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Weighted subdirect sum of matrices: Definition and properties for positivity classes

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The concept of subdirect sum of matrices (a type of sum of matrices with overlapping blocks) was introduced in 1999 and has been broadly studied. In this paper we extend this concept by introducing the weighted subdirect sum: a sum of matrices with overlapping blocks allowing to give a different weight to the overlapped blocks. This concept naturally arises in applications with overlapping regions such as overlapping graphs in multilayer networks or in iterative methods to solve linear systems of equations based on overlapping blocks. We analyze the same positivity classes of matrices that were studied for the usual subdirect sum in the seminal paper of 1999. We extend those previous results on positivity classes to the new weighted subdirect sum, and we also introduce a methodology that can be useful to extend these results to other classes of matrices. We illustrate the theoretical results with some small examples and we show the potential applicability of the new concept in two lines of future research.

KEYWORDS

clique, multilayer networks, overlap, positive matrices, subdirect sum, supraadjacency matrix, union of graphs

MSC CLASSIFICATION

15B48, 05C50

1 | INTRODUCTION

The concept of subdirect sum of matrices was introduced in Fallat and Johnson[1] motivated by the appearance of these kind of sums in some graph-related problems (see Drew and Johnson[2]). In Fallat and Johnson[1] some properties of this new sum are analyzed for some positive classes of matrices. Since then, these kind of studies have been extended to a wide variety of matrix classes: inverses of M-matrices,[3] S-Strictly Diagonally Dominant matrices,[4] Doubly Diagonally Dominant matrices,[5] P-matrices that are also Strictly Diagonally Dominant,[6] H-matrices,[7] Accretive, Dissipative and Benzi-Golub matrices,[8] Inverse-Positive matrices,[9,10] B-matrices and doubly B-matrices,[11] generalization to linear operators on Hilbert spaces,[12] Nekrasov matrices,[13] Weakly Chained Diagonally Dominant matrices,[14] QN-matrices,[15] p-norm Strictly Diagonally Dominant matrices,[16] Doubly Strictly Diagonally Dominant matrices,[17] Σ-Strictly Diagonally Dominant matrices,[18] and Dashnic–Zusmanovich matrices[19]; see Pedroche[20] for more details on these results.

In this paper we introduce a generalization of the subdirect sum of matrices, which we call weighted subdirect sum, that is motivated by two problems. One is the description of overlapping motifs[21] or cliques[22–24] in complex networks (see, e.g., previous works[25,26]), and the other is the analysis of the convergence of iterative methods for solving linear systems based on the technique of overlapping blocks.[27,28] We believe that the new concept can be applied to some other problems in which overlapping subdomains are involved. In this paper we analyze some properties of the weighted subdirect sum for those positive classes of matrices that were analyzed in the seminal paper.[1] We show that the results of Fallat and Johnson[1] can be extended to the new weighted subdirect sum, and we illustrate the results with some small examples. We also outline how this new concept can be included in the formulation of two standard lines of research: overlapping graphs in complex networks and iterative methods with overlapping blocks for solving linear systems of equations.

The structure of the paper is the following. In Section 2 we recall some basic definitions and set the nomenclature of the paper. In Section 3 we define the weighted subdirect sum of matrices and give immediate properties. In Section 4 we establish the four questions that are the goal of the paper and we give some theoretical results that are the key to develop the paper. In Section 5 we analyze the properties of the weighted subdirect sum for some positivity classes of matrices. In Section 6 we show two applications of the weighted subdirect sum, and finally, we give some conclusions in Section 7.

2 PRELIMINARIES

In this section we give the basic definitions and we establish the kind of matrices that we will consider.

Given two block matrices

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{1}$$

with A_{22} and B_{11} of size $k \times k$, the sum

$$C = \begin{bmatrix} A_{11} & A_{12} & O \\ A_{21} & A_{22} + B_{11} & B_{12} \\ O & B_{21} & B_{22} \end{bmatrix} \tag{2}$$

is called the *k-subdirect sum* (or simply the *subdirect sum*) of A and B, and it is denoted as $C = A \oplus_k B$. This sum was introduced in Fallat and Johnson.[1] The particular case $k = 0$ can be considered as the usual direct sum of matrices.

Given a matrix class $\mathcal{S}$, the questions that were addressed in Fallat and Johnson[1] are the following.

1. If A and B belong to $\mathcal{S}$, does $A \oplus_k B$ belong to the same class $\mathcal{S}$, for $k = 1$?
2. A matrix of the form

$$C = \begin{bmatrix} C_{11} & C_{12} & O \\ C_{21} & C_{22} & C_{23} \\ O & C_{32} & C_{33} \end{bmatrix}$$

that belongs to a class $\mathcal{S}$ can be written as $A \oplus_1 B$ with A and B in the same class as C, for $k = 1$?

3. Question 1 with $k > 1$.
4. Question 2 with $k > 1$.

In this paper we address these questions but referred to the weighted subdirect sum that we define in the next section. All through the paper we focus on real matrices.

3 WEIGHTED SUBDIRECT SUM

Definition 1. Given two square matrices A and B of size n_1 and n_2, respectively, written by blocks as in (1), we define the weighted subdirect sum of A and B, with weights $\alpha \geq 0$ and $\beta \geq 0$ to be the matrix

$$A \oplus_k^{\alpha,\beta} B := \begin{bmatrix} A_{11} & A_{12} & O \\ A_{21} & \alpha A_{22} + \beta B_{11} & B_{12} \\ O & B_{21} & B_{22} \end{bmatrix}$$

where we recall that k is the size of A_{22} and B_{11}. Note that the relation with the k-subdirect sum is

$$A \oplus_k^{\alpha,\beta} B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & \alpha A_{22} \end{bmatrix} \oplus_k \begin{bmatrix} \beta B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Trivial properties

- When $n_1 = n_2 = k$ then $A = A_{22}$ and $B = B_{11}$, and therefore, when $\alpha = \beta = 1$, we have $A \oplus_k^{1,1} B = A + B$, the usual sum of matrices.
- When $k = 0$ it is assumed that $A = A_{11}$ and $B = B_{22}$, and therefore, for any α, β, we have $A \oplus_0^{\alpha,\beta} B = A \oplus B$, the usual direct sum of matrices.
- When $\alpha = \beta = 1$ we have $A \oplus_k^{1,1} B = A \oplus_k B$, the usual k-subdirect sum of matrices.

Remark 1. Let $A \in \mathbb{R}^{n_1 \times n_1}$ and $B \in \mathbb{R}^{n_2 \times n_2}$ be two matrices partitioned as in (1), and let A_γ and B^δ be of the same size as, and partitioned conformably to, A and B, respectively, and defined by

$$A_\gamma = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & \gamma A_{22} \end{bmatrix}, \quad B^\delta = \begin{bmatrix} \delta B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{3}$$

with $\gamma > 0$ and $\delta > 0$. Then

$$A \oplus_k B = A_\gamma \oplus_k^{\gamma^{-1},\delta^{-1}} B^\delta$$

for any $k = 1, 2, \ldots, \min\{n_1, n_2\}$

In the next sections we study more properties and focus on positive classes of matrices. Before enter into these classes we can state a result about the inverse of the weighted subdirect sum. We do this in the next section.

3.1 | The inverse of the weighted subdirect sum

By following a similar technique as in theorem 2.1 of Bru et al[3] we can prove the following result.

Proposition 1. *Given two matrices A and B partitioned as in (1), let*

$$A_\alpha = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & \alpha A_{22} \end{bmatrix}, \quad B^\beta = \begin{bmatrix} \beta B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{4}$$

with $\alpha \geq 0$ and $\beta \geq 0$. Let us assume that A_α and B^β are nonsingular matrices and let us partition their inverses conformably to (4) and denote the blocks as

$$(A_\alpha)^{-1} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \quad (B^\beta)^{-1} = \begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \end{bmatrix}$$

then it holds that

$$A \oplus_k^{\alpha,\beta} B = \begin{bmatrix} A_\alpha & O \\ O & I_{n-n_1} \end{bmatrix} \begin{bmatrix} I_{n_1-k} & \hat{A}_{12} & O \\ O & \hat{H} & \hat{B}_{12} \\ O & O & I_{n_2-k} \end{bmatrix} \begin{bmatrix} I_{n-n_2} & O \\ O & B^\beta \end{bmatrix}$$

with $\hat{H} = \hat{A}_{22} + \hat{B}_{11}$.

Proof. Let us denote $C = A \oplus_k^{\alpha,\beta} B$. Note the following equality

$$\begin{bmatrix} (A_\alpha)^{-1} & O \\ O & I_{n-n_1} \end{bmatrix} C \begin{bmatrix} I_{n-n_2} & O \\ O & (B^\beta)^{-1} \end{bmatrix} =$$

$$
\begin{aligned}
&= \begin{bmatrix} (A_\alpha)^{-1} & O \\ O & I_{n-n_1} \end{bmatrix} \left(\begin{bmatrix} A_\alpha & O \\ O & O \end{bmatrix} + \begin{bmatrix} O & O \\ O & B^\beta \end{bmatrix} \right) \begin{bmatrix} I_{n-n_2} & O \\ O & (B^\beta)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} I_{n_1-k} & O & O \\ O & I_k & O \\ O & O & O \end{bmatrix} \begin{bmatrix} I_{n-n_2} & O & O \\ O & \hat{B}_{11} & \hat{B}_{12} \\ O & \hat{B}_{21} & \hat{B}_{22} \end{bmatrix} + \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & O \\ \hat{A}_{21} & \hat{A}_{22} & O \\ O & O & I_{n-n_1} \end{bmatrix} \begin{bmatrix} O & O & O \\ O & I_k & O \\ O & O & I_{n_2-k} \end{bmatrix} \\
&= \begin{bmatrix} I_{n_1-k} & O & O \\ O & \hat{B}_{11} & \hat{B}_{12} \\ O & O & O \end{bmatrix} + \begin{bmatrix} O & \hat{A}_{12} & O \\ O & \hat{A}_{22} & O \\ O & O & I_{n_2-k} \end{bmatrix} \\
&= \begin{bmatrix} I_{n_1-k} & \hat{A}_{12} & O \\ O & \hat{H} & \hat{B}_{12} \\ O & O & I_{n_2-k} \end{bmatrix}
\end{aligned}
$$

and, since $A \oplus_k^{\alpha,\beta} B = A_\alpha \oplus_k B^\beta$, the proof follows. □

Corollary 1. *The determinant of $A \oplus_k^{\alpha,\beta} B$ is given by*

$$
det(A \oplus_k^{\alpha,\beta} B) = det(A_\alpha) det(\hat{H}) det(B^\beta)
$$

Corollary 2. *The inverse of $C = A \oplus_k^{\alpha,\beta} B$, when it exists, is given by*

$$
C^{-1} = \begin{bmatrix} I_{n-n_2} & O \\ O & (B^\beta)^{-1} \end{bmatrix} \begin{bmatrix} I_{n-n_2} & -\hat{A}_{12}(\hat{H})^{-1} & \hat{A}_{12}(\hat{H})^{-1}\hat{B}_{12} \\ O & (\hat{H})^{-1} & -(\hat{H})^{-1}\hat{B}_{12} \\ O & O & I_{n-n_1} \end{bmatrix} \begin{bmatrix} (A_\alpha)^{-1} & O \\ O & I_{n-n_1} \end{bmatrix}
$$

Remark 2. As we have used in the above theorem, since $A \oplus_k^{\alpha,\beta} B = A_\alpha \oplus_k B^\beta$ the known properties of the subdirect sum can be analyzed in an easy way to check whether they are also properties of the weighted subdirect sum or not. We formalize this in the next section.

4 | METHODOLOGY

Our main goal in this paper is to answer the following questions.

Q-I If A and B belong to a class $\mathcal{S}$, do exist some $\alpha > 0$ and $\beta > 0$ such that $A \oplus_k^{\alpha,\beta} B$ belongs to the same class $\mathcal{S}$, for $k = 1$?

Q-II A matrix of the form

$$
C = \begin{bmatrix} C_{11} & C_{12} & O \\ C_{21} & C_{22} & C_{23} \\ O & C_{32} & C_{33} \end{bmatrix}
$$

that belongs to a class $\mathcal{S}$ can be written as $A \oplus_k^{\alpha,\beta} B$, for $k = 1$, with some $\alpha > 0$ and $\beta > 0$, and with A and B in the same class as C?

Q-III Question Q-I with $k > 1$.

Q-IV Question Q-II with $k > 1$.

The questions Q-I (when $k = 1$) and Q-III (when $k > 1$) referring to k-subdirect sum can be extended in a natural way to the weighted subdirect sum by using the next result.

Lemma 1. *Let $A \in \mathbb{R}^{n_1 \times n_1}$ and $B \in \mathbb{R}^{n_2 \times n_2}$ be two matrices of the same class $\mathcal{S}$ and partitioned as in (1). Let A_α and B^β be of the same size as, and partitioned conformably to, A and B, respectively, and defined by (4). Let us assume that question Q-I (Q-III) is verified for the usual subdirect sum for matrices A and B in the class $\mathcal{S}$ and there exist $\alpha > 0$ and $\beta > 0$ such that A_α and B^β are also in the class $\mathcal{S}$. Then question Q-I (Q-III) is verified for the weighted subdirect sum.*

Proof. Since question Q-I (Q-III) is verified for the usual subdirect sum we know that given A and B in the class $\mathcal{S}$ then $A \oplus_k B$ is in $\mathcal{S}$. Therefore, since A_α and B^β are also in the class $\mathcal{S}$ we conclude that $A_\alpha \oplus_k B^\beta$ is in the class $\mathcal{S}$. By noting that

$$A_\alpha \oplus_k B^\beta = \begin{bmatrix} A_{11} & A_{12} & O \\ A_{12} & \alpha A_{22} + \beta B_{11} & B_{12} \\ O & B_{12}^T & B_{22} \end{bmatrix} = A \oplus_k^{\alpha,\beta} B$$

with $\alpha > 0$ and $\beta > 0$, we conclude that question Q-I (Q-III) is affirmative for the weighted subdirect sum. □

The questions Q-II (when $k = 1$) and Q-IV (when $k > 1$) referred to k-subdirect sum can be extended in a natural way to the weighted subdirect sum by using the next result.

Lemma 2. *Let*

$$C = \begin{bmatrix} C_{11} & C_{12} & O \\ C_{21} & C_{22} & C_{23} \\ O & C_{32} & C_{33} \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{5}$$

be a matrix in some class $\mathcal{S}$, with C_{11} and C_{33} square matrices and with C_{22} a square matrix of size k. Let us assume that C verifies question Q-II (Q-IV) for the usual subdirect sum and can be written as $A' \oplus_k B'$, with $A' \in \mathbb{R}^{n_1\times n_1}$ and $B' \in \mathbb{R}^{n_2\times n_2}$ both in the class $\mathcal{S}$ and partitioned as in (1). Let A'_γ and B'^δ, with $\gamma > 0$ and $\delta > 0$, be of the same size as, and partitioned conformably to, A' and B', respectively, and defined as in (3). Let us assume that A'_γ and B'^δ are in the class $\mathcal{S}$. Then question Q-II (Q-IV) is verified for the weighted subdirect sum.

Proof. Note that by Remark 1,

$$C = A' \oplus_k B' = A'_\gamma \oplus_k^{\gamma^{-1},\delta^{-1}} B'^\delta$$

is in the class $\mathcal{S}$. Denoting $A = A'_\gamma$, $B = B'^\delta$, $\alpha = \gamma^{-1}$ and $\beta = \delta^{-1}$ we have that $C = A \oplus_k^{\alpha,\beta} B$, for some $\alpha > 0$ and $\beta > 0$ with A and B in the class $\mathcal{S}$. Therefore, we conclude that question Q-II (Q-IV) is affirmative for the weighted subdirect sum. □

Some results in Fallat and Johnson[1] answer to questions Q-I and Q-II (or Q-III and Q-IV) for the usual subdirect sum at the same time by writing propositions with necessary and sufficient conditions. In the present paper we are restricted by the conditions of A_α and B^β to be in the class $\mathcal{S}$. Since the conditions that we manage are sufficient conditions we prefer to answer the questions one by one, for the sake of clarity in the presentations.

Remark 3. Note that by using these two previous lemmas the key point to answer questions Q-I to Q-IV for the weighted subdirect sum by using the results of Fallat and Johnson,[1] will be to give conditions such that A_α, and B^β, are in the same class as their corresponding matrices A and B.

When we have a matrix class $\mathcal{S}_C$ that is closed under addition (i.e., when A, B are in the class then $A + B$ is in the class), we can use the following result to answer questions Q-I and Q-III

Lemma 3. *Let $\mathcal{S}_C$ be a matrix class that is closed under addition and such that for any matrix A and B in S_c it holds that*

1. *αA is in $\mathcal{S}_C$ for any $\alpha \geq 0$.*
2. *Any principal submatrix of A is in $\mathcal{S}_C$.*
3. *Any matrix of the form*

$$\begin{bmatrix} O_{p\times p} & O_{p\times k} & O_{p\times q} \\ O_{k\times p} & H & O_{k\times q} \\ O_{q\times p} & O_{q\times k} & O_{q\times q} \end{bmatrix}$$

is in $\mathcal{S}_C$ when H is in $\mathcal{S}_C$.

4. *$A \oplus_k B$ is in $\mathcal{S}_C$ for some k.*

then $A \oplus_k^{\alpha,\beta} B$ is in the class $\mathcal{S}_C$ for those values of k, and for any $\alpha \geq 1$ and $\beta \geq 1$.

Proof. Note that we can write the weighted subdirect sum as

$$A \oplus_k^{\alpha,\beta} B = A \oplus_k B + \begin{bmatrix} O_{(n_1-k)\times(n_1-k)} & O_{(n_1-k)\times k} & O_{(n_1-k)\times q} \\ O_{k\times(n_1-k)} & A_{22}(\alpha-1)+B_{11}(\beta-1) & O_{k\times(n_2-k)} \\ O_{(n_2-k)\times(n_1-k)} & O_{(n_2-k)\times k} & O_{(n_2-k)\times(n_2-k)} \end{bmatrix}$$

Now, by the Hypothesis 2, A_{22} and B_{11} are in $\mathcal{S}_C$ and since the class is closed under addition and by Hypothesis 1, is clear that $A_{22}(\alpha-1)+B_{11}(\beta-1)$ is also in $\mathcal{S}_C$ when $\alpha \geq 1$ and $\beta \geq 1$. Now, by using the Hypotheses 3 and 4 and that the class is closed under addition, the proof follows. □

When $\mathcal{S}_C$ is closed under addition with the property that given any matrix A in $\mathcal{S}_C$ then any principal submatrix of A is in $\mathcal{S}_C$, we can use the following result to give a restricted answer questions Q-II and Q-IV.

Lemma 4. *Let*

$$C = \begin{bmatrix} C_{11} & C_{12} & O \\ C_{21} & C_{22} & C_{23} \\ O & C_{32} & C_{33} \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{6}$$

be in $\mathcal{S}_C$, with C_{11} and C_{33} square matrices and with C_{22} a square matrix of size k. Let $\mathcal{S}_C$ be closed under addition, and with the property that given any matrix A in $\mathcal{S}_C$ then any principal submatrix of A is in $\mathcal{S}_C$. Then it holds that we can write $C = A \oplus_k^{\alpha,1-\alpha} B$ for any $0 < \alpha < 1$ and where A and B are in $\mathcal{S}_C$.

Proof. Note that we can write C as

$$\begin{aligned} C &= \begin{bmatrix} C_{11} & C_{12} & O \\ C_{21} & \alpha C_{22} & O \\ O & O & O \end{bmatrix} + \begin{bmatrix} O & O & O \\ O & (1-\alpha)C_{22} & C_{23} \\ O & C_{32} & C_{33} \end{bmatrix} \\ &= \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \oplus_k^{\alpha,1-\alpha} \begin{bmatrix} C_{22} & C_{23} \\ C_{32} & C_{33} \end{bmatrix} \end{aligned} \tag{7}$$

and therefore, $C = A \oplus_k^{\alpha,1-\alpha} B$, with A and B the matrices

$$A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \quad B = \begin{bmatrix} C_{22} & C_{23} \\ C_{32} & C_{33} \end{bmatrix}$$

that are in $\mathcal{S}_C$, since they are principal submatrices of C. □

5 | RESULTS FOR POSITIVITY CLASSES

5.1 | Positive definite (PD) and positive semidefinite (PSD) matrices

We recall (see, e.g., Meyer[29] or Horn and Johnson[30]) that a symmetric real matrix A of size n is a positive definite matrix, denoted as PD, if it holds that

$$\mathbf{x}^T A\mathbf{x} > 0, \quad \text{for all} \quad \mathbf{x} \neq \mathbf{0} \in \mathbb{R}^n$$

Analogously, a symmetric real matrix A of size n is a positive semidefinite matrix, denoted as PSD, if it holds that

$$\mathbf{x}^T A\mathbf{x} \geq 0, \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n$$

We recall that Fallat and Johnson[1] give affirmative answer for the classes *PD* and *PSD* for questions Q-I to Q-IV for the usual subdirect sum.

Our goal in this section is to answer questions Q-I to Q-IV for the weighted subdirect sum. To do this, we need to recall some basic results, that we present as remarks, and two basic results that we present as lemmas.

Remark 4. [30] Any principal submatrix of a PD matrix is a PD matrix.

Remark 5. [30] Any principal submatrix of a PSD matrix is a PSD matrix.

Remark 6. [30] The sum of two PD matrices of the same size is a PD matrix. Any nonnegative linear combination of PSD matrices is a PSD matrix.

Lemma 5. *Let*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}, \quad A_\alpha = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & \alpha A_{22} \end{bmatrix} \tag{8}$$

with A a (real symmetric) PD matrix and let $\alpha \geq 1$. Then A_α is a PD matrix.

Proof. We must show that $\mathbf{x}^T A_\alpha \mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0} \in \mathbb{R}^n$. The case $\alpha = 1$ is clear since $A_\alpha = A$. For the case $\alpha > 1$, let $\mathbf{x} \neq \mathbf{0}$ and let us compute

$$\begin{aligned}
\mathbf{x}^T A \mathbf{x} &= \begin{bmatrix} \mathbf{x}_1^T & \mathbf{x}_2^T \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{x}_1^T A_{11} + \mathbf{x}_2^T A_{12}^T & \mathbf{x}_1^T A_{12} + \mathbf{x}_2^T A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \\
&= \mathbf{x}_1^T A_{11} \mathbf{x}_1 + \mathbf{x}_2^T A_{12}^T \mathbf{x}_1 + \mathbf{x}_1^T A_{12} \mathbf{x}_2 + \mathbf{x}_2^T A_{22} \mathbf{x}_2 \\
&< \mathbf{x}_1^T A_{11} \mathbf{x}_1 + \mathbf{x}_2^T A_{12}^T \mathbf{x}_1 + \mathbf{x}_1^T A_{12} \mathbf{x}_2 + \alpha \mathbf{x}_2^T A_{22} \mathbf{x}_2 \\
&= \begin{bmatrix} \mathbf{x}_1^T A_{11} + \mathbf{x}_2^T A_{12}^T & \mathbf{x}_1^T A_{12} + \alpha \mathbf{x}_2^T A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \\
&= \mathbf{x}^T A_\alpha \mathbf{x}
\end{aligned}$$

where we have used that $\mathbf{x}_2^T A_{22} \mathbf{x}_2 < \alpha \mathbf{x}_2^T A_{22} \mathbf{x}_2$ since A_{22} is PD, by Remark 4, and $\alpha > 1$. Since $\mathbf{x}^T A \mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0} \in \mathbb{R}^n$, the proof follows. □

Analogously, it is easy to prove the following lemmas, and we omit the proof.

Lemma 6. *Let*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}, \quad A_\alpha = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & \alpha A_{22} \end{bmatrix} \tag{9}$$

with A a (real symmetric) PSD matrix and let $\alpha \geq 1$. Then A_α is a PSD matrix.

Lemma 7. *Let*

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix}, \quad B^\beta = \begin{bmatrix} \beta B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix} \tag{10}$$

with B a PD (PSD) matrix and $\beta \geq 1$. Then B^β is a PD (PSD) matrix.

Now we have all the ingredients to answer questions Q-I (i.e., when $k = 1$) and Q-III (when $k > 1$) for the matrix class *PD* (and *PSD*).

Proposition 2. *Question Q-I (and question Q-III, taking $k > 1$) for the classes PD and PSD have affirmative answer for the weighted subdirect sum.*

Proof. By Fallat and Johnson,[1] we know that Q-I and Q-III have an affirmative answer for the usual subdirect sum. By using Lemma 5 (for *PD* class), Lemma 6 (for *PSD*), Lemma 7 (*PD* and *PSD*) and Lemma 1 we conclude that when $\alpha \geq 1$ and $\beta \geq 1$, question Q-I (and question Q-III, taking $k > 1$) have affirmative answer for the weighted subdirect sum. □

Note that when $0 \leq \alpha < 1$ or $0 \leq \beta < 1$, we cannot assure that the weighted subdirect sum is also in the *PD* or *PSD* classes, as the following example shows.

Example 1. The matrix

$$A = \begin{bmatrix} 1 & 3 \\ 3 & a \end{bmatrix}$$

is a *PSD* matrix when $a = 9$ and a *PD* matrix when $a > 9$. Let us take $B = A$ and compute

$$A \oplus_1^{\frac{1}{2},1} B = \begin{bmatrix} 1 & 3 & 0 \\ 3 & \frac{1}{2}a + 1 & 3 \\ 0 & 3 & a \end{bmatrix}$$

It is easy to see that when $a = 9$, $A \oplus_1^{\frac{1}{2},1} B$ is not a *PSD* matrix, and when $a \in \left[9, 8 + \sqrt{82}\right]$, $A \oplus_1^{\frac{1}{2},1} B$ is not a *PD* matrix.

Remark 7. It is known (see, e.g., Horn and Johnson[30]) that the inverse of a PD matrix is also a PD matrix. Therefore, the inverse of $A \oplus_k^{\alpha,\beta} B$ given by Proposition 2 does exist and it is given by Corollary 2.

Analogously, we have all the ingredients to answer questions Q-II (when $k = 1$) and Q-IV (when $k > 1$) for the matrix classes *PD* and *PSD*.

Proposition 3. *Question Q-II (and question Q-IV, taking $k > 1$) for the matrix classes PD and PSD have affirmative answer for the weighted subdirect sum.*

Proof. By Fallat and Johnson,[1] we know that Q-II and Q-IV have an affirmative answer for the matrix class *PD* (*PSD*) for the usual subdirect sum. By using Lemma 5 (for *PD* class) (Lemma 6 for *PSD*) and Lemma 7 (*PD* and *PSD*) and Lemma 2 we conclude that when $\gamma \geq 1$ and $\delta \geq 1$, that is to say, when $\alpha \leq 1$ and $\beta \leq 1$ question Q-II (and question Q-IV, taking $k > 1$) have affirmative answer for the weighted subdirect sum. □

Example 2.

$$C = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 19 & 3 \\ 0 & 3 & 9 \end{bmatrix}$$

is a *PSD* matrix. We can write $C = A \oplus_1^{\frac{1}{2},1} B$ by taking the PSD matrices

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 36 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

Note that taking $\alpha > 1$ or $\beta > 1$, we cannot be certain of finding matrices A and B in the PSD class. For example, if $\alpha = 2.5$ and $\beta = 1$, we have the equality

$$C = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 19 & 3 \\ 0 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 2.5a_{22} + b_{11} & 3 \\ 0 & 3 & 9 \end{bmatrix} \tag{11}$$

that is,

$$A = \begin{bmatrix} 1 & 3 \\ 3 & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & 3 \\ 3 & 9 \end{bmatrix}$$

and since we want A to be in the class *PSD*, we need $det(A) \geq 0$ that is to say $a_{22} \geq 9$ but, from (11), it is needed that $19 = 2.5a_{22} + b_{11}$, and therefore,

$$b_{11} = 19 - 2.5a_{22} < 0$$

and then B is not in the *PSD* class.

5.2 | Symmetric M matrices (SM)

Symmetric M matrices (SM) are the intersection of Z matrices and PD matrices.

We recall that Fallat and Johnson[1] give affirmative answer for the class SM for questions Q-I to Q-IV for the usual subdirect sum. Note that Lemma 5 and Lemma 7 applied over Z-matrices can be used to give conditions on A_α and B^β to be in the class SM. Therefore, we can use an analogous result to Proposition 2 to the SM class.

Proposition 4. *Question Q-I (and question Q-III, taking $k > 1$) for the class SM have affirmative answer for the weighted subdirect sum.*

Proof. By Fallat and Johnson,[1] we know that Q-I and Q-III have an affirmative answer for the class SM and the usual subdirect sum. By using Lemma 5 and Lemma 7 over Z-matrices and Lemma 1 we conclude that when $\alpha \geq 1$ and $\beta \geq 1$, question Q-I (and question Q-III, taking $k > 1$) have affirmative answer for the weighted subdirect sum for this class. □

When $0 < \alpha < 1$ or $0 < \beta < 1$, we cannot assure that the weighted subdirect sum is also in the SM class, as the following example shows.

Example 3. Consider the SM matrix

$$A = \begin{bmatrix} 1 & -3 \\ -3 & 10 \end{bmatrix}$$

Let us take $B = A$ and compute

$$A \oplus_1^{\frac{1}{2},\frac{1}{2}} B = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix}$$

that it is not an SM-matrix since $det\left(A \oplus_1^{\frac{1}{2},\frac{1}{2}} B\right) < 0$.

To answer question Q-II (and question Q-IV, taking $k > 1$), we can use an analogous result to Proposition 3 applied to the SM class.

Proposition 5. *Question Q-II (and question Q-IV, taking $k > 1$) for the matrix class SM have affirmative answer for the weighted subdirect sum.*

Proof. By Fallat and Johnson,[1] we know that Q-II and Q-IV have an affirmative answer for the matrix class SM for the usual subdirect sum. By using Lemma 5 and Lemma 7 over Z-matrices and Lemma 2, we conclude that when $\gamma \geq 1$ and $\delta \geq 1$, that is to say, when $\alpha \leq 1$ and $\beta \leq 1$ question Q-II (and question Q-IV, taking $k > 1$) have affirmative answer for the weighted subdirect sum for this class. □

5.3 | Completely positive (CP) matrices and double nonnegative (DN) matrices

Recall (see, e.g., previous works[31,32]) that a square (symmetric) matrix A is called completely positive when it can be written as $A = RR^T$ with $R \geq 0 \in \mathbb{R}^{n \times r}$. A square matrix A is said to be double nonnegative (DN) when it is PSD and $A \geq 0$. It is known (see, e.g., Fallat and Johnson[1]) that CP is a subset of DN and that the classes CP and DN are closed under addition.

We recall that Fallat and Johnson[1] give affirmative answer for the classes CP and DN for questions Q-I and Q-III when using the usual subdirect sum.

To analyze the CP class as in the previous sections, we need to give general conditions on α and β such that A_α and B^β are in the class CP. To do this, we need the following results. First, we recall (see, e.g., Fallat and Johnson[1]) that any principal submatrix of a CP matrix is also a CP matrix.

Lemma 8. [33] *If A is an $n \times n$ CP matrix, and H is an $m \times n$ nonnegative matrix, then HAH^T is CP.*

Proof. Since there exists $B \geq 0$ such that $A = BB^T$, then $HAH^T = HBB^TH^T = (HB)(HB)^T$. □

Lemma 9. *Let*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}, \quad A_\alpha = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & \alpha A_{22} \end{bmatrix} \tag{12}$$

with A an $n \times n$ CP matrix and let $\alpha \geq 1$. Then A_α is a CP matrix.

Proof. Let A_{22} be of size $k \times k$, and let I_r the identity matrix of size r. Note that

$$\begin{bmatrix} I_{n-k} & O \\ O & \alpha I_k \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \begin{bmatrix} I_{n-k} & O \\ O & \alpha I_k \end{bmatrix} = \alpha \begin{bmatrix} \frac{1}{\alpha} A_{11} & A_{12} \\ A_{12}^T & \alpha A_{22} \end{bmatrix}$$

is a CP matrix, by Lemma 8. Therefore,

$$\begin{bmatrix} \frac{1}{\alpha} A_{11} & A_{12} \\ A_{12}^T & \alpha A_{22} \end{bmatrix}$$

is also a CP matrix. Note now that

$$A_\alpha = \begin{bmatrix} \frac{1}{\alpha} A_{11} & A_{12} \\ A_{12}^T & \alpha A_{22} \end{bmatrix} + \begin{bmatrix} \frac{\alpha-1}{\alpha} A_{11} & O \\ O & O \end{bmatrix}$$

and since $\begin{bmatrix} \frac{\alpha-1}{\alpha} A_{11} & O \\ O & O \end{bmatrix} = \frac{\alpha-1}{\alpha} \begin{bmatrix} I_{n-k} \\ O \end{bmatrix} A_{11} \begin{bmatrix} I_{n-k} & O \end{bmatrix}$ is a CP matrix and the class CP is closed under addition, the proof follows. □

Analogously, it is easy to prove the following lemma, and we omit the proof.

Lemma 10. *Let*

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix}, \quad B^\beta = \begin{bmatrix} \beta B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix} \tag{13}$$

with B an $n \times n$ CP matrix and let $\beta \geq 1$. Then B^β is a CP matrix.

Proposition 6. *Question Q-I (and question Q-III, taking $k > 1$) for the class CP have affirmative answer for the weighted subdirect sum.*

Proof. By Fallat and Johnson,[1] we know that Q-I and Q-III have an affirmative answer for the class *CP* and the usual subdirect sum. By using Lemma 9, Lemma 10, and Lemma 1 we conclude that when $\alpha \geq 1$ and $\beta \geq 1$, question Q-I (and question Q-III, taking $k > 1$) have affirmative answer for the weighted subdirect sum for this class. □

To answer question Q-II (and question Q-IV, taking $k > 1$), we can use the same technique as in the previous sections.

Proposition 7. *Question Q-II (and question Q-IV, taking $k > 1$) for the matrix class CP have affirmative answer for the weighted subdirect sum.*

Proof. By Fallat and Johnson,[1] we know that Q-II and Q-IV have an affirmative answer for the matrix class *CP* for the usual subdirect sum. By using Lemma 9, Lemma 10, and Lemma 2 we conclude that when $\gamma \geq 1$ and $\delta \geq 1$, that is to say, when $0 < \alpha \leq 1$ and $0 < \beta \leq 1$ question Q-II (and question Q-IV, taking $k > 1$) have affirmative answer for the weighted subdirect sum for this class. □

Note that since the class CP is closed under addition, we could use Lemma 3 to prove Proposition 6. Now we use Lemma 4 to obtain the following result.

Proposition 8. *Let*

$$C = \begin{bmatrix} C_{11} & C_{12} & O \\ C_{12}^T & C_{22} & C_{23} \\ O & C_{23}^T & C_{33} \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{14}$$

be an n by n symmetric CP-matrix, with C_{11} and C_{33} square matrices and with C_{22} a square matrix of size k. Then it holds that $C = A \oplus_k^{\alpha,1-\alpha} B$ for any $0 < \alpha < 1$ and where A and B are CP-matrices.

Proof. It is a consequence of Lemma 4. □

In general terms, when $0 \le \alpha < 1$ or $0 \le \beta < 1$, we cannot assure that the weighted subdirect sum is also in the *CP* class, as the following example shows.

Example 4. The matrix

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

is a *CP* matrix. Let us take $B = A$ and compute

$$A \oplus_1^{\frac{1}{4},1} B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \oplus_1 \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

that is not in the *CP* class, since $det\left(A \oplus_1^{\frac{1}{4},1} B\right) < 0$.

Despite the above example, we can still find subclasses of *CP* matrices for which the weighted subdirect sum is in the class, for particular values of $0 \le \alpha < 1$ and $0 \le \beta < 1$, as the next proposition claims.

Proposition 9. *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$ be CP matrices of the following form*

$$A = \begin{bmatrix} O_{(n-1)\times 1} & \boldsymbol{a}_{(n-1)\times 1} \\ \boldsymbol{b}_{1\times 1} & \boldsymbol{b}_{1\times 1} \end{bmatrix} \begin{bmatrix} O_{1\times(n-1)} & \boldsymbol{b}_{1\times 1} \\ \boldsymbol{a}^T_{1\times(n-1)} & \boldsymbol{b}_{1\times 1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{aa}^T & \boldsymbol{ab} \\ \boldsymbol{ba}^T & 2\boldsymbol{bb} \end{bmatrix}$$

$$B = \begin{bmatrix} \boldsymbol{c}_{1\times 1} & \boldsymbol{c}_{1\times 1} \\ O_{(m-1)\times 1} & \boldsymbol{d}_{(m-1)\times 1} \end{bmatrix} \begin{bmatrix} \boldsymbol{c}_{1\times 1} & O_{1\times(m-1)} \\ \boldsymbol{c}_{1\times 1} & \boldsymbol{d}^T_{1\times(m-1)} \end{bmatrix} = \begin{bmatrix} 2\boldsymbol{cc} & \boldsymbol{cd}^T \\ \boldsymbol{dc} & \boldsymbol{dd}^T \end{bmatrix}$$

for any nonnegative vectors $\boldsymbol{a} \in \mathbb{R}^{(n-1)\times 1}$, $\boldsymbol{d} \in \mathbb{R}^{(m-1)\times 1}$, and nonnegative scalars $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^{1\times 1}$. Then $A \oplus_1^{\frac{1}{2},\frac{1}{2}} B$ is also a CP matrix.

Proof. A direct computation gives

$$A \oplus_1^{\frac{1}{2},\frac{1}{2}} B = \begin{bmatrix} \mathbf{aa}^T & \mathbf{ab} \\ \mathbf{ba}^T & \mathbf{bb} \end{bmatrix} \oplus_1 \begin{bmatrix} \mathbf{cc} & \mathbf{cd}^T \\ \mathbf{dc} & \mathbf{dd}^T \end{bmatrix} \tag{15}$$

and noting that

$$\begin{bmatrix} \mathbf{aa}^T & \mathbf{ab} \\ \mathbf{ba}^T & \mathbf{bb} \end{bmatrix} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \begin{bmatrix} \mathbf{a}^T & \mathbf{b} \end{bmatrix} \in CP$$

and

$$\begin{bmatrix} \mathbf{cc} & \mathbf{cd}^T \\ \mathbf{dc} & \mathbf{dd}^T \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix} \begin{bmatrix} \mathbf{c} & \mathbf{d}^T \end{bmatrix} \in CP$$

we have that $A \oplus_1^{\frac{1}{2},\frac{1}{2}} B$ given by (15) is in the *CP* class since in Fallat and Johnson[1] is proved that the 1-subdirect sum of *CP* matrices is a *CP* matrix. □

Let us now focus on the class *DN*. Since the matrices in this class are nonnegative and PSD, it is clear that we can extend Lemma 6 and Lemma 7 to prove the following results (and we omit the proofs).

Lemma 11. *Let*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}, \quad A_\alpha = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & \alpha A_{22} \end{bmatrix} \tag{16}$$

with A an $n \times n$ DN matrix and let $\alpha \geq 1$. Then A_α is a DN matrix.

Lemma 12. *Let*

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix}, \quad B^\beta = \begin{bmatrix} \beta B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix} \tag{17}$$

with B an $n \times n$ DN matrix and let $\beta \geq 1$. Then B^β is a DN matrix.

Regarding the class *DN*, we recall that Fallat and Johnson[1] give positive answer, when using the usual subdirect sum, to questions Q-I, Q-II and Q-III, but not to question Q-IV. Now we can give a positive answer to questions Q-I (when $k = 1$) and Q-III (when $k > 1$) for the matrix class *DN* following the same technique as before (we could also use Lemma 3).

Proposition 10. *Question Q-I (and question Q-III, taking $k > 1$) for the class DN have affirmative answer for the weighted subdirect sum.*

Proof. By Fallat and Johnson,[1] we know that Q-I and Q-III have an affirmative answer for the class *DN* and the usual subdirect sum. By using Lemma 11, Lemma 12, and Lemma 1 we conclude that when $\alpha \geq 1$ and $\beta \geq 1$, question Q-I (and question Q-III, taking $k > 1$) has affirmative answer for the weighted subdirect sum for this class. □

Regarding question II, we can use Lemma 2 to give an affirmative answer for $k = 1$.

Proposition 11. *Question Q-II for the matrix class DN has affirmative answer for the weighted subdirect sum.*

Proof. By Fallat and Johnson,[1] we know that Q-II has an affirmative answer for the matrix class *DN* for the usual subdirect sum. By using Lemma 11, Lemma 12, and Lemma 2 we conclude that when $\gamma \geq 1$ and $\delta \geq 1$, that is to say, when $0 < \alpha \leq 1$ and $0 < \beta \leq 1$ question Q-II has affirmative answer for the weighted subdirect sum for this class. □

Furthermore, since *DN* is closed under addition we can use Lemma 4 to give a restricted answer to questions Q-II and Q-IV. Note that Fallat and Johnson[1] did not give an affirmative answer for question Q-IV but in our case, the weighted subdirect sum avoids the contradiction explained in Fallat and Johnson,[1] see Example 5 below.

Proposition 12. *Let*

$$C = \begin{bmatrix} C_{11} & C_{12} & O \\ C_{12}^T & C_{22} & C_{23} \\ O & C_{23}^T & C_{33} \end{bmatrix} \in \mathbb{R}^{n \times n} \tag{18}$$

be an n by n symmetric DN-matrix, with C_{11} and C_{33} square matrices and with C_{22} a square matrix of size k. Then it holds that $C = A \oplus_k^{\alpha,1-\alpha} B$ for any $0 < \alpha < 1$ and where A and B are DN-matrices.

Proof. It is a consequence of Lemma 4. □

Example 5. The *DN* matrix (taken from Fallat and Johnson[1])

$$C = \begin{bmatrix} 4 & 2 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 0 & 4 & 3 & 0 \\ 0 & 0 & 3 & 4 & 2 \\ 0 & 2 & 0 & 2 & 4 \end{bmatrix},$$

can be written as $C = A \oplus_3^{\frac{1}{2},\frac{1}{2}} B$ with

$$A = \begin{bmatrix} 4 & 2 & 2 & 0 \\ 2 & 4 & 0 & 0 \\ 2 & 0 & 4 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 & 0 & 2 \\ 0 & 4 & 3 & 0 \\ 0 & 3 & 4 & 2 \\ 2 & 0 & 2 & 4 \end{bmatrix},$$

where A and B are DN matrices. Note that the usual subdirect sum does not allow such decomposition since we can write $C = A' \oplus_3 B'$ with

$$A' = \begin{bmatrix} 4 & 2 & 2 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 3/2 \\ 0 & 0 & 3/2 & 2 \end{bmatrix}, \quad B' = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 3/2 & 0 \\ 0 & 3/2 & 2 & 2 \\ 2 & 0 & 2 & 4 \end{bmatrix},$$

but A' and B' are not DN matrices (since they have negative eigenvalues).

5.4 | M, P, and P_0 matrices

Recall the definition of M-matrix given in Fallat and Johnson[1]: a square matrix A is an M-matrix if it is a matrix with nonpositive off-diagonal entries (i.e., a Z-matrix) and with (all) positive principal minors. We also recall that an equivalent definition is that the square matrix A is a Z-matrix and there exists a positive diagonal matrix D such that AD is strictly diagonally dominant.[34]

We recall that Fallat and Johnson[1] gives affirmative answer for the class M for questions Q-I, Q-II, and Q-IV when using the usual subdirect sum, but does not give affirmative answer to question Q-III for the usual subdirect sum.

In our case, to give affirmative answer to question Q-I we need two technical Lemmas.

Lemma 13. *Let*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbb{R}^{n \times n}$$

be an M-matrix, with M_{22} a square matrix of size k. Then it holds that

$$M_\alpha = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & \alpha M_{22} \end{bmatrix} \tag{19}$$

with $\alpha \geq 1$ is an M-matrix.

Proof. Let us denote by m_{ij} the entries of M. Note that the entries of M_α are αm_{ij} when i and j are greater than or equal to $n - k + 1$.

Since M is an M-matrix there exists a positive diagonal matrix D, with entries d_{ij}, such that MD is strictly diagonally dominant. That is, for each $i = 1, 2, \ldots,$ it holds

$$|m_{ii} d_{ii}| > \sum_{j \neq i} |m_{ij} d_{jj}| \tag{20}$$

Let us construct a diagonal matrix $\hat{D}$ with entries $\hat{d}_{ij}$ such that $\hat{d}_{ij} = 0$ when $i \neq j$ and

$$\hat{d}_{ii} = \begin{cases} d_{ii} & \text{if } i < n - k + 1 \\ \frac{1}{\alpha} d_{ii} & \text{if } i \geq n - k + 1 \end{cases} \tag{21}$$

Now we show that $M_\alpha \hat{D}$, with entries r_{ij}, is strictly diagonally dominant and therefore M_α is an M-matrix. To do this, we first consider the case $i < n - k + 1$. We have, by using (21) and (20)

$$|r_{ii}| = |m_{ii} \hat{d}_{ii}| = |m_{ii} d_{ii}| > \sum_{j \neq i} |m_{ij} d_{jj}| = \sum_{j < i} |m_{ij} d_{jj}| + \sum_{j > i} |m_{ij} d_{jj}|$$

and, using that $\alpha \geq 1$, and again (21), we have

$$\sum_{j<i} |m_{ij}d_{jj}| + \sum_{j>i} |m_{ij}d_{jj}| \geq \sum_{j<i} |m_{ij}d_{jj}| + \sum_{j>i} |\frac{1}{\alpha} m_{ij}d_{jj}| = \sum_{j<i} |r_{jj}| + \sum_{j>i} |r_{jj}|$$

and therefore,

$$|r_{ii}| > \sum_{j\neq i} |r_{ij}| \tag{22}$$

Finally, considering the case $i \geq n-k+1$, we have, by using (21) and (20),

$$|r_{ii}| = |\alpha m_{ii}\hat{d}_{ii}| = |\alpha m_{ii}\frac{1}{\alpha}d_{ii}| = |m_{ii}d_{ii}| > \sum_{j\neq i} |m_{ij}d_{jj}|$$

that can be written as

$$|r_{ii}| > \sum_{j<i} |m_{ij}d_{jj}| + \sum_{j>i} |m_{ij}d_{jj}| = \sum_{j<i} |r_{ij}| + \sum_{j>i} |r_{ij}| = \sum_{j\neq i} |r_{ij}| \tag{23}$$

By equations (22) and (23), we conclude that $M_\alpha \hat{D}$ is strictly diagonally dominant and the proof follows. □

Following the same technique, it is straightforward to prove the following result, and hence, we omit the proof.

Lemma 14. *Let*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{24}$$

be an M-matrix, with M_{11} a square matrix of size k. Then it holds that

$$M^\beta = \begin{bmatrix} \beta M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

with $\beta \geq 1$ is an M-matrix.

Now we can given an affirmative answer to question Q-I for the class M for the weighted subdirect sum.

Proposition 13. *Question Q-I for the class M has an affirmative answer for the weighted subdirect sum.*

Proof. By Fallat and Johnson,[1] we know that Q-I has an affirmative answer for the usual subdirect sum for this class. By using Lemma 19, Lemma 24, and Lemma 1 we conclude that when $\alpha \geq 1$ and $\beta \geq 1$, question Q-I has an affirmative answer for the weighted subdirect sum. □

We cannot use Lemma 1 to give an affirmative answer in general terms for question Q-III for the weighted subdirect sum since Fallat and Johnson[1] do not give an affirmative answer for this question. We cannot either use Lemma 3 since the class M is not closed under addition.

Regarding questions Q-II and Q-IV for the class M, we can give an affirmative answer for the weighted subdirect sum.

Proposition 14. *Question Q-II (and question Q-IV, taking $k > 1$) for the matrix class M have affirmative answer for the weighted subdirect sum.*

Proof. By Fallat and Johnson,[1] we know that Q-II and Q-IV have an affirmative answer for the matrix class M for the usual subdirect sum. By using Lemma 19, Lemma 24, and Lemma 2 we conclude that when $\gamma \geq 1$ and $\delta \geq 1$, that is to say, when $0 < \alpha \leq 1$ and $0 < \beta \leq 1$ question Q-II (and question Q-IV, taking $k > 1$) have affirmative answer for the weighted subdirect sum. □

Let us focus on P-matrices. Recall that a P-matrix has all its principal minors with positive value. We recall that Fallat and Johnson[1] give affirmative answer for the class P for questions Q-I, Q-II, and Q-IV when using the usual subdirect sum, but does not give affirmative answer to question Q-III for the usual subdirect sum.

In our case, to give affirmative answer to question Q-I we need two technical Lemmas.

Lemma 15. *Let*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & p \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{25}$$

be a P-matrix, with $p \in \mathbb{R}$. Then it holds that

$$P_\alpha = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & \alpha p \end{bmatrix}$$

with $\alpha \geq 1$ is a P-matrix.

Proof. Note that p is a real number, and therefore, P and P_α only have one different entry. Let us denote by $P[\gamma|\delta]$ the submatrix of P with rows indexed by γ and columns indexed by δ. For principal submatrices we denote $P[\gamma|\gamma] \equiv P[\gamma]$. Since P is a P-matrix it holds that

$$det(P[\gamma]) > 0$$

for any set of indices $\gamma \subseteq \{1, 2, \dots, n\}$.

Since

$$det(P_\alpha[\gamma]) = det(P[\gamma])$$

for $\gamma \subseteq \{1, 2, \dots, n-1\}$, we only need to pay attention to the cases in which some index is the index n. Denoting by p_{ij} the entries of P, the principal minors of P_α in which the nth column (and row) is included are of the form

$$det(P_\alpha[\gamma]) = det \left(\begin{array}{cccc|c} p_{i_1,i_1} & p_{i_1,i_2} & \cdots & p_{i_1,i_t} & p_{i_1,n} \\ p_{i_2,i_1} & p_{i_2,i_2} & \cdots & p_{i_2,i_t} & p_{i_2,n} \\ \vdots & & \cdots & & \vdots \\ p_{i_t,i_1} & p_{i_t,i_2} & \cdots & p_{i_t,i_t} & p_{i_t,n} \\ \hline p_{n,i_1} & p_{n,i_2} & \cdots & p_{n,i_t} & \alpha p \end{array} \right)$$

and calculating this determinant by using the Laplace expansion along the last column we have that it is equal to

$$\pm p_{i_1,n} det(P[\widehat{i_1}|\hat{n}]) \mp p_{i_2,n} det(P[\widehat{i_2}|\hat{n}]) \pm \dots - p_{i_t,n} det(P[\widehat{i_t}|\hat{n}]) + \alpha p det(P[\hat{n}|\hat{n}]) \tag{26}$$

where we have noted by $P[\hat{i}|\hat{j}]$ the principal submatrix in P with rows in $\{i_1, i_2, \dots, i_t, n\}$ except for the index i, and columns in $\{i_1, i_2, \dots, i_t, n\}$ except for the index j. Since $\alpha \geq 1$ we have that the quantity (26) is greater than or equal to

$$\geq \pm p_{i_1,n} det(P[\widehat{i_1}|\hat{n}]) \mp p_{i_2,n} det(P[\widehat{i_2}|\hat{n}]) \pm \dots - p_{i_t,n} det(P[\widehat{i_t}|\hat{n}]) + p det(P[\hat{n}|\hat{n}]) \tag{27}$$

and this quantity is positive since P is a P-matrix. Therefore, we have shown that any principal minor of P_α is positive, and therefore, P_α is a P-matrix. □

It is clear that following a similar reasoning, but expanding the determinant along the first column, it is straightforward to prove the following result, and we omit the details.

Lemma 16. *Let*

$$P = \begin{bmatrix} p & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{28}$$

be a P-matrix, with $p \in \mathbb{R}$. Then it holds that

$$P^\beta = \begin{bmatrix} \beta p & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

with $\beta \geq 1$ is a P-matrix.

As before, we can given an affirmative answer to question Q-I for the class P for the weighted subdirect sum.

Proposition 15. *Question Q-I for the class P has an affirmative answer for the weighted subdirect sum.*

Proof. By Fallat and Johnson,[1] we know that Q-I has an affirmative answer for the usual subdirect sum for this class. By using Lemma 15, Lemma 16 and Lemma 1 we conclude that when $\alpha \geq 1$ and $\beta \geq 1$, question Q-I has an affirmative answer for the weighted subdirect sum for this class. □

We cannot use Lemma 1 to give an affirmative answer in general terms for question Q-III for the weighted subdirect sum since Fallat and Johnson[1] does not give an affirmative answer for this question. We cannot either use Lemma 3 since the class P is not closed under addition.

Regarding questions Q-II and Q-IV for the class P we can give an affirmative answer for the weighted subdirect sum.

Proposition 16. *Question Q-II (and question Q-IV, taking $k > 1$) for the matrix class P have affirmative answer for the weighted subdirect sum.*

Proof. By Fallat and Johnson,[1] we know that Q-II and Q-IV have an affirmative answer for the matrix class P for the usual subdirect sum. By using Lemma 15, Lemma 16, and Lemma 2, we conclude that when $\gamma \geq 1$ and $\delta \geq 1$, that is to say, when $0 < \alpha \leq 1$ and $0 < \beta \leq 1$ question Q-II (and question Q-IV, taking $k > 1$) have affirmative answer for the weighted subdirect sum. □

Example 6. The P matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ -1 & -1 & 2 & 1 \\ 0 & -1 & -1 & -1 \end{bmatrix},$$

can be written as $C = A' \oplus_2 B'$ with

$$A' = B' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

Then it is clear that we can also write $C = A \oplus_2^{\frac{1}{3}\frac{1}{2}} B$, with the P matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 3 \\ -1 & -3 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ -1 & -1 & 1 \end{bmatrix}.$$

To end this section, we extend Proposition 15 and Proposition 16 to the matrix class P_0. Recall that a matrix is P_0 when has all its principal minors nonnegative. It is clear that following a similar reasoning as in Lemma 15 and Lemma 16 we can prove that given A and B in the class P_0 then the corresponding A_α and B^β are also in P_0 for $\alpha \geq 1$ and $\beta \geq 1$, respectively. Therefore, it is clear that we can prove the two following results in a similar fashion as Proposition 15 and Proposition 16, and we omit the details.

Proposition 17. *Question Q-I for the class P_0 has an affirmative answer for the weighted subdirect sum.*

Example 7. Consider the P_0 matrices

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 14 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Then it is easy to check that

$$A \oplus_1^{2,3} B = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 9 & 14 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

is a P_0 matrix, but

$$A \oplus_2^{2,3} B = \begin{bmatrix} 3 & 40 & 1 \\ 2 & 9 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

is not a P_0-matrix.

Proposition 18. *Question Q-II (and question Q-IV, taking $k > 1$) for the matrix class P_0 have affirmative answer for the weighted subdirect sum.*

5.5 | *TN* matrices

Recall that a totally nonnegative (*TN*) matrix has all its minors nonnegative.

We recall that Fallat and Johnson[1] give affirmative answer for the class *TN* for questions Q-I, Q-II, and Q-IV when using the usual subdirect sum, but does not give affirmative answer to question Q-III for the usual subdirect sum.

As in the previous sections, in our case, to give affirmative answer to question Q-I we need two technical Lemmas.

Lemma 17. *Let*

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & p \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{29}$$

be a TN matrix, with $p \in \mathbb{R}$. Then it holds that

$$T_\alpha = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & \alpha p \end{bmatrix}$$

with $\alpha \geq 1$ is a TN matrix.

Proof. The proof is similar to that of Lemma 15. We have to prove that all the minors of T_α are nonnegative. Note that the majority of the minors of T_α are the same as the minors of T and therefore are nonnegative. We only have to focus on minors of T_α of the form $T_\alpha[\gamma|\delta]$ with the index n included in both sets γ and δ. To study each of these minors we proceed as in Lemma 15 and we compute the determinant along the column n. Since $\alpha p \geq p$ it is easy to see that these minors of T_α are greater than or equal to the corresponding minors of T, and therefore, they are nonnegative. □

It is clear that following the same reasoning (but developing the determinant along the first column) it is easy to prove the following result

Lemma 18. *Let*

$$T = \begin{bmatrix} p & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{30}$$

be a TN matrix, with $p \in \mathbb{R}$. Then it holds that

$$T^\beta = \begin{bmatrix} \beta p & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

with $\beta \geq 1$ is a TN matrix.

As before, we can given an affirmative answer to question Q-I for the class *TN* for the weighted subdirect sum.

Proposition 19. *Question Q-I for the class TN has an affirmative answer for the weighted subdirect sum.*

Proof. By Fallat and Johnson,[1] we know that Q-I has an affirmative answer for the usual subdirect sum for this class. By using Lemma 17, Lemma 18 and Lemma 1 we conclude that when $\alpha \geq 1$ and $\beta \geq 1$, question Q-I has an affirmative answer for the weighted subdirect sum for this class. □

In the same way as we commented on previous classes, we cannot use Lemma 1 to give an affirmative answer in general terms for question Q-III for the weighted subdirect sum since Fallat and Johnson[1] does not give an affirmative answer for this question. We cannot either use Lemma 3 since the class *TN* is not closed under addition.

Example 8. Consider the *TN* matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

By taking $B = A$ it is easy to check that

$$A \oplus_1^{2,3} B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

is a *TN* matrix, but

$$A \oplus_2^{2,3} B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 \\ 1 & 5 & 5 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

is not a *TN*-matrix.

Regarding questions Q-II and Q-IV for the class *TN* we can give an affirmative answer for the weighted subdirect sum.

Proposition 20. *Question Q-II (and question Q-IV, taking $k > 1$) for the matrix class TN have affirmative answer for the weighted subdirect sum.*

Proof. By Fallat and Johnson,[1] we know that Q-II and Q-IV have an affirmative answer for the matrix class *TN* for the usual subdirect sum. By using Lemma 17, Lemma 18 and Lemma 2 we conclude that when $\gamma \geq 1$ and $\delta \geq 1$, that is to say, when $0 < \alpha \leq 1$ and $0 < \beta \leq 1$ question Q-II (and question Q-IV, taking $k > 1$) have affirmative answer for the weighted subdirect sum. $\square$

Example 9. The *TN* matrix

$$C = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 2 & 0 & 0 & 0 \\ 0 & 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

can be written as $A \oplus_2^{\frac{1}{3},\frac{1}{5}} B$, with

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 25/2 & 10 & 0 & 0 & 0 \\ 15 & 20 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

that are also *TN* matrices.

In the next section we comment on two applications where the concept of weighted subdirect sum can play a role.

6 | EXAMPLES OF APPLICATIONS

As we have stated in Section 1, a natural application in where the concept of subdirect sum appears can be found in problems related with overlapping subdomains, subgraphs, motifs (a connected subgraph[21]) or cliques (a complete subgraph[21]). In this section we motivate how the new concept of weighted subdirect sum may also appear by using two examples of applications.

6.1 | Overlapping cliques

The analysis of overlapping subgraphs in complex networks is an active line of research; see, for example, previous works.[35–39]

We call $G = \{V, E\}$ a simple graph. That is to say undirected, with no loops, and no multiple edges (there are no pairs of vertices connected by more than one edge). We denote as $V = \{v_1, v_2, \ldots, v_n\}$ the set of vertices, or *nodes*, and its cardinality is denoted as $n = |V|$. Two vertices (v_i, v_j) joined by an edge are said to be *adjacent*. Each pair (v_i, v_j) of adjacent nodes define an edge. We denote by E the set of edges, or *links*, with cardinality $m = |E|$. The weighted adjacency matrix of the graph G is defined as the square matrix of size n denoted by A with entries a_{ij}, $i, j \in \{1, 2, \dots, n\}$ defined by

$$a_{ij} = \begin{cases} w_{ij} & \text{if } (v_i, v_j) \in E \text{ with weight } w_{ij} \\ 0 & \text{otherwise} \end{cases}$$

Given two simple weighted graphs $G_1 = \{V_1, E_1\}$, $G_2 = \{V_2, E_2\}$, let us denote $n_1 = |V_1|$, and $n_2 = |V_2|$. Let us recall that the union $G_1 \cup G_2$ is given by[40]

$$G_1 \cup G_2 = \{V_1 \cup V_2, E_1 \cup E_2\}$$

Let us denote the number of common vertices by $k \equiv |V_1 \cap V_2|$. Let A_1 be the adjacency matrix of the graph G_1 and let A_2 be the adjacency matrix of the graph G_2 and such that we have labeled the indices $v_1, v_2, \dots, v_{n_1}$ in such a way that the last k indices in the graph G_1 correspond to the vertices of $V_1 \cap V_2$. That is $(v_{n_1-k+1}, v_{n_1-k+2}, \dots, v_{n_1-1}, v_{n_1})$ are common vertices for G_1 and G_2. Regarding the graph G_2 we label the vertices such that the common vertices are in the first k positions (with the same order taken for G_1). That is, the first k vertices of G_2 are labeled as $(v_{n_1-k+1}, v_{n_1-k+2}, \dots, v_{n_1-1}, v_{n_1})$ and the following vertices are labeled as $v_{n_1+1}, v_{n_1+2}, \dots, v_{n_1+n_2-k}$. The next step is to write the adjacency matrices A_1 and A_2 in block form as

$$A_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_2 = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{31}$$

with A_{22} and B_{11} of size $k \times k$. Therefore, it is clear that the (weighted) adjacency matrix of $G_1 \cup G_2$ is given by $A_1 \oplus_k A_2$.

In Figure 1 we show two weighted graphs with an overlapping clique (i.e., a complete subgraph) of four vertices. The weighted adjacency matrices are given by

$$A_1 = \left[\begin{array}{cc|cccc} 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ \hline 2 & 0 & 0 & 5 & 1 & 6 \\ 0 & 0 & 5 & 0 & 7 & 3 \\ 0 & 0 & 1 & 7 & 0 & 4 \\ 0 & 3 & 6 & 3 & 4 & 0 \end{array}\right], \quad A_2 = \left[\begin{array}{cccc|cc} 0 & 2 & 3 & 2 & 0 & 0 \\ 2 & 0 & 2 & 1 & 3 & 0 \\ 3 & 2 & 0 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 3 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 4 & 0 \end{array}\right]$$

and the (weighted) adjacency matrix of $G_1 \cup G_2$ is given by $A_1 \oplus_4 A_2$.

6.1.1 | Giving a different weight depending on the origin of the overlap

Now we want to construct the union of the previous two graphs by giving a different weight to the cliques depending whether they come from G_1 (that we are going to weight by a quantity α) or from G_2 (that we weight with β). This would

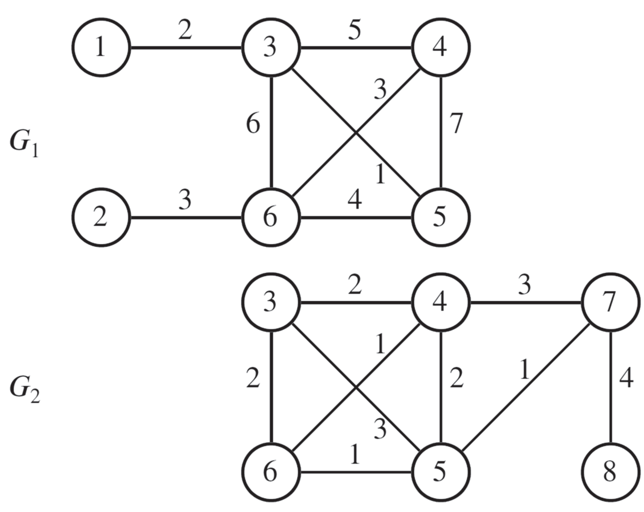

FIGURE 1 Two graphs with an overlapping clique

be the case, for example, when we are dealing with a multiplex composed of two layers, one representing the connections in Facebook and the other one representing the connections in Twitter. We are interested in constructing a union that takes into account whether the users in the overlapping clique come from one social network or the other one. In this case the adjacency matrix of the union $G_1 \cup G_2$ is given by

$$A_1 \oplus_4^{\alpha,\beta} A_2 = \left[\begin{array}{cc|cccc|cc} 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ \hline 2 & 0 & 0 & 5\alpha+2\beta & \alpha+3\beta & 6\alpha+2\beta & 0 & 0 \\ 0 & 0 & 5\alpha+2\beta & 0 & 7\alpha+2\beta & 3\alpha+\beta & 3 & 0 \\ 0 & 0 & \alpha+3\beta & 7\alpha+2\beta & 0 & 4\alpha+1\beta & 1 & 0 \\ 0 & 3 & 6\alpha+2\beta & 3\alpha+\beta & 4\alpha+\beta & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \end{array}\right]$$

Note that, in the practice, before computing the weighted subdirect sum it is necessary to label the vertices according to the rules explained above. Clearly, these method can be applied to some overlapping cliques by ordering in a consecutive way the cliques. Note that the weighted subdirect sum is different from the subdirect sum of the *scaled* weighted adjacency matrices A_1 and A_2 (that is to say $\alpha A_1 \oplus_k \beta A_2$).

Note that this example can be understood in terms of a multilayer composed by graphs G_1 and G_2, while the graph $G_1 \cup G_1$, with cliques weighted by α and β, can be considered as a representation of the multilayer (when $\alpha = \beta = 1$ this graph is usually called the *projected graph*). Note, for example, that some properties of this multilayer (like, e.g., centrality measures) may depend on the values of α and β. The future application of this idea is an open line of research.

6.2 | Additive Schwarz iterative method

The additive Schwarz iterative method with overlapping subdomains to solve the linear system $Ax = b$ can be written in the form

$$x^{k+1} = Tx^k + c, \qquad k = 0, 1, \ldots$$

with the iteration matrix[28]

$$T = I - \theta \sum_{i=1}^{p} R_i^T A_i^{-1} R_i A$$

where p is the number of overlapping subdomains, θ is a relaxation parameter and the restriction operators are

$$R_i = [I_i | O]\pi_i, \quad i = 1, 2, \ldots p$$

with I_i the identity matrix of size i and π_i is a permutation matrix of the same size as A.

It is not difficult to see that the matrix $B := \sum_{i=1}^{p} R_i^T A_i^{-1} R_i$ can be written as a matrix sum of subdirect sums of the form

$$A_i^{-1} \oplus_{s(i)} A_{i+1}^{-1} \tag{32}$$

where $s(i)$ is the size of the overlapping between the subdomains i and $i+1$.

It can be shown (see[28]) that the convergence of the iterative method depends on the values of θ (for a particular type of the initial matrix A). Therefore, it is natural to think that the convergence properties of an iterative method based on overlapping subdomains can be affected if we give a different weight to each subdomain. That is, if we change the formulation given by the subdirect sums in (32) by the sums according to the weighted subdirect sums of the form

$$A_i^{-1} \oplus_{s(i)}^{\alpha(i),\beta(i)} A_{i+1}^{-1} \tag{33}$$

where $\alpha(i)$ and $\beta(i)$ are the weights corresponding to the weighted subdirect sum of the subdomains i and $i+1$. In this method, it is important that matrix B is invertible (see Frommer and Szyld[41]), and therefore, it is important to know when this sum is invertible when using weighted overlapping domains; note that Corollary 1 and Corollary 2 can help to solve this problem, jointly with the analyzed properties of the matrix classes (i.e., if a matrix class is invertible and the weighted subdirect sum is also in the class, then it is also invertible). As far as we know, the convergence properties of iterative methods for solving linear systems based on weighted overlapping subdomains is an open problem.

7 | CONCLUSIONS

In this paper we have extended the concept of subdirect sum defined in 1999 in Fallat and Johnson[1] to the concept of weighted subdirect sum by allowing a weight of the overlapping blocks. The paper is organized as in Fallat and Johnson,[1] and therefore, we answer four natural questions on weighted subdirect sums. These questions are focused on whether the weighted subdirect sum is in the same class as the given matrices and, the other way round, if given a matrix in a particular class (with a structure compatible with a subdirect sum) then it can be written as a weighted subdirect sum of matrices in the same class. We have shown that all the results on positivity classes of real matrices studied in Fallat and Johnson[1] can be extended to the weighted subdirect sum. Another contribution of the paper is that we develop a methodology that can be applied to study the properties of the weighted subdirect sum of other classes of matrices. We also remark that we have been able to give positive answer to some of the questions that had negative answer in Fallat and Johnson.[1] In more detail, we have shown that question Q-IV for the class DN has a positive answer for the weighted subdirect sum. Finally, we have shown two applications in which the weighted subdirect sum can play a role. These applications are certainly a motivation for the weighted subdirect sum introduced in this paper and also represent a basis for future research.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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