New interaction formula for plastic resistance of channel sections under combinations of bending moments $M_{y,Ed}$, $M_{z,Ed}$ and bimoment B_{Ed}

A. Agüero [a,*], I. Baláž [b], Y. Koleková [c], M. Lázaro [a]

[a] Dept. of Continuous Medium Mechanics and Theory of Structures, Universitat Politècnica de València, c/Camino de Vera s/n, 46022 Valencia, Spain
[b] Slovak University of Technology in Bratislava, Faculty of Civil Engineering, Department of Metal and Timber Structures, Bratislava, Slovak Republic
[c] Slovak University of Technology in Bratislava, Faculty of Civil Engineering, Department of Structural Mechanics, Bratislava, Slovak Republic

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ABSTRACT

The interaction formulae for the plastic resistance of channel sections under combination of 4 internal forces: the bending moment about strong axis $M_{y,Ed}$, the bending moment about weak axis $M_{z,Ed}$, the bimoment B_{Ed} and the axial force N_{Ed} were created and analyzed. The exact interaction curves obtained by the linear programming for 4 internal forces have enabled to create and to verify the proposed approximate interaction formula for 3 internal forces.

The approximate interaction formula for the design plastic resistance of channel sections under combination of 3 internal forces: the bending moment about strong axis $M_{y,Ed}$, the bending moment about weak axis $M_{z,Ed}$ and the bimoment B_{Ed} was analyzed and verified. The explicit interaction formula that takes into account these 3 internal forces is missing in Eurocodes. A large parametrical study was performed for the rolled channel profiles within the size range UPE 80 UPE 400. The differences between the results of the approximate interaction formula and the exact interaction curves were analyzed and summarized.

Till today nobody presented results of the parametrical study of the channel sections under the combination of 4 internal forces.

1. Introduction

1.1. Overview of the former investigations

The theory in the fundamental Vlasov's books [1–3] relating to the elastic behaviour of the thin walled beams was known to the rest of the world only after translations of Vlasov's books in the period 1961–1965 [4]. Consequently Strel'bickaja [5–7] immediately started the investigation of the thin walled structures in the elastic–plastic state [5]. Some of Strel'bickaja's results were later published by Mrázik in the books [8,9]. In [5] may be found the interaction formulae for the plastic resistance of I-sections and U-sections subjected to the combination of: a) 5 internal forces M_y, V_z, B, T_w, and T_t valid for the plastic limit state without strengthening and b) 2 internal forces M_y, B valid for the plastic limit state with strengthening. In the bi-linear stress–strain diagram in the case a) $E_1 = 0$ MPa and in the case b) $E_1 > 0$ MPa (details see in [10,11]). Strel'bickaja verified her theoretical results within her large experimental project published in Strel'bickaja [7]. She tested altogether 33 cantilevers with I- or U- sections: a) 3 cantilevers with I-sections subjected to the torsion, b) 8 cantilevers with I-sections subjected to biaxial bending, c) 18 cantilevers subjected to bending and torsion (9 with I-sections, 9 with U-sections), d) 4 cantilevers subjected to biaxial bending and torsion (1 with I-section, 3 with U-sections). In the book [6] Strel'bickaja investigated also the members subjected to the combination of the internal forces including the axial force N. The relevant interaction formulae may be found also in [8,9]. For example her formula for I-section subjected to N, M_y, M_z, B is in [8] on the page 123.

For channel section Strel'bickaja investigated combination M_y, B only for the case of the negative bimoment B (Fig. 1). She derived formulae (38.26) in [5] on page 251 (the formulae (40) in [11]). These formulae were rearranged in the formula (45) in [11], here the formulae (1) and (2):

$$\frac{|B_{Ed}|}{B_{pl,Rk}} = K_1 \left(\frac{M_{y,Ed}}{M_{pl,y,Rk}} \right)^2 + K_2 \frac{M_{y,Ed}}{M_{pl,y,Rk}} + 1 \tag{1}$$

where

$$K_1 = \left(4a_{wf} + 1\right)^2 / \left(16a_{wf}^2 + 8a_{wf} \quad 1\right) \; K_2 = 2 / \left(16a_{wf}^2 + 8a_{wf} \quad 1\right) \; a_{wf} = A_f / A_w = \left(b_f t_f\right) / \left(h_f t_w\right) \tag{2}$$

Nomenclature

γ_{M0}	is the partial safety factor for the resistance of cross section whatever the class is
f_y	yield stress
b	width of a cross-section
h	depth of a cross-section
h_w	depth of a web
t_f	flange thickness
t_w	web thickness
y, z	section coordinates along y and z axes
ω	warping function
$M_{y,Ed}$	design value of the bending moment about the y axis
$M_{z,Ed}$	design value of the bending moment about the z axis
B_{Ed}	design value of the bimoment
$M_{pl,y,Rd}$	$M_{pl,Rd}$ design value of the plastic moment resistance about the y axis
$M_{w,pl,y,Rd}$	design value of the plastic moment resistance of the web about the y axis
$M_{pl,z,Rd}$	design value of the plastic moment resistance about the z axis
$B_{pl,Rd}$	design value of the plastic bimoment resistance
$N_{Tfl,Ed}$	design value of the axial force of the top flange
$N_{Bfl,Ed}$	design value of the axial force of the bottom flange
$N_{w,Ed}$	design value of the axial force of the web
$M_{Tfl,z,Ed}$	design value of the bending moment of the top flange about z axis
$M_{Bfl,z,Ed}$	design value of the bending moment of the bottom flange about z axis
$M_{w,y,Ed}$	design value of the bending moment of the web about y axis
$N_{Tfl,Rd}$	design value of the plastic axial force resistance of the top flange
$N_{Bfl,Rd}$	design value of the plastic axial force resistance of the bottom flange
$N_{w,Rd}$	design value of the plastic axial force resistance of the web
$M_{Tfl,z,Rd}$	design value of the plastic bending moment resistance of the top flange about z axis
$M_{Bfl,z,Rd}$	design value of the plastic bending moment resistance of the bottom flange about z axis
$M_{fl,z,Rd}$	design value of the plastic bending moment resistance of the flange about z axis
$M_{w,y,Rd}$	design value of the plastic bending moment resistance of the web about y axis

For the positive bimoment B according to [11] the approximate linear equation (3) may be used.

$$\frac{M_{y,Ed}}{M_{pl,y,Rd}}+\frac{B_{Ed}}{B_{pl,Rd}}\leqslant 1.0 \tag{3}$$

where the value $M_{pl,y,Rd} < M_{pl,y,Rd,max}$ may be calculated e.g. by the program QST-TSV-3Blech [12] for B = 0. The value $M_{pl,y,Rd,max}$ may be achieved only together with relevant value of a bimoment. In the newer statical tables or databases of some computer programs the both values $M_{pl,y,Rd}$ and $M_{pl,y,Rd,max}$ may be found. In older statical tables there is only $M_{pl,y,Rd,max}$ value. If the $M_{pl,y,Rd}$ value is not available, it may be replaced in the equation (3) by $M_{el,y,Rd}$ value being slightly on the safe side (formula (47) in [11]). For example for UPE 360 section (DIN1026-2: October 2002) the value $M_{pl,y,Rd}$ = 199.52 kNm may be replaced by $M_{el,y,Rd}$ = 197.81 kNm (Fig. 2).

The channel section UPE 360 was investigated in Baláž and Koleková [11]. Fig. 2 shows the comparisons of the results of various calculations for the case $M_{z,Ed}/M_{N,pl,z,Rd} = 0$. The influence of the values of the relative bending moment $M_{z,Ed}/M_{pl,z,Rd}$ = 0; 0.2; 0.4; 0.6; 0.8 on the plastic resistance of UPE 360 obtained for f_y = 235 MPa is shown in Fig. 2 too.

Strel'bickaja in [5] derived the general interaction formula for both I- and U- sections subjected to the combination of the internal forces M_y, V_z, B, T_w and T_t (formula (31.25) on p.212). She published in [5] also several partial cases of the general formula (31.25) e.g. the formula valid for combination of M_y and B (the formula in Table 50 on page 216).

The fundamental book by Kindmann and Frickel [13] contains a lot of solutions including the investigation of the channel profiles. They developed the original TSV method called Teilschnittgrößenverfahren (Partial-Internal-Forces-Method) which is basis for several efficient computer programs, e.g. [12]. The experimental and theoretical investigations of the channels were published by Höss et al. [14] and by La

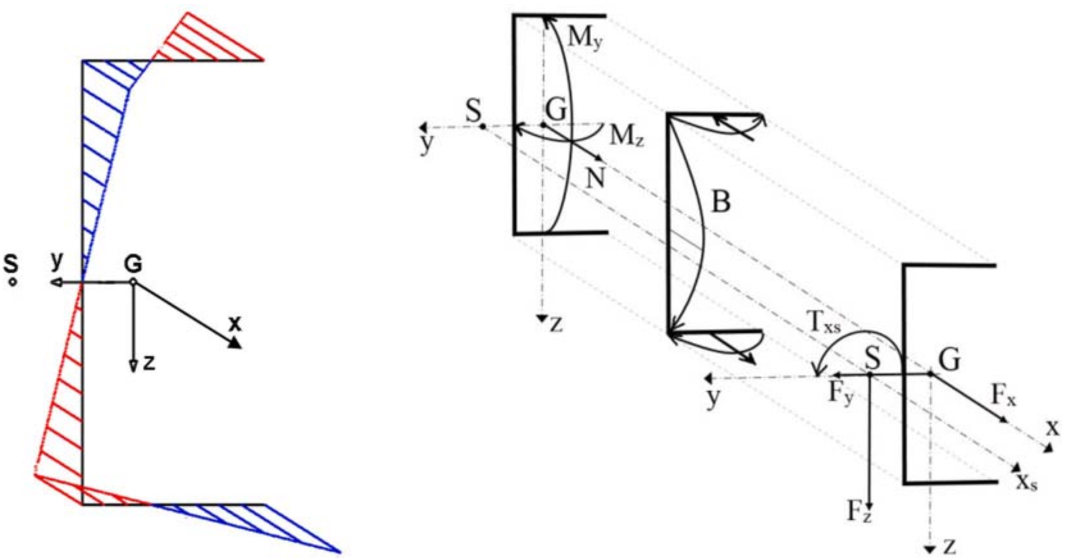

Fig. 1. UPE section. Sign convention. Distribution of sectorial coordinate ω(s). Positive external forces and moment F_x, F_y, F_z, T_{xs} and internal force and moments N, M_y, M_z, B.

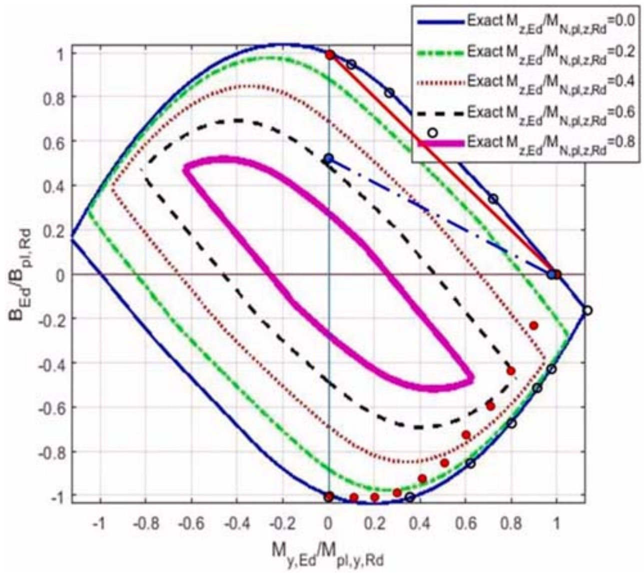

Fig. 2. UPE 360 [25,26]. S235. Relative plastic resistances. $M_{pl.y.Rd}$ = 199.52 kNm for B_{Ed} = 0, $M_{pl.y.Rd.max}$ = 225.45 kNm, for B_{Ed} = -1.0144 kNm^2, $B_{pl.Rd}$ = 6.7920 kNm^2. Results of: a) authors proposal for $M_{z,Ed}/M_{N.pl.z.Rd}$ = 0; 0,2; 0,4; 0,6; 0,8; compared for $M_{z,Ed}/M_{N.pl.z.Rd}$ = 0 with results of: b) QST-TSV-3Blech [12] – empty circles; c) formula (1) for B_{Ed} < 0 – full red circles; d) formula (3) for B_{Ed} > 0 – red solid straight line; e) elastic resistance – blue dot- and dashed straight line. The exact results of b) are drawn only in the 1st and 4th quadrants. The approximate results of c), d), e) are only in the 1st quadrant.

Poutré et al. [15–17].

The large research project (Table 1) of three universities (RWTH Aachen, TU Berlin, RU Bochum) [18] deals with the effects of torsion and their influence on the cross-sectional and member resistance of rolled steel profiles. The ultimate resistances were analysed experimentally and theoretically. The experimental part comprised tests on the cross-sectional resistance of profiles subject to bending and torsion. Other tests concerned the member resistance of profiles subject to bending and torsion with and without axial compression. The specimens were made of profiles commonly used from the series IPE 200, UPE 200 and HEB 200 (Table 1). Suitable design models were developed to determine the cross-sectional resistance and the member resistance of steel profiles taking into account of torsion. Their applicability was verified by comparison with the experimental and numerical data. The feasibility of the Partial-Internal-Forces-Method (PIM) for the verification of the cross-sectional resistance with elastically determined internal forces and moments were verified.

The comparisons of the experimental results [18] with the theoretical results obtained with the own and Kindmann's computer programs show good agreement, which will be published in the next paper.

Table 1
Overview of the tests performed in the frame of the large project [18].

	Cross-section resistance. S355 RWTH Aachen			Member resistance. S355 TU Berlin. [RU Bochum]		
Action	IPE 200	UPE 200	HEB 200	IPE 200	UPE 200	HEB 200
F_z	1	1	1	–	–	–
F_z, F_y	2	2	1	–	–	–
F_z, T_x	2	3	2	2	2	6
F_z, F_y, T_x	2	2	3	3	6	–
N, F_z	–	–	–	–	–	[5]
N, F_z, T_x	–	–	–	–	[8 + 5]	–
N, F_y, T_x	–	–	–	–	–	[4]
N, M_z, M_y	–	–	–	–	–	[8]

The channel profiles were investigated also by Beyer et al in [19–21]. Beyer [19] proposed: - for the interaction between $M_{z,Ed}$ and B_{Ed}:

$$M_{B,pl,z,Rd} = M_{pl,z,Rd}\left(1.1 \quad \frac{B_{Ed}}{B_{pl,Rd}}\right) \le M_{pl,z,Rd} \tag{4}$$

- for the interaction between $M_{y,Ed}$ and B_{Ed}:

$$M_{B,pl,y,Rd} = M_{pl,y,Rd}, \text{ if } B_{Ed} \le B_{Mpl,y,Rd} \tag{5}$$

$$M_{B,pl,y,Rd} = M_{pl,y,Rd} \quad \left(M_{pl,y,Rd} \quad 2M_{Bmax,y}\right)\left(\frac{B_{Ed} \quad B_{Mpl,y,Rd}}{B_{pl,Rd} \quad B_{Mpl,y,Rd}}\right)^{1.5} \text{ otherwise.} \tag{6}$$

1.2. *Calculation of the factor ξ*

ξ is the factor by which internal forces and moments must be multiplied to reach the plastic resistance of the cross-sections. In order to find the factor ξ $\{\xi N_{Ed}, \xi M_{y,Ed}, \xi M_{z,Ed}, \xi B_{Ed}\}$, the lower bound theorem can be used.

1.2.1. ***Method A*** *(according to Osterrieder et al. [22,23])*

In the Method A the factor ξ and the associated normal stresses σ at each element are calculated. It consists of several steps.

Step 1. Dividing section into elements.

Step 2. Considering linear constraints by Eqs. (7 to 11):

Limitation of the normal stresses:

$$f_y \leqslant \sigma_i \leqslant f_y \tag{7}$$

Equilibrium equations:

$$\sum_{i=1} \sigma_i A_i = \xi N_{Ed} \tag{8}$$

$$\sum_{i=1} \sigma_i A_i y_i = \xi M_{z,Ed} \tag{9}$$

$$\sum_{i=1} \sigma_i A_i \omega_i = \xi B_{Ed} \quad \sum_{i=1} \sigma_i A_i \omega_i = \xi B_{Ed}$$

$$\sum_{i=1} \sigma_i A_i \omega_i = \xi B_{Ed} \tag{10}$$

$$\sum_{i=1} \sigma_i A_i z_i = \xi M_{y,Ed} \tag{11}$$

Step 3. Calculation of the maximum value of the factor ξ:

$$\max(\xi) \tag{12}$$

1.2.2. ***Method B*** *(similar method was used by rubin [24] and Kindmann and Frickel [25])*

Computer programs [12] and [26] give similar results.

The channel section consists of three parts: top flange – index *Tfl*, bottom flange – index *Bfl* and web – index *w*.

The steps of the Method B are as follows:

Step 1. Dividing section into three parts.

Calculation of the axial and bending resistances of all three parts:

$$N_{Tfl,pl,Rd} = N_{Bfl,pl,Rd} = b \cdot t_f \cdot f_y; N_{wpl,Rd} = h_w \cdot t_w \cdot f_y$$

$$M_{Tfl,pl,z,Rd} = M_{Bfl,pl,z,Rd} = \frac{\quad}{4}; M_{w,pl,y,Rd} = \frac{w}{4}$$

Step 2. Considering constraints:

Nonlinear interaction top flange:

$$\left(\frac{N_{Tfl,Ed}}{N_{Tfl,pl,Rd}}\right)^2 + \left|\frac{M_{Tfl,z,Ed}}{M_{Tfl,pl,z,Rd}}\right| \leqslant 1 \tag{14}$$

Nonlinear interaction bottom flange:

$$\left(\frac{N_{Bfl,Ed}}{N_{Bfl,pl,Rd}}\right)^2+\left|\frac{M_{Bfl,z,Ed}}{M_{Bfl,pl,z,Rd}}\right|\leqslant 1 \quad (15)$$

Nonlinear interaction web:

$$\left(\frac{N_{w,Ed}}{N_{w,pl,Rd}}\right)^2+\left|\frac{M_{w,y,Ed}}{M_{w,pl,y,Rd}}\right|\leqslant 1 \quad (16)$$

Equilibrium equations:

$$N_{Tfl,Ed}+N_{Bfl,Ed}+N_{w,Ed}=\xi N_{Ed} \quad (17)$$

$$M_{Tfl,z,Ed}+M_{Bfl,z,Ed} \quad N_{w,Ed}\left(\frac{b}{2} \quad d_g\right)+\left(N_{Tfl,Ed}+N_{Bfl,Ed}\right)d_g=\xi M_{z.Ed} \quad (18)$$

$$d_s M_{w,y,Ed}+\left(\frac{h \quad t_f}{2}\right)\left(M_{Tfl,z,Ed} \quad M_{Bfl,z,Ed}\right) \quad \left(d_s \quad \frac{b}{2}\right)\left(\frac{h \quad t_f}{2}\right)\left(N_{Tfl,Ed} \quad N_{Bfl,Ed}\right)$$

$$=\xi B_{Ed} \quad (19)$$

$$M_{w,y,Ed}+\left(\frac{h \quad t_f}{2}\right)\left(N_{Bfl,z,Ed} \quad N_{Tfl,z,Ed}\right)=\xi M_{y,Ed} \quad (20)$$

where:

d_s is the distance of the shear center to the web midline,

d_g is the horizontal distance between the centroid and the flange's centroid.

Step 3. Calculation of the maximum value of the factor ξ:

$$\max(\xi) \quad (21)$$

1.3. Research significance

The Steps to obtain the exact curves are described.

For the beams with channel sections the interaction formulae for the calculation of the plastic resistance of cross-sections under bending moment about strong axis $M_{y,Ed}$, bending moment about weak axis $M_{z,Ed}$ and bimoment B_{Ed}, are proposed. They are derived and presented in Section 3.

2. Calculation of the exact interaction curves for combinations of four internal forces

The Method A is applied to obtain the interaction curves with the parameter $M_{z,Ed}$

Step 1. Calculation of $M_{pl,z,Rd}$ for the given $N_{Ed} = \alpha A f y$, $M_{y,Ed} = 0$ and $B_{Ed} = 0$.

Considering constraints by Eqs. ((22) to (25)):

$$f_y\leqslant\sigma_i\leqslant f_y \quad (22)$$

$$\sum_{i=1}\sigma_i A_i = N_{Ed} = \alpha\cdot A\cdot fy \quad (23)$$

$$\sum_{i=1}\sigma_i A_i z_i = M_{y,Ed} = 0.0 \quad (24)$$

$$\sum_{i=1}\sigma_i A_i \omega_i = B_{Ed} = 0.0 \quad (25)$$

Calculation of the maximum value of $M_{z,Ed}$:

$$M_{pl,z,Rd,\alpha}(N_{Ed}) = \max\left(\sum_{i=1}\sigma_i A_i y_i\right) \quad (26)$$

Step 2. Calculation of $B_{Ed,\gamma}$ to achieve the plastic resistance for the given $N_{Ed} = 0$, $M_{y,Ed} = 0$, $M_{z,Ed} = \gamma M_{pl,z,Rd}$; γ [0 to 1].

Constraints Eqs. (27 to 30):

$$f_y\leqslant\sigma_i\leqslant f_y \quad (27)$$

$$\sum_{i=1}\sigma_i A_i = N_{Ed} = \alpha\cdot A\cdot fy \quad (28)$$

$$\sum_{i=1}\sigma_i A_i z_i = M_{y,Ed} = 0.0 \quad (29)$$

$$\sum_{i=1}\sigma_i A_i y_i = M_{z,Ed} = \gamma\cdot M_{pl,z,Rd} \quad (30)$$

Calculation of the maximum value of $B_{pl,Rd,\gamma}$:

$$B_{pl,Rd,\alpha,\gamma}\left(N_{Ed};M_{z,Ed}\right) = \max\left(\sum_{i=1}\sigma_i A_i \omega_i\right) \quad (31)$$

Step 3. Calculation of $M_{y,Ed,\gamma,\eta}$ to achieve the plastic resistance for the given $N_{Ed} = 0$, $B_{Ed} = \eta B_{pl,Rd,\gamma}$, $M_{z,Ed} = \gamma M_{pl,z,Rd}$; γ [0 to 1] and η [0 to 1].

Considering constraints by Eqs. (32 to 35):

$$f_y\leqslant\sigma_i\leqslant f_y \quad (32)$$

$$\sum_{i=1}\sigma_i A_i = N_{Ed} = \alpha\cdot A\cdot fy \quad (33)$$

$$\sum_{i=1}\sigma_i A_i y_i = M_{z,Ed} = \gamma M_{pl,z,Rd} \quad (34)$$

$$\sum_{i=1}\sigma_i A_i \omega_i = B_{Ed} = \eta B_{pl,Rd,\gamma} \quad (35)$$

Calculation of maximum $M_{yEd,\alpha,\gamma,\eta}$:

$$M_{y,Ed,\alpha,\gamma,\eta}\left(N_{Ed};M_{z,Ed};B_{Ed}\right) = \max\left(\sum_{i=1}\sigma_i A_i z_i\right) \quad (36)$$

Interaction of the axial force, bending moments about strong and weak axes and bimoment.

3. Interaction formulae of the authors' proposal (approximate curves)

In the following part an approximate interaction formula for channel section plastic resistance taking into account three internal forces: $M_{y,Ed}$, $M_{z,Ed}$ and B_{Ed} is presented.

The definitions of the dimensionless design bending moments and bimoment are as follows.

$$m_y=\frac{M_{y,Ed}}{M_{pl,y,Rd}},\ m_z=\frac{M_{z,Ed}}{M_{pl,z,Rd}},\ m_\omega=\frac{B_{Ed}}{B_{pl,Rd}} \quad (37)$$

The interaction diagrams with axes $\{m_y, m_\omega\}$, assume m_z as a parameter and zero axial force $N_{x,Ed} = 0$. The family of the interaction diagrams, after varying the third parameter m_z in the range $0 \leq m_z < 1$, are shown in Fig. 3(it is ≤ 0.98), where 50 curves are plotted for the UPE 220 section.

For the each value of m_z a closed antisymmetric interaction curve arises. Each diagram is formed by two branches which start and end from a separation line, represented by the curve defined by circles in Fig. 3 and which is named *supporting curve*. The graph of this curve lies in general in the 2nd and the 4th quadrant. The points of the 4th quadrant have the form $(+m_{y0},\ m_{\omega0})$, while those of the 2nd quadrant are pointsymmetric respect to the origin, therefore they can be written as $(\ m_{y0},\ +m_{\omega0})$. For channel-type sections, both parameters m_{y0} and $m_{\omega0}$

are positive and depend on the ratio r and on the relative bending moment m_z. The parameter r is defined as follows.

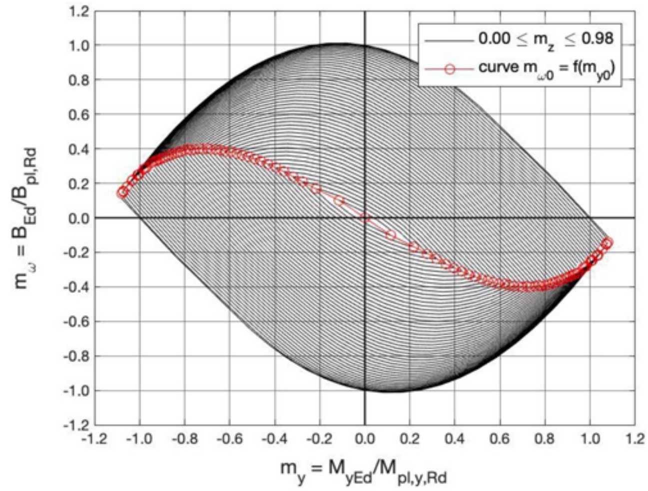

Fig. 3. Exact interaction diagrams (m_y,m_ω) for 50 different values of m_z between 0 and 0.98. Points on the red curve are named (m_{y0},$m_{\omega 0}$) for each channel. Such curve will be represented as m_{y0} = f($m_{\omega 0}$). The results correspond to the UPE 220 [25,26].

$$r = \frac{A_w}{A_f} = \frac{h_w t_w}{2bt_f} \tag{38}$$

where $h_w = h - 2t_f$ is the depth of the web. For given value of the bending moment m_z and given a channel section by its parameter r, the analytic definition of the supporting curve must be constructed such that the resulting points ($-m_{y0}$, $m_{\omega 0}$) and (m_{y0}, $-m_{\omega 0}$) lie on the exact curve (obtained by optimization-based procedures) with a minimum error. The proposed model for the determination of $m_{y0} > 0$ and $m_{\omega 0} > 0$ consists of the following r and m_z dependent functions.

$$m_{y0} = \mathrm{A}(r, m_z), m_{\omega 0} = \mathrm{B}(r, m_{y0}) \tag{39}$$

where the two-variable functions A (r, η) and B (r, η) are defined as

$$\mathrm{A}(r, \eta) = \sqrt{a_0(r) + a_1(r)\eta + a_2(r)\eta^2} \tag{40}$$

$$\mathrm{B}(r, \eta) = b_0(r) + b_1(r)\eta + b_2(r)\eta^2 + b_3(r)\eta^3 \tag{41}$$

Coefficients $a_j(r)$, $b_j(r)$ where j = 0, 1, 2, 3 are polynomials in the variable r given in Table 2 and in Fig. 4. Our proposed approximate interaction curve has been constructed using the support curve points as starting and end points and using certain shape functions which will be described below. The general expression of the proposed interaction diagram (top-curve) can be written using the following terms.

$$\Psi^+(r, m_y, m_z) = m_{\omega 0}\mathrm{N}_\omega(\xi) + m_{y0}[\mathrm{N}_{\alpha 1}(\xi)\cdot\alpha_1 + \mathrm{N}_{\alpha 2}(\xi)\cdot\alpha_2] \tag{42}$$

where the parameter $\xi = \frac{m_y}{m_{y0}}$. In practice, the internal forces combination (m_y, m_z, m_ω) lies inside of the interaction diagram if.

$$\Psi^-(r, m_y, m_z) \leqslant m_\omega \leqslant \Psi^+(r, m_y, m_z) \tag{43}$$

In the Eq. (44),(45) and (46) the shape functions $\mathrm{N}_\omega(\xi)$, $\mathrm{N}_{\alpha 1}(\xi)$, $\mathrm{N}_{\alpha 2}(\xi)$ are defined in the range $-1 \le \xi \le 1$ as follows.

$$\mathrm{N}_\omega(\xi) = \frac{1}{2}\xi(\xi^2 - 3) \tag{44}$$

$$\mathrm{N}_{\alpha 1}(\xi) = -\frac{1}{4}(1-\xi)(1+\xi)^2 \tag{45}$$

$$\mathrm{N}_{\alpha 2}(\xi) = +\frac{1}{4}(1-\xi)^2(1+\xi) \tag{46}$$

Finally, the parameters α_1 and α_2 can be obtained from the following expressions.

$$\alpha_1 = \mathrm{P}(r, m_z), \alpha_2 = \mathrm{Q}(r, m_z) \tag{47}$$

where

$$\mathrm{P}(r, \eta) = p_0(r) + p_1(r)\eta + p_2(r)\eta^2 + p_3(r)\eta^3 \tag{48}$$

$$\mathrm{Q}(r, \eta) = q_0(r) + q_1(r)\eta + q_2(r)\eta^2 + q_3(r)\eta^3 \tag{49}$$

The η-coefficients $p_j(r)$, $q_j(r)$ j = 0, 1, 2, 3 are third-order polynomials in the parameter r, whose r-coefficients have been fitted using as reference the opitimization-based interaction curves. Fitting results are shown in Table 2 and in Fig. 4. The validity range of the areas ratio is approximately.$0.35 \le r \le 1.20$

Some illustrative examples are in Figs. 5, 6. To quantify the accuracy of the proposed approach, the errors in B_{Ed} values leading to the plastic resistance of the cross-section for the given $M_{y,Ed}$ and $M_{z,Ed}$ are computed for the investigated sections UPE 220 and UPE 360 in Table 3. The mean values and standard deviations are shown in Table 3 for each profile, and for five different values of the parameter $M_{z,Ed}/M_{pl,z,Rd}$ = 0.0, 0.2, 0.4, 0.6 and 0.8. The absolute error are defined as follows.

$$Error = \frac{B_{Ed}^{Proposal} - B_{Ed}^{Exact}}{B_{pl,Rd}} \tag{50}$$

3.1. Interaction of internal forces M_y, M_z, B, N

In the following parametric study the special UPE 160 section produced in the past by the company Salzgitter AG, Peine, Germany is analysed because such profile was investigated in [13], [14] and [15]. The input values are $h = 160$ mm, $b = 70$ mm, $t_f = 10$ mm, $t_w = 6.5$ mm. The results of the study obtained for $f_y = 240$ MPa, $\gamma_{M0} = 1$ are presented in Figs. 7 to 15. Diagrams based on relative dimesionless quantities may be used for any value of yield stress f_y.

The interaction curves for M_y, B for different values of M_z are given in each plot for the given value of N. Each diagram consists of five curves for constant axial force and several values of the relative bending moments about the weak axis ratio in steps of 0.2 values.

In order to see the influence of the cross section geometry on the plot shape, an additional plot for partial case $N_{Ed} = 0$ is shown in Fig. 16 for UPE sections with dimensions according to DIN 1026:2002 [27] and DIN EN 10279:2000 [28] for section sizes from the interval UPE 80 to UPE 400.

4. Conclusions

Previous research done by the authors was focused on I- and H-

Table 2

Coefficients of polynomials $a_j(r) = \sum_{n\ge 0} a_j^{(n)} r^n$. $b_j(r) = \sum_{n\ge 0} b_j^{(n)} r^n$, $p(r) = \sum p^{(n)} r^n$, $q_j(r) = \sum_{n\ge 0} q_j^{(n)} r^n$.

		1	r	r^2	r^3
A (r, η)	$a_0(r)$	1.067646	0.160046	0	0
	$a_1(r)$	0.777362	0.481218	0.639469	0
	$a_2(r)$	0.235722	0.081018	0.441733	0
B (r, η)	$b_0(r)$	0.002528	0.019823	0	0
	$b_1(r)$	0.885348	3.076211	1.135336	0
	$b_2(r)$	3.059028	5.033789	1.856962	0
	$b_3(r)$	2.110460	2.303023	0.815989	0
P (r, η)	$p_0(r)$	2.165417	0.510201	0	0
	$p_1(r)$	8.872098	29.559197	33.980863	11.481748
	$p_2(r)$	35.619142	119.850166	134.134977	46.855384
	$p_3(r)$	32.406269	105.962456	114.711123	39.969293
Q (r, η)	$q_0(r)$	1.688882	0.244785	0	0
	$q_1(r)$	6.038917	17.606250	18.609629	6.435804
	$q_2(r)$	27.269861	75.982452	81.465216	29.131783
	$q_3(r)$	22.998906	71.242492	76.108743	27.436445

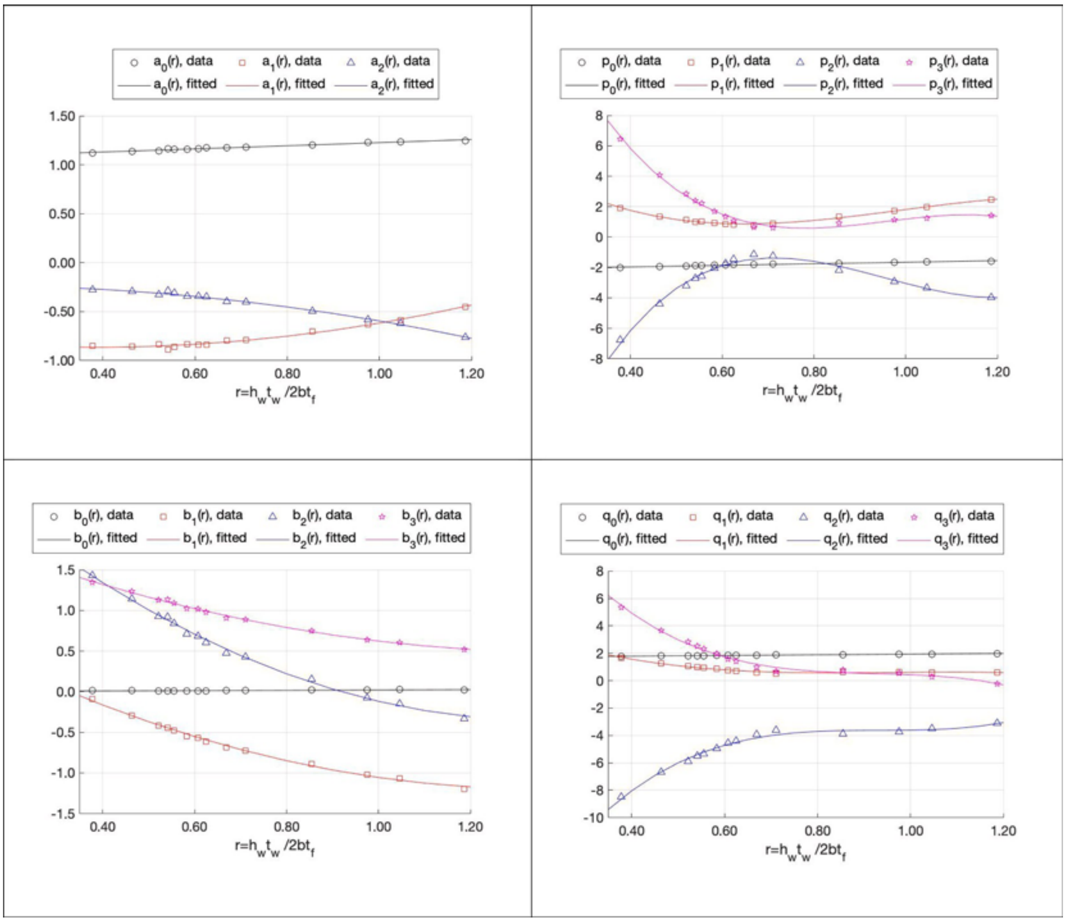

Fig. 4. Left column: Data and fitting coefficients of parameters $a_j(r)$, j = 0, 1, 2 (fitting model of m_{y0}) and $b_j(r)$, j = 0, 1, 2, 3 (fitting model of $m_{\omega0}$). Fig. 4 right column: fitting coefficients in r of r-polynomials $p_j(r)$, $q_j(r)$.

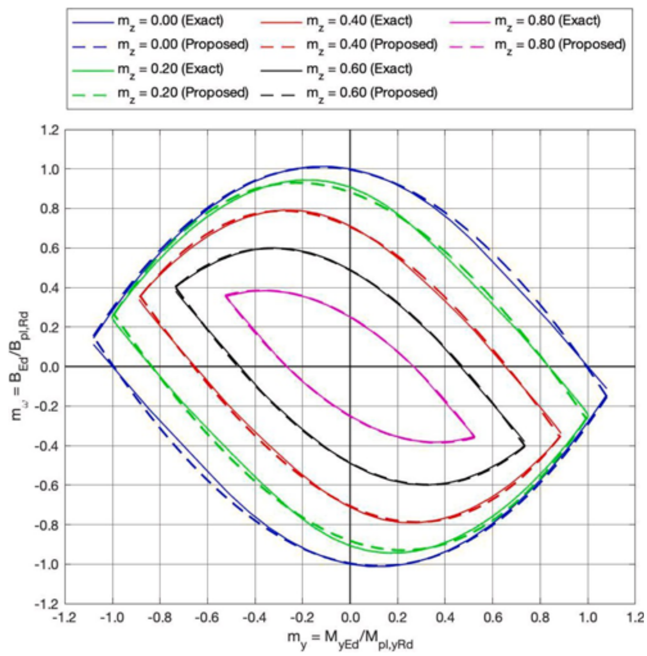

Fig. 5. The interaction diagrams for channel section UPE 220 [25,26] with concomitant forces m_z = 0.0, 0.2, 0.4, 0.6, 0.8 and n_x = 0. Exact result: solid lines, proposed approximation: dashed lines.

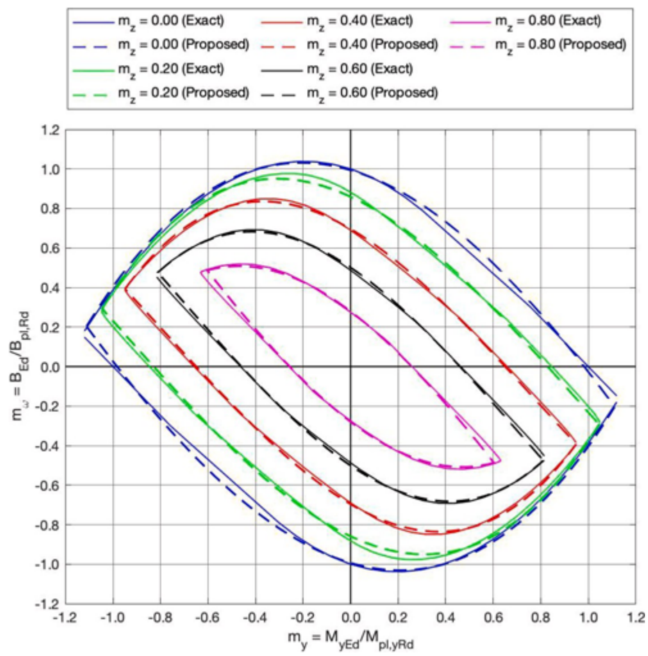

Fig. 6.T. he interaction diagrams for UPE 360 [25,26] with concomitant forces m_z = 0.0, 0.2, 0.4, 0.6, 0.8 and n_x = 0. Exact result: solid lines, proposed approximation: dashed lines.

Table 3
The mean value μ and the standard deviation σ of the absolute error divided by $B_{pl,Rd}$ for UPE 220 and UPE 360 profiles and $\frac{M_{Z,Ed}}{M_{pl,z,Rd}} = \{0.0; 0.2; 0.4; 0.6; 0.8\}$.

Profile	UPE 220		UPE 360	
$M_{z,Ed}/M_{pl,z,Rd}$	μ	σ	μ	σ
0.00	0.0119	0.0166	0.0117	0.0237
0.20	0.0043	0.0180	0.0043	0.0194
0.40	0.0052	0.0105	0.0007	0.0125
0.60	0.0022	0.0052	0.0035	0.0120
0.80	0.0022	0.0026	0.0101	0.0149

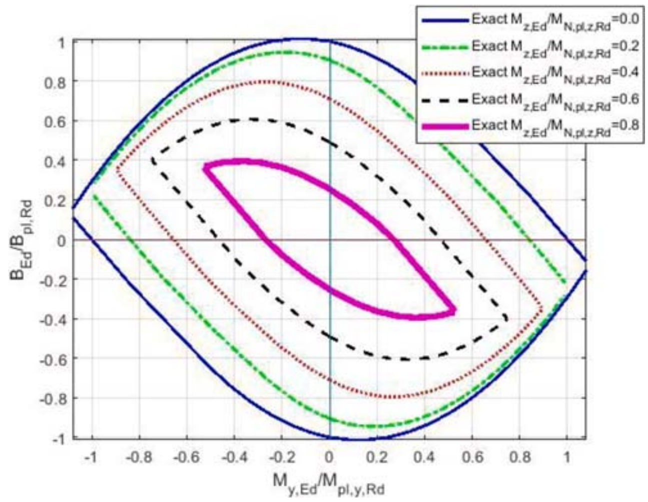

Fig. 7. UPE 160 Salzgitter AG section. $N_{Ed} = 0$, $M_{N,pl,z,Rd}/M_{pl,z,Rd} = 1$, $N_{pl,Rd} = 542.8500\text{kN}$, $M_{pl,z,Rd} = 10.1003\text{kNm}$, $M_{pl,y,Rd} = 29.4089\text{kNm}$, $B_{pl,Rd} = 0.6539\text{kNm}^2$, $M_{N,pl,y,Rd,max}/M_{pl,y,Rd} = 1.0900$, $B_{N,pl,Rd,max}/B_{pl,Rd} = 1.0131$.

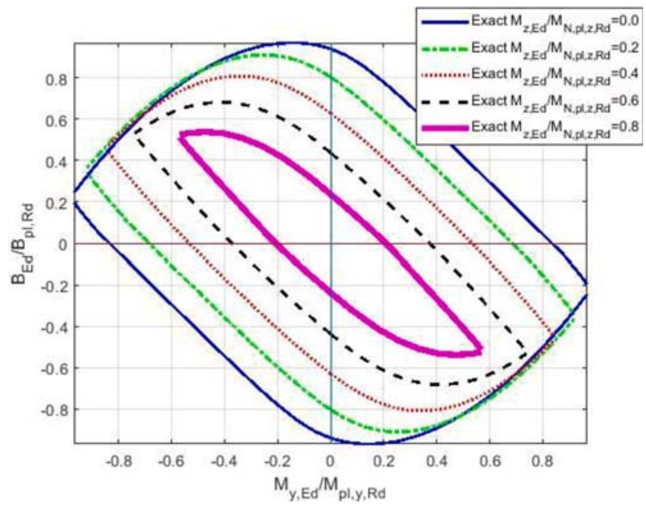

Fig. 8. UPE 160 Salzgitter AG section. $N_{Ed} = 0.2N_{pl,Rd}$, $M_{N,pl,z,Rd}/M_{pl,z,Rd} = 0.8363$, $M_{N,pl,y,Rd,max}/M_{pl,y,Rd} = 0.9763$, $B_{N,pl,Rd,max}/B_{pl,Rd} = 0.9689$.

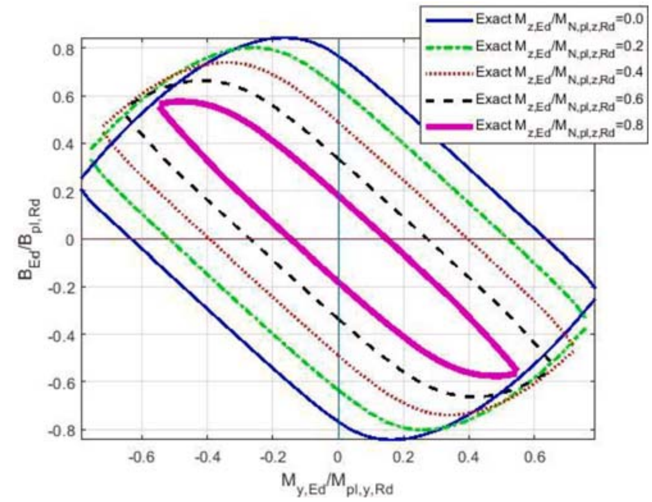

Fig. 9. UPE 160 Salzgitter AG section. $N_{Ed} = 0.4N_{pl,Rd}$, $M_{N,pl,z,Rd}/M_{pl,z,Rd} = 0.6471$, $M_{N,pl,y,Rd,max}/M_{pl,y,Rd} = 0.7927$, $B_{N,pl,Rd,max}/B_{pl,Rd} = 0.8435$.

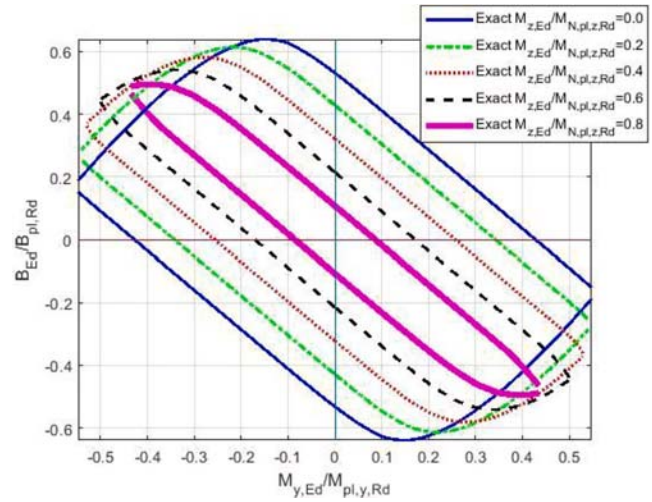

Fig. 10. UPE 160 Salzgitter AG section. $N_{Ed} = 0.6N_{pl,Rd}$, $M_{N,pl,z,Rd}/M_{pl,z,Rd} = 0.4486$, $M_{N,pl,y,Rd,max}/M_{pl,y,Rd} = 0.5522$, $B_{N,pl,Rd,max}/B_{pl,Rd} = 0.6395$.

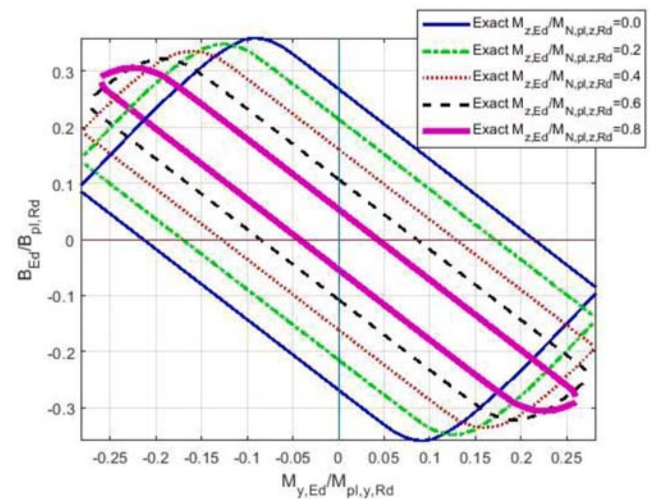

Fig. 11. UPE 160 Salzgitter AG section. $N_{Ed} = 0.8N_{pl,Rd}$, $M_{N,pl,z,Rd}/M_{pl,z,Rd} = 0.2243$, $M_{N,pl,y,Rd,max}/M_{pl,y,Rd} = 0.2847$, $B_{N,pl,Rd,max}/B_{pl,Rd} = 0.3593$.

sections [29]. The proposed paper deals with channel sections. The solution of the Z-sections is in the preparation. The paper devoted to channel sections [30] was accepted by reviewers and it will be published in the conference proceedings. It shows in some points comparisons of exact solutions with the analytical expressions offered for the designers in practice.

The reasons why the channel sections are investigated in the proposed paper are as follows:

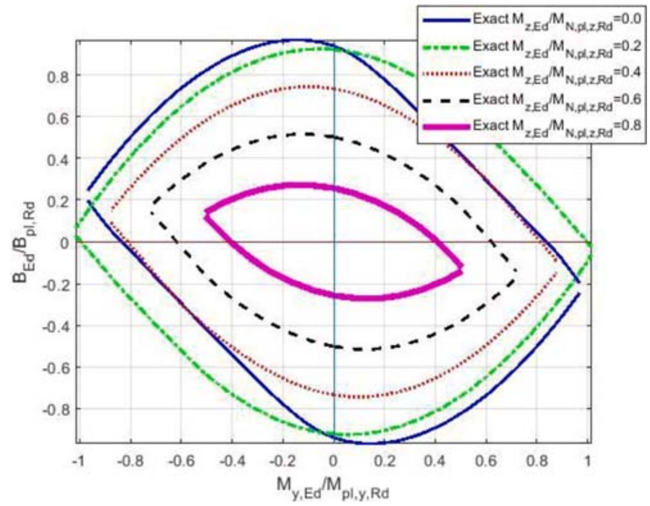

Fig. 12. UPE 160 Salzgitter AG section. $N_{Ed} = 0.2N_{pl,Rd}$, $M_{N,pl,z,Rd}/M_{pl,z,Rd} = 1.0485$, $M_{N,pl,y,Rd,max}/M_{pl,y,Rd} = 1.0252$, $B_{N,pl,Rd,max}/B_{pl,Rd} = 0.9689$.

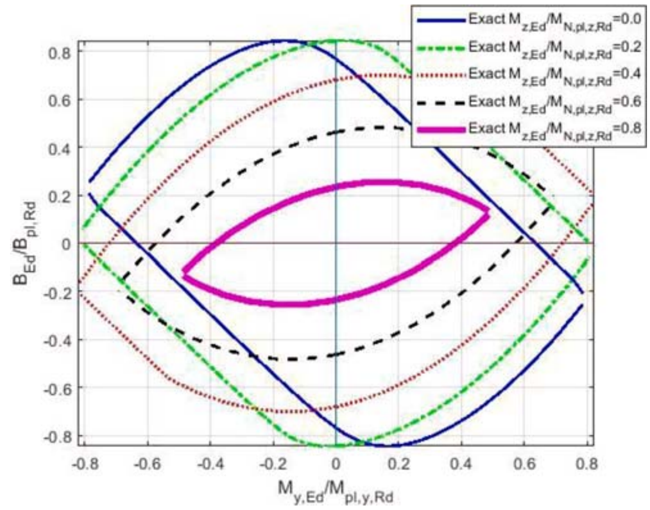

Fig. 13. UPE 160 Salzgitter AG section. $N_{Ed} = 0.4N_{pl,Rd}$, $M_{N,pl,z,Rd}/M_{pl,z,Rd} = 0.9728$, $M_{N,pl,y,Rd,max}/M_{pl,y,Rd} = 0.8289$, $B_{N,pl,Rd,max}/B_{pl,Rd} = 0.8436$.

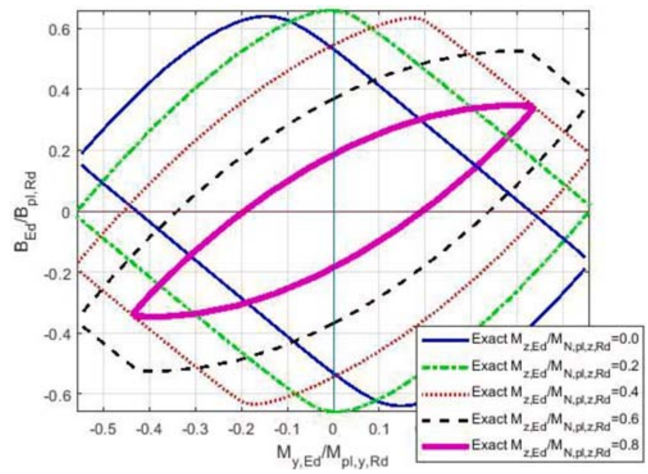

Fig. 14. UPE 160 Salzgitter AG section. $N_{Ed} = -0.6N_{pl,Rd}$, $M_{N,pl,z,Rd}/M_{pl,z,Rd} = 0.7717$, $M_{N,pl,y,Rd,max}/M_{pl,y,Rd} = 0.5633$, $B_{N,pl,Rd,max}/B_{pl,Rd} = 0.6596$.

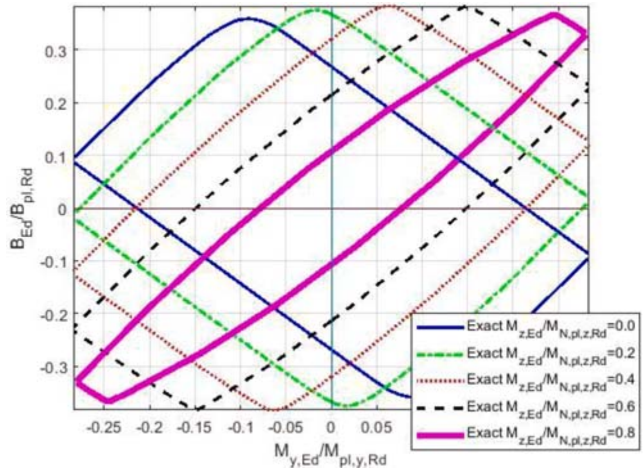

Fig. 15. UPE 160 Salzgitter AG section. $N_{Ed} = 0.8N_{pl,Rd}$, $M_{N,pl,z,Rd}/M_{pl,z,Rd} = 0.4474$, $M_{N,pl,y,Rd,max}/M_{pl,y,Rd} = 0.2860$, $B_{N,pl,Rd,max}/B_{pl,Rd} = 0.3834$.

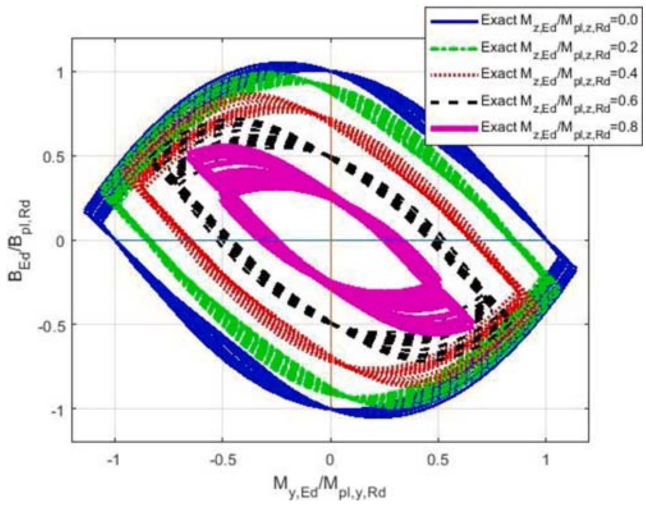

Fig. 16. UPE sections [25,26] from UPE 80 to UPE 400. $N_{Ed} = 0$.

a) for the channel sections there are in the both current and future metal Eurocodes (for steel and aluminum structures) only basic properties, e.g. the shear area; the plastic shear resistance reduced from Vpl, Rd to Vpl,T,Rd; b) only conservative approximation for the various shapes of profiles of all cross-sectional classes, a linear summation of the utilization ratios for each stress resultant is available; c) in the metal Eurocodes there are formulae for the plastic resistance of the cross-section under various combinations of the internal forces, but they are not valid for the channel sections; d) even in the scientific publications there are no explicit formulae for any shape of rolled sections under 3 or more internal forces.

Till today nobody presented results of the parametrical study of the channel sections under combination of 4 internal forces. The authors present own exact solution of the plastic resistant of the channel section under 4 internal forces N_{Ed}, $M_{y,Ed}$, $M_{z,Ed}$, B_{Ed} in the form of diagrams and compare the results with the results of other procedures valid for partial cases. Moreover also the procedure using (42) is given to obtain the approximate values of the plastic channel sections resistances under 3 internal forces $M_{y,Ed}$, $M_{z,Ed}$, B_{Ed}. It is realized in approximate way using the third order polynomials. The numerical values may be easily obtained with the help of some mathematical program (e.g. MATHCAD). The accuracy of the proposed approach is quantified in the illustrative examples in Figs. 5 and 6. The mean values and standard deviations are shown in Table 3 for each profile, and for five different values of the parameter $M_{z,Ed}/M_{pl,z,Rd}$ = 0.0, 0.2, 0.4, 0.6 and 0.8. The large parametric study shows that the approximate values fit very well the exact values given in diagrams.

In the future an attempt will be performed to simplify proposed approximate procedure with acceptable loss of accuracy.

CRediT authorship contribution statement

A. Agüero: Conceptualization, Investigation, Methodology, Software, Writing – original draft, Writing – review & editing. **I. Baláž:** Conceptualization, Investigation, Methodology, Software, Writing – original draft, Writing – review & editing. **Y. Koleková:** Conceptualization, Investigation, Methodology, Software, Writing – original draft, Writing – review & editing. **M. Lázaro:** Conceptualization, Investigation, Methodology, Software, Writing – original draft, Writing – review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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