Finite groups whose prime graph on class sizes is a block square

V. Sotomayor*

Abstract

Let G be a finite group, and let $\Delta(G)$ be the prime graph built on its set of conjugacy class sizes: this is the (simple undirected) graph whose vertices are the prime numbers dividing some conjugacy class size of G, and two distinct vertices p, q are adjacent if and only if pq divides some class size of G. In this paper, we characterise the structure of those groups G whose prime graph $\Delta(G)$ is a *block square*.

Keywords Finite groups · Conjugacy classes · Prime graph

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1 Introduction

Throughout this paper, all groups considered are finite. Within finite group theory, the influence of the arithmetical properties of the conjugacy class sizes of a group on its algebraic structure is a research area that has attracted the interest of several authors over the last decades. The *prime graph* built on the set of class sizes of a group G, which we denote by $\Delta(G)$, is a useful tool that is gaining an increasing interest for analysing the arithmetical properties of this set. This (simple undirected) graph has as vertex set $V(G)$ the prime divisors of the conjugacy class sizes of G, and its edge set $E(G)$ contains pairs $\{p, q\} \subseteq V(G)$ such that pq divides some class size of G. In this framework, two relevant question that arise are: which graphs can occur as $\Delta(G)$ for some finite group G, and how is the structure of G affected by the graph-theoretical properties of $\Delta(G)$?

Interestingly, non-adjacency between vertices of $\Delta(G)$ highly restricts the structure of G, which suggests that $\Delta(G)$ tends to have "many" edges. In fact, the extreme case

*Departamento de Matemáticas y Estadística, Centro Universitario EDEM, Muelle de la Aduana s/n, La Marina de Valencia, 46024 Valencia, Spain.
✉: vsotomayor@edem.es

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when $\Delta(G)$ is disconnected happens if and only if G is a *$\mathcal{D}$-group*, that is, $G = AB$ where $A \trianglelefteq G$ and B are abelian subgroups of coprime orders, $\mathbf{Z}(G) \leqslant B$, and the factor group $G/\mathbf{Z}(G)$ is a Frobenius group with kernel $A\mathbf{Z}(G)/\mathbf{Z}(G)$ (see Theorem 4 of [3]). In this situation, G has three class sizes, which are $\{1, |A|, |B/\mathbf{Z}(G)|\}$. So the vertex sets of the (two) connected components of $\Delta(G)$ turn out to be the sets of prime divisors of the orders of A and $B/\mathbf{Z}(G)$, respectively, and both sets are *cliques* (i.e. they induce complete subgraphs) of $\Delta(G)$.

In [1], C. Casolo *et al.* studied the structure of those finite groups G such that $\Delta(G)$ has no complete vertices. Moreover, they characterised those groups whose prime graph on class sizes is non-complete and regular, and they are basically direct products of certain $\mathcal{D}$-groups. In particular, if $\Delta(G)$ is a square with $V(G) = \{p, q, r, s\}$ and $E(G) = \{\{p, r\}, \{p, s\}, \{q, r\}, \{q, s\}\}$, then from their result it follows that (up to abelian direct factors) $G = A \times B$ where A and B are $\mathcal{D}$-groups of orders divisible by $\{p, q\}$ and $\{r, s\}$, respectively.

A natural way to generalise a square graph is to replace each vertex by a set of vertices. In this spirit, a graph is called a *block square* if its vertex set can be written as a union of four disjoint, non-empty subsets $\pi_1, \pi_2, \pi_3, \pi_4$, where no prime in π_1 is adjacent to any prime in π_4 and no prime in π_2 is adjacent to any prime in π_3, and there exist vertices in both π_1 and π_4 that are adjacent to vertices in π_2 and in π_3. Certainly, any direct product $G = A \times B$ of two coprime $\mathcal{D}$-groups yields a block square $\Delta(G)$. So the question that naturally arises is whether there exist other types of groups whose prime graph on class sizes is a block square. The main result of this paper shows that in fact this is the unique way of obtaining groups with such class-size prime graph.

Theorem A. *Let G be a finite group. Then $\Delta(G)$ is a block square if and only if, up to an abelian direct factor, $G = A \times B$ where A and B are $\mathcal{D}$-groups of coprime orders.*

As a consequence, we have attained a characterisation of the block square graphs that can occur as $\Delta(G)$ for some finite group G.

Corollary B. *Let Δ be a block square graph. Then there exists a finite group G such that $\Delta(G) = \Delta$ if and only if all the primes in $\pi_1 \cup \pi_4$ are adjacent to all the primes in $\pi_2 \cup \pi_3$.*

Frequently, the results on the class-size context have a dual version in the context of degrees of irreducible characters. It is worth mentioning that M.L. Lewis and Q. Meng introduced in [4] the concept of block square graphs, and in that paper they carried out an analysis of block squares for the prime graph built on the character degrees of *soluble* groups. Among other things, they proved an analogous version of Theorem A for the character-degree prime graph in the particular case that the group possesses two normal non-abelian Sylow subgroups.

2 Preliminaries

In the sequel, if x is an element of a group G, then we denote by x^G the conjugacy class of x in G, and its size is $|x^G| = |G : \mathbf{C}_G(x)|$. For a positive integer n, we write $\pi(n)$ for the set of prime divisors of n, and in particular $\pi(G)$ is the set of prime divisors of $|G|$. As usual, given a prime p, the set of all Sylow p-subgroups of G is denoted by $\mathrm{Syl}_p(G)$, and $\mathrm{Hall}_\pi(G)$ is the set of all Hall π-subgroups of G for a set of primes π. A group is called p-nilpotent if it has a normal Hall p'-subgroup. The remaining notation and terminology used is standard in the framework of finite group theory.

The following elementary properties will be used without further reference.

Lemma 2.1. *Let G be a group. Then the following conclusions hold.*

(a) *If either $x, y \in G$ have coprime orders and they commute, or $x \in M$ and $y \in N$ with M and N normal subgroups of G such that $M \cap N = 1$, then $\pi(|x^G|) \cup \pi(|y^G|) \subseteq \pi(|(xy)^G|)$.*

(b) *A given prime p does not lie in $V(G)$ if and only if G has a central Sylow p-subgroup.*

As it was mentioned in the Introduction, non-adjacency between vertices significantly constrains the structure of the group. The next result also illustrates this fact. It is Theorem C of [2].

Proposition 2.2. *Let G be a group. If π is a set of vertices which are all non-adjacent in $\Delta(G)$ to a vertex p, then G is π-soluble with abelian Hall π-subgroups, and the vertices in π are pairwise adjacent.*

Observe that if $\Delta(G)$ is a block square, then it certainly has no complete vertices. Therefore the result below, which is Theorem C of [1], yields a reduction on the structure of such a group G.

Proposition 2.3. *Let G be group. Assume that no vertex of $\Delta(G)$ is complete. Then, up to an abelian direct factor, $G = KL$ with $K \trianglelefteq G$ and L abelian subgroups of coprime orders. Moreover, $K = G'$, $K \cap \mathbf{Z}(G) = 1$, and both $\pi(K)$ and $\pi(L)$ are cliques of $\Delta(G)$.*

We close this section with the next key fact, which is partially Proposition 3.1 of [2].

Proposition 2.4. *Let G be a group, and p, q non-adjacent vertices of $\Delta(G)$. Let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, and let M be a non-trivial abelian normal subgroup of G such that $|M|$ is a power of a suitable prime r. Assume that $\mathbf{C}_M(P) = 1$, and that M has a complement in G. Then $\mathbf{O}_q(G) = Q \cap \mathbf{C}_G(M) \leqslant \mathbf{Z}(\mathbf{C}_G(M))$.*

3 Proof of main results

Proof of Theorem A. First, recall that the class sizes of G are the same that those of $G \times A$, where A is an abelian group. Therefore, we may assume that G has no abelian direct factors, and in particular $\pi(G) = V(G)$.

If $G = A \times B$ where A and B are $\mathcal{D}$-groups of coprime orders, then certainly $\Delta(G)$ is a block square, where π_1 and π_4 are respectively the set of prime divisors of the Frobenius kernel and complement of $A/\mathbf{Z}(A)$, and π_2 and π_3 are the set of prime divisors of the Frobenius kernel and complement of $B/\mathbf{Z}(B)$.

Therefore, the remainder of the proof is devoted to show that if $\Delta(G)$ is a block square, then $G = A \times B$ where A and B are $\mathcal{D}$-groups of coprime orders. In virtue of Proposition 2.2, we have that π_i is a clique of $\Delta(G)$ and there exists an abelian Hall π_i-subgroup H_i of G for every $i \in \{1, 2, 3, 4\}$. In particular, all the Sylow subgroups of G are abelian.

Since there is no complete vertices in $\Delta(G)$, then by Proposition 2.3 it follows that $G = KL$ where $K \trianglelefteq G$ and L are abelian subgroups of coprime orders, $K = G'$, $K \cap \mathbf{Z}(G) = 1$, and both $\pi(K)$ and $\pi(L)$ induce complete subgraphs in $\Delta(G)$. In particular, $V(G) = \pi(K) \cup \pi(L)$.

Without loss of generality, we may assume that there exists a prime $p \in \pi_1 \cap \pi(K)$, so G has a normal (abelian) Sylow p-subgroup. As there is no edge in $\Delta(G)$ between π_1 and π_4, and $\pi(K)$ is a clique of $\Delta(G)$, then $\pi_4 \subseteq \pi(L)$. Further, as $\pi(L)$ is also a clique of $\Delta(G)$, then necessarily it holds that $\pi_1 \subseteq \pi(K)$, so $H_1 = \mathbf{O}_{\pi_1}(G) \leqslant K$. Arguing analogously, we may suppose that $\pi_3 \subseteq \pi(K)$ and $\pi_2 \subseteq \pi(L)$. It follows that $K = H_1 \times H_3$ and, up to conjugation, $L = H_2 \times H_4$.

Next we proceed in three steps.

Step 1: For each $s \in \pi_4$, there exists $p \in \pi_1$ such that $[P, S] \neq 1$, where $P \in \mathrm{Syl}_p(H_1)$ and $S \in \mathrm{Syl}_s(H_4)$. Besides, for each $r \in \pi_2$, there exists $q \in \pi_3$ such that $[Q, R] \neq 1$, where $Q \in \mathrm{Syl}_q(H_3)$ and $R \in \mathrm{Syl}_r(H_2)$.

In order to prove the first assertion, and arguing by contradiction, let us assume that $[S, H_1] = 1$. Since L is abelian and $s \in V(G)$, then there necessarily exists $q \in \pi_3$ and $Q \in \mathrm{Syl}_q(H_3)$ such that $[Q, S] \neq 1$. As $Q \trianglelefteq G$, there must exist some $y \in S$ with $q \in \pi(|y^G|)$.

Let $r \in \pi_2$ and $R \in \mathrm{Syl}_r(H_2)$. We claim that $[R, H_1] = 1$. If not, then R does not centralise some $P \in \mathrm{Syl}_p(H_1)$ for some $p \in \pi_1$. If r does not divide $|x^G|$ for each element $x \in P$, then $x \in \mathbf{C}_G(R^{g_x})$ for some $g_x \in G$, and we may suppose that $g_x \in K$. But then $x = x^{g_x^{-1}} \in \mathbf{C}_G(R)$ because $x \in P \leqslant \mathbf{Z}(K)$. Since this is valid for all the elements $x \in P$, we get that $P \leqslant \mathbf{C}_G(R)$, a contradiction. Hence we may take an element $x \in P \leqslant H_1$ with $r \in \pi(|x^G|)$. As $[S, H_1] = 1$, then qr divides $|(xy)^G|$, which is a contradiction because $q \in \pi_3$ and $r \in \pi_2$.

Since the previous argument holds for each prime $r \in \pi_2$, we deduce that H_2 centralises H_1. But $H_1 \leqslant \mathbf{Z}(K)$, so there exists $t \in \pi_4$ and $T \in \mathrm{Syl}_t(H_4)$ with $[H_1, T] \neq 1$ (in particular $t \neq s$). So we can take a suitable prime $p \in \pi_1$ such that $[P, T] \neq 1$ for $P \in \mathrm{Syl}_p(H_1)$. In particular, p divides $|w^G|$ for some $w \in T$.

Next we claim $\mathbf{C}_Q(T) = 1$. Let us suppose that there exists a non-trivial element $x \in \mathbf{C}_Q(T)$. Certainly $K \leqslant \mathbf{C}_G(x)$, and since $K \cap \mathbf{Z}(G) = 1$, then there exists a prime $u \in \pi_2 \cup \pi_4$ such that u divides $|x^G|$. Recall that $[Q, S] \neq 1$ by the first paragraph, so we can pick an element $z \in Q$ with $s \in \pi(|z^G|)$. If Q centralises T, then $|(wz)^G|$ is divisible by both $p \in \pi_1$ and $s \in \pi_4$, a contradiction. Thus Q does not centralise T, and therefore there exists an element $w_2 \in T$ with $q \in \pi(|w_2^G|)$. Now we distinguish two cases: if $u \in \pi_2$, then the class size of xw_2 is divisible by u and $q \in \pi_3$, a contradiction; if $u \in \pi_4$, then $\{u, p\} \subseteq \pi(|(xw)^G|)$, which is also a contradiction. Hence $\mathbf{C}_Q(T) = 1$.

Recall that Q is an abelian normal Sylow q-subgroup of G, so it is complemented in G. Since $\{p, t\} \notin E(G)$ for every $p \in \pi_1$, then Proposition 2.4 leads to $P \leqslant \mathbf{Z}(\mathbf{C}_G(Q))$ for $P \in \mathrm{Syl}_p(G)$, and this is valid for every prime $p \in \pi_1$. It follows $\mathbf{C}_G(Q) \leqslant \mathbf{C}_G(H_1)$.

Note that π_1 and π_2 are adjacent in $\Delta(G)$ by hypothesis, so there exist $v_1 \in \pi_1$ and $v_2 \in \pi_2$ such that $v_1 v_2 \in \pi(|g^G|)$ for some $g \in G$. We can decompose $g = g_k g_l$ in such way that $g_k \in K$, $g_l \in L$ up to conjugation, and $g_k g_l = g_l g_k$. In particular, since additionally g_k^G and g_l^G have coprime sizes, then $|g^G|$ is the product of $|g_k^G|$ and $|g_l^G|$. Therefore $v_1 \in \pi(|g_l^G|)$ and $v_2 \in \pi(|g_k^G|)$. As $\mathbf{C}_G(Q) \leqslant \mathbf{C}_G(H_1)$ by the previous paragraph, then certainly $g_l \notin \mathbf{C}_G(Q)$. This means that q divides the class size of g_l, so $|g^G|$ is divisible by both $v_2 \in \pi_2$ and $q \in \pi_3$, a contradiction.

The first assertion of Step 1 is already proved. Observe that the second part analogously follows, since the roles of π_1 and π_4 are symmetric with respect to π_3 and π_2.

Step 2: H_1 centralises H_2, and H_3 centralises H_4.

For proving that $[H_1, H_2] = 1$, let us suppose that there exist $P \in \mathrm{Syl}_p(H_1)$ and $R \in \mathrm{Syl}_r(H_2)$ such that they do not commute, and we aim to reach a contradiction. Note that we can then take an element $z_1 \in R$ such that p divides $|z_1^G|$.

We claim $\mathbf{C}_P(R) = 1$. Otherwise, there exists a non-trivial element $x \in \mathbf{C}_P(R)$, and since $K \cap \mathbf{Z}(G) = 1$, then $\pi(|x^G|)$ contains a suitable prime $u \in \pi_2 \cup \pi_4$. Using Step 1, there is a prime $q \in \pi_3$ and $Q \in \mathrm{Syl}_q(G)$ such that Q does not centralise R. Hence q divides the class size of certain element $z_2 \in R$. It follows that the class sizes of xz_1 and xz_2 are divisible by pu and qu, respectively. As $u \in \pi_2 \cup \pi_4$, then this contradicts our assumptions. Therefore $\mathbf{C}_P(R) = 1$.

Since $\{q, r\} \notin E(G)$ for every $q \in \pi_3$, then by Proposition 2.4 we get that $Q \leqslant \mathbf{Z}(\mathbf{C}_G(P))$, for $Q \in \mathrm{Syl}_q(G)$, and for all primes $q \in \pi_3$. It follows that $\mathbf{C}_G(P) \leqslant \mathbf{C}_G(H_3)$.

By assumptions, we can take an element $g \in G$ such that $v_3 v_4 \in \pi(|g^G|)$, where $v_3 \in \pi_3$ and $v_4 \in \pi_4$. This element can be written as a product of two suitable commutative elements $g_k \in K$ and $g_l \in L$. Since $|g^G|$ is the product of the class sizes of g_k and g_l, then necessarily we obtain that $v_3 \in \pi(|g_l|)$ and $v_4 \in \pi(|g_k|)$. Thus, in virtue of the above paragraph, g_l cannot centralise P, and therefore $p \in \pi(|g_l^G|)$. It follows that pv_3v_4 divides $|g^G|$, which is a contradiction because $p \in \pi_1$.

Hence $[H_1, H_2] = 1$, and analogously it can be proved $[H_3, H_4] = 1$. These two facts together with Step 1 yield $G = A \times B$, where $A := H_1H_4$ with $H_1 \trianglelefteq A$ and $B := H_3H_2$ with $H_3 \trianglelefteq B$. Note that A and B have coprime orders.

Step 3: $A = H_1H_4$ and $B = H_3H_2$ are $\mathcal{D}$-groups.

We will show that $A = H_1H_4$ is a $\mathcal{D}$-group, and the same arguments are analogously valid for B. Note that H_1 and H_4 are abelian groups of coprime orders. Moreover, $\mathbf{Z}(A) \leqslant H_4$ since the Hall π_1-subgroup of $\mathbf{Z}(A)$ is contained in $K \cap \mathbf{Z}(G) = 1$.

Let $Z := \mathbf{Z}(A)$. We claim that A/Z is a Frobenius group with Frobenius kernel H_1Z/Z. By coprime action, it is enough to prove that $\mathbf{C}_{H_4}(x) \leqslant Z$ for every non-trivial element $x \in H_1$. If this does not hold, then we can take an element $y \in \mathbf{C}_{H_4}(x) \smallsetminus Z$, so $|y^A|$ is divisible by some prime $p \in \pi_1$. Since $x \notin Z$, then its class size in A is divisible by certain prime $s \in \pi_4$. Therefore $ps \in \pi(|(xy)^A|) = \pi(|(xy)^G|)$, which contradicts our assumptions. □

Proof of Corollary B. In virtue on Theorem A, the necessity of the condition is clear. Hence, let us suppose that Δ is a block square graph where all the primes in $\pi_1 \cup \pi_4$ are adjacent to all the primes in $\pi_2 \cup \pi_3$, and we aim to show that there exists a suitable group G such that $\Delta(G) = \Delta$.

Let m_i denote the size of each π_i, for $i \in \{1, 2, 3, 4\}$. Let $n_1 := p_1p_2 \cdots p_{m_1}$ where the p_j are pairwise distinct prime numbers. Let $s_1, s_2, \ldots, s_{m_4}$ be distinct primes such that $n_4 := s_1s_2 \cdots s_{m_4}$ is congruent to 1 modulo n_1; we point out that they exist by Dirichlet's theorem on primes in an arithmetic progression. Let K_1 and L_1 be cyclic groups of orders n_1 and n_4, respectively. Consider the semidirect product $A = K_1 \rtimes L_1$ with respect to a Frobenius action of L_1 on K_1. Certainly A is a $\mathcal{D}$-group.

Now let $n_3 := q_1q_2 \cdots q_{m_3}$ where $q_k \notin \pi(A)$ for each $k \in \{1, \ldots, m_3\}$ and they are pairwise distinct primes. Consider a set $\{r_1, r_2, \ldots, r_{m_2}\}$ of pairwise distinct primes such that none of them lies in $\pi(A)$ and $n_2 := r_1r_2 \cdots r_{m_2}$ is congruent to 1 modulo n_3; again they exist by the aforementioned theorem due to Dirichlet. Let K_2 and L_2 be cyclic groups of orders n_3 and n_2, respectively. Consider the semidirect product $B = K_2 \rtimes L_2$ with respect to a Frobenius action of L_2 on K_2, so B is a $\mathcal{D}$-group.

Note that A and B have coprime orders. Let $G = A \times B$. Hence it easily follows that $\Delta(G)$ is a block square graph where all the vertices in $\pi(K_1) \cup \pi(L_1)$ are adjacent to all the vertices in $\pi(K_2) \cup \pi(L_2)$. Thus $\Delta(G) = \Delta$. □

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