GENERALISED MUTUALLY PERMUTABLE PRODUCTS AND SATURATED FORMATIONS

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Abstract. We say that a group $G = AB$ is the weakly mutually permutable product of the subgroups A and B, if A permutes with every subgroup of B containing $A \cap B$ and B permutes with every subgroup of A containing $A \cap B$. We prove that some known results for mutually permutable products remain true for weakly mutually permutable ones. Moreover, if G' is nilpotent, A permutes with every Sylow subgroup of B and B permutes with every Sylow subgroup of A, we show that $G^{\mathfrak{F}} = A^{\mathfrak{F}}B^{\mathfrak{F}}$, where $\mathfrak{F}$ is a saturated formation containing $\mathfrak{U}$, the class of supersoluble groups. This generalises the corresponding result on mutually permutable products.

1. Introduction

All groups considered here will be finite.

If a group $G = AB$ is a product of two subgroups A and B, the question arises what can be said about the structure of the factorised group G if the structure of the two subgroups A and B is known. There are many group theoretical properties that do not carry over from the factors A and B to the factorised group G. Indeed, if one experiments with properties such as nilpotency, supersolubility and solubility, one soon realises the difficulty of using the factorisation to obtain information about the structure of the group. This problem is much more treatable if the subgroups of the factorised group are connected by certain permutability properties. In a seminal paper [2], Asaad and Shaalan introduced the notion of mutually permutable products and since that time many people have considered such products, usually imposing additional conditions on A and B in order to see how the structure of G is further restricted (see [3], [5], [6], [8]).

Recall that a group G is the *mutually permutable* product of the subgroups A and B if $G = AB$ and A permutes with every subgroup of B and B permutes with every subgroup of A.

If G is a mutually permutable product of the subgroups A and B and U and V are subgroups of A and B respectively such that either $A \cap B \leq U$ or $A \cap B \leq V$, then U permutes with V (see [3, Proposition 4.1.16(2)]). Therefore the behaviour of mutually permutable products with respect to saturated formations containing the class $\mathfrak{U}$ of all supersoluble groups depends heavily on the family of subgroups of A and B containing $A \cap B$ (see [3]).

The main object of the present work is to introduce and study a new type of products which helps to better understand the structure of mutually permutable products.

Definition 1.1. *Let A and B be two subgroups of a group G such that $G = AB$. We say that G is the weakly mutually permutable product of A and B if A permutes with*

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every subgroup V of B such that $A \cap B \leqslant V$, and B permutes with every subgroup U of A such that $A \cap B \leqslant U$.

Mutually permutable products are weakly mutually permutable, whilst the converse is not necessarily true as the following example shows.

Example 1.2. *Let $G = \Sigma_4$ be the symmetric group of degree 4. Consider a maximal subgroup A of G which is isomorphic to Σ_3, the symmetric group of degree 3, and $B = A_4$, the alternating group of degree 4. Then $G = AB$ is the weakly mutually permutable product of the subgroups A and B. However, G is not a mutually permutable product of A and B because A does not permute with a subgroup of order 2 of B.*

We study here the behaviour of the residuals associated to saturated formations containing $\mathfrak{U}$ in weakly mutually permutable products.

Recall that if $\mathfrak{F}$ is a saturated formation, the $\mathfrak{F}$-*residual* $G^{\mathfrak{F}}$ of a group G is the smallest normal subgroup of G with quotient in $\mathfrak{F}$.

The first and third named authors proved the following result:

Theorem 1.3. [4, Theorem A] *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$, the class of all supersoluble groups. Let the group $G = AB$ be the mutually permutable product of the subgroups A and B. If G' is nilpotent, then $G^{\mathfrak{F}} = A^{\mathfrak{F}}B^{\mathfrak{F}}$.*

However, Theorem 1.3 does not hold for weakly mutually permutable products even when the saturated formation is the class of all supersoluble groups as the following example shows:

Example 1.4. *We are constructing a group $G = AB$ which is the weakly mutually permutable product of A and B such that $G^{\mathfrak{U}} = \langle A^{\mathfrak{U}}, B^{\mathfrak{U}}\rangle \neq A^{\mathfrak{U}}B^{\mathfrak{U}}$. Consider $H = A_4 = VS$ the alternating group on $\{1,2,3,4\}$, being $V = \langle v_1, v_2\rangle$ with $v_1 = (12)(34)$, $v_2 = (13)(24)$ and $S = \langle x\rangle$ with $x = (123)$. Let M be the natural permutation module for Alt(4) over $\mathbb{F}_2$ with permutation basis $\{x_1, x_2, x_3, x_4\}$. We set*

$$y_1 = x_1x_2,\ y_2 = x_1x_3,\ z = x_1x_2x_3x_4 \in M,$$

$Y = \langle y_1, y_2\rangle$, $Z = \langle z\rangle$ and $W = \langle y_1, y_2, z\rangle = Y \times Z$ which is an H-submodule of M. Let $G = [W]H$ be the corresponding semidirect product. More precisely H acts on W as follows:

$$y_1^x = y_1y_2,\ y_2^x = y_1,\ z^x = z; y_1^{v_1} = y_1,\ y_2^{v_1} = y_2z,\ z^{v_1} = z; y_1^{v_2} = y_1z,\ y_2^{v_2} = y_2,\ z^{v_2} = z.$$

In particular we have that: $Z = Z(G)$, and Y is a nontrivial irreducible S-submodule of W.

Let $A = WS = ZYS$ and $B = ZVS = ZH$. Then $G = AB$, $A^{\mathfrak{U}} = Y, B^{\mathfrak{U}} = V$ and $G^{\mathfrak{U}} = WV = \langle A^{\mathfrak{U}}, B^{\mathfrak{U}}\rangle \neq A^{\mathfrak{U}}B^{\mathfrak{U}}$. Moreover $A \cap B = ZS$. We prove that A and B are weakly mutually permutable. Since Y is an irreducible S-module, it is clear B permutes with every subgroup of A containing ZS, they are ZS or A. Also $B = Z \times$ Alt(4), therefore the unique subgroups of B containing ZS are ZS and B and A permutes with them. It is clear that G' which is a 2-group is nilpotent.

We show the following result.

Theorem A. *Let $\mathfrak{F}$ be a formation. Assume that either $\mathfrak{F} = \mathfrak{U}$ or $\mathfrak{F}$ is a saturated Fitting formation containing $\mathfrak{U}$. Let $G = AB$ be the weakly mutually permutable product of the subgroups A and B. If G' is nilpotent, then $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle$.*

Unfortunately, Theorem A does not hold for saturated formations containing $\mathfrak{U}$ as the following example shows:

Example 1.5. *Define a formation function f as follows:*

$$f(p) = \begin{cases} \{G : G \text{ abelian group of exponent dividing } p-1\} & \text{if } p \neq 17, \\ \{G : G \text{ abelian group of exponent dividing } 48\} & \text{if } p = 17. \end{cases}$$

Consider the subgroup-closed saturated formation $\mathfrak{F} = LF(f)$ locally defined by f. It is clear that $\mathfrak{F}$ contains $\mathfrak{U}$. Let B be a cyclic group of order 9. Then B has an irreducible module V over the finite field of 17 elements of dimension 2 which is also irreducible for the maximal subgroup C of B (see [7, B, Theorem 9.8]*). Let $G = [V]B$ be the corresponding semidirect product. Then $G = AB$ is the weakly mutually permutable product of $A = VC$ and B. A and B belong to $\mathfrak{F}$ but G is not an $\mathfrak{F}$-group.*

An additional condition allows us to obtain an extension of Theorem 1.3.

Theorem B. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Let the group $G = AB$ be the weakly mutually permutable product of the subgroups A and B. Suppose that A permutes with every Sylow subgroup of B and B permutes with every Sylow subgroup of A. If G' is nilpotent, then $G^{\mathfrak{F}} = A^{\mathfrak{F}} B^{\mathfrak{F}}$.*

2. Preliminary Results

We first show that factor groups of weakly mutually permutable products are also weakly mutually permutable products.

Lemma 2.1. *Let $G = AB$ be the weakly mutually permutable product of A and B and let N be a normal subgroup of G. Then $G/N = (AN/N)(BN/N)$ is the weakly mutually permutable product of AN/N and BN/N.*

Proof. We have that $G/N = (AN/N)(BN/N)$. Suppose that H/N is a subgroup of AN/N such that $AN/N \cap BN/N \leqslant H/N$. Then $U = H \cap A$ is a subgroup of A such that $H = UN$ and $A \cap B \leq U$. Since U permutes with B and $H = UN$, it follows that H permutes with BN. Analogously, it can be showed that AN/N permutes with every subgroup of BN/N containing $AN/N \cap BN/N$ and therefore G/N is the weakly mutually permutable product of AN/N and BN/N. □

Lemma 2.2. *Let $G = AB$ be the weakly mutually permutable product of A and B.*

(a) *If H is a subgroup of A such that $A \cap B \leqslant H$ and K is a subgroup of B such that $A \cap B \leqslant K$, then HK is a weakly mutually permutable product of H and K.*
(b) *If $A \cap B = 1$, then G is the totally permutable product of the subgroups A and B, that is, every subgroup of A permutes with every subgroup of B.*

Proof. We have that B permutes with every subgroup L of H such that $A \cap B \leqslant L$ and A permutes with every subgroup M of K such that $A \cap B \leqslant M$. Then $LM = L(A \cap B)M = (A \cap LB)M = AM \cap LB = MA \cap BL = M(A \cap BL) = M(A \cap B)L = ML$. Hence L permutes with M and HK is the weakly mutually permutable product of H and K.

Statement (b) holds immediately from (a). □

Lemma 2.3. *Let $G = AB$ be the product of the subgroups A and B. If A permutes with every Sylow subgroup of B and B permutes with every Sylow subgroup of A, then $A \cap B$ also permutes with every Sylow subgroup of A and B. In particular, $A \cap B$ is a subnormal subgroup of G.*

Proof. Let A_p be a Sylow p-subgroup of A. Then B permutes with A_p and so BA_p is a subgroup of G. Furthermore, $BA_p \cap A = A_p(A \cap B)$. Therefore $A \cap B$ permutes with A_p. We have shown that $A \cap B$ permutes with every Sylow subgroup of A. Applying [3, Theorem 1.2.14(3)], $A \cap B$ is a subnormal subgroup of both A and B. By [3, Theorem 1.1.7], we have that $A \cap B$ is a subnormal subgroup of G. □

Lemma 2.4 ([4, Lemma 1])**.** *Let $\mathfrak{F}$ be a saturated formation. Let $G = AB$ be the product of the subgroups A and B. If G' is nilpotent, and $G \in \mathfrak{F}$, then $A \in \mathfrak{F}$ and $B \in \mathfrak{F}$.*

Corollary 2.5. *Let $\mathfrak{F}$ be a saturated formation. Let G be a group. If G' is nilpotent, then $H^{\mathfrak{F}} \leq G^{\mathfrak{F}}$ for every subgroup H of G.*

Proof. The group $G/G^{\mathfrak{F}}$ is the product of the subgroups $G/G^{\mathfrak{F}}$ and $HG^{\mathfrak{F}}/G^{\mathfrak{F}}$. Since $G/G^{\mathfrak{F}} \in \mathfrak{F}$, it follows that $H/H \cap G^{\mathfrak{F}} \cong HG^{\mathfrak{F}}/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$ by Lemma 2.4. Hence $H^{\mathfrak{F}} \leq G^{\mathfrak{F}}$.

□

Lemma 2.6. *Let $\mathfrak{F}$ be a formation. Assume that either $\mathfrak{F} = \mathfrak{U}$ or $\mathfrak{F}$ is a saturated Fitting formation containing $\mathfrak{U}$. Let $G = AB$ be the weakly mutually permutable product of the subgroups A and B. Assume that G' is nilpotent. If $A \in \mathfrak{F}$ and $B \in \mathfrak{F}$, then $G \in \mathfrak{F}$.*

Proof. Suppose that the theorem is not true and let G be a minimal counterexample. Then A and B are proper subgroups of G. Let N be a minimal normal subgroup of G. Then G/N is the weakly mutually permutable product of the subgroups AN/N and BN/N by Lemma 2.1, $AN/N \in \mathfrak{F}$, $BN/N \in \mathfrak{F}$ and $(G/N)'$ is nilpotent. By the minimality of G, it follows that $G/N \in \mathfrak{F}$. Since $\mathfrak{F}$ is saturated, we have that G is a primitive soluble group. Then $\mathbf{C}_G(N) = N = \mathbf{F}(G) = G^{\mathfrak{F}} = G'$ is the unique minimal normal subgroup of G, and N is a Sylow p-subgroup of G which is complemented in G by an abelian Hall p'-subgroup of G. Moreover, by [1, Lemma 1.3.2], there exist Hall p'-subgroups $A_{p'}$ and $B_{p'}$ of A and B respectively such that $H = A_{p'}B_{p'}$ is a Hall p'-subgroup of G. Then $G = NH$ and H is abelian.

Note that $X = AN = A(X \cap B)$ is the weakly mutually permutable product of the $\mathfrak{F}$-subgroups A and $X \cap B$, and $Y = NB$ is the weakly mutually permutable product of the $\mathfrak{F}$-subgroups B and $Y \cap A$ by Lemma 2.2 and Corollary 2.5. Since $\mathbf{C}_G(N) = N$, it follows that $\mathbf{O}_{p'p}(X) = \mathbf{O}_{p'p}(Y) = N$. Assume that X and Y are both proper subgroups of G. Then X and Y belong to $\mathfrak{F}$ by the minimal choice of G. Then $A_{p'} \cong X/N \in F(p)$ and $B_{p'} \cong Y/N \in F(p)$, where F is the canonical local definition of $\mathfrak{F}$. Since H is abelian and $F(p)$ is a formation, we have that $H \in F(p)$. Therefore, $G \in \mathfrak{S}_pF(p) = F(p) \subseteq \mathfrak{F}$ (see [7, IV, Proposition 3.8(a)]). This contradiction shows that either $G = X$ or $G = Y$. Assume that $G = Y$. Then $N \cap B = 1$ and B is an abelian Hall p'-subgroup of G. In particular, N is contained in A and so $A = N(A \cap B)$. Moreover, every subgroup of B belongs to $\mathfrak{F}$ by [7, IV, Theorem 1.14].

Let N_1 be a minimal normal subgroup of A contained in N. Then $N_1(A \cap B)$ is a subgroup of A and N_1B is a subgroup of G. Since B is a maximal subgroup of G, it follows that $G = N_1B$ and $N = N_1$. Hence N is a minimal normal subgroup of A and

$A \cap B$ is a maximal subgroup of A. Moreover, since N is a faithful and irreducible B-module, it follows that B is cyclic by [7, B, Corollary 9.4]. Assume that $|B|$ is not a prime power. Let C be a Sylow subgroup of B. Then AC is a proper subgroup of G which is the weakly mutually permutable product of the $\mathfrak{F}$-subgroups A and $(A \cap B)C$ by Lemma 2.2. Then $AC \in \mathfrak{F}$, and so $AC/\mathbf{O}_{p'p}(AC) = AC/N \in F(p)$. Hence $(A \cap B)C \in F(p)$. By [7, IV, Theorem 1.14], it follows that $C \in F(p)$. Since B is a direct product of its Sylow subgroups, it follows that $B \in F(p)$. Thus $G \in \mathfrak{S}_p F(p) = F(p) \subseteq \mathfrak{F}$, a contradiction. Hence B is a cyclic group of prime power order. Assume that $A \cap B = 1$. Then G is the totally permutable product of A and B. By [5], $G \in \mathfrak{F}$. Therefore $A \cap B \neq 1$. Let M be the maximal subgroup of B. Then AM is a maximal subgroup of G which is the weakly mutually permutable product of the $\mathfrak{F}$-subgroups A and M by Lemma 2.2. By the choice of G, $MA \in \mathfrak{F}$ and so $M \cong AM/\mathbf{O}_{p'p}(AM) = AM/N \in F(p)$.

Assume that $\mathfrak{F} = \mathfrak{U}$. Then M is a cyclic group of exponent dividing $p-1$. Since N is an irreducible M-module, it follows that N is of order p by [7, B, Theorem 9.8]. Consequently, G is supersoluble, a contradiction.

Assume that $\mathfrak{F}$ is a saturated Fitting formation containing $\mathfrak{U}$. Then $F(p)$ is a subgroup-closed Fitting formation by [7, IV, Proposition 3.16]. Since $1 \neq M \in F(p)$, we can apply [7, IX, Lemma 1.8] to conclude that $B \in F(p)$. Hence $G \in \mathfrak{S}_p F(p) = F(p) \subseteq \mathfrak{F}$, a contradiction. Therefore no counterexample exists. □

Assume that $G = AB$ is the weakly mutually permutable product of the subgroups A and B. Assume further that A permutes with every Sylow subgroup of B and B permutes with every Sylow subgroup of A. Then, by Lemma 2.3, $A \cap B$ is subnormal in G. In this case, we cannot have a minimal configuration as in Lemma 2.6. Therefore we have:

Lemma 2.7. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Let $G = AB$ be the weakly mutually permutable product of the subgroups A and B. Suppose that A permutes with every Sylow subgroup of B and B permutes with every Sylow subgroup of A. If G' is nilpotent and A and B belong to $\mathfrak{F}$, then G belongs to $\mathfrak{F}$.*

3. Main Results

Proof of Theorem A. Suppose that the result is not true and let (G, A, B) be a minimal counterexample. Then A and B are proper subgroups of G. Let N be a minimal normal subgroup of G. Then G/N is the weakly mutually permutable product of the subgroups AN/N and BN/N by Lemma 2.1, and $(G/N)'$ is nilpotent. By the minimality of G, we have that $G^{\mathfrak{F}}N = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle N$. Since $G/G^{\mathfrak{F}} = (AG^{\mathfrak{F}}/G^{\mathfrak{F}})(BG^{\mathfrak{F}}/G^{\mathfrak{F}}) \in \mathfrak{F}$, we have that $AG^{\mathfrak{F}}/G^{\mathfrak{F}} \in \mathfrak{F}$ and $BG^{\mathfrak{F}}/G^{\mathfrak{F}} \in \mathfrak{F}$ by Corollary 2.5. So $\langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle \subseteq G^{\mathfrak{F}}$. Hence $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle (G^{\mathfrak{F}} \cap N)$. Therefore if $G^{\mathfrak{F}} \cap N = 1$, then $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle$, a contradiction. We may assume that $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle N$ for every minimal normal subgroup N of G. This means that $\mathbf{Core}_G(\langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle) = 1$.

On the other hand, $G^{\mathfrak{F}}$ is contained in G' which is nilpotent. So $A^{\mathfrak{F}}$ and $B^{\mathfrak{F}}$ are subnormal subgroups of G. By [7, A, Theorem 14.4], $\langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle$ is a subnormal subgroup of G. Using [7, A, Lemma 14.3], we have that $\mathbf{Soc}(G) \subseteq \mathbf{N}_G(\langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle)$ and so $\langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle$ is a normal subgroup of $G^{\mathfrak{F}}$.

Let p be a prime such that p divides $|N|$ for a minimal normal subgroup N of G. Since G is soluble, N is an abelian p-group. Then $G^{\mathfrak{F}}/(\langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle)$ is an abelian p-group and so $\mathbf{O}^p(G^{\mathfrak{F}})(G^{\mathfrak{F}})'$ is contained in $\langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle$. Since $\mathbf{O}^p(G^{\mathfrak{F}})(G^{\mathfrak{F}})'$ is a normal subgroup of G and $\mathbf{Core}_G(\langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle) = 1$, it follows that $\mathbf{O}^p(G^{\mathfrak{F}})(G^{\mathfrak{F}})' = 1$ and so $G^{\mathfrak{F}}$ is an abelian

p-group. Since every minimal normal subgroup of G is contained in $G^{\mathfrak{F}}$, we have that $\mathbf{F}(G) = \mathbf{O}_p(G)$ and $\mathbf{F}(G)$ is the unique Sylow p-subgroup of G since $G' \leqslant \mathbf{F}(G)$.

Let A_p and B_p denote the Sylow p-subgroups of A and B, respectively. By [1, Lemma 1.3.2], $\mathbf{F}(G) = A_pB_p$. Consider $A_p(A \cap B)$ and $B_p(A \cap B)$. These are normal subgroups of A and B containing $A \cap B$, respectively. Hence A_pB and AB_p are subgroups of G by Lemma 2.2.

Assume that $B \notin \mathfrak{F}$. Suppose that $A_pB < G$. Then $G' \leqslant \mathbf{F}(G) \leqslant A_pB$ and so A_pB is a normal subgroup of G. Since A_pB is the weakly mutually permutable product of $A_p(A \cap B)$ and B by Lemma 2.2, we have that $\langle (A_p(A \cap B))^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle = (AB_p)^{\mathfrak{F}}$ by the minimality of G. Note that $(A_p(A \cap B))^{\mathfrak{F}} \leqslant A^{\mathfrak{F}}$ by Corollary 2.5, and $(AB_p)^{\mathfrak{F}}$ is a normal subgroup of G. Hence $(AB_p)^{\mathfrak{F}} = \langle (A_p(A \cap B))^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle \leqslant \mathbf{Core}_G(\langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle) = 1$ and so $B \in \mathfrak{F}$, against our assumption. Consequently, $A_pB = G$.

Let M be a maximal subgroup of G containing B. Then M is the weakly mutually permutable product of $M \cap A$ and B by Lemma 2.2. By the minimality of G, $M^{\mathfrak{F}} = \langle (M \cap A)^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle$. Applying Corollary 2.5, we have that $(M \cap A)^{\mathfrak{F}} \leq A^{\mathfrak{F}}$. Hence $M^{\mathfrak{F}}$ is contained in $\langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle$. Moreover, $A = A_p(M \cap A)$, because $G = A_pM = \mathbf{F}(G)M$.

Suppose that $G^{\mathfrak{F}}$ is not contained in M. Then $G = MG^{\mathfrak{F}}$. Since $G^{\mathfrak{F}}$ is abelian, $M^{\mathfrak{F}}$ is normal in $G^{\mathfrak{F}}$. It follows that $M^{\mathfrak{F}}$ is normal in G since $M^{\mathfrak{F}}$ is a normal subgroup of M. But $M^{\mathfrak{F}} \leqslant \mathbf{Core}_G(\langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle) = 1$ and so $B \in \mathfrak{F}$, against our supposition. This means that $G^{\mathfrak{F}}$ is contained in M.

Let N be a minimal normal subgroup of G. Then $N \leqslant G^{\mathfrak{F}} \leqslant M$. Since $G = M\mathbf{F}(G)$ and $\mathbf{F}(G)$ centralises N, it follows that N is a minimal normal subgroup of M. If $N \cap M^{\mathfrak{F}} = N$, then $N \leqslant \langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle$ and $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle$, a contradiction. Hence $N \cap M^{\mathfrak{F}} = 1$ and $NM^{\mathfrak{F}}/M^{\mathfrak{F}}$ is a minimal normal subgroup of $M/M^{\mathfrak{F}} \in \mathfrak{F}$. Moreover, N is $\mathfrak{F}$-central in M and so N is $\mathfrak{F}$-central in G. Using [7, V, Theorem 3.2], we have that N is contained in every $\mathfrak{F}$-normalizer of G. Let E be one of these $\mathfrak{F}$-normalizers. Then $G = G^{\mathfrak{F}}E$ and $E \cap G^{\mathfrak{F}} = 1$ by [7, IV, Theorem 4.2 and Theorem 5.18]. However, $N \leqslant E \cap G^{\mathfrak{F}}$, a contradiction.

Consequently, $B \in \mathfrak{F}$. Arguing analogously with the subgroup A, we conclude that $A \in \mathfrak{F}$. By Lemma 2.6, $G \in \mathfrak{F}$, our final contradiction. □

Proof of Theorem B. Suppose that the result is false and derive a contradiction. Let (G, A, B) be a counterexample with $|G|+|G : A|+|G : B|$ as small as possible. Then G is soluble and A and B are proper subgroups of G. Let N be a minimal normal subgroup of G. Then G/N is the weakly mutually permutable product of the subgroups AN/N and BN/N by Lemma 2.1, and $(G/N)'$ is nilpotent. By [3, Lemma 4.1.10], AN/N permutes with every Sylow subgroup of BN/N and BN/B permutes with every Sylow subgroup of AN/N. Our assumption about G gives $G^{\mathfrak{F}}N = A^{\mathfrak{F}}B^{\mathfrak{F}}N$. Moreover, $A^{\mathfrak{F}}$ and $B^{\mathfrak{F}}$ are both contained in $G^{\mathfrak{F}}$ by Corollary 2.5. Consequently, $\mathbf{Soc}(G)$ is contained in $G^{\mathfrak{F}}$ and $G^{\mathfrak{F}} = A^{\mathfrak{F}}B^{\mathfrak{F}}N$ for every minimal normal subgroup N of G.

Let p be a prime divisor of $|N|$ for an arbitrary minimal normal subgroup N of G. We claim $G^{\mathfrak{F}}$ is a p-group. Suppose that $G^{\mathfrak{F}}$ is not a p-group. Since $G^{\mathfrak{F}}$ is nilpotent, it has a unique normal Hall p'-subgroup $(G^{\mathfrak{F}})_{p'} \neq 1$ and $(G^{\mathfrak{F}})_{p'}$ is the product of the Hall p'-subgroup $(A^{\mathfrak{F}})_{p'}$ of $A^{\mathfrak{F}}$ and the Hall p'-subgroup $(B^{\mathfrak{F}}N)_{p'}$ of $B^{\mathfrak{F}}N$. Since N is a p-group, we have that $(B^{\mathfrak{F}}N)_{p'} = (B^{\mathfrak{F}})_{p'}$ is a Hall p'-subgroup of $B^{\mathfrak{F}}$. Now $(G^{\mathfrak{F}})_{p'} = (A^{\mathfrak{F}})_{p'})((B^{\mathfrak{F}})_{p'})$ is normal in G. In particular, $A^{\mathfrak{F}}(G^{\mathfrak{F}})_{p'} = A^{\mathfrak{F}}(B^{\mathfrak{F}})_{p'}$ is a subgroup of G. Analogously, $A^{\mathfrak{F}}(G^{\mathfrak{F}})_p = A^{\mathfrak{F}}(B^{\mathfrak{F}})_p$ is a subgroup of G, where $(B^{\mathfrak{F}})_p$ is the unique Sylow p-subgroup of $B^{\mathfrak{F}}$. This implies that $A^{\mathfrak{F}}B^{\mathfrak{F}}$ is a subgroup of G. Let N_1 be a minimal normal subgroup of G contained in $(G^{\mathfrak{F}})_{p'}$. Then $G^{\mathfrak{F}} = A^{\mathfrak{F}}B^{\mathfrak{F}}N_1 = A^{\mathfrak{F}}B^{\mathfrak{F}}N$. Since N is a p-group and N_1 is a p'-group, it follows that $G^{\mathfrak{F}} = A^{\mathfrak{F}}B^{\mathfrak{F}}$. This contradicts the choice of G. Hence $G^{\mathfrak{F}}$ is a p-group for some prime p.

Since $\mathbf{Soc}(G)$ is contained in $G^{\mathfrak{F}}$, it follows that $\mathbf{O}_{p'}(G) = 1$ and $\mathbf{F}(G) = \mathbf{O}_p(G)$. Hence G' is a p-group and $\mathbf{F}(G)$ is the unique normal Sylow p-subgroup of G. Moreover, by Lemma 2.3, $A \cap B$ is a subnormal subgroup of G and, by [3, Proposition 4.1.16], $A \cap B$ permutes with every Sylow subgroup of A and B. Let A_p and B_p be the Sylow p-subgroups of A and B respectively such that $\mathbf{F}(G) = A_pB_p$.

Assume that $B \notin \mathfrak{F}$. The subgroups $A_p(A \cap B)$ and B of G permute and so $A_pB = A_p(A \cap B)B$ is a subgroup of G which is the weakly mutually permutable product of the subgroups $A_p(A \cap B)$ and B by Lemma 2.2. Since $G' \leqslant A_pB$, it follows that A_pB is a normal subgroup of G. Let X be a Sylow subgroup of B. Since $A \cap B$ permutes with X, we have that $(A_p(A \cap B))X = (A_p(A \cap B))((A \cap B)X) = (X(A \cap B))(A_p(A \cap B)) = X(A_p(A \cap B))$, that is, $A_p(A \cap B)$ permutes with every Sylow subgroup of B. Also, since $A \cap B$ is a subnormal subgroup of $T = A_p(A \cap B)$ and $\mid T : A \cap B \mid$ is a p-number, we have that $A \cap B$ contains every Sylow q-subgroup of T for every prime $q \neq p$ and hence B permutes with every Sylow subgroup of $T = A_p(A \cap B)$. Therefore $A_p(A \cap B)$ satisfies the hypotheses of the theorem. If $A_pB < G$, then $(A_p(A \cap B))^{\mathfrak{F}}B^{\mathfrak{F}} = (A_pB)^{\mathfrak{F}}$ by the choice of G. Since $(A_pB)^{\mathfrak{F}}$ is a normal subgroup of G and $(A_p(A \cap B))^{\mathfrak{F}}$ is a subgroup of $A^{\mathfrak{F}}$ by Corollary 2.5, we have that

$$\begin{aligned} A^{\mathfrak{F}}B^{\mathfrak{F}} &= A^{\mathfrak{F}}(A_p(A \cap B))^{\mathfrak{F}}B^{\mathfrak{F}} \\ &= A^{\mathfrak{F}}(A_pB)^{\mathfrak{F}} \\ &= (A_pB)^{\mathfrak{F}}A^{\mathfrak{F}} \\ &= B^{\mathfrak{F}}(A_p(A \cap B))^{\mathfrak{F}}A^{\mathfrak{F}} \\ &= B^{\mathfrak{F}}A^{\mathfrak{F}}. \end{aligned}$$

Therefore $A^{\mathfrak{F}}B^{\mathfrak{F}}$ is a subgroup of G and $(A_pB)^{\mathfrak{F}} \leq A^{\mathfrak{F}}B^{\mathfrak{F}}$. Since $B \notin \mathfrak{F}$ and $B^{\mathfrak{F}} \leq (A_pB)^{\mathfrak{F}}$ by Corollary 2.5, $A^{\mathfrak{F}}B^{\mathfrak{F}}$ contains a minimal normal subgroup of G, which implies that $G^{\mathfrak{F}} = A^{\mathfrak{F}}B^{\mathfrak{F}}$. This contradicts the choice of G. Hence $G = A_pB$. Since $A \cap B$, A_p and B_p are subnormal subgroups of G, we have that A is a subnormal subgroup of G by [7, A, Theorem 14.4].

Let M be a normal maximal subgroup of G containing A. Then $M = A(M \cap B)$ is the weakly mutually permutable product of A and $M \cap B$ by Lemma 2.2. Since $M \cap B$ is normal in B, we have that every Sylow subgroup X of $M \cap B$ is of the form $X = M \cap Y$ for some Sylow subgroup Y of B. Therefore $AX = A(M \cap Y) = M \cap AY = M \cap YA = XA$ and A permutes with every Sylow subgroup of $M \cap B$. Now, if Z is a Sylow subgroup of A, then Z permutes with B. Hence $Z(M \cap B) = M \cap ZB = M \cap BZ = (M \cap B)Z$ and $M \cap B$ permutes with every Sylow subgroup of A. Consequently, M satisfies the hypotheses of the theorem. The minimal choice of G implies that $M^{\mathfrak{F}} = A^{\mathfrak{F}}(M \cap B)^{\mathfrak{F}}$. Assume that $M^{\mathfrak{F}} \neq 1$. Then $M^{\mathfrak{F}}$ contains a minimal normal subgroup of G because it is normal in G. Consequently, $G^{\mathfrak{F}} = M^{\mathfrak{F}}B^{\mathfrak{F}} = A^{\mathfrak{F}}(M \cap B)^{\mathfrak{F}}$. By Corollary 2.5, $(M \cap B)^{\mathfrak{F}} \leq B^{\mathfrak{F}}$. Hence $G^{\mathfrak{F}} = A^{\mathfrak{F}}B^{\mathfrak{F}}$. This contradicts our supposition. Thus $M \in \mathfrak{F}$ and so $A \in \mathfrak{F}$ by Corollary 2.5. In particular, $G^{\mathfrak{F}} = B^{\mathfrak{F}}N$ for every minimal normal subgroup N of G. This implies that $(G^{\mathfrak{F}})' \leq B^{\mathfrak{F}}$ and so $(G^{\mathfrak{F}})' = 1$. Hence $G^{\mathfrak{F}}$ is abelian. By [7, IV, Theorem 4.2 and Theorem 5.18 and V, Theorem 3.2], $G^{\mathfrak{F}}$ does not contain any $\mathfrak{F}$-central chief factor of G.

Assume that $\mathbf{Core}_G(B) \neq 1$ and let N be a minimal normal subgroup of G contained in $\mathbf{Core}_G(B)$. Since $G = \mathbf{F}(G)B$ and $\mathbf{F}(G)$ centralises N, we have that N is a minimal normal subgroup of B. Now $G^{\mathfrak{F}} \neq B^{\mathfrak{F}}$ implies that $B^{\mathfrak{F}} \cap N = 1$ and N is an $\mathfrak{F}$-central chief factor of B. Therefore N is $\mathfrak{F}$-central in G, a contradiction. Hence $\mathbf{Core}_G(B) = 1$.

Write $E = \mathbf{Core}_A(A \cap B) \neq 1$. Then the normal closure $E^G = E^B$ is a normal subgroup of G contained in B. Thus $E^G \leq \mathbf{Core}_G(B) = 1$. In particular, $E = 1$ and $A \cap B$ is nilpotent by [3, Theorem 1.2.14]. The subnormality of $A \cap B$ yields $A \cap B \leq \mathbf{F}(G)$. Hence A is a p-group.

Let L be a maximal subgroup of G containing B. Then $L = B(A \cap L)$ is the weakly mutually permutable product of B and $A \cap L$ by Lemma 2.2. The same arguments to those used above with M show that L satisfies the hypotheses of the theorem. The minimality of G yields $L^{\mathfrak{F}} = B^{\mathfrak{F}}(A \cap L)^{\mathfrak{F}} = B^{\mathfrak{F}}$.

Assume that C is a subgroup of L containing $A \cap L$. Then $C = (B \cap C)(A \cap L)$ and $A \cap B \leq B \cap C$. Since A permutes with $B \cap C$, we have that A permutes with C. Let D be a Sylow q-subgroup of L for some prime $q \neq p$. Then there exist a Sylow q-subgroup J of B and elements $b \in B$ and $a \in A \cap L$ such that $D = J^{ba}$. By hypothesis, AJ^b is a subgroup of G and so is AD. Therefore A permutes with D. Since A permutes with the Sylow p-subgroup of L, we have that A permutes with every Sylow subgroup of L. Now, if K is a subgroup of A containing $A \cap L$, then K contains $A \cap B$ and so K permutes with B and so K permutes with L. We have shown that the triple (G, A, L) satisfies the hypotheses of the theorem. If B were a proper subgroup of L, we would be have that $G^{\mathfrak{F}} = A^{\mathfrak{F}} L^{\mathfrak{F}} = L^{\mathfrak{F}} = B^{\mathfrak{F}}$ by the choice of G. This contradicts our assumption. Consequently, $B = L$ is a core-free maximal subgroup of G and G is a primitive soluble group. Then $\mathbf{F}(G)$ is a minimal normal subgroup of G and $B \cap \mathbf{F}(G) = 1$. In particular, $A \cap B = 1$ and G is the totally permutable product of A and B. Applying [3, Theorem 5.2.7], $G^{\mathfrak{F}} = A^{\mathfrak{F}} B^{\mathfrak{F}}$, a contradiction.

Consequently $B \in \mathfrak{F}$. Arguing analogously with A, we conclude that $A \in \mathfrak{F}$. By Lemma 2.7, $G \in \mathfrak{F}$. This final contradiction proves the theorem.

□

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