analogues of spaces of entire functions, but these are defined with Gaussian weight functions, which is not required here. Thus, here we keep the term Bergman space also for the entire functions.) Consider $R = 1$ or $R = \infty$. We study holomorphic functions $f : R \cdot \mathbb{D} \to \mathbb{C}$ where $R \cdot \mathbb{D} = \mathbb{D}$ if $R = 1$ and $R \cdot \mathbb{D} = \mathbb{C}$ if $R = \infty$. Let $\hat{f}(k)$ be the Taylor coefficients of f, i.e., $f(z) = \sum_{k=0}^\infty \hat{f}(k) z^k$. We

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take a non-atomic positive bounded Borel measure μ on $[0, R[$ such that $\mu([r, R[) > 0$ for every $r > 0$ and $\int_0^R r^n d\mu(r) < \infty$ for all $n > 0$. Put, for $1 \leq p < \infty$,

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^R \int_0^{2\pi} |f(re^{i\varphi})|^p d\varphi d\mu(r) \right)^{1/p}$$

and let

$$A^p_\mu = \{f : R \cdot \mathbb{D} \to \mathbb{C} : f \text{ holomorphic with } \|f\|_p < \infty\}.$$

We will also consider the weighted spaces

$$H^\infty_v = \{f : \mathbb{D} \to \mathbb{C} \text{ holomorphic} : \|f\|_v = \sup_{z \in \mathbb{D}} v(z)|f(z)| < \infty\},$$

where the weight $v : \mathbb{D} \to (0, \infty)$ is a continuous and radial ($v(z) = v(|z|)$) function which is decreasing with respect to $r = |z|$, and $\lim_{r \to 1^-} v(r) = 0$.
Let A be a vector space of holomorphic functions on $R \cdot \mathbb{D}$ containing the polynomials. The *solid core* is defined as

$$s(A) = \{f \in A : g \in A \text{ for all holomorphic } g \text{ with } |\hat{g}(k)| \leq |\hat{f}(k)| \text{ for all } k\}$$

and the *solid hull* as

$$S(A) = \{g : \mathbb{D} \to \mathbb{C} \text{ holomorphic} : \text{there is } f \in A \text{ with } |\hat{g}(k)| \leq |\hat{f}(k)| \text{ for all } k\}.$$

The space A is called *solid* if $A = S(A)$. The concept of a solid hull will also be discussed at the beginning of Section "On solid hulls."
Here, in Theorem 2.4, we will transfer Theorem 4.1 of [5] to the case $p = 1$. This result concerns the characterization of solid Bergman spaces A^1_μ, and it is motivated by the fact that such spaces indeed exist in the case $R = \infty$ only (see Example 2.7 and Corollary 2.6). In Theorem 2.8, we determine the solid cores for all Bergman spaces A^1_μ.
In Section "On solid hulls," we also present how the duality theory can be used for new results on certain solid hulls (see the beginning of Section "On solid hulls" for detailed definitions). In particular, we construct the solid hull $S_{BK}(A^1_\mu)$ of A^1_μ for $R = 1$ by using the known solid core of the space H^∞_v in [4]. This result is more special than the previous one for solid cores since we need to restrict to the case μ is the weighted Lebesgue measure $d\mu = vdA = v\pi^{-1} r dr d\varphi$, where the weight v needs to satisfy some special assumptions in addition to those mentioned above. Examples of such weights include important cases like exponentially decreasing weights.
For a holomorphic g and $r > 0$, we define

$$M_p(g, r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\varphi})|^p d\varphi \right)^{1/p}$$

and denote the Dirichlet projections by $P_n g(z) = \sum_{k=0}^n \hat{g}(k) z^k$, $n \in \mathbb{N}$. It is well known that, for $1 < p < \infty$, there are constants $c_p > 0$, not depending on g, n or r, such that $M_p(P_n g, r) \leq c_p M_p(g, r)$. Moreover, we have $\lim_{n\to\infty} M_p(g - P_n g, r) = 0$. Hence, we obtain

$$\|P_n f\|_p \leq c_p \|f\|_p \text{ for all } f \in A^p_\mu \text{ and all } n \text{ and } \lim_{n\to\infty} \|f - P_n f\|_p = 0.$$

In particular, we see that the monomials $z \mapsto z^n$, $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\} = \mathbb{N} \cup \{0\}$, form a (Schauder) basis of A^p_μ if $1 < p < \infty$. On the other hand, denoting by H^1 the Hardy space of all holomorphic functions on $\mathbb{D}$ which are bounded under $\sup_{0 \leq r < 1} M_1(\cdot, r)$, it is well known that the operator norm of $P_n : H^1 \to H^1$ tends to infinity as $n \to \infty$. (See details in [9] and [17].) For the terminology and definitions on bases in Banach spaces, see also [13].
In the remaining part of the article, $[r]$ denotes the largest integer smaller or equal to $r > 0$. By c, c_1, c_2, C, C', etc., we denote generic positive constants, the actual value of which may vary depending on the place.

Solid core and examples of solid A^1_μ-spaces

In this section, we extend Theorem 4.1 of [5] concerning the characterization of solid Bergman spaces to the case $p = 1$ and also determine the solid cores for all spaces A^1_μ. We consider both cases $R = 1$ and $R = \infty$ unless otherwise specified. First, we recall a fundamental result from [10], which concerns equivalent representations of the norm of the space A^1_μ.

Theorem 2.1 *There are sequences $0 < s_1 < s_2 < \ldots < R$ and $0 = m_0 < m_1 < m_2 < \ldots$, non-negative numbers d_n, $t_{n,k}$ (with $n \in \mathbb{N}$ and $[m_{n-1}] < k \le [m_{n+1}]$) and constants $c_1, c_2 > 0$ such that for all $g(z) = \sum_{k=0}^\infty \alpha_k z^k$ we have*

$$c_1\|g\|_1 \le \sum_{n=0}^\infty M_1(T_n g, s_n)d_n \le c_2\|g\|_1, \tag{2.1}$$

where

$$T_n g = \sum_{[m_{n-1}]+1}^{[m_{n+1}]} t_{n,k}\alpha_k z^k. \tag{2.2}$$

We will need the following consequence of this result.

Corollary 2.2 *Let $(n_j)_{j=1}^\infty$ be an increasing sequence of indices such that $n_{j+1} - n_j \ge 2$ for all j and let $h_j(z) = \sum_{k=[m_{n_j}]+1}^{[m_{n_j+1}]} \alpha_k z^k$ be a polynomial. We have*

$$\|h\|_1 \le \sum_{j=0}^\infty \|h_j\|_1 \le C\|h\|_1 \text{ for all } h = \sum_{j=1}^\infty h_j \in A^1_\mu. \tag{2.3}$$

Proof Applying (2.1) to h_j yields that $\|h_j\|_1$ and

$$M_1(T_{n_j}h_j, s_{n_j})d_{n_j} + M_1(T_{n_j+1}h_j, s_{n_j+1})d_{n_j+1}$$

are proportional quantities. Moreover, $T_n h = 0$, if n is not equal to n_j or $n_j + 1$ for any j, and

$$T_{n_j}h = T_{n_j}h_j,\ \ T_{n_j+1}h = T_{n_j+1}h_j \text{ for all } j. \tag{2.4}$$

Hence, by another application of (2.1),

$$\|h\|_1 \le \sum_{j=0}^\infty \|h_j\|_1 \le C\sum_{j=0}^\infty \left(M_1(T_{n_j}h_j, s_{n_j})d_{n_j} + M_1(T_{n_j+1}h_j, s_{n_j+1})d_{n_j+1}\right)$$

$$= C\sum_{n=0}^\infty M_1(T_n h, s_n)d_n \le C'\|h\|_1.$$

Let us make a remark concerning the numbers and constants in the above results.

Remark 2.3 1°. Theorem 2.1 is a reformulation of Theorem 1.3 of [10], where the sequences $(s_n)_{n=1}^\infty$ and $(m_n)_{n=0}^\infty$ were chosen, by using induction, such that, for all $n \in \mathbb{N}$,

$$\int_0^{s_n} r^{m_n}d\mu = b\int_{s_n}^R r^{m_n}d\mu \quad \text{and} \quad \int_0^{s_n} r^{m_{n+1}}d\mu = \int_{s_n}^R r^{m_{n+1}}d\mu.$$

where $b > 5$ is some constant. Then, the numbers d_n were set to be

$$d_n = \left(\int_0^{s_n} \left(\frac{r}{s_n}\right)^{m_n} d\mu + \int_{s_n}^{R} \left(\frac{r}{s_n}\right)^{m_{n+1}} d\mu \right). \tag{2.5}$$

As proven in Section 5 of [10], it is always possible to find these sequences, although calculating them exactly for given concrete weights seems to be difficult in general.

2°. If $R = 1$ and $d\mu = rv(r)drd\theta$ with $v(r) = \exp\left(-\alpha(1-r^{\ell})^{-\beta}\right)$ for some constants $\alpha, \beta, \ell > 0$, then the numbers m_n and s_n, $n \in \mathbb{N}$, $(m_0 = 0)$, were calculated by a different method than in 1° in Propositions 3.1 and 3.3.(ii) of [6]:

$$m_n = \ell\beta^2\left(\frac{\beta}{\alpha}\right)^{1/\beta} n^{2+2/\beta} - \ell\beta^2 n^2 \quad \text{and} \quad s_n = \left(1 - \left(\frac{\alpha}{\beta}\right)^{1/\beta} n^{-2/\beta}\right)^{1/\ell}. \tag{2.6}$$

3°. In the citations mentioned in 1° and 2°, the numbers $t_{n,k}$ were chosen as the coefficients of certain de la Valleé Poussin operators, more precisely,

$$t_{n,k} = \begin{cases} \frac{k-[m_n]}{[m_n]-[m_{n-1}]}, & \text{if } m_{n-1} < |k| \le m_n, \\[2ex] \frac{[m_{n+1}]-k}{[m_{n+1}]-[m_n]}, & \text{if } m_n < |k| \le m_{n+1}. \end{cases} \tag{2.7}$$

Next let us state our result on the characterization of solid A^1_μ-spaces.

Theorem 2.4 *The following are equivalent*:

(i) A^1_μ *is solid*,
(ii) $s(A^1_\mu) = A^1_\mu$,
(iii) *The monomials* $(z^n)_{n=0}^\infty$ *are an unconditional basis of* A^1_μ,
(iv) *The normalized monomials* $(z^n/\|z^n\|_1)_{n=0}^\infty$ *are equivalent to the unit vector basis of* ℓ^1,
(v) $\sup_{n\in\mathbb{N}}(m_{n+1} - m_n) < \infty$ *for the numbers* m_n *in Theorem* 2.1.

In the following, we retain the numbers m_n, s_n of Theorem 2.1 and consider the Dirichlet projections P_n.

Lemma 2.5 *Assume that* $\limsup_{n\to\infty}(m_{n+1} - m_n) = \infty$. *Then, for every* $N > 0$, *there exists an arbitrarily large* $n \in \mathbb{N}$, *an index* $M < m_{n+1}$ *and a polynomial* $f(z) = \sum_{k=[m_n]+1}^{[m_{n+1}]} \alpha_k z^k$ *with* $\|f\|_1 \le 1$ *but* $\|P_M f\|_1 = \|(P_M - P_{[m_n]})f\|_1 \ge N$.

Proof Due to the unboundedness of the operator norms of P_n on H^1 (see Section "Introduction and preliminaries"), we find an index K and a polynomial $g(z) = \sum_{j=0}^{L} \beta_j z^j$ with $M_1(g, 1) = 1$ but $M_1(P_K g, 1) > N$. By assumption, we find $n \in \mathbb{N}$, as large as we wish, such that $m_{n+1} - m_n > L + 1$. Then put

$$f(z) = \sum_{k=[m_n]+1}^{[m_n]+L+1} \beta_{k-[m_n]-1} \frac{1}{s_n^k} z^k.$$

We obtain

$$M_1(f, s_n) = M_1(g, 1) = 1 \quad \text{and} \quad M_1(P_{m_n+K} f, s_n) = M_1(P_K g, 1) > N.$$

Put $M = K + [m_n] + 1$ and use Theorem 2.1 to complete the proof of the lemma. We have $P_M f = (P_M - P_{[m_n]})f$ just by the choice of f.

Proof of Theorem 2.4.

$(i) \Leftrightarrow (ii)$: follows from the definition.

$(iv) \Rightarrow (iii) \Rightarrow (ii)$: these are obvious.

$(ii) \Rightarrow (v)$: Assume that $\limsup_{n\to\infty}(m_{n+1} - m_n) = \infty$. For every $j \in \mathbb{N}$ we find, by Lemma 2.5, a polynomial $f_j \in \operatorname{span}\{z^{[m_{n_j}]+1}, \dots, z^{[m_{n_{j+1}}]}\}$ for some m_{n_j} with

$$\|f_j\|_1 = 2^{-j} \text{ and } \left\|P_{k_j} f_j\right\|_1 \geq 1 \text{ for some } k_j \in (m_{n_j}, m_{n_j+1}). \tag{2.8}$$

We may assume that $n_{j+1} - n_j \geq 2$. Put $f = \sum_j f_j$ and $g = \sum_j P_{k_j} f_j = \sum_j (P_{k_j} - P_{[m_{n_j}]}) f_j$. Then, $f \in A^1_\mu$ but in view of (2.8), (2.3) we have $g \notin A^1_\mu$. Hence $f \notin s(A^1_\mu)$.

$(v) \Rightarrow (iv)$: Let $g(z) = \sum_{k=[m_{n-1}]+1}^{[m_{n+1}]} \alpha_k z^k$. By (v) we obtain a constant independent of n, r, and g with

$$M_1(g, r) \leq \sum_{k=[m_{n-1}]+1}^{[m_{n+1}]} |\alpha_k| r^k \leq c M_1(g, r).$$

Then, (2.1) yields numbers $\delta_k = t_{n,k} s_n^k d_n$ such that for all functions

$$f(z) = \sum_{k=0}^{\infty} \alpha_k z^k \in A^1_\mu$$

we have, with the universal constants c_1, c_2,

$$c_1 \|f\|_1 \leq \sum_{k=0}^{\infty} \delta_k |\alpha_k| \leq c_2 \|f\|_1.$$

This proves (iv).

Corollary 2.6 *If $R < \infty$ then A^1_μ is never solid.*

Proof It follows from Proposition 2.1. of [10] that in this case we always have $\limsup_{n\to\infty}(m_{n+1} - m_n) = \infty$.

Example 2.7 There are indeed examples where A^1_μ is solid. Let $R = \infty$ and $d\mu(r) = \exp(-\log^2(r))dr$. It was illustrated in [10], Example 2a that here $\sup_n (m_{n+1} - m_n) < \infty$.

Theorem 2.8 *Let m_n, s_n and d_n be the numbers of Theorem* 2.1. *The solid core of A^1_μ equals*

$$s(A^1_\mu) = \left\{ g : R \cdot \mathbb{D} \to \mathbb{C} : g(z) = \sum_{k=0}^{\infty} \hat{g}(k) z^k \right.$$
$$\left. \text{with } \sum_{n=1}^{\infty} d_n \Bigg(\sum_{k=[m_n]+1}^{[m_{n+1}]} |\hat{g}(k)|^2 s_n^{2k} \Bigg)^{1/2} < \infty \right\}. \tag{2.9}$$

Proof For a holomorphic function $g(z) = \sum_{k=0}^{\infty} \hat{g}(k) z^k$ we write

$$g_n(z) = \sum_{k=[m_n]+1}^{[m_{n+1}]} \hat{g}(k) z^k \quad \text{and} \quad g_I(z) = \sum_{n=0}^{\infty} g_{2n}(z), \quad g_{II}(z) = \sum_{n=0}^{\infty} g_{2n+1}(z).$$

Let us denote by V the function space on the right-hand side of (2.9). Moreover, for all n, let $\Delta_n = \{+1, -1\}^{[m_{n+1}]-[m_n]}$, and for $\Theta_n = (\theta_{[m_n]+1}, \dots, \theta_{[m_{n+1}]}) \in \Delta_n$ put

$$g_{\Theta_n}(z) = \sum_{k=[m_n]+1}^{[m_{n+1}]} \theta_k \hat{g}(k) z^k.$$

First, assume that $g \in V$. Then $g_I, g_{II} \in V$. Let f be holomorphic with $|\hat{f}(k)| \leq |\hat{g}_I(k)|$ for all k. By (2.3) and Theorem 2.1

$$\begin{aligned}\|f\|_1 \le \sum_{n=0}^{\infty} \|f_{2n}\|_1 \le c \sum_{n=0}^{\infty} d_{2n} M_1(f_{2n}, s_{2n}) \\ \le c \sum_{n=0}^{\infty} d_{2n} M_2(f_{2n}, s_{2n}) \le c \sum_{n=0}^{\infty} d_{2n} M_2(g_{2n}, s_{2n}) < \infty\end{aligned}$$

where $c > 0$ is a universal constant. We also used the definition of the space V in the last step. Hence $f \in A^1_\mu$, in particular $g_I \in A^1_\mu$. We conclude $g_I \in s(A^1_\mu)$. The same proof shows that g_{II} and hence $g \in s(A^1_\mu)$.

Conversely, let $g \in s(A^1_\mu)$. Then $g_I, g_{II} \in s(A^1_\mu)$. Let $\tilde{\Theta}_n \in \Delta_n$ be such that

$$a_1 \left(\sum_{k=[m_n]+1}^{[m_{n+1}]} |\hat{g}(k)|^2 \right)^{1/2} \le \frac{1}{2^{[m_{n+1}]-[m_n]}} \sum_{\Theta_n \in \Delta_n} M_1(g_{\Theta_n}, s_n) \le M_1(g_{\tilde{\Theta}_n}, s_n).$$

Here we used the Khintchine inequality (see [18], Ch. V, Thm. 8.4) with the Khintchine constant a_1. Put $h_I = \sum_{n=0}^{\infty} g_{\tilde{\Theta}_{2n}}$. Then we obtain $|\hat{h}_I(k)| = |\hat{g}_I(k)|$ for all k. Hence $h_I \in A^1_\mu$. The choice of $\tilde{\Theta}_n$ and Theorem 2.1 applied to h_I yield

$$\begin{aligned}\sum_{n=0}^{\infty} d_{2n} \Big(\sum_{k=[m_{2n}]+1}^{[m_{2n+1}]} |\hat{g}(k)|^2 s_{2n}^{2k} \Big)^{1/2} = \sum_{n=0}^{\infty} d_{2n} \Big(\sum_{k=[m_{2n}]+1}^{[m_{2n+1}]} |\hat{h}_I(k)|^2 s_{2n}^{2k} \Big)^{1/2} \\ \le \frac{1}{a_1} \sum_{n=0}^{\infty} d_{2n} M_1(g_{\tilde{\Theta}_{2n}}, s_{2n}) \le \frac{c_2}{a_1} \|h_I\|_1 < \infty.\end{aligned}$$

Here, c_2 is the constant of Theorem 2.1. We conclude $g_I \in V$, and similarly we see that $g_{II} \in V$. Hence $g \in V$, which implies $V = s(A^1_\mu)$.

On solid hulls

In this section, we assume $R = 1$. We start off with the remark that in addition to the definition of a solid hull (see Section "Introduction and preliminaries"), there exist two other priori different definitions in the literature: in [1], the solid hull $S_{\mathrm{vect}}(X)$ of a space X of analytic functions on $\mathbb{D}$ is defined as the intersection of all solid *vector spaces* of analytic functions on $\mathbb{D}$. Obviously, $S(X)$ is a vector space if and only if for every $f, g \in X$ there is $h \in X$ such that the Taylor coefficients satisfy $|\hat{f}(k)| + |\hat{g}(k)| \le |\hat{h}(k)|$ for all k.

Another variant appears in the theory of the so-called BK-spaces. By definition, a BK-space is a vector space of complex sequences $f = (f_k)_{k=0}^{\infty}$ endowed with a norm which makes it into a Banach space, such that the coordinate functionals become bounded operators. In the theory of BK-spaces (see [8]), the solid hull $S_{BK}(X)$ of a BK-space X is defined as the intersection of all solid BK-spaces containing X. By using the Taylor coefficients, we consider Banach spaces of analytic functions on $\mathbb{D}$ as BK-spaces, and, in particular, we will characterize in the sequel the solid hull $S_{BK}(A^1_\mu)$ although we will avoid using the terminology of BK-spaces, except for the proof of Proposition 3.1. It is obvious that

$$S(X) \subset S_{\mathrm{vect}}(X) \subset S_{BK}(X) \tag{3.1}$$

for a BK-space X as above. All results on solid hulls $S(X)$ in the literature known to the authors are vector spaces which can be endowed with norms making them into solid BK-spaces. Thus, in all of these cases, one actually has $S(X) = S_{BK}(X)$. Our aim is to use the known duality relations between the weighted A^1 and H^∞-spaces and existing results of the solid core of H^∞_v in order to find the solid hull $S_{BK}(A^1_\mu)$. We focus on the case where the measure μ is the weighted Lebesgue measure vdA with a radial weight v making the Bergman space into a "large" one: the admissible weights include the exponentially decreasing weights (see Example 3.3 below).

We start with some general considerations.

Given a sequence $\theta = (\theta_k)_{k=0}^{\infty}$ with $|\theta_k| \le 1$ for all k, we denote by M_θ the operator $M_\theta \sum_{k=0}^{\infty} \hat{f}(k) z^k = \sum_{k=0}^{\infty} \theta_k \hat{f}(k) z^k$. We will need to consider analytic function spaces on $\mathbb{D}$ such that the norm of the space satisfies

$$\|M_\theta f\| \le \|f\| \tag{3.2}$$

for all $f = \sum_{k=0}^{\infty} \hat{f}(k) z^k \in X$ and all sequences $\theta = (\theta_k)_{k=0}^{\infty}$ with $|\theta_k| \le 1$ for all k.
The following result is essentially known.

Proposition 3.1 *If $(X, \|\cdot\|_X)$ is a Banach space of analytic functions on the unit disc $\mathbb{D}$ such that all coordinate functionals $f \mapsto \hat{f}(k)$ are bounded operators, then its solid hull $S_{BK}(X)$ can be endowed with a norm $\|\cdot\|_S$ such that*

(i) *the embedding $X \hookrightarrow S_{BK}(X)$ is continuous,*
(ii) *the norm $\|\cdot\|_S$ satisfies (3.2),*
(iii) *if $p : S_{BK}(X) \to \mathbb{R}_0^+$ is any norm with (3.2) such that $p(f) \le \|f\|_X$ for all $f \in X$, then $p(f) \le C\|f\|_S$ for a constant $C > 0$ and all $f \in S_{BK}(X)$,*
(iv) *the normed space $\big(S_{BK}(X), \|\cdot\|_S\big)$ is complete, and*
(v) *if the subspace of polynomials $\mathcal{P}$ is dense in X, then it is dense in $\big(S_{BK}(X), \|\cdot\|_S\big)$, too.*

Proof Let us explain the way the claims follow from the theory of BK-spaces; see [7, 8]. For the sake of the simplicity of the notation, let us consider X as a BK-sequence space in the following, that we can do by assumption. We denote by $y \cdot f$ the coordinatewise product of two complex sequences y and f. The space $\ell^\infty \widehat{\otimes} X$ is defined in [7] to consist of sequences $g = (g_k)_{k=0}^{\infty}$ having a coordinatewise convergent representation

$$g = \sum_{j=1}^{\infty} y^{(j)} \cdot f^{(j)} \quad \text{with } y^{(j)} = \big(y_k^{(j)}\big)_{k=0}^{\infty} \in \ell^\infty, f^{(j)} = \big(f_k^{(j)}\big)_{k=0}^{\infty} \in X \ \forall j \tag{3.3}$$

such that

$$\sum_{j=1}^{\infty} \|y^{(j)}\|_{\ell^\infty} \|f^{(j)}\|_X < \infty. \tag{3.4}$$

The norm $\|\cdot\|_S$ of $g \in \ell^\infty \widehat{\otimes} X$ is defined by taking the infimum of the quantity (3.4) over all possible representations (3.3) of g. Theorem 3 of [7] yields that the resulting space is complete, and Theorem 8 of [8] says that $\ell^\infty \widehat{\otimes} X$ equals the solid hull $S_{BK}(X)$. The completeness of the space is included in the same reference; hence, property (*iv*) holds.

If $f \in X$, then we have $\mathrm{e} \cdot f = f$, where $\mathrm{e} = (1, 1, 1, \dots) \in \ell^\infty$, and in view of the previous definition of the norm $\|\cdot\|_S$, this implies that $\|f\|_S \le \|f\|_X$ for all $f \in X$ so that the embedding of X into $\big(S_{BK}(X), \|\cdot\|_S\big)$ is continuous.

Also, if $g \in S_{BK}(X)$ has a representation (3.3) and θ is given as in (3.2), then $M_\theta g$ has a coordinatewise convergent representation

$$M_\theta g = \sum_{j=1}^{\infty} (M_\theta y^{(j)}) \cdot f^{(j)}, \tag{3.5}$$

and property (*ii*) follows from the definition of $\|\cdot\|_{BK}$.

In the proof of Theorem 3 of [7], it is shown that if p is the norm of any BK-space containing $S_{BK}(X)$, then there exists $C > 0$ such that

$$p(y \cdot f) \le C\|y\|_\infty \|f\|_X$$

for all $y \in \ell^\infty$, $f \in X$. This implies

$$p\Big(\sum_{j=1}^{\infty} y^{(j)} \cdot f^{(j)} \Big) \le C \sum_{j=1}^{\infty} \|y^{(j)}\|_\infty \, \|f^{(j)}\|_X \ \text{ for all } g = \sum_{j=1}^{\infty} y^{(j)} \cdot f^{(j)} \in \ell^\infty \widehat{\otimes} X,$$

and property (iii) follows from the definition of $\|\cdot\|_S$. Finally, as for property (v), it follows from Theorem 2 of [7] that finite linear combinations of functions $y \cdot f$, $y \in \ell^\infty$, $f \in X$, form a dense subspace of $\ell^\infty \widehat{\otimes} X = S(X)$. If y, f, and $\varepsilon > 0$ are given, we use the assumption in (v) to find a polynomial h such that $\|f - h\|_X < \varepsilon/(1 + \|y\|_\infty)$. Then, $y \cdot h$ is a polynomial, which satisfies

$$p(y \cdot f - y \cdot h) = p(y \cdot (f - h)) \le \|y\|_\infty \|f - h\|_X \le \varepsilon.$$

Property (v) follows from these arguments.

Lemma 3.2 *Let X be a Banach space of analytic functions on the unit disc $\mathbb{D}$ such that the subspace $\mathcal{P}$ of polynomials is dense in X, and let w be a radial weight function on $\mathbb{D}$. Let Y be the space of all analytic functions g on the disc such that*

$$\sup_{f \in B_X} |\langle f, g \rangle| < \infty, \quad \text{where} \quad \langle f, g \rangle = \int_{\mathbb{D}} f \overline{g} w dA \tag{3.6}$$

and B_X denotes the unit ball of X. If X is solid and there exists a constant $C > 0$ such that

$$\|M_\theta f\|_X \le C \|f\|_X \tag{3.7}$$

for all numerical sequences $\theta = (\theta_k)_{k=0}^\infty$ with $|\theta_k| \le 1$, then Y is solid, too.

We point out that given the Banach space X as in the assumption, it is not in general known whether its dual space has a representation as a space of analytic functions with dual norm coming from (3.6).

Proof If $g = \sum_{k=0}^\infty \hat{g}(k) z^k \in Y$ and θ is as above, then for $M_\theta g$ we have by (3.7)

$$\begin{aligned} \sup_{f \in B_X} |\langle f, M_\theta g \rangle| &= \sup_{f \in B_X} \sum_{k=0}^{\infty} \bar{\theta}_k \hat{f}(k) \overline{\hat{g}(k)} \int_0^1 r^{2k+1} w(r) dr \\ &= \sup_{f \in B_X} |\langle M_{\bar{\theta}} f, g \rangle| \le \sup_{\substack{f \in X \\ \|f\|_X \le C}} |\langle f, g \rangle| < \infty. \end{aligned} \tag{3.8}$$

Thus, $M_\theta g \in Y$.

Next we recall an elementary fact concerning Banach sequence spaces. Assume that the sequences $(\beta_k)_{k=0}^\infty$ and $(\gamma_k)_{k=0}^\infty$ of positive numbers are given and $\alpha_k = \gamma_k \beta_k^{-1}$ for all k. Let also $(\mu_n)_{n=0}^\infty$ be an increasing, unbounded sequence of non-negative numbers; denote $\mu_{-1} = -1$ and let

$$A = \Big\{ a = (a_k)_{k=0}^\infty \ : \ \|a\|_A = \sum_{n \in \mathbb{N}} \max_{\mu_{n-1} < k \le \mu_n} \alpha_k |a_k| < \infty \Big\}, \tag{3.9}$$

$$B = \Big\{ b = (b_k)_{k=0}^\infty \ : \ \|b\|_B = \sup_{n \in \mathbb{N}} \sum_{\mu_{n-1} < k \le \mu_n} \beta_k |b_k| < \infty \Big\}. \tag{3.10}$$

Then, B is the dual of A with respect to the dual pairing

$$\langle a, b \rangle = \sum_{k=0}^\infty \gamma_k b_k \overline{a_k}, \quad \text{where } a = (a_k)_{k=0}^\infty \in A, \ b = (b_k)_{k=0}^\infty \in B. \tag{3.11}$$

From now on, we consider radial weights $v : \mathbb{D} \to \mathbb{R}^+$ satisfying two following assumptions.

(I) We have

$$v(z) = \exp(-\varphi(z)), \tag{3.12}$$

where φ belongs to the class $\mathcal{W}_0$ of [11].
We will not need a detailed definition of $\mathcal{W}_0$, but recall that $\varphi \in \mathcal{W}_0$, if it is a twice continuously differentiable real valued function with $\Delta\varphi > 0$ on $\mathbb{D}$ and there exists a function $\rho : \mathbb{D} \to \mathbb{R}$ and a constant $C > 0$ such that

$$\frac{1}{C}\rho(z) \le \frac{1}{\sqrt{\Delta\varphi(z)}} \le C\rho(z) \;\; \forall z \in \mathbb{D}; \tag{3.13}$$

the function ρ must also satisfy the Hölder-property

$$\sup_{z,w\in\mathbb{D},z\neq w} \frac{|\rho(z) - \rho(w)|}{|z-w|} < \infty \tag{3.14}$$

as well as the Lipschitz property

$$\forall\, \varepsilon > 0 \,\exists \text{ compact } E \subset \mathbb{D} : \;|\rho(z) - \rho(w)| \le \varepsilon|z-w| \;\forall\, z, w \in \mathbb{D} \setminus E. \tag{3.15}$$

For more details, see [11]. Note that the considerations in [11] are not restricted to radial weights, contrary to our situation. According to [11], Theorem 4.3, if the weight v satisfies the condition (I), then the space H_v^∞ is the dual of A_v^1 with respect to the dual pairing

$$\langle f, g\rangle = \int_{\mathbb{D}} f\overline{g}v^2 dA. \tag{3.16}$$

The second requirement is the following:
(II) The weight v satisfies the condition (b) of [3, 4].
Recall that the weight v satisfies the condition (b) if there exist numbers $b > 2$, $K > b$ and $0 < \mu_1 < \mu_2 < \dots$ with $\lim_{n\to\infty} \mu_n = \infty$ such that

$$b \le \left(\frac{r_{\mu_n}}{r_{\mu_{n+1}}}\right)^{\mu_n} \frac{v(r_{\mu_n})}{v(r_{\mu_{n+1}})}, \left(\frac{r_{\mu_{n+1}}}{r_{\mu_n}}\right)^{\mu_{n+1}} \frac{v(r_{\mu_{n+1}})}{v(r_{\mu_n})} \le K, \tag{3.17}$$

where $r_m \in]0, 1[$ denotes the global maximum point of the function $r^m v(r)$ for any $m > 0$. Theorem 2.4 of [4] states that the solid core of the space H_v^∞ equals

$$s(H_v^\infty) = \Big\{(b_k)_{k=0}^\infty \;:\; \|b\|_{v,s} = \sup_{n\in\mathbb{N}} v(r_{\mu_n}) \sum_{\mu_n<k\le\mu_{n+1}} |b_k|\sigma_k < \infty\Big\}, \tag{3.18}$$

where we denote $\sigma_k = r_{\mu_n}^k$. Let us define for every $k \in \mathbb{N}_0$ the number

$$S_k = \frac{\int_0^1 r^{2k+1} v(r)^2 dr}{v(r_{\mu_n})\sigma_k}\;, \tag{3.19}$$

where n is the unique number such that $\mu_n < k \le \mu_{n+1}$.

Example 3.3 According to [4], all weights $v(r) = \exp\big(-\alpha/(1-r^2)^\beta\big)$ with $\alpha, \beta > 0$, satisfy the condition (b), and it is easy to see that they also satisfy the assumption (I).

Theorem 3.4 *Let the weight v satisfy the assumptions (I) and (II). Then, we have*

$$S_{BK}(A^1_\mu) = \Big\{ b = (b_k)_{k=0}^\infty \ : \ \|b\|_{\mu,S} = \sum_{n=0}^\infty \sup_{\mu_n < k \le \mu_{n+1}} |b_k| S_k < \infty \Big\}, \tag{3.20}$$

and the norm $\|\cdot\|_S$ *given by Proposition* 3.1 *is equivalent with* $\|\cdot\|_{\mu,S}$.

Proof Let the solid hull $S_{BK}(A^1_\mu)$ be endowed with the norm $\|\cdot\|_S$ of Proposition 3.1, and let us denote the Banach space on the right-hand side of (3.20) by Z.

We note that by the duality relations explained above (see (3.18) for the definition of $\|\cdot\|_{v,s}$), we have for all $f \in A^1_\mu$

$$\|f\|_1 = \sup_{\substack{g \in H^\infty_v \\ \|g\|_{H^\infty_v} \le 1}} |\langle f, g\rangle| \quad \text{and} \quad \|f\|_{\mu,S} = \sup_{\substack{g \in s(H^\infty_v) \\ \|g\|_{v,s} \le 1}} |\langle f, g\rangle|. \tag{3.21}$$

It is proved in [4], Eq. (2.4) and at the very end of the proof of Theorem 2.4, that $\|g\|_{H^\infty_v} \le C\|g\|_{v,s}$ for $g \in s(H^\infty_v)$. Therefore $\|f\|_1 \ge C\|f\|_{\mu,S}$ for all $f \in A^1_\mu$. This implies in particular that $A^1_\mu \subset Z$. Clearly, Z is a solid Banach space and the coordinate functionals are continuous; thus, it contains the space $S_{BK}(A^1_\mu)$. Moreover, we obtain $\|f\|_S \ge C\|f\|_{\mu,S}$ for $f \in S_{BK}(A^1_\mu)$ from Proposition 3.1.(*iii*).

We show that the norms $\|\cdot\|_{\mu,S}$ and $\|\cdot\|_S$ are equivalent in $S_{BK}(A^1_\mu)$. For this purpose, we prove that $C\|f\|_{\mu,S} \ge \|f\|_S$. Note that the space (3.18) is the dual space of Z in the dual pairing (3.16). Indeed, if $f = \sum_k \hat f(k) z^k$ and $g = \sum_k \hat g(k) z^k$ are polynomials, then, by a direct calculation,

$$\langle f, g\rangle = \sum_{k=0}^\infty \hat f(k)\overline{\hat g(k)} \int_0^1 r^{2k+1} v(r)^2 dr. \tag{3.22}$$

The result follows from (3.9)–(3.10), in addition to the definitions (3.16)–(3.20).

Let us suppose that by antithesis $\|\cdot\|_{\mu,S}$ and $\|\cdot\|_S$ are non-equivalent norms so that we can find a sequence $(f_n)_{n=1}^\infty \subset S_{BK}(A^1_\mu)$ such that

$$\|f_n\|_{\mu,S} \le 2^{-n}\|f_n\|_S \ \text{ and } \ \|f_n\|_S = 1 \ \forall\, n \in \mathbb{N}. \tag{3.23}$$

By property (v) in Proposition 3.1, we can assume that f_n's are polynomials. We claim that it is possible to find polynomials $\tilde f_n$, $n \in \mathbb{N}$, with property (3.23) such that they have distinct degrees, more precisely

$$\tilde f_n(z) = \sum_{k=K_n}^{K_{n+1}-1} \hat f(n,k) z^k, \quad n \in \mathbb{N}, \tag{3.24}$$

for some unbounded sequence $0 = K_0 < K_1 < \ldots$ and some $\hat f(n,k) \in \mathbb{C}$. Assume that $N \in \mathbb{N}$ and that such polynomials $\tilde f_n$ have been found for $n \le N$, and let $M \in \mathbb{N}$ be the highest degree of these polynomials. Since $\mathcal{P}_M$ (the $M+1$-dimensional space of polynomials of degree at most M) is finite dimensional, all norms are equivalent there and we thus find a constant $K = K(M) > 0$ such that

$$\|f\|_S \le K\|f\|_{\mu,S} \tag{3.25}$$

for all $f \in \mathcal{P}_M$. We pick up the polynomial f_L as in (3.23) with $L = M + K$ and write $f_1 = P_M f_L$, $f_2 = f_L - f_1$, where P_M is the Mth Dirichlet projection from $S_{BK}(A^1_\mu)$ onto $\mathcal{P}_M$ (see Section Introduction and preliminaries). Then, we have $\|f_2\|_S \ge \frac{1}{2}\|f_L\|_S$, since otherwise we get by (3.25) and the triangle inequality

$$\|f_L\|_{\mu,S} \ge \|f_1\|_{\mu,S} \ge \frac{1}{K}\|f_1\|_S \ge \frac{1}{2K}\|f_L\|_S > \frac{1}{2L}\|f_L\|_S$$

which contradicts (3.23). Now we get

$$\|f_2\|_{\mu,S} \le \|f_L\|_{\mu,S} \le 2^{-L}\|f_L\|_S \le 2^{-L+1}\|f_2\|_S. \tag{3.26}$$

Taking $f_2\|f_2\|_S^{-1}$ for $\tilde{f}_{N+1}$, the claim is proved.

Finally, for every n, we set

$$T_n := \left(P^{(n)}\left(S_{BK}(A^1_\mu)\right), \|\cdot\|_S\right) \quad \text{with} \quad P^{(n)} = P_{K_{n+1}-1} - P_{K_n} \tag{3.27}$$

and then, using the Hahn-Banach theorem, pick up a polynomial

$$g_n = \sum_{k=K_n}^{K_{n+1}-1} \hat{g}(n,k)z^k$$

which defines a bounded functional on $(T_n, \|\cdot\|_S)$ with respect to the dual pairing (3.22), such that

$$\langle \tilde{f}_n, g_n\rangle = 1, \quad \|g_n\|_{n,*} := \sup_{\substack{f\in T_n\\ \|f\|_S\le 1}} |\langle f, g_n\rangle| = 1 \tag{3.28}$$

Then, we observe that g_n extends via (3.22) to a functional on $S_{BK}(A^1_\mu) =: S$ such that

$$\sup_{\substack{f\in S\\ \|f\|_S\le 1}} |\langle f, g_n\rangle| = \sup_{\substack{f\in S\\ \|f\|_S\le 1}} |\langle P^{(n)}f, g_n\rangle| = \sup_{\substack{f\in T_n\\ \|f\|_S\le 1}} |\langle f, g_n\rangle| = 1, \tag{3.29}$$

since the norm $\|\cdot\|_S$ of $S_{BK}(A^1_\mu)$ satisfies (ii) of Proposition 3.1 and thus $\|P^{(n)}f\|_S \le \|f\|_S$ for all $f \in S_{BK}(A^1_\mu)$. Consequently,

$$g = \sum_{n\in\mathbb{N}} \frac{1}{n^2} g_n \tag{3.30}$$

is an analytic function which also is a bounded functional on $\left(S_{BK}(A^1_\mu), \|\cdot\|_S\right)$ in the dual pairing (3.22). However, g is not a bounded functional on Z, since

$$\langle 2^n\tilde{f}_n, g\rangle = \frac{2^n}{n^2}\langle \tilde{f}_n, g_n\rangle = \frac{2^n}{n^2} \tag{3.31}$$

and by (3.23), the Z-norm $\|2^n\tilde{f}_n\|_{\mu,S}$ is still at most 1.

The space Y of all analytic functions on $\mathbb{D}$, which also are bounded functionals on $\left(S_{BK}(A^1_\mu), \|\cdot\|_S\right)$ in the dual pairing (3.22), equals the space Y in Lemma 3.2, when $X := S_{BK}(A^1_\mu)$. Hence, Y is solid. Due to the characterization of H^∞_v as the dual of A^1_v, see (3.16), we also have $Y \subset H^\infty_v$. On the other hand, at the beginning of the proof, we observed that the solid core $s(H^\infty_v)$ (see (3.18)) equals the dual of Z in the pairing (3.22). The properties of the function g, (3.30), show that $s(H^\infty_v) \subsetneq Y$, which contradicts the definition of a solid core. We conclude that $C\|f\|_{\mu,S} \ge \|f\|_S$ for all $f \in S_{BK}(A^1_\mu)$.

We come to the conclusion that the norms $\|\cdot\|_S$ and $\|\cdot\|_{\mu,S}$ are equivalent; hence, the spaces $S_{BK}(A^1_\mu)$ and Z coincide, since they both are complete.

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Declarations

Ethics approval The research has been ethically conducted.

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