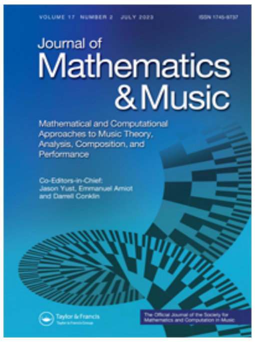



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# Type and class vectors and matrices in $\mathbb{Z}_n$. Application to $\mathbb{Z}_6$, $\mathbb{Z}_7$, and $\mathbb{Z}_{12}$


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# Type and class vectors and matrices in $\mathbb{Z}_n$. Application to $\mathbb{Z}_6$, $\mathbb{Z}_7$, and $\mathbb{Z}_{12}$

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In post-tonal theory, set classes are normally elements of $\mathbb{Z}_{12}$ and are characterized by their interval-class vector. Those being non-inversionally-symmetrical can be split into two set types related by inversion, which can be characterized by their trichord-type vector. In this paper, I consider the general case of set classes and types in $\mathbb{Z}_n$ and their $m$-class and $m$-type vectors, $m$ ranging from 0 to $n$, which are properly grouped into matrices. As well, three relevant cases are considered: $\mathbb{Z}_6$ (hexachords), $\mathbb{Z}_7$ (heptatonic scales), and $\mathbb{Z}_{12}$ (chromatic scale), where all those type and class matrices are computed and provided in supplementary files; and, in the first two cases, also in the form of tables. This completes the corresponding information given in previous publications on this subject and can directly be used by researchers and composers. Moreover, two computer programs, written in MATLAB, are provided for obtaining the above-mentioned and other related matrices in the general case of $\mathbb{Z}_n$. Additionally, several theorems on type and class matrices are provided, including a complete version of the hexachord theorem. These theorems allow us to obtain the type and class matrices by different procedures, thus providing a broader perspective and better understanding of the theory.

**Keywords:** Complementary order; normal order; interval-class vector; trichord-type vector; trichord-class vector; type vector; class vector; type matrix; class matrix

## 1. Introduction

Tonal music is based on major and minor scales. They are heptatonic scales whose notes show great acoustical affinity among them. Nowadays, the prevalent tuning system is *twelve-tone equal temperament* (12-TET), which divides the octave into 12 equal parts, from which the notes of any scale are taken. In the *musical set theory*, the notes are *pitch classes* that are represented by integers of $\mathbb{Z}_{12}$, where 0 corresponds to note C and 11 to note B. Similarly, for the general case when the octave is divided into $n$ parts, the set theory in $\mathbb{Z}_n$ can be used. From another point of view, and following Tymoczko (2011, 4.1), the different heptatonic scales can be studied in a simplified and unified way in $\mathbb{Z}_7$, considering the divisions of the scale unequally sized. In this case, however, a chord type usually represents pitch-class sets with different sonorities.

Regarding atonal music, it is not based on major or minor scales, or their typical harmonies, and some composers began to experiment with it by the early twentieth century. In particular, Schönberg develops a composition technique based on 12-tone series, where all notes are equally relevant (Perle 1991 [1962]). Around the same time, Hauer develops

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another 12-tone composition technique, based on so-called "tropes", which are pairs of complementary hexachords (Šedivý 2011). As a further development, O'Gallagher (2013) creates a system for improvising with 12-tone rows, which are actually tropes made up from the six "all-combinatorial" hexachords. Therefore, in these cases, the set theory both in $\mathbb{Z}_{12}$ and $\mathbb{Z}_6$ are helpful. Morris (2007) gives a historical overview of the mathematics involved in the development of the twelve-tone system, starting from the work by Babbitt (1955).

The study of pitch-class sets and set classes in a more general context was carried out by Forte (1973), who published the first list of set classes in $\mathbb{Z}_{12}$, introducing his set-class names and including the interval-class vectors, which list the 2-note set classes contained in each set class. His work was continued by Rahn (1980) in a more formal mathematical language, whereas Lewin (1987) is a significant work laying the foundations for a systematic study and giving numerous relevant results. Straus (1991) is a primer on this subject under the name of *atonal set theory* and Straus (2016) is an undergraduate textbook under the name of *post-tonal theory*. Recently, Nuño (2021) arranged the set classes in $\mathbb{Z}_{12}$ in a compact periodic table and provided a more detailed list of set classes and types, including the trichord-type vectors, which list the 3-note set types contained in each set type. As explained there, from these vectors it is easy to obtain the corresponding trichord-class vectors.

In the light of the foregoing, in this paper the concepts of interval-class, trichord-type, and trichord-class vectors, which correspond to cardinalities two and three, are extended to the rest of cardinalities. To cover the general case, the study is carried out in $\mathbb{Z}_n$. But, due to its major relevance, the particular case of $\mathbb{Z}_{12}$ received special attention and the corresponding $m$-type and $m$-class vectors, for $0 \leq m \leq 12$, after grouping them into appropriate matrices, are given in a supplementary file. And the same has been done for the cases of $\mathbb{Z}_6$ and $\mathbb{Z}_7$; but, because they are quite simple, the corresponding type vectors and matrices are also given in two tables, which clearly show the full picture and, moreover, serve for illustrating part of the theory here developed. As well, they are significant examples of $n$ being composite and prime numbers. Furthermore, two computer programs, written in MATLAB, are provided for obtaining the above-mentioned and other related matrices in the general case of $\mathbb{Z}_n$. For reasons of generality, we will consider that, in all cases, the divisions of the octave are not necessarily equally sized. The reader is assumed to be familiar with Forte names and set classes (Forte 1973), although some of the main concepts are also explained here.

## 2. Basic concepts

A *pitch class* represents all pitches that are a whole number of octaves apart. If the octave is divided into $n$ parts, not necessarily equal, and here called *steps*, each pitch class can be represented by an integer of $\mathbb{Z}_n$. A *pitch-class set* is then a subset of $\mathbb{Z}_n$ and can be represented either by the corresponding *set of integers* or by the *characteristic function*. The number of pitch classes in the pitch-class set is its *cardinality*. For example, in twelve-tone equal temperament (12-TET), we have that $n = 12$, a step is a semitone, pitch class C is assigned the integer 0, and a pitch-class set such as [G,B,D,F], that is, G7, can be represented either by the set of integers [7,11,2,5] or by the characteristic function [001001010001] (both written here in square brackets), its cardinality being 4. Additionally, it is common (and useful!) to envision those integers as hours on a clockface, where "12" is substituted with "0". The *transposition* and *inversion operations*[1] in $\mathbb{Z}_n$

[1] The *transposition* of a pitch-class set consists in adding the same integer to all its pitch classes. For example, in $\mathbb{Z}_{12}$, by adding 3 to G7 = [7,11,2,5], we obtain [10,2,5,8] (mod 12), that is, B♭7 (3 semitones above G7). And the *inversion* of a pitch-class set consists in changing the sign of all its pitch classes. Thus, in $\mathbb{Z}_{12}$, the inversion of B♭7 = [10,2,5,8] is [-10,-2,-5,-8] or [2,10,7,4] (mod 12), that is, $E^{ø}$. If we envision those integers as a set of hours on a clockface, the inversion is simply obtained by turning them around the "0 – 6" axis (backside front).

are defined in a similar way as in $\mathbb{Z}_{12}$, as well as the *degrees of transpositional and inversional symmetries.*[2] In the same way, given a pitch-class set, the pitch classes excluded in it compose its *complement*. Therefore, the characteristic function of the complement is obtained from the original one by simply substituting every 1 with a 0 and vice versa. As can easily be proved, the degrees of transpositional and inversional symmetries of a pitch-class set, its inversion, and its complement are the same. If $n$ is even, the pitch-class sets with cardinality $n/2$ constitute a special case and we will call them *hemichords*.

All pitch-class sets related by transposition form a *set type* or *chord type*. If the steps are equally sized, all those pitch-class sets will have the same "sonority". A chord type will be represented by any of its pitch-class sets, either by its set of integers or its characteristic function. Regarding the set of integers, it is common to choose them in increasing order, starting from zero, and with the rest of them being reasonably low, which is known as the *normal form*. There are actually two commonly used normal forms, one given by Forte (1973) and the other by Rahn (1980), which in most cases are the same. This is the case for the previous example (G7), where both normal forms are [0,3,6,8], which corresponds to A♭7. Although the two normal forms were defined for $\mathbb{Z}_{12}$, they can be generalized to $\mathbb{Z}_n$ straightforwardly. A third option for representing a chord type is to generalize the *intervallic form* given by Nuño (2021), which will be the sequence of intervals, in steps, between every two adjacent pitch classes, including the interval between the last and the first ones; or any of its circular shifts. In the current example, it is {3,3,2,4} (here written in curly brackets) or any of its circular shifts. Note that the sum of all integers in any intervallic form in $\mathbb{Z}_n$ is, obviously, $n$. The least of all possible circular shifts of an intervallic form, with respect to the lexicographic order, is the *normal intervallic form*. In this case, {2,4,3,3}. The intervallic form has several important properties. Thus, given the intervallic form of a chord type, it is simple to obtain the one of its inversion, its complement, and, therefore, the inversion of its complement. Additionally, it allows to easily obtain the degrees of transpositional and inversional symmetries. Examples are given in Nuño (2021).

On the other hand, all pitch-class sets related by transposition or inversion form a *set class* or *chord class*, which will be represented by any of its pitch-class sets. Thus, a chord class generally consists of two types related by inversion, which we will call $a$ and $b$, following Nuño (2021), their sonorities being different. In this case, the least of the two normal forms or the two normal intervallic forms, with respect to the lexicographic order, are, respectively, the *prime form* or the *prime intervallic form* (which do not necessarily correspond to the same type!). But, when a chord type and its inversion are the same, the chord type is *inversionally symmetrical*, thus being actually a chord class. For simplicity, a chord type with cardinality $c$ will be called a $c$-type, and a chord class with cardinality $c$, a $c$-class.

## 3. Type and class vectors and matrices

The next definition is closely related to the *embedding number* introduced by Lewin (1987, 5.3.1), with the difference that here an "unweighted" version is also considered, which has interesting properties and will be used for obtaining relevant results, such as Theorems 5.1 and

[2] The *degree of transpositional symmetry* $s_T$ of a pitch-class set is the number of different transpositions by which the pitch-class set maps into itself. It is at least 1 (which corresponds to adding the integer 0 to all its pitch classes), but may be greater (up to $n$ in $\mathbb{Z}_n$ and necessarily a divisor of $n$). For example, in $\mathbb{Z}_{12}$, every augmented triad has $s_T = 3$ ($\mathsf{C+} = [0,4,8] = \mathsf{E+} = \mathsf{G\sharp+}$). And the *degree of inversional symmetry* $s_I$ of a pitch-class set is the number of different transpositions, performed after an inversion, by which the pitch-class set maps into itself. It may be 0, but if it is different from 0, then equals $s_T$ and the pitch-class set is said to be *inversionally symmetrical*. For example, in $\mathbb{Z}_{12}$, every diminished triad has $s_I = 1$ (Bdim = [11,2,5], whose inversion is [-11,-2,-5] or [1,10,7]; and, by adding 4 to all the pitch classes, we obtain [5,2,11], which is again Bdim). As well, every major triad has $s_I = 0$ and every augmented triad, $s_I = 3$.

7.5. As well, it is easier to compute and two different mathematical expressions are provided for obtaining it.

*Definition 3.1 (Type and class contents):* Let $S$ and $T$ be two chord types in $\mathbb{Z}_n$. The *unweighted type content* of $S$ with respect to $T$, $R(S/T)$, is the number of times the former contains the latter, multiplied by the degree of transpositional symmetry of $T$.

For example, if $S$ is the diminished triad (3-type) and $T$ the tritone (2-type), then $R(S/T) = 2$, because a diminished triad contains *one* tritone and the degree of transpositional symmetry of a tritone is *two*.

To provide a mathematical formula for computing $R(S/T)$, let us use the same letters $S$ and $T$ for representing the characteristic functions of the two chord types, and let $c$ and $m$ be their corresponding cardinalities. Then, by defining

$$p_m(j) = \sum_{k=0}^{n-1} S(k)T(k+j), \quad j = 0, \dots, n-1, \tag{1}$$

where $k + j$ is understood modulo $n$, $R(S/T)$ will be the number of times $p_m(j) = m$ for $j = 0, \dots, n-1$, which can be expressed as

$$\begin{aligned} R(S/T) &= \sum_{j=0}^{n-1} \left\lfloor \frac{p_m(j)}{m} \right\rfloor, \quad m \neq 0, \\ R(S/T) &= n, \quad m = 0, \end{aligned} \tag{2}$$

where "$\lfloor x \rfloor$" is the floor function (the greatest integer less than or equal to $x$, for $x \in \mathbb{R}$).

Note that, if $T$ has a degree of transpositional symmetry $s_T$, then the *same* vector is used $s_T$ times when computing $p_m(j)$ in (1). So, the *actual* number of times $S$ contains $T$ is $R(S/T)/s_T$, which we will call the *weighted type content* or, simply, the *type content* of $S$ with respect to $T$ (this is the Lewin's embedding number "EMB$(T, S)$" for "CANON" being the group of transposition operations[3]). Thus, if $S$ is the diminished triad and $T$ the tritone ($s_T = 2$), the type content of $S$ with respect to $T$ will be $R(S/T)/s_T = 1$, indicating that a diminished triad contains just *one* tritone. As $s_T$ is required for obtaining the type contents, some immediate values of it are given below:

$$\begin{aligned} &s_T = n \text{ for } m = 0 \text{ or } m = n, \\ &s_T = 1 \text{ for } m = 1,\ m = n-1, \text{ or if } m \text{ is not a divisor of } n, \\ &s_T = 1 \text{ for } m = 2 \text{ or } m = n-2, \text{ except if } n \text{ is even} \\ &\text{and } T = [0, n/2] \text{ or its complement, respectively, in} \\ &\text{which cases } s_T = 2. \end{aligned} \tag{3}$$

Alternatively, we can represent the chord type $T$ by a set of integers $[i_1, \cdots, i_m]$ (usually, in normal form), which allows computing $R(S/T)$ as

$$R(S/T) = \sum_{j=0}^{n-1} S(j+i_1) \cdots S(j+i_m), \tag{4}$$

where $j + i_k$, $1 \leq k \leq m$ are understood modulo $n$. The equivalence with the previous definition is clear by substituting $T(k+j)$ with $T(k-j)$ in (1), which does not modify $R(S/T)$ in (2). In this case, for $m \neq 0$, $p_m(j) = m$, or equivalently $\lfloor p_m(j)/m \rfloor = 1$, if and only if $S(j+i_1) \cdots S(j+i_m) = 1$; and, if $p_m(j) < m$, both $\lfloor p_m(j)/m \rfloor$ and $S(j+i_1) \cdots S(j+i_m)$ are 0.

[3] See Lewin (1987, 5.2.1, 5.2.2, and 5.3.1).

Additionally, for $m = 0$, we define $S(j + i_1) \cdots S(j + i_m) = 1$. Equation (4) is equivalent to the definition of "$k$-deck" given in Mandereau et al. (2011), with $k = m$.

Then, the following results are easily obtained:

$$\begin{aligned} R(S/T) &= n \text{ for } m = 0, \\ R(S/T) &= c \text{ for } m = 1, \\ R(S/T) &= 0 \text{ for } m > c, \\ R(S/T) &= n \text{ for } c = n, \\ R(S/T) &= n - m \text{ for } c = n - 1. \end{aligned} \tag{5}$$

Similar definitions can be given for $T$ being a chord class and $S$ being either a chord class or a chord type. Then, if $T$ is inversionally symmetrical, the *unweighted class content* of $S$ with respect to $T$ is equal to $R(S/T)$ as defined in (1) and (2), or (4) (using any representative of $S$). Otherwise, it is equal to $R(S/T_a) + R(S/T_b)$, $T_a$ and $T_b$ being the two types of chord class $T$ (using the same representative of $S$ in both computations). In both cases, if we divide $R(S/T)$ by the degree of transpositional symmetry of $T$, $s_T$, we will obtain the corresponding *(weighted) class content* (if both $S$ and $T$ are chord classes, this is the Lewin's embedding number "EMB$(T, S)$" for "CANON" being the group of transposition and inversion operations[4]).

For the next definition, it is necessary to arrange all the different $m$-types in $\mathbb{Z}_n$, for which there are several procedures (see below for the $c$-types). Determining the number of $m$-types in $\mathbb{Z}_n$, $N_n(m)$, as well as $m$-classes, $\tilde{N}_n(m)$, are not easy problems, but they are fully explained and solved in Hook (2007). Evidently, the number of $m$-types in $\mathbb{Z}_n$ equals the number of their complements, that is, $N_n(m) = N_n(m')$, where $m' = n - m$; and, similarly, $\tilde{N}_n(m) = \tilde{N}_n(m')$. As well, the following results are easily obtained:

$$\begin{aligned} &N_n(0) = \tilde{N}_n(0) = N_n(n) = \tilde{N}_n(n) = 1, \quad n \geq 0, \\ &N_n(1) = \tilde{N}_n(1) = N_n(n-1) = \tilde{N}_n(n-1) = 1, \quad n \geq 1, \\ &N_n(2) = \tilde{N}_n(2) = N_n(n-2) = \tilde{N}_n(n-2) = \lfloor n/2 \rfloor, \quad n \geq 2. \end{aligned} \tag{6}$$

From (6), non-inversionally-symmetrical chord classes can only appear for $n \geq 6$. In these cases, for every $c$, $3 \leq c \leq n - 3$, there is at least one such chord class. It is the one with prime intervallic form $\{1, \cdots, 1, 2, n - c\}$, where the number of "1" is $c - 2$ and $n - c \geq 3$, its Forte ordinal being 2. Examples will be seen in next section for $n = 6$ and $n = 7$.

*Definition 3.2 (Type and class vectors):* Let $S$ be a chord type and let $T_l$, $1 \leq l \leq N_n(m)$ be all the different $m$-types in $\mathbb{Z}_n$, their degrees of transpositional symmetry being $s_l^m$. The *unweighted m-type vector* of $S$, $P_m(S)$, is the vector whose elements are $R(S/T_l)$, $1 \leq l \leq N_n(m)$. And the *(weighted) m-type vector* of $S$, $V_m(S)$ or $m$TV$(S)$, is the vector whose elements are $R(S/T_l)/s_l^m$, $1 \leq l \leq N_n(m)$. Similar definitions can be given for $T_l$ being chord classes and $S$ being either a chord class or a chord type. Then, the *(weighted) m-class vector* of $S$ will be called $\tilde{V}_m(S)$ or $m$CV$(S)$, its length being $\tilde{N}_n(m)$ (this is the "$k$-vector" defined in Mandereau et al. (2011), with $k = m$). If both $S$ and $T_l$ are chord types or chord classes, the corresponding weighted vectors are the Lewin's "M-class vector",[5] with M $= m$.

For example, in 12-TET there are six different 2-types, the dyads, which are actually chord classes. Their degrees of transpositional symmetry are 1, except for the tritone, which is 2. And the 2-class vector (2CV) is the well-known *interval-class vector* or ICV.

Vectors are here understood as column vectors, while row vectors are indicated by superscript "$t$" (transpose). In fact, type and class vectors will normally be written as row vectors, that is,

[4] See Lewin (1987, 5.2.1, 5.2.2, and 5.3.1).

[5] See Lewin (1987, 5.3.3).

$P^t_m(S)$, $V^t_m(S)$ and $\tilde{V}^t_m(S)$. Logically, the sum of the elements of $V^t_m(S)$ or $\tilde{V}^t_m(S)$ equals the number of combinations of $c$ notes chosen $m$ at a time, that is, $\binom{c}{m}$, for $c \geq m$.

**Procedures for arranging the *c*-types.** All the different $c$-types in $\mathbb{Z}_n$ can be arranged by following several criteria or a combination of them, the most common being:

> *Criterion 1:* To arrange the $c$-types based on their own structures. Usually, by their normal or normal intervallic forms, in increasing lexicographic order. It is common to first arrange the $c$-classes and then split those being non-inversionally-symmetrical into two types. In any case, the corresponding $c$-type vectors (weighted or unweighted) will only have one non-zero element and are arranged in decreasing lexicographic order.
>
> *Criterion 2:* To arrange the $c$-types based on their relations with other $m$-types, $m < c$. Usually, by their $m$-class or $m$-type vectors, in decreasing lexicographic order. Obviously, this requires that those $m$-classes or $m$-types are previously arranged (normally, by Criterion 1). Note that Criterion 1 can be viewed as a Criterion 2 with $m = c$.

For example, Forte (1973) arranges the 2-classes in $\mathbb{Z}_{12}$ by Criterion 1, using their normal forms (which, in this case, is equivalent to using their normal intervallic forms), while for $c \geq 3$ the chord classes are arranged by Criterion 2 using the 2-class vector (2CV or ICV, there called *interval vector*). The ties are the pairs of *Z-related* chord classes, which only appear for $c \geq 4$ and one member of each pair is placed at the end of the corresponding group. In contrast, at this point, Nuño (2021) arranges the two members of a Z-related pair by Criterion 2 using the 3-class vector (3CV, there called *trichord-class vector* or TCV), the first class being called *hard* and the second *soft*. Then, the non-inversionally-symmetrical 3-classes are split into two types and arranged by Criterion 1 (using any of the two forms); and, for $c \geq 4$, the two types of the same class are arranged by Criterion 2 using the 3-type vector (3TV, there called *trichord-type vector* or TTV). In both cases, the first one is called *type a* and the second *type b*. Finally, there is still one more tie: the two types of chord class 6-14, which have the same 3TV. Then, they are arranged by Criterion 1 using the normal intervallic form (but, in this case, contrary to the normal form!), which coincides, in this case, with Criterion 2 using the 4-type vector (4TV). This is the procedure followed for developing both the detailed list and the periodic table of set classes given in that work, and the one considered here in Sections 4 and 6.

*Definition 3.3 (Type and class matrices):* Let $S_k$, $1 \leq k \leq N_n(c)$ be all the different $c$-types and $T_l$, $1 \leq l \leq N_n(m)$ all the different $m$-types in $\mathbb{Z}_n$. The *unweighted type matrix* $Q_{c,m}$ is the matrix whose rows are the unweighted $m$-type row vectors $P^t_m(S_k)$, $1 \leq k \leq N_n(c)$. And the *(weighted) type matrix* $M_{c,m}$ is the matrix whose rows are the $m$-type row vectors $V^t_m(S_k)$, $1 \leq k \leq N_n(c)$. Note that the dimensions of both matrices are not $c \times m$, but $N_n(c) \times N_n(m)$. Similar definitions can be given for $T_l$ being chord classes and $S_k$ being either chord classes or chord types. If both are chord classes, we will represent the corresponding *(weighted) class matrix* as $\tilde{M}_{c,m}$, its dimensions being $\tilde{N}_n(c) \times \tilde{N}_n(m)$.

*Definition 3.4 (Weighting matrix):* Let $T_l$, $1 \leq l \leq N_n(m)$ be all the different $m$-types in $\mathbb{Z}_n$, their degrees of transpositional symmetry being $s^m_l$. The *weighting matrix* $W_m$ is a diagonal matrix with dimensions $N_n(m) \times N_n(m)$, whose elements in the diagonal are $1/s^m_l$, $1 \leq l \leq N_n(m)$.

The weighting matrix allows us to relate matrices $Q_{c,m}$ and $M_{c,m}$ as $M_{c,m} = Q_{c,m} W_m$.

*Definition 3.5 (Type-adding and type-deleting matrices):* Let $T_l$, $1 \leq l \leq N_n(m)$ be all the different $m$-types in $\mathbb{Z}_n$. The *type-adding matrix* $A_m$ is an "almost diagonal matrix" with dimensions $N_n(m) \times \tilde{N}_n(m)$ that, when right-multiplying a matrix like $M_{c,m}$, modifies it by adding every pair of columns corresponding to the two types of the same chord class, leaving the rest of the columns unchanged. As well, the *type-deleting matrix* $D_m$ is an "almost diagonal matrix" with dimensions $N_n(m) \times \tilde{N}_n(m)$ that, when right-multiplying a matrix like $M_{c,m}$, modifies it by deleting one column of each pair corresponding to the two types of the same chord class, leaving the rest of the columns unchanged. Logically, left-multiplying by the transpose of those matrices (with the appropriate dimensions) has the same effect on the rows.

These matrices allow us to relate type and class matrices $M_{c,m}$ and $\tilde{M}_{c,m}$ as $\tilde{M}_{c,m} = D_c^t M_{c,m} A_m$.

Examples of weighting, type-adding, and type-deleting matrices, as well as their use, are given by the end of Section 5.

*Definition 3.6 (Full n-type and n-class matrices):* In $\mathbb{Z}_n$, the *full unweighted n-type matrix* is the square matrix consisting of submatrices $Q_{c,m}$, $0 \leq c \leq n$, $0 \leq m \leq n$. And the *full (weighted) n-type matrix* is the square matrix consisting of submatrices $M_{c,m}$, $0 \leq c \leq n$, $0 \leq m \leq n$. A similar definition can be given for the *full (weighted) n-class matrix*, considering submatrices $\tilde{M}_{c,m}$.

Table 1. Chord types and type vectors and matrices in $\mathbb{Z}_6$. Columns (left to right): (1) Cardinality $c$ and general ordinal. (2) Extended Forte name. (3) Normal Intervallic Form (NIF). (4)–(10) $m$-type vectors, $0 \leq m \leq 6$, and the corresponding matrices. The rectangle shows the full 6-type matrix.

| Chord | Type | NIF | 0TV | 1TV | 2TV | 3TV | 4TV | 5TV | 6TV |
|---|---|---|---|---|---|---|---|---|---|
| $c$=0 | | | | | | | | | |
| 0-$^{6}$ | 0-1 | - | 1 | 0 | 000 | 0000 | 000 | 0 | 0 |
| $c$=1 | | | | | | | | | |
| 1- | 1-1 | 0 | 1 | 1 | 000 | 0000 | 000 | 0 | 0 |
| $c$=2 | | | | | | | | | |
| 2- | 2-1 | 15 | 1 | 2 | 100 | 0000 | 000 | 0 | 0 |
| 3- | 2-2 | 24 | 1 | 2 | 010 | 0000 | 000 | 0 | 0 |
| 4-$^{2}$ | 2-3 | 33 | 1 | 2 | 001 | 0000 | 000 | 0 | 0 |
| $c$=3 | | | | | | | | | |
| 5- | 3-1 | 114 | 1 | 3 | 210 | 1000 | 000 | 0 | 0 |
| 6 | 3-2a | 123 | 1 | 3 | 111 | 0100 | 000 | 0 | 0 |
| 7 | 3-2b | 132 | 1 | 3 | 111 | 0010 | 000 | 0 | 0 |
| 8-$^{3}$ | 3-3 | 222 | 1 | 3 | 030 | 0001 | 000 | 0 | 0 |
| $c$=4 | | | | | | | | | |
| 9- | 4-1 | 1113 | 1 | 4 | 321 | 2110 | 100 | 0 | 0 |
| 10- | 4-2 | 1122 | 1 | 4 | 231 | 1111 | 010 | 0 | 0 |
| 11-$^{2}$ | 4-3 | 1212 | 1 | 4 | 222 | 0220 | 001 | 0 | 0 |
| $c$=5 | | | | | | | | | |
| 12- | 5-1 | 11112 | 1 | 5 | 442 | 3331 | 221 | 1 | 0 |
| $c$=6 | | | | | | | | | |
| 13-$^{6}$ | 6-1 | 111111 | 1 | 6 | 663 | 6662 | 663 | 6 | 1 |

Table 2. Chord types and type vectors and matrices in $\mathbb{Z}_7$. Columns (left to right): (1) Cardinality $c$ and general ordinal. (2) Extended Forte name. (3) Normal Intervallic Form (NIF). (4)–(11) $m$-type vectors, $0 \le m \le 7$, and the corresponding matrices. The rectangle shows the full 7-type matrix.

| Chord | Type | NIF | 0TV | 1TV | 2TV | 3TV | 4TV | 5TV | 6TV | 7TV |
|---|---|---|---|---|---|---|---|---|---|---|
| $c$=0 | | | | | | | | | | |
| 0-$^7$ | 0-1 | - | 1 | 0 | 000 | 00000 | 00000 | 000 | 0 | 0 |
| $c$=1 | | | | | | | | | | |
| 1- | 1-1 | 0 | 1 | 1 | 000 | 00000 | 00000 | 000 | 0 | 0 |
| $c$=2 | | | | | | | | | | |
| 2- | 2-1 | 16 | 1 | 2 | 100 | 00000 | 00000 | 000 | 0 | 0 |
| 3- | 2-2 | 25 | 1 | 2 | 010 | 00000 | 00000 | 000 | 0 | 0 |
| 4- | 2-3 | 34 | 1 | 2 | 001 | 00000 | 00000 | 000 | 0 | 0 |
| $c$=3 | | | | | | | | | | |
| 5- | 3-1 | 115 | 1 | 3 | 210 | 10000 | 00000 | 000 | 0 | 0 |
| 6 | 3-2a | 124 | 1 | 3 | 111 | 01000 | 00000 | 000 | 0 | 0 |
| 7 | 3-2b | 142 | 1 | 3 | 111 | 00100 | 00000 | 000 | 0 | 0 |
| 8- | 3-3 | 133 | 1 | 3 | 102 | 00010 | 00000 | 000 | 0 | 0 |
| 9- | 3-4 | 223 | 1 | 3 | 021 | 00001 | 00000 | 000 | 0 | 0 |
| $c$=4 | | | | | | | | | | |
| 10- | 4-1 | 1114 | 1 | 4 | 321 | 21100 | 10000 | 000 | 0 | 0 |
| 11 | 4-2a | 1123 | 1 | 4 | 222 | 11011 | 01000 | 000 | 0 | 0 |
| 12 | 4-2b | 1132 | 1 | 4 | 222 | 10111 | 00100 | 000 | 0 | 0 |
| 13- | 4-3 | 1213 | 1 | 4 | 213 | 01120 | 00010 | 000 | 0 | 0 |
| 14- | 4-4 | 1222 | 1 | 4 | 132 | 01102 | 00001 | 000 | 0 | 0 |
| $c$=5 | | | | | | | | | | |
| 15- | 5-1 | 11113 | 1 | 5 | 433 | 32221 | 21110 | 100 | 0 | 0 |
| 16- | 5-2 | 11122 | 1 | 5 | 343 | 22213 | 11102 | 010 | 0 | 0 |
| 17- | 5-3 | 11212 | 1 | 5 | 334 | 12232 | 01121 | 001 | 0 | 0 |
| $c$=6 | | | | | | | | | | |
| 18- | 6-1 | 111112 | 1 | 6 | 555 | 44444 | 33333 | 222 | 1 | 0 |
| $c$=7 | | | | | | | | | | |
| 19-$^7$ | 7-1 | 1111111 | 1 | 7 | 777 | 77777 | 77777 | 777 | 7 | 1 |

## 4. Application to $\mathbb{Z}_6$ and $\mathbb{Z}_7$

Tables 1 and 2 give the chord types and their type vectors and matrices in $\mathbb{Z}_6$ and $\mathbb{Z}_7$, respectively. The information is given in several columns containing the following:

1) Cardinality $c$ and general ordinal. Chord types being inversionally symmetrical are indicated by a hyphen after the ordinal. The degree of transpositional symmetry is given as a superscript, when greater than one.
2) Extended Forte name, consisting of two numbers separated by a hyphen, the first one being the cardinality and the second an ordinal for the chord classes, which is followed by the type ($a$ or $b$) when applicable. Both the chord classes and types are arranged as in Nuño (2021) (see the previous section), with the difference that now there are no ties.
3) Normal intervallic form (NIF).
4) The rest of columns show the $m$-type vectors ($m$TV), $0 \leq m \leq n$, and the corresponding matrices. The rectangle shows the full $n$-type matrix.

As explained above, for $n \geq 6$ and every $c$, $3 \leq c \leq n-3$, there is at least one non-inversionally-symmetrical chord class, whose Forte ordinal is 2. In these cases, they are 3-2 in $\mathbb{Z}_6$ (Table 1) and 3-2 and 4-2 in $\mathbb{Z}_7$ (Table 2).

These tables provide all type contents of every chord type and can easily be verified. In a sense, they are simplified versions of $\mathbb{Z}_{12}$ (which will be seen in Section 6). For example, the four basic triads in $\mathbb{Z}_{12}$, which are the chord types 3-10 (diminished), 3-11a (minor), 3-11b (major), and 3-12 (augmented), in $\mathbb{Z}_7$ they all are the chord class 3-4. Regarding class vectors and matrices, they are quite simple to obtain in these cases and some examples are given by the end of next section. On the other hand, these tables are significant examples of $n$ being composite and prime numbers, and will also serve to illustrate the theory developed in next section.

## 5. Properties and examples

Type and class vectors and matrices, as well as full $n$-type and $n$-class matrices, have important and interesting properties, which are described below. The first three follow from (3) and (5).

*Property 1.* All type matrices $M_{c,c}$ are identity matrices and all type matrices $M_{c,m}$, $m > c$, are zero matrices. Therefore, a full $n$-type matrix is lower triangular and all elements in its main diagonal are equal to 1.

*Property 2.* In the first column of a full $n$-type matrix, we have matrices $M_{c,0}$, $0 \leq c \leq n$, whose elements are all equal to 1. In the last row, we have matrices $M_{n,m}$, $0 \leq m \leq n$, whose elements are equal to $n/s_l^m$, $1 \leq l \leq N_n(m)$.

*Property 3.* In the second column of a full $n$-type matrix, we have matrices $M_{c,1}$, $0 \leq c \leq n$, whose elements are equal to $c$. In the last but one row, we have matrices $M_{n-1,m}$, $0 \leq m \leq n$, whose elements are equal to $(n-m)/s_l^m$, $1 \leq l \leq N_n(m)$.

*Property 4.* As explained after Definition 3.2, the sum of the elements of every row in $M_{c,m}$, as well as in $\tilde{M}_{c,m}$, $c \geq m$, is $\binom{c}{m}$. Therefore, the sum of the elements of a row corresponding to a cardinality $c$ in a full $n$-type or $n$-class matrix is $\sum_{m=0}^{c} \binom{c}{m} = 2^c$.

Theorem 5.1 (Complementary reciprocity): *Let $S$ and $T$ be two chord types and $S'$ and $T'$ their corresponding complements. Then,*

$$R(S/T) = R(T'/S'). \tag{7}$$

*Proof:* Let $c$ and $m$ be the cardinalities of $S$ and $T$, respectively, and $c' = n - c$ and $m' = n - m$ those of $S'$ and $T'$. Using (1) with $S(k) = 1 - S'(k)$, $T(k) = 1 - T'(k)$ yields

$$p_m(j) = \sum_{k=0}^{n-1} [1 - S'(k)][1 - T'(k+j)] = n - m' - c' + \sum_{k=0}^{n-1} S'(k)T'(k+j). \tag{8}$$

Defining

$$q_{c'}(j) = \sum_{k=0}^{n-1} S'(k)T'(k+j) = \sum_{k=0}^{n-1} T'(k)S'(k-j) \tag{9}$$

allows us to compute $R(T'/S')$ as the number of times $q_{c'}(j) = c'$ for $j = 0, \dots, n-1$, that is,

$$\begin{aligned} R(T'/S') &= \sum_{j=0}^{n-1} \left\lfloor \frac{q_{c'}(j)}{c'} \right\rfloor, \quad c' \neq 0, \\ R(T'/S') &= n, \quad c' = 0, \end{aligned} \tag{10}$$

according to (2). Since, from (8),

$$q_{c'}(j) = p_m(j) - m + c', \tag{11}$$

the number of times $q_{c'}(j) = c'$ equals the number of times $p_m(j) = m$, both for $j = 0, \dots, n-1$, which gives (7).

This result allows us to relate the unweighted type matrices $Q_{c,m}$ and $Q_{m',c'}$, but the next definition is required.

*Definition 5.1 (Complementary order):* The chord types in $\mathbb{Z}_n$ are arranged in *complementary order* if we follow these steps*:*

1) Group the chord types by their cardinality $c$ (1CV).
2) For $0 \leq c \leq \lfloor n/2 \rfloor$, arrange the chord types having the same cardinality by any criteria (see the most common in Section 3).
3) For $\lfloor n/2 \rfloor + 1 \leq c \leq n$, arrange the chord types having the same cardinality in the same order as their corresponding complements, which were arranged in the previous step.

A similar definition can be given for chord classes.

*Property 5.* From Theorem 5.1 (Complementary reciprocity), if the chord types in $\mathbb{Z}_n$ are arranged in complementary order, then, excluding the hemichord types ($c = n/2$ or $m = n/2$, which requires that $n$ be even),

$$Q_{c,m} = Q^t_{m',c'} \tag{12}$$

where superscript "$t$" means transpose. In particular, if $m = c'$, the square matrices $Q_{c,c'}$ are symmetric. And this is also true if $c = c' = n/2$, since in this case the matrix is diagonal.

In general, however, (12) cannot be guaranteed if $c = n/2$ or $m = n/2$, because the hemichord types are not necessarily self-complementary, although (7) is always applicable. Anyway, excluding the hemichords, (12) is also accomplished if, for $\lfloor n/2 \rfloor + 1 \leq c \leq n$, the two types of every non-inversionally-symmetrical chord class are interchanged, as can be justified from the next lemma. In fact, this was already done for $\mathbb{Z}_7$ in Table 2, where there are no hemichords. Regarding $\mathbb{Z}_6$, the hemichords happen to be self-complementary, except the only two types $a$ and $b$, which are precisely the complements of each other (see Table 1). For these reasons, all type matrices in $\mathbb{Z}_6$, as well as in $\mathbb{Z}_7$, satisfy (12).

The next lemma is given without proof, because it is evident by envisioning the chord types as sets of hours on a clockface with $n$ hours.

Lemma 5.1 (Type-content symmetry): *Let $S_a$ and $S_b$ be the two types of a chord class and let $T_a$ and $T_b$ be the two types of another (or the same) chord class. Then,*

$$R(S_a/T_a) = R(S_b/T_b), \; R(S_a/T_b) = R(S_b/T_a). \tag{13}$$

*Definition 5.2 (Normal order):* The chord types in $\mathbb{Z}_n$ are arranged in *normal order* if we first arrange them in complementary order and then, for $\lfloor n/2 \rfloor + 1 \leq c \leq n$, we interchange the two types of every non-inversionally-symmetrical chord class.

This is the way the chord types in $\mathbb{Z}_7$ are arranged in Table 2 and those in $\mathbb{Z}_{12}$ will be arranged in next section. Logically, regarding chord classes, there is no difference between normal and complementary orders. Forte (1973) himself arranged the chord classes in this way.

*Definition 5.3 (Left-weighted type and class matrices):* Given an unweighted type matrix $Q_{c,m}$, its corresponding *left-weighted type matrix* is $L_{c,m} = W_c Q_{c,m}$, where $W_c = W_c^t$ is a weighting matrix. As well, the corresponding *left-weighted class matrix* is $\tilde{L}_{c,m} = A_c^t L_{c,m} D_m$, where $A_c$ and $D_m$ are a type-adding and a type-deleting matrices, respectively. Left-weighted matrices are useful for simplifying some expressions.

*Property 6:* If the chord types in $\mathbb{Z}_n$ are arranged in complementary or normal order, then, excluding the hemichord types,

$$M_{c,m} = L^t_{m',c'} \tag{14}$$

This is obtained by simply right-multiplying (12) by $W_m$, which is equal to $W_{m'}$ (because a chord type and its complement have the same degree of transpositional symmetry).

*Property 7:* If the chord classes in $\mathbb{Z}_n$ are arranged in complementary or normal order, then, excluding the hemichord classes,

$$\tilde{M}_{c,m} = \tilde{L}^t_{m',c'} \tag{15}$$

This is obtained from (14) by left-multiplying by type-deleting matrix $D_c^t$, right-multiplying by type-adding matrix $A_m$, and taking into account that $D_c = D_{c'}$ and $A_m = A_{m'}$.

Let us now give some examples. In Table 1, the chord types in $\mathbb{Z}_6$ are arranged in complementary (or normal) order; and, as observed above, all unweighted type matrices satisfy (12). Therefore, they will also satisfy (14). Thus, for example, for $c = 3$, $m = 2$, and for $c = 4$, $m = 3$, we have

$$M_{3,2} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix}, \quad M_{4,3} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \end{bmatrix}. \tag{16}$$

As the involved weighting matrices are

$$W_2 = W_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/3 \end{bmatrix}, \tag{17}$$

we obtain

$$L_{3,2} = W_3 M_{3,2} W_2^{-1} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_{4,3} = W_4 M_{4,3} W_3^{-1} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \tag{18}$$

Therefore, $M_{3,2} = L^t_{4,3}$ and $M_{4,3} = L^t_{3,2}$.

As well,

$$Q_{4,2} = M_{4,2} W_2^{-1} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 4 \end{bmatrix}, \tag{19}$$

which is symmetric, as stated in (12) for $m = c'$.

In Table 2, the chord types in $\mathbb{Z}_7$ are arranged in normal order; and, as observed above, all unweighted type matrices satisfy (12). Therefore, they will also satisfy (14). Regarding matrices

$W_m$, they are identity matrices, except for $m = 0$ and $m = 7$. Therefore, $M_{3,2} = M^t_{5,4}$, $M_{4,2} = M^t_{5,3}$, and matrices $M_{4,3}$, $M_{5,2}$ are symmetric, as can be seen in the table.

To show how to obtain class matrices from type matrices, let us obtain, in $\mathbb{Z}_7$, class matrix $\tilde{M}_{5,3}$ from type matrix $M_{5,3}$ and type-adding matrix $A_3$:

$$\tilde{M}_{5,3} = M_{5,3}A_3 = \begin{bmatrix} 3 & 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 & 3 \\ 1 & 2 & 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 4 & 2 & 1 \\ 2 & 4 & 1 & 3 \\ 1 & 4 & 3 & 2 \end{bmatrix}. \tag{20}$$

To obtain class matrix $\tilde{M}_{4,3}$, we need, apart from $M_{4,3}$ and $A_3$, type-deleting matrix $D_4$:

$$\begin{aligned} \tilde{M}_{4,3} &= D^t_4 M_{4,3} A_3 \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}. \end{aligned} \tag{21}$$

Note that $M_{4,3}$ is symmetric, but $\tilde{M}_{4,3}$ is not.

## 6. Application to $\mathbb{Z}_{12}$ and software

For $\mathbb{Z}_{12}$, the information on chord types and their corresponding type and class vectors and matrices is too large to give it in a table, as was done for $\mathbb{Z}_6$ and $\mathbb{Z}_7$. Thus, it is provided as an Online Supplement, in a MATLAB file named "TwelveToneMatrices.mat". It contains 234 matrices, whose names start with letter "T" (for "Twelve") and, to avoid any misunderstanding, integers 10, 11 and 12 are represented by letters A, B and C, respectively. The chord types are defined in the following matrices:

Chord Types: TT$x, 0 \le x \le C$

These matrices have $N_{12}(x)$ rows and $x + 1$ columns. The first column contains the chord type descriptors, which are real numbers, their absolute integer parts being the Forte ordinals of the corresponding chord classes. When a chord class is Z-related to another one, its descriptor is negative, otherwise positive. Non-inversionally-symmetrical chord classes are split into two types, the descriptor of type $a$ having a fractional part ".1" and type $b$, ".2". For inversionally-symmetrical chord classes, the descriptors are integer numbers. The rest of columns contain natural numbers, which are the normal intervallic forms (for $x \ge 1$).

As can be seen, the chord classes and types are arranged as in Nuño (2021) (see Section 3), except for the Z-related pairs of 7- and 8-classes, where the hard and soft classes are interchanged. This only affects to 4 pairs of chord classes and makes all chord classes and types to be arranged in normal order.

The following matrices are given in the same file:

Weighting Matrices $W_x$: TW$x, 0 \le x \le C$

Type-Adding Matrices $A_x$: TA$x, 0 \le x \le C$

Type-Deleting Matrices $D_x$: TD$x, 0 \le x \le C$

Type Matrices $M_{x,y}$: TM$xy,\ 0 \le x \le C,\ 0 \le y \le x$

Class Matrices $\tilde{M}_{x,y}$: TN$xy,\ 0 \le x \le C,\ 0 \le y \le x$

The rows of matrices $M_{x,2} = \tilde{M}_{x,2}$ are actually the 2CV, which were first published by Forte (1973), there called interval vectors. And the rows of matrices $M_{x,3}$ are the 3TV, which were published by Nuño (2021), there called trichord-type vectors or TTV. Now, the rest of $n$TV and $n$CV are given, thus completing the information for $\mathbb{Z}_{12}$.

Regarding the corresponding matrices for $\mathbb{Z}_6$ and $\mathbb{Z}_7$, they are given in other two MATLAB files, named, respectively, "HexaToneMatrices.mat" and "SevenToneMatrices.mat". The notation for the matrices in these cases is similar to the previous case, except that now their names start with letters "H" (for "Hexa") and "S" (for "Seven") instead of "T".

Apart from its relevance and usefulness in music theory and composition, these matrices can also be used to verify and better interpret the theorems given in next section.

On the other hand, a program for computing type and class matrices is provided, its name being "TypeClassMatrices.m". It can be used for obtaining both the above results and other new ones, as it is valid for any value of $n$. For running the program, the user has to supply two input matrices, with names S and T, in the same directory as the program. They must have the same format as matrices TT$x$ described above. In each of these matrices, all chord types must have the same cardinality, although it is not necessary to include all the different chord types with that cardinality. However, types $a$ and $b$ of the same chord class must be included both jointly, in consecutive rows, and their descriptors must only differ in the fractional parts ".1" and ".2" (which can be assigned arbitrarily to the two types). Apart from this, the chord classes can be given in any order and their descriptors do not need to include the minus sign for Z-related chord classes. As well, the intervallic forms do not need to be normal. Then, the program gives the (weighted) type matrix TM and (weighted) class matrix CM of chord types and chord classes in matrix S with respect to those in matrix T.

Furthermore, a program for computing weighting, type-adding, and type-deleting matrices is also provided, its name being "WADMatrices.m", which is also valid for any value of $n$. Now, the user has to supply only one input matrix, with name T, in the same directory as the program. Its format must be as described in the previous paragraph, but in case it contains the chord type with cardinality 0, the user also has to supply the value of $n$ through the variable zn, in the same directory as the program. Then, the program gives the weighting W, type-adding A, and type-deleting D matrices related to chord types in matrix T.

By the end of the programs, all auxiliary variables are deleted, a list of which can be seen in the last but one line of each program.

Both programs are self-explanatory and the user can easily adapt or modify them as desired.

## 7. Theorems on type and class matrices

Theorems in this section provide relations among weighted and left-weighted type and class matrices, which allow obtaining them by different alternatives.

Theorem 7.1 (Type matrix contraction): *The type matrices satisfy the following relation*

$$\binom{c-q}{c-p} M_{c,q} = M_{c,p} M_{p,q}, \quad c \geq p \geq q \tag{22}$$

*(irrespective of how the chord types are arranged). This allows obtaining $M_{c,q}$ from $M_{c,p}$.*

Before giving the proof, let us analyze a particular case. For example, in $\mathbb{Z}_7$ let us consider $c = 5$, $q = 2$, and $p = 4$. The left- and right-hand sides of (22) are, respectively (see Table 2),

$$\binom{c-q}{c-p} M_{c,q} = \binom{3}{1} M_{5,2} = 3 \begin{bmatrix} 4 & 3 & 3 \\ 3 & 4 & 3 \\ 3 & 3 & 4 \end{bmatrix} \tag{23}$$

and

$$M_{c,p} M_{p,q} = M_{5,4} M_{4,2} = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 9 & 9 \\ 9 & 12 & 9 \\ 9 & 9 & 12 \end{bmatrix}, \tag{24}$$

which clearly coincide.

For the 5-type 5-1, the first row in $M_{5,4}$, that is, $\begin{bmatrix} 2 & 1 & 1 & 1 & 0 \end{bmatrix}$, lists the number of times each 4-type is contained in 5-1. Its product with matrix $M_{4,2}$ is the row vector $\begin{bmatrix} 12 & 9 & 9 \end{bmatrix}$, which lists the number of times each 2-type is contained in all those 4-types. This is different from the number of times each 2-type is contained in 5-1, which are listed in the first row of $M_{5,2}$, that is, $\begin{bmatrix} 4 & 3 & 3 \end{bmatrix}$. And this difference is just the factor 3. Let us see why. Each 2-type contained in 5-1 will also be contained in all the 4-types in 5-1 formed by the 2 notes of the 2-type plus two more notes from 5-1 (chosen from the remaining 3), which gives a total of $\binom{3}{2} = \binom{3}{1} = 3$ combinations. The same can be said for the 5-types 5-2 and 5-3, which explains the result. The proof of the theorem is, simply, the generalization of this reasoning.

*Proof.* Let $S$ be a $c$-type. The row vector $V_p^t(S)$ lists the number of times each $p$-type is contained in $S$; and the product $V_p^t(S) M_{p,q}$ is the row vector listing the number of times each $q$-type is contained in all those $p$-types. This is different from the row vector $V_q^t(S)$, which lists the number of times each $q$-type is contained in $S$. And this difference is just a factor that can be obtained as follows: any $q$-type contained in $S$ will belong to as many $p$-types in $S$ as can be composed with the $q$ notes of the $q$-type plus other $p - q$ notes from $S$ (chosen from the remaining $c - q$), which gives a total of $\binom{c-q}{p-q} = \binom{c-q}{c-p}$ combinations, provided that $c \geq p \geq q$. Therefore,

$$\binom{c-q}{c-p} V_q^t(S) = V_p^t(S) M_{p,q}, \quad c \geq p \geq q. \tag{25}$$

And, by considering all the $c$-types in $\mathbb{Z}_n$, we obtain (22).

Note that no particular arrangement was required for the complements of the chord types. Formula (25) for the particular case $n = 12$, $p = 3$, $q = 2$ was given by Nuño (2021).

Theorem 7.2: *The left-weighted type matrices satisfy the following relation*

$$\binom{q-m}{p-m} L_{q,m} = L_{q,p} L_{p,m}, \quad m \le p \le q \tag{26}$$

*(irrespective of how the chord types are arranged). This allows obtaining $L_{q,m}$ from $L_{p,m}$.*

*Proof.* Renaming in (22) $c$ to $m$ and right-multiplying by $W_q^{-1}$ gives

$$\binom{m-q}{m-p} Q_{m,q} = M_{m,p} Q_{p,q}, \quad m \ge p \ge q. \tag{27}$$

And, because $M_{m,p}Q_{p,q} = Q_{m,p}W_pQ_{p,q} = Q_{m,p}L_{p,q}$, left-multiplying (27) by $W_m$ gives

$$\binom{m-q}{m-p} L_{m,q} = L_{m,p} L_{p,q}, \quad m \ge p \ge q. \tag{28}$$

Finally, interchanging $m$ with $q$ and considering that $\binom{q-m}{q-p} = \binom{q-m}{p-m}$, we obtain (26).

Lemma 7.1 (Class-content symmetry): *From Lemma 5.1 (Type-content symmetry), if in matrix $M_{p,q}$ we add every pair of columns corresponding to the two types of the same chord class, that is, we obtain $M_{p,q}A_q$, then, in this resulting matrix, any two rows corresponding to the two types of the same chord class are the same. Similarly, if in matrix $L_{q,p}$ we add every pair of rows corresponding to the two types of the same chord class, that is, we obtain $A_q^t L_{q,p}$, then, in this resulting matrix, any two columns corresponding to the two types of the same chord class are the same.*

Theorem 7.3 (Class matrix contraction): *The class matrices satisfy the following relation*

$$\binom{c-q}{c-p} \tilde{M}_{c,q} = \tilde{M}_{c,p} \tilde{M}_{p,q}, \quad c \ge p \ge q \tag{29}$$

*(irrespective of how the chord classes are arranged). This allows obtaining $\tilde{M}_{c,q}$ from $\tilde{M}_{c,p}$.*

*Proof.* This result is similar to (22), but is not obtained straightforwardly from it and requires further reasoning. First, right-multiplying (25) by type-adding matrix $A_q$ yields

$$\binom{c-q}{c-p} \tilde{V}_q^t(S) = V_p^t(S) M_{p,q} A_q, \quad c \ge p \ge q, \tag{30}$$

$\tilde{V}_q^t(S)$ being the corresponding row of $\tilde{M}_{c,q}$. Then, from Lemma 7.1 (Class-content symmetry), in the resulting matrix $M_{p,q}A_q$ any two rows, say $i_a$ and $i_b$, corresponding to the two types of the same chord class, say $i$, are the same. Thus, when multiplying vector $V_p^t(S)$ by column $j$ of matrix $M_{p,q}A_q$, the corresponding elements give

$$\begin{aligned} &[V_p^t(S)]_{i_a}[M_{p,q}A_q]_{i_a,j} + [V_p^t(S)]_{i_b}[M_{p,q}A_q]_{i_b,j} \\ &\quad = \{[V_p^t(S)]_{i_a} + [V_p^t(S)]_{i_b}\}[M_{p,q}A_q]_{i_a,j} \\ &\quad = [V_p^t(S)A_p]_i[D_p^t M_{p,q}A_q]_{i,j} = [\tilde{V}_p^t(S)]_i[\tilde{M}_{p,q}]_{i,j} \end{aligned} \tag{31}$$

where $\tilde{V}_p^t(S)$ is the corresponding row of $\tilde{M}_{c,p}$ and type-adding $A_p$ and type-deleting $D_p$ matrices were used. Thus,

$$\binom{c-q}{c-p} \tilde{V}_q^t(S) = \tilde{V}_p^t(S) \tilde{M}_{p,q}, \quad c \ge p \ge q. \tag{32}$$

And, by considering all the $c$-classes in $\mathbb{Z}_n$, we obtain (29).

Relations (22) and (29) for only one element of matrices $M_{c,q}$ and $\tilde{M}_{c,q}$, respectively, were given by Lewin (1987, 5.3.5.2). Curiously enough, he gives a proof based on probability theory.

Choosing an arbitrary column of (26) and reasoning as in the previous proof, but interchanging rows with columns, we obtain the next theorem.

Theorem 7.4: *The left-weighted class matrices satisfy the following relation*

$$\binom{q-m}{p-m}\tilde{L}_{q,m} = \tilde{L}_{q,p}\tilde{L}_{p,m}, \quad m \le p \le q \tag{33}$$

*(irrespective of how the chord classes are arranged). This allows obtaining* $\tilde{L}_{q,m}$ *from* $\tilde{L}_{p,m}$.

Corollary 7.1 (Nesting): *From (25), if two c-types have the same p-type vector,* $p \le c$, *then they also have the same q-type vectors,* $0 \le q \le p$. *From* (32), *the same applies for c-classes and class vectors*. These results are also given by Mandereau et al. (2011), but from a different approach.

Corollary 7.2: *From Corollary 7.1 (Nesting), if two c-types have different q-type vectors (* $q \le c$*), then all their p-type vectors,* $q \le p \le c$, *will also be different. The same applies for c-classes and class vectors.*

Theorem 7.5 (Type content of the complement): *Let S and S′ be the characteristic functions of a c-type and its complement, and let T be an m-type,* $m \ge 1$. *Then,*

$$\begin{aligned} &R(S'/T) = (-1)^m R(S/T) + (-1)^{m-1} P^t_{m-1}(S) V_{m-1}(T) \\ &+(-1)^{m-2} P^t_{m-2}(S) V_{m-2}(T) + \cdots - P^t_1(S) V_1(T) + P^t_0(S) V_0(T), \end{aligned} \tag{34}$$

*the last two terms being* $-cm + n$. *Of course, this can also be written in the compact form*

$$R(S'/T) = (-1)^m R(S/T) + \sum_{l=0}^{m-1} (-1)^l P^t_l(S) V_l(T), \quad m \ge 1. \tag{35}$$

The case $m = 1$ is, simply, $c' = -c + n$. And the case $m = 0$ also corresponds to (35) but without the summation, that is, $R(S'/T) = R(S/T) = n$, which is not relevant.

*Proof*. Let $T$ be represented by the set of integers $[i_1, \cdots, i_m]$. Then,

$$\begin{aligned} R(S/T) &= \textstyle\sum_{j=0}^{n-1} S(j+i_1)\cdots S(j+i_m), \\ R(S'/T) &= \textstyle\sum_{j=0}^{n-1} S'(j+i_1)\cdots S'(j+i_m). \end{aligned} \tag{36}$$

As $S'(j) = 1 - S(j)$,

$$R(S'/T) = \sum_{j=0}^{n-1} [1 - S(j+i_1)] \cdots [1 - S(j+i_m)]. \tag{37}$$

Calling, for simplicity, $S_k = S(j+i_k)$, $1 \le k \le m$, considering them as the roots of a polynomial of degree $m$, so that $x^m + A_1 x^{m-1} + A_2 x^{m-2} + \cdots + A_m = (x - S_1)\cdots(x - S_m)$, using Vieta's formulas relating the coefficients of a polynomial with its roots (see, for example, Bronshtein et al. 2007, 43–44), and taking $x = 1$, the expression in the summation in (37) can be written as

$$[1 - S(j + i_1)] \cdots [1 - S(j + i_m)] = \prod_{k=1}^{m} (1 - S_k) = \sum_{l=0}^{m} A_l \tag{38}$$

where

$$\begin{aligned}
&A_0 = 1\\
&A_1 = -(S_1 + S_2 + \cdots + S_m) = -\sum_{p=1}^{m} S_p\\
&A_2 = S_1S_2 + S_1S_3 + \cdots + S_{m-1}S_m = \sum_{\substack{p,q=1\\p<q}}^{m} S_pS_q\\
&A_3 = -(S_1S_2S_3 + S_1S_2S_4 + \cdots + S_{m-2}S_{m-1}S_m) = -\sum_{\substack{p,q,r=1\\p<q<r}}^{m} S_pS_qS_r\\
&\vdots\\
&A_m = (-1)^m S_1S_2 \cdots S_m
\end{aligned} \tag{39}$$

Thus, substituting (38) in (37) gives

$$R(S'/T) = \sum_{j=0}^{n-1} \sum_{l=0}^{m} A_l = \sum_{l=0}^{m} \sum_{j=0}^{n-1} A_l \tag{40}$$

Each coefficient $A_l$, $0 \le l \le m$, given in (39), involves all possible $l$-types chosen from $[i_1, \cdots, i_m]$, that is, from $T$. For example, the terms in $A_3$ involve all the 3-types $[i_p, i_q, i_r]$ in $T$ (the condition $p < q < r$ guarantees that the 3 notes are different and that no one combination is repeated). Therefore, the summation $\sum_{j=0}^{n-1} A_3$ in (40) equals, apart from the minus sign, the sum of the unweighted type contents of $S$ with respect to all the 3-types in $T$. And this can be written as the product of row vector $P_3^t(S)$ with column vector $V_3(T)$. The same applies to all the summations $\sum_{j=0}^{n-1} A_l$, the case $l = m$ giving $\sum_{j=0}^{n-1} A_m = (-1)^m R(S/T)$. Consequently, (40) equals (35).

Formula (34) for the particular case $n = 12$, $m = 2$, that is, $R(S'/T) = R(S/T) - 2c + 12$ was first published by Lewin (1960), who considered $T$ as any interval from 0 to 11 semitones.

Theorem 7.6 (Complete hexachord theorem): *The type vectors satisfy the following relation*

$$V_m^t(S') = (-1)^m V_m^t(S) + \sum_{l=0}^{m-1} (-1)^l V_l^t(S) L_{m,l}^t, \quad m \ge 1. \tag{41}$$

*And, similarly, the class vectors satisfy the following relation*

$$\tilde{V}_m^t(S') = (-1)^m \tilde{V}_m^t(S) + \sum_{l=0}^{m-1} (-1)^l \tilde{V}_l^t(S) \tilde{L}_{m,l}^t, \quad m \ge 1. \tag{42}$$

*Proof*. From Theorem 7.5 (Type content of the complement), and considering all $m$-types in $\mathbb{Z}_n$, we obtain

$$P_m^t(S') = (-1)^m P_m^t(S) + \sum_{l=0}^{m-1} (-1)^l P_l^t(S) M_{m,l}^t, \quad m \ge 1. \tag{43}$$

The product in the summation $P_l^t(S)M_{m,l}^t = P_l^t(S)(Q_{m,l}W_l)^t = P_l^t(S)W_lQ_{m,l}^t = V_l^t(S)Q_{m,l}^t$. So, right-multiplying (43) by $W_m$ gives (41). Then, right-multiplying (41) by $A_m$, writing the product in the summation as $V_l^t(S)L_{m,l}^t A_m = V_l^t(S)(A_m^t L_{m,l})^t$, using Lemma 7.1 (Class-content symmetry), and reasoning as in Theorem 7.3 (Class matrix contraction), we obtain (42).

Equation (41) allows us to obtain the $m$-type vector $V_m(S')$ from $V_m(S)$, although $V_l(S)$, $0 \leq l \leq m-1$, are also required. This equation for the particular case $n = 12$, $c = 6$, $m = 2$, is known as the "hexachord theorem". And, for any $n$ and $c$, but $m = 2$, it is known as the "generalized hexachord theorem". In both cases, the values of $s_i^2$, $1 \leq i \leq N_n(2)$, required for computing $W_2$ and then $L_{2,l}^t$, are given in (3). Rahn (1980, 105–107) gives these two theorems, but leaves the proofs to the reader. In contrast, Blau (1999) gives interesting proofs based on geometrical representations, as well as historical notes. On the other hand, Nuño (2021) gives the corresponding formula for $n = 12$, any $c$, and $m = 3$. Now, a complete version of the hexachord theorem for any $n$, $c$, and $m$, is given. This theorem, together with Corollary 7.1 (Nesting), provide other two important corollaries:

Corollary 7.3: *If two c-types have the same m-type vector, then their complements also have the same m-type vector. The same applies for c-classes and class vectors.*

Corollary 7.4: *If all c-types with the same* $(m-1)$*-type vector are arranged by their m-type vectors in increasing or decreasing lexicographic order, then their complements will be arranged by their m-type vectors in the same or reverse order, depending on whether m is even or odd, respectively. The same applies for c-classes and class vectors.*

For example, if we arrange the $c$-classes (which have the same 1CV) by decreasing 2CV, then their complements will also be arranged by decreasing 2CV (because 2 is even). This is why a chord class and its complement are assigned the same Forte ordinal (except for the hemichord classes). In the particular case of $\mathbb{Z}_{12}$, it turns out that no more than two chord classes have the same 2CV and each chord class has a unique 3CV. Thus, if two chord classes with the same 2CV are arranged, for example, by their 3CV in decreasing lexicographic order and we call the first one hard and the second soft, then the complement of the hard class will be soft and vice versa (because 3 is odd). Additionally, every chord type in $\mathbb{Z}_{12}$ has a unique 3TV, except 6-14a and 6-14b. Thus, excluding these ones, the two types of the same chord class (which have the same 2CV or 2TV) can be arranged by their 3TV in decreasing lexicographic order. And, if we call the first one type $a$ and the second type $b$, then the complement of a type $a$ will be a type $b$ and vice versa (again because 3 is odd).

Theorem 7.7 (Complementary hemichords): *Let us consider* $\mathbb{Z}_n$ *with n being even and let S and* $S'$ *be two complementary hemichord types, their cardinalities being* $c = n/2$*. If they have the same* $(m-1)$*-type vector with m being even, then they also have the same m-type vector. The same applies for complementary hemichord classes and class vectors.*

*Proof*. If $m-1 > n/2$, the $(m-1)$- and $m$-type vectors are zero vectors, and the theorem, although valid, is not relevant. And, if $m-1 \leq n/2$, from Corollary 7.1 (Nesting), $S$ and $S'$ have the same $l$-type vectors, $0 \leq l \leq m-1$. Then, if we interchange $S$ with $S'$ in (41), the summations in both cases are the same. Thus, subtracting the two corresponding equations gives $V_m^t(S') - V_m^t(S) = (-1)^m[V_m^t(S) - V_m^t(S')]$. And, if $m$ is even, that is, $(-1)^m = 1$, we obtain $V_m^t(S') = V_m^t(S)$. A similar proof can be given for class vectors using (42) instead of (41).

Corollary 7.5 (Complementary hemichords and Z-relation): *If two hemichords, either types or classes, are complementary, then they are Z-related. (In general, the reverse is not true). Consequently, if a hemichord class is not Z-related to any other one, then it is self-complementary.*

*Proof*. As $S$ and $S'$ have the same 1CV (cardinality), they also have the same 2CV (because 2 is even), that is, they are Z-related.

In $\mathbb{Z}_{12}$, every group of Z-related 6-classes (hemichords) has two members: a hard and a soft class (as explained just after Corollary 7.4). Since their complements are in the same group (because they are Z-related to the former) and the complement of a hard class is soft and vice versa, they are the complements of each other. But this is not applicable to 6-types, since there are many pairs being Z-related but not complementary.[6] And the situation becomes more complex for $n \geq 16$, as Z-related groups with more than two classes appear (see, for example, the survey by Jedrzejewski and Johnson 2013).

Corollary 7.6 (Self-complementary hemichords): *If a group of hemichord types with their complements have the same $(m-1)$-type vector but they all have different m-type vectors, with m being even, then they all are self-complementary. The same applies for hemichord classes and class vectors.*

For example, in $\mathbb{Z}_{12}$, the hexachord types 6-14a and 6-14b belong to a self-complementary class and have the same 3TV. As they have different 4TV and 4 is even, they are self-complementary.

The next theorems are derived from Theorem 7.6 (Complete hexachord theorem) and require that the chord types and chord classes are exclusively arranged in complementary order, which is indicated by an asterisk on the corresponding matrices. Accordingly, these theorems are not generally valid for the hemichords. In these cases, however, (41) and (42) are always applicable.

Theorem 7.8: *Let the $c'$-types be arranged in complementary order with respect to the c-types and let $M_{c,m}$ and $M^*_{c',m}$ be two type matrices. These matrices are related by*

$$M^*_{c',m} = (-1)^m M_{c,m} + \sum_{l=0}^{m-1} (-1)^l M_{c,l} L^t_{m,l}, \quad m \geq 1, \tag{44}$$

*where matrices $M_{c,l}$ and $L^t_{m,l}$, $0 \leq l \leq m-1$, are also required. Normally, this theorem is used for $c' > c$. Similarly,*

$$\tilde{M}^*_{c',m} = (-1)^m \tilde{M}_{c,m} + \sum_{l=0}^{m-1} (-1)^l \tilde{M}_{c,l} \tilde{L}^t_{m,l}, \quad m \geq 1. \tag{45}$$

*Proof*. Formula (44) is directly obtained from (41), and (45) from (42).

Formula (44) for the particular case $n = 12$, $m = 3$, was given by Nuño (2021).

Theorem 7.9: *Let the $m'$-types be arranged in complementary order with respect to the m-types and let $L_{c,m}$ and $L^*_{c,m'}$ be two left-weighted type matrices. These matrices are related by*

$$L^*_{c,m'} = (-1)^{n-c} L_{c,m} + (-1)^n \sum_{l=c+1}^{n} (-1)^l M^t_{l,c} L_{l,m}, \quad c \leq n-1, \tag{46}$$

*where matrices $L_{l,m}$ and $M^t_{l,c}$, $c+1 \leq l \leq n$, are also required. Normally, this theorem is used for $m' < m$. Similarly,*

$$\tilde{L}^*_{c,m'} = (-1)^{n-c} \tilde{L}_{c,m} + (-1)^n \sum_{l=c+1}^{n} (-1)^l \tilde{M}^t_{l,c} \tilde{L}_{l,m}, \quad c \leq n-1. \tag{47}$$

*Proof*. Considering that the $c'$-types are arranged in complementary order with respect to the $c$-types, relating all of them by (43), and taking into account that $M_{m,m}$ is an identity matrix, we

[6] For example, 6-11a and 6-40a, or 6-12b and 6-41b. A special example is 6-14a and 6-14b, the chord class 6-14 being self-complementary.

can write

$$Q^*_{c',m} = \sum_{l=0}^{m} (-1)^l Q_{c,l} M^t_{m,l} \tag{48}$$

(which is also valid for $m = 0$). Transposing this matrix equation, applying (12) and (14), and using index $l' = n - l$ in the summation, yields

$$Q^*_{m',c} = \sum_{l'=m'}^{n} (-1)^{n-l'} L^t_{l',m'} Q_{l',c'} \tag{49}$$

By interchanging $m'$ with $c$, therefore considering that the $m'$-types are arranged in complementary order with respect to the $m$-types, and renaming $l'$ as $l$, we obtain

$$Q^*_{c,m'} = \sum_{l=c}^{n} (-1)^{n-l} L^t_{l,c} Q_{l,m} \tag{50}$$

Now, left-multiplying by $W_c$, the product in the summation becomes $W_c L^t_{l,c} Q_{l,m} = W_c (W_l Q_{l,c})^t Q_{l,m} = W_c Q^t_{l,c} L_{l,m} = (Q_{l,c} W_c)^t L_{l,m} = M^t_{l,c} L_{l,m}$. Thus, taking into account that $M^t_{c,c}$ is an identity matrix and writing $(-1)^{-l} = (-1)^l$ gives (46).

Then, left-multiplying (46) by $A^t_c$, the matrix product in the summation is $A^t_c M^t_{l,c} L_{l,m} = (M_{l,c} A_c)^t L_{l,m}$. From Lemma 7.1 (Class-content symmetry), any two columns of $(M_{l,c} A_c)^t$, say $j_a$ and $j_b$, corresponding to the two types of the same chord class, say $j$, are the same. Thus, reasoning as in Theorem 7.3 (Class matrix contraction), but interchanging rows with columns, we obtain that $(M_{l,c} A_c)^t L_{l,m} = (M_{l,c} A_c)^t D_l A^t_l L_{l,m} = (D^t_l M_{l,c} A_c)^t A^t_l L_{l,m} = \tilde{M}^t_{l,c} A^t_l L_{l,m}$. So, right-multiplying by $D_m = D_{m'}$ gives (47).

Theorem 7.10: *Let the $c'$-types be arranged in complementary order with respect to the $c$-types and let $M_{c,m}$ and $M^*_{c',m}$ be two type matrices. These matrices are related by*

$$M^*_{c',m} = M_{c,m} \left[ (-1)^m I_m + \sum_{l=0}^{m-1} \frac{(-1)^l}{\binom{c-l}{c-m}} M_{m,l} L^t_{m,l} \right], \quad c \geq m \geq 1, \tag{51}$$

*where $I_m$ is the identity matrix of size $N_n(m)$. Now, the required matrices are $M_{m,l}$ and $L^t_{m,l}$, $0 \leq l \leq m-1$. Normally, this theorem is used for $c' > c$.*

This theorem is obtained by simply substituting (22) in (44). As well, by substituting (29) in (45), we obtain

$$\tilde{M}^*_{c',m} = \tilde{M}_{c,m} \left[ (-1)^m \tilde{I}_m + \sum_{l=0}^{m-1} \frac{(-1)^l}{\binom{c-l}{c-m}} \tilde{M}_{m,l} \tilde{L}^t_{m,l} \right], \quad c \geq m \geq 1, \tag{52}$$

where $\tilde{I}_m$ is the identity matrix of size $\tilde{N}_n(m)$.

Formula (51) for the particular case $n = 12$, $m = 3$, was given by Nuño (2021).

Theorem 7.11: *Let the $m'$-types be arranged in complementary order with respect to the $m$-types and let $L_{c,m}$ and $L^{*}_{c,m'}$ be two left-weighted type matrices. These matrices are related by*

$$L^{*}_{c,m'} = \left[ (-1)^{n-c} I_c + (-1)^n \sum_{l=c+1}^{n} \frac{(-1)^l}{\binom{l-m}{c-m}} M^t_{l,c} L_{l,c} \right] L_{c,m}, \quad m \leq c \leq n-1, \tag{53}$$

*where $I_c$ is the identity matrix of size $N_n(c)$. Now, the required matrices are $L_{l,c}$ and $M^t_{l,c}$, $c+1 \leq l \leq n$. Normally, this theorem is used for $m' < m$.*

This theorem is obtained by simply substituting (26) in (46). As well, by substituting (33) in (47), we obtain

$$\tilde{L}^{*}_{c,m'} = \left[ (-1)^{n-c} \tilde{I}_c + (-1)^n \sum_{l=c+1}^{n} \frac{(-1)^l}{\binom{l-m}{c-m}} \tilde{M}^t_{l,c} \tilde{L}_{l,c} \right] \tilde{L}_{c,m}, \quad m \leq c \leq n-1, \tag{54}$$

where $\tilde{I}_c$ is the identity matrix of size $\tilde{N}_n(c)$.

## 8. Conclusions

The concepts of embedding number and M-class vector introduced by Lewin are here extended to type and class matrices, which provide a broader overview of the type and class contents of chord types and chord classes, respectively. The corresponding full pictures are given by the full $n$-type and $n$-class matrices. The results for the particular cases of $\mathbb{Z}_6$ (hexachords), $\mathbb{Z}_7$ (heptatonic scales), and $\mathbb{Z}_{12}$ (chromatic scale) are given in supplementary files; and, in the first two cases, the type matrices are also given in the form of tables. Therefore, this information can directly be used by researchers and composers; but, additionally, two computer programs, written in MATLAB, are provided for obtaining the above-mentioned and other related matrices in the general case of $\mathbb{Z}_n$. Furthermore, all this material can be used for better interpreting and understanding the properties and theorems here included.

Theorem 5.1 (Complementary reciprocity) served to show the symmetry relation between weighted and left-weighted type and class matrices, thus giving rise to several properties in Section 5. As well, it led to the definitions of complementary and normal orders, which are key concepts in this study. Theorems 7.1–7.4 (Type and class matrix contractions) allow us to obtain the weighted and left-weighted type and class matrices recursively, and provide a simple way to derive the nesting property (Corollaries 7.1 and 7.2). Theorems 7.5 and 7.6 give a complete version of the hexachord theorem (valid for any $n$, $c$, and $m$), supplemented by Corollaries 7.3 and 7.4. Theorem 7.7 reveals a curious property of complementary hemichords, which is supplemented by Corollaries 7.5 and 7.6. Finally, Theorems 7.8–7.11 (based on Theorem 7.6) relate type and class matrices of chord types and chord classes with those of their complements. All these theorems and corollaries show different alternatives for obtaining the type and class matrices, thus providing a wider and deeper insight of the whole theory.

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## References


Babbitt, M. 1955. "Some Aspects of Twelve-Tone Composition." *The Score and I. M. A. Magazine* 12: 53–61.

Blau, Steven K. 1999. "The Hexachordal Theorem: A Mathematical Look at Interval Relations in Twelve-Tone Composition." *Mathematics Magazine* 72: 4.

Bronshtein, Ilja N., Konstantin A. Semendyayev, Gerhard Musiol, and Heiner Muehlig. 2007. *Handbook of Mathematics*. 5th ed. Berlin: Springer-Verlag.

Forte, A. 1973. *The Structure of Atonal Music*. New Haven: Yale University Press.

Hook, J. 2007. "Why Are There Twenty-Nine Tetrachords? A Tutorial on Combinatorics and Enumeration in Music Theory." *Music Theory Online* 13: 4.

Jedrzejewski, F., and T. Johnson. 2013. "The Structure of Z-Related Sets." In *MCM 2013. LNAI 7937*, edited by J. Yust, J. Wild, and J. A. Burgoyne, 128–137. Heidelberg: Springer.

Lewin, D. 1960. "The Intervallic Content of a Collection of Notes, Intervallic Relations Between a Collection of Notes and Its Complement: An Application to Schoenberg's Hexachordal Pieces." *Journal of Music Theory* 4 (1): 98–101.

Lewin, D. 1987. *Generalized Musical Intervals and Transformations*. New Haven: Yale University Press.

Mandereau, J., D. Ghisi, E. Amiot, M. Andreatta, and C. Agon. 2011. "Discrete Phase Retrieval in Musical Structures." *Journal of Mathematics and Music* 5 (2): 99–116.

Morris, R. D. 2007. "Mathematics and the Twelve-Tone System: Past, Present, and Future." *Perspectives of New Music* 45 (2): 76–107.

Nuño, L. 2021. "A Detailed List and a Periodic Table of Set Classes." *Journal of Mathematics and Music* 15 (3): 267–287.

O'Gallagher, John. 2013. *Twelve-Tone Improvisation. A Method for Using Tone Rows in Jazz*. Rottenburg: Advance Music.

Perle, G. 1991 [1962]. *Serial Composition and Atonality: An Introduction to the Music of Schönberg, Berg and Webern*. 6th ed. Berkeley, CA: University of California Press.

Rahn, J. 1980. *Basic Atonal Theory*. New York: Schirmer Books.

Straus, J. N. 1991. "A Primer for Atonal Set Theory." *College Music Symposium* 31: 1–26.

Straus, J. N. 2016. *Introduction to Post-Tonal Theory*. 4th ed. New York: W. W. Norton.

Šedivý, Dominik. 2011. *Serial Composition and Tonality. An Introduction to the Music of Hauer and Steinbauer*. Vienna: Mono/Monochrom.

Tymoczko, Dmitri. 2011. *A Geometry of Music: Harmony and Counterpoint in the Extended Common Practice*. New York: Oxford University Press.