Overdispersion Effects on Reliability Test Planning

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***Abstract*—The impact of overdispersion on the design of optimal reliability demonstration test plans for beta-binomial models with Weibull, gamma and lognormal lifetime distributions is analyzed. Assuming limited producer and consumer risks, fixed-duration test plans with minimal sample sizes and optimal-duration test plans with minimum costs based on failure count data are determined by solving the corresponding nonlinear programming problems when the fluctuation of the failure probability is described by a beta distribution. If the test time is fixed, the overdispersion effect on the optimal sample size and decision criterion is important in most situations. The influence is less relevant when the engineer also wants to find the minimum-cost test time. Optimal-duration plans usually outperform fixed-duration schemes in terms of costs and robustness against overdispersion. The use of lot inspection schemes with optimal reliability test times is strongly recommended when the presence of overdispersed failure count data is suspected. Applications of the developed methodology to the manufacturing of microelectronic chips and semiconductor lasers are provided for illustrative and comparative purposes.**

***Index Terms*—Reliability demonstration test plans, beta-binomial model, fixed and optimal test durations, constrained nonlinear optimization, limited producer and consumer risks, Weibull, log-normal and gamma distributions.**

I. Introduction

PLANNING reliability tests to judge the acceptability of manufacturing processes and batches is a critical issue in most industries worldwide. In practice, a device is considered acceptable if its reliability at a given conforming lifetime reaches a certain minimum level. Reliability engineers typically minimize sample sizes or cost functions subject to various quality and risk prerequisites in order to find optimal test plans. Generally, test times are prefixed by the analyst, whereas optimal decision criteria and sample sizes are determined by solving constrained optimization problems.

The construction of test plans for reliability demonstration has been widely addressed in diverse fields. Some recent references are Kantam *et al.* [1], Balamurali and Jun [2], Guo and Liao [3], Hsieh and Lu [4], Li and Lin [5], Yang [6], Fernández [7], Wu and Huang [8], Wang *et al.* [9], Aslam *et al.* [10], Lewitschnig and Fanzott [11], Chen *et al.* [12] and Wu *et al.* [13], [14].

Decision criteria in many reliability tests are based on the number of failures observed in the experiment. Numerous papers have considered failure count data, including Guo *et al.* [15], Yalcin and Eryilmaz [16], Qin *et al.* [17], Lu *et al.* [18], Li *et al.* [19], Bouslah *et al.* [20], Ge *et al.* [21], Pérez-González *et al.* [22], [23] and Zhao and Yun [24].

Reliability test durations are often fixed in advance. If t denotes the test time, n represents the number of devices randomly selected from the submitted lot and $X_{n,t}$ designates the number of observed failures at time t, then the entire lot is accepted if and only if (iff) $X_{n,t}$ is at most the so-called acceptance number, k. Hence, a t-duration reliability test plan is described by the pair (n, k). Recently, Fernández [25] has proposed lot inspection schemes with optimal test times by minimizing cost functions. In this case, a reliability test plan is characterized by the triple (n, k, t).

In general, it is assumed that the distribution of $X_{n,t}$ is binomial with sample size n and constant failure probability p. However, p often fluctuates from batch to batch, resulting in greater variability than the binomial distribution. In many situations, the empirical variance is larger than specified by a binomial model. A convenient approach to capture the heterogeneity of the test devices is to consider that p is also a random variable with a determined beta distribution. In such a case, the beta-binomial distribution of $X_{n,t}$ has an additional dispersion parameter d. This alternative model reduces to the standard binomial distribution when $d = 0$ and provides a better fit to the observed failure count data when there is overdispersion.

Typical lifetime models in reliability testing are the Weibull, gamma and lognormal distributions. Reliability test plans for Weibull models have been proposed by Jun *et al.* [26], Chen *et al.* [27], Arizono *et al.* [28], Tsai *et al.* [29], Seo *et al.* [30], Aslam *et al.* [31], Fernández [32], [33], Roy [34] and Gao *et al.* [35]. If the Weibull shape parameter is 1, the distribution reduces to the exponential model, which plays an important role in reliability engineering; see, e.g., Aslam [36], Dey and Chakraborty [37], Fernández [38], [39], [40] and Lee *et al.* [41], [42]. If the shape parameter is 2, the distribution is Rayleigh, which has also a wide range of applications in reliability; e.g., Soliman and Al-Aboud [43], Lee *et al.* [44], Dey and Dey [45], MirMostafaee *et al.* [46] and Asgharzadeh *et al.* [47]. Gamma sampling plans have been discussed, among others, by Lu and Tsai [48], Tseng *et al.* [49], Fernández [50], [51] and Fernández *et al.* [52], and references therein. Test plans for lognormal lifetime distributions have been presented in Gupta [53], Schneider [54], Alhadeed and Yang [55], Wu and Lu [56], Naqvi and Aslam [57], Fernández [58] and Pérez-González *et al.* [59].

Assuming beta-binomial failure count data, this paper de-

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termines the best reliability demonstration test (RDT) plans with fixed and optimal test durations by solving the corresponding constrained optimization problems. The impact of overdispersion on the optimal sample size, decision rule and test time is then studied when the lifetime variables follow Weibull, gamma and lognormal distributions. Fixed-duration RDT plans (n, k) minimize sample sizes, whereas optimal-duration inspection schemes (n, k, t) minimize cost functions depending on test time and sample size. In both cases, producer and consumers risks are controlled and the failure probability p randomly varies according to a beta distribution.

The remainder of our study is structured as follows. Section II deals with RDT planning with fixed test time when the decision criterion is based on the uniformly most powerful beta-binomial test. The probability of lot acceptance is derived in terms of the reliability level. Fixed-duration RDT plans with minimal sample size and limited risks are determined in Section III by solving the underlying constrained optimization problem, whereas Section IV analyzes the influence of the overdispersion on the optimal sample size and acceptance number assuming Weibull, gamma and lognormal lifetime distributions. Next, Section V presents the optimal-duration RDT plan with minimum cost and controlled risks. Mixed integer nonlinear programming problems are solved to find optimal inspection schemes. The impact of the dispersion on the best plan is also studied. Illustrative examples concerning microelectronic chips and semiconductor lasers are provided in Section VI. Section VII then offers a brief discussion and some concluding remarks.

II. RDT PLANS WITH FIXED TEST TIME

Sampling plans for lot acceptance are frequently designed in practice to assure that a certain device has achieved the desired reliability at a given time.

Suppose that a large lot of devices from a given manufacturing process is submitted to judge its acceptability based on the number of failed units in a reliability test. If the random variable T represents the lifetime of the device, the reliability at time $t > 0$ is then defined as $R_T(t) = \Pr(T \geq t)$. Moreover, the minimum lifetime of a conforming unit is denoted by $v > 0$. In such a case, reliability tests are usually developed for determining whether the reliability of the device at the conforming lifetime $R = R_T(v)$ is sufficiently high.

Assuming that R_0 and R_1 are the acceptable and rejectable reliability levels, respectively, specified by the producer and the consumer, then a lot is deemed acceptable if $R \geq R_0$, and rejectable if $R \leq R_1$. A random sample of n units is often chosen from the submitted lot with the purpose of deciding between the null hypothesis $H_0 : R \geq R_0$ and the alternative hypothesis $H_1 : R \leq R_1$. These devices are then simultaneously place on life test for a certain prefixed time t and the number of observed failures, designated by $X_{n,t}$, is noted.

In our case, if $p = 1 - R_T(t)$ denotes the unreliability of the device at time t, the conditional distribution of $X_{n,t}$ given p has a binomial $Bin(n, p)$ distribution with parameters $n \in \mathbb{N} = \{1, 2, \ldots\}$ and $p \in (0, 1)$, i.e. $X_{n,t} \mid p \sim Bin(n, p)$. Hence,

$$\Pr(X_{n,t} = x \mid p) = \binom{n}{x} p^x (1-p)^{n-x}$$

for $x \in \Omega_n = \{0, 1, \ldots, n\}$. In addition, it is assumed that the random failure probability p follows a $Beta(a, b)$ distribution with mean $u = 1 - R_T(t)$ and variance $u(1-u)d/(1+d)$, where $a, b > 0$, and $d = 1/(a+b)$ represents a dispersion parameter. Since $p \sim Beta(u/d, (1-u)/d)$, its probability density function (pdf) is given by

$$g(p) = \frac{p^{u/d-1}(1-p)^{(1-u)/d-1}}{B[u/d, (1-u)/d]}, \quad 0 < p < 1,$$

where $B[\cdot, \cdot]$ denotes the beta function. Therefore,

$$\Pr(X_{n,t} = x) = \int_0^1 \Pr(X_{n,t} = x \mid p) g(p) \mathrm{d}p$$

for $x \in \Omega_n$, which implies that the pdf of $X_{n,t}$ is defined by

$$\Pr(X_{n,t} = x) = \frac{\binom{n}{x} B[p/d + x, (1-p)/d + n - x]}{B[p/d, (1-p)/d]}$$

for $x \in \Omega_n$. That is, the unconditional distribution of $X_{n,t}$ is Beta-Binomial with parameters n, p and d, which is denoted as $X_{n,t} \sim BetaBin(n, p, d)$.

The mean and variance of $X_{n,t}$ are

$$E[X_{n,t}] = np \quad \text{and} \quad V[X_{n,t}] = np(1-p)(nd+1)/(d+1),$$

respectively. Essentially, the distribution of $X_{n,t}$ is $Bin(n, p)$ when $d = 0$. If $d \to 0^+$, then $V[X_{n,t}]$ converges to $np(1-p)$, which coincides with the variance of a $Bin(n, p)$ distribution, whereas $V[X_{n,t}]$ converges to $n^2p(1-p)$ as $d \to \infty$.

The uniformly most powerful beta-binomial test would reject the submitted lot when $X_{n,t}$ was large enough; say, if and only if (iff) $X_{n,t} > k$, where $k \in \Omega_{n-1}$ is the so-called acceptance number.

Given the fixed test time $t > 0$, an RDT plan is represented by a pair $S_t = (n, k)$, where n and k are integers such that $0 \leq k < n$. The inspection scheme for lot acceptance $S_t = (n, k)$ can be stated as follows: *Step 1:* Choose a sample of n units at random from the submitted lot. *Step 2:* Place the n selected devices on life test for time t. *Step 3:* Count the number of failed units at time t, $X_{n,t}$. *Step 4:* Accept the lot if $X_{n,t} \leq k$, and reject it, otherwise.

In our situation, the operating characteristic (OC) function of a given RDT plan provides the probability of lot acceptance versus the reliability level. Hence, the OC function of the inspection scheme $S_t = (n, k)$ at $R = R_T(v)$ is the probability of observing k failures or less by time t. Thus, the OC function associated with the plan S_t at the reliability level R is defined as

$$A(R) \equiv A(R; n, k, t) = \Pr(X_{n,t} \leq k), \quad 0 < R < 1. \quad (1)$$

Since $p = 1 - R_T\left(tR_T^{-1}(R)/v\right)$, the OC function of S_t at R can be expressed as $A(R) = H[p]$, where $H[p] \equiv H[p;n,k,d]$ is given by

$$H[p] = \sum_{i=0}^{k} \frac{\binom{n}{i} B\left[\frac{p}{d}+i, \frac{1-p}{d}+n-i\right]}{B\left[\frac{p}{d}, \frac{1-p}{d}\right]} \tag{2}$$

for $0 < R < 1$. If the time-censored reliability test is finished at time $t = cv$, it is then deduced that $p = 1 - R_T\left(cR_T^{-1}(R)\right)$, where the censoring factor $c = t/v$ is fixed by the decision maker. The OC function (1) is decreasing in n and t, and increasing in k and R. As the OC function describes the discriminatory power of the inspection scheme, a quality manager can compare OC curves in order to select the suitable RDT plan.

III. Design of fixed-duration RDT plans

Suppose that a determined test time t is fixed by the analyst. In such a case, the design of RDT plans in industry usually assumes minimum sample size, as well as the required protections to customers and manufacturers. That is, in addition to minimize the sampling effort, the best t-duration RDT plan has to assure that the probability of rejecting good lots (producer risk) and the probability of accepting bad lots (consumer risk) are low enough.

In our framework, the producer and consumer risks associated with the inspection scheme $S_t = (n,k)$ are defined by

$$\max_{R \geq R_0}\{1 - A(R;n,k,t)\} \quad \text{and} \quad \max_{R \leq R_1}\{A(R;n,k,t)\},$$

respectively. Hence, as the OC function is increasing in R, the corresponding producer and consumer risks are given by

$$1 - A(R_0;n,k,t) \quad \text{and} \quad A(R_1;n,k,t).$$

Traditionally, it is assumed an agreement between the customer and the manufacturer on the specifications of the maximum tolerated producer and consumer risks, denoted as α_0 and α_1, where $0 < \alpha_0, \alpha_1 < 0.5$. Of course, α_0 and α_1 are usually very small in most applications. An RDT plan $S_t = (n,k)$ that satisfies the nonlinear inequality constraints

$$1 - A(R_0;n,k,t) \leq \alpha_0 \quad \text{and} \quad A(R_1;n,k,t) \leq \alpha_1. \tag{3}$$

is said to be feasible. Thus, the optimal t-duration RDT plan would be the feasible inspection scheme $S_t^* = (n^*,k^*)$ with minimal sample size.

The constrained minimization problem to find $S_t^* = (n^*,k^*)$ is an integer nonlinear programming problem, which can be stated as follows:

$$\begin{aligned} &\text{Minimize} && n \\ &\text{Subject to} && A(R_0;n,k,t) \geq 1 - \alpha_0, \\ &&& A(R_1;n,k,t) \leq \alpha_1, \\ &&& n,k \in \mathbb{N}_0,\ n > k, \end{aligned} \tag{4}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, in which $\mathbb{N}$ is the set of natural numbers. More compactly, the optimization problem (4) can be formulated as

$$\min\{n \in \mathbb{N} : (n,k) \in D_t\},$$

where D_t denotes the feasible region associated with (4).

Assuming that

$$m_k = \min\{n \in \mathbb{N} : A(R_1;n,k,t) \leq \alpha_1, \quad n > k\}$$

for $k \in \mathbb{N}_0$, it is then deduced that the minimal sample size is $n^* = m_{k^*}$, where

$$k^* = \min\{k \in \mathbb{N}_0 : A(R_0;m_k,k,t) \geq 1 - \alpha_0,\ 0 \leq k < m_k\}$$

is the optimal acceptance number. That is, the RDT plan with fixed test time t and minimal sample size is given by $S_t^* = (m_{k^*}, k^*)$.

IV. Influence of the overdispersion

The impact of the dispersion parameter d in reliability test planning with prefixed duration is analyzed in this section. Weibull, gamma and lognormal distributions are assumed to describe the random behavior of the lifetime variable T. These distributions are relevant probabilistic models in reliability engineering due to their flexibility in fitting different types of time-to-failure data. Some tables and figures are provided to quantify the overdispersion effects on the optimal sample size and acceptance number.

Suppose first that the reliability function of T is given by $R_T(t) = \exp(-\lambda t^s)$ for $t > 0$, where $s, \lambda > 0$. In this case, T has a Weibull $W(s,\lambda)$ distribution with parameters s and λ. In practice, the Weibull shape parameter s is often tied to the device failure mechanism [25]. If $S_t = (n,k)$ is a Weibull RDT plan, the corresponding OC function can be expressed as

$$A_W(R) \equiv A_W(R;n,k,t) = H[p_W], \quad 0 < R < 1, \tag{5}$$

where $p_W = 1 - R^{t^s/v^s}$ and $H[\cdot]$ was defined in (2).

Table I presents the optimal (minimum sample size) $W(s,\lambda)$ RDT plans (n^*,k^*) with fixed test time for $\alpha_0 = 0.05,\ \alpha_1 = 0.10,$ acceptable reliability level $R_0 = 0.98, 0.99,$ rejectable reliability level $R_1 = 0.88, 0.90,$ shape parameter $s = 1, 2, 3,$ censoring factor $c = 0.8, 1.2$ and dispersion parameter $d = 0.000, 0.002, 0.004, 0.006$. The failure rate is constant over time if s is 1, whereas the device wear-out is linear and quadratic when s is 2 and 3, respectively.

Next, assume that the lifetime of the device, T, follows a gamma $G(r,\theta)$ distribution with shape parameter $r > 0$ and scale parameter $\theta > 0$. Since the pdf of $T \sim G(r,\theta)$ is

$$f_T(t) = \frac{t^{r-1}\exp(-t/\theta)}{\theta^r \Gamma(r)}, \quad t > 0,$$

the reliability function of T is defined as

$$R_T(t) = 1 - I_r[t/\theta], \quad t > 0,$$

TABLE I
OPTIMAL RELIABILITY TEST PLANS (n^*, k^*) FOR WEIBULL $W(s, \lambda)$ DISTRIBUTIONS WHEN $\alpha_0 = 0.05$ AND $\alpha_1 = 0.10$.

			$s=1$				$s=2$				$s=3$			
			$c=0.8$		$c=1.2$		$c=0.8$		$c=1.2$		$c=0.8$		$c=1.2$	
R_0	R_1	d	n^*	k^*	n^*	k^*	n^*	k^*	n^*	k^*	n^*	k^*	n^*	k^*
0.98	0.88	0.000	67	3	46	3	84	3	38	3	104	3	32	3
		0.002	70	3	47	3	87	3	39	3	110	3	33	3
		0.004	72	3	48	3	110	4	40	3	140	4	33	3
		0.006	90	4	49	3	135	5	41	3	200	6	34	3
	0.90	0.000	81	3	55	3	101	3	46	3	126	3	39	3
		0.002	102	4	68	4	128	4	57	4	162	4	47	4
		0.004	125	5	70	4	159	5	58	4	262	7	48	4
		0.006	150	6	84	5	242	8	70	5	442	12	50	4
0.99	0.88	0.000	39	1	26	1	49	1	22	1	60	1	19	1
		0.002	40	1	27	1	50	1	22	1	87	2	19	1
		0.004	57	2	27	1	72	2	23	1	92	2	19	1
		0.006	59	2	39	2	75	2	32	2	96	2	19	1
	0.90	0.000	65	2	44	2	80	2	37	2	100	2	31	2
		0.002	67	2	45	2	84	2	37	2	106	2	31	2
		0.004	70	2	46	2	89	2	38	2	113	2	32	2
		0.006	73	2	47	2	119	3	39	2	153	3	32	2

where $I_r[\cdot]$ is the incomplete gamma function given by

$$I_r[y] = \frac{1}{\Gamma(r)} \int_0^y x^{r-1} \exp(-x) \mathrm{d}x, \quad y > 0.$$

In view of (2), the OC function associated with the gamma RDT plan $S_t = (n, k)$ is defined by

$$A_G(R) \equiv A_G(R; n, k, t) = H\left[p_G\right], \quad 0 < R < 1, \qquad (6)$$

where $p_G = I_r\left[tI_r^{-1}[1-R]/v\right]$.

The optimal gamma $G(r, \theta)$ RDT plans (n^*, k^*) with fixed test time for $\alpha_0 = 0.05$, $\alpha_1 = 0.10$, $R_0 = 0.98, 0.99$, $R_1 = 0.88, 0.90$, $r = 1, 2, 3$, $c = 0.8, 1.2$ and $d = 0.000, 0.002, 0.004, 0.006$ are reported in Table II. Evidently, the optimal Weibull and gamma RDT plans coincide in the exponential case (i.e., when $s = r = 1$).

The reliability engineer considers now that the log-lifetime variable $Y = \log(T)$ has a normal distribution with mean μ and variance $\sigma^2 > 0$, which implies that the time-to-failure variable T follows a lognormal $LN(\mu, \sigma)$ distribution. If $\Phi[\cdot]$ denotes the standard normal cumulative distribution function, the reliability function of $T \sim LN(\mu, \sigma)$ is then given by

$$R_T(t) = 1 - \Phi[(\log(t) - \mu)/\sigma], \quad t > 0.$$

From (2), it turns out that the OC function of the lognormal RDT plan $S_t = (n, k)$ can be expressed as

$$A_L(R) \equiv A_L(R; n, k, t) = H\left[p_L\right], \quad 0 < R < 1, \qquad (7)$$

where $p_L = \Phi\left[\log(t/v)/\sigma + \Phi^{-1}[1-R]\right]$.

Table III shows the optimal lognormal $LN(\mu, \sigma)$ RDT plans (n^*, k^*) with fixed test time for $\alpha_0 = 0.05$, $\alpha_1 = 0.10$, $R_0 = 0.98, 0.99$, $R_1 = 0.88, 0.90$, $\sigma = 1, 2, 3$, $c = 0.8, 1.2$ and $d = 0.000, 0.002, 0.004, 0.006$.

In view of Tables I, II and III, as the dispersion parameter d grows, the optimal sample size n^* and acceptance number k^* tend to increase in most cases, especially n^*. Similarly, n^* and k^* grow when the prefixed test duration is reduced. The influence of d on the optimal design S_t^* is often considerable. Generally, as d increases, n^* grows rapidly, while k^* is more stable. For example, if $R_0 = 0.98$, $R_1 = 0.90$, $s = 1$ and $c = 0.8$, it is seen in Table I that n^* is 81, 102, 125 and 150 and k^* is 3, 4, 5 and 6 when d is 0.000, 0.002, 0.004 and 0.006, respectively. Identical results are observed in Table II when the lifetime distribution is gamma with $r = 1$. In the lognormal case with $\sigma = 1$, it follows from Table III that n^* is 100, 105, 134 and 189 and k^* is 3, 3, 4 and 6 when d is 0.000, 0.002, 0.004 and 0.006, respectively.

For comparative purposes, Table IV reports the optimal fixed-duration RDT plans (n^*, k^*) for Weibull $W(s, \lambda)$, gamma $G(r, \theta)$ and log-normal $LN(\mu, \sigma)$ distributions with identical coefficients of variation when $s = 2$, $\alpha_0 = 0.05$ and $\alpha_1 = 0.10$.

From Table IV, it is clear that n^* and k^* tend to increase as the dispersion degree becomes higher and/or the test time is reduced. According to Table IV, the optimal RDT plans $S_t^* = (n^*, k^*)$ when $d = 0$ and the prefixed test duration is $t = 0.8v$, where v is the minimal conforming lifetime, are $(101, 3)$, $(119, 3)$ and $(128, 2)$ in the Weibull, gamma and lognormal cases. The corresponding best inspection schemes when $d = 0.006$ are $(242, 8)$, $(325, 9)$ and $(471, 9)$. In this situation, the high impact of the overdispersion on the reliability test planning is evident.

As graphical illustrations, Figs. 1 and 2 display the optimal sample size n^* and acceptance number k^* versus the dispersion parameter d assuming Weibull $W(s, \lambda)$, gamma $G(r, \theta)$ and lognormal $LN(\mu, \sigma)$ lifetime distributions with identical coefficients of variation when $s = 2$, $R_0 = 0.99$, $R_1 = 0.90$, $c = 0.8$, $\alpha_0 = 0.05$ and $\alpha_1 = 0.10$. Clearly, both n^* and

TABLE II
Optimal reliability test plans (n^*, k^*) for gamma $G(r,\theta)$ distributions when $\alpha_0 = 0.05$ and $\alpha_1 = 0.10$.

			$r=1$				$r=2$				$r=3$			
			$c=0.8$		$c=1.2$		$c=0.8$		$c=1.2$		$c=0.8$		$c=1.2$	
R_0	R_1	d	n^*	k^*	n^*	k^*	n^*	k^*	n^*	k^*	n^*	k^*	n^*	k^*
0.98	0.88	0.000	67	3	46	3	79	3	40	3	72	2	37	3
		0.002	70	3	47	3	83	3	41	3	95	3	37	3
		0.004	72	3	48	3	86	3	42	3	120	4	38	3
		0.006	90	4	49	3	109	4	43	3	126	4	47	4
	0.90	0.000	81	3	55	3	96	3	48	3	110	3	52	4
		0.002	102	4	68	4	122	4	59	4	141	4	54	4
		0.004	125	5	70	4	150	5	61	4	175	5	55	4
		0.006	150	6	84	5	205	7	73	5	268	8	66	5
0.99	0.88	0.000	39	1	26	1	46	1	23	1	52	1	21	1
		0.002	40	1	27	1	48	1	24	1	55	1	21	1
		0.004	57	2	27	1	68	2	33	2	79	2	30	2
		0.006	59	2	39	2	71	2	34	2	82	2	31	2
	0.90	0.000	65	2	44	2	77	2	38	2	64	1	35	2
		0.002	67	2	45	2	80	2	39	2	93	2	35	2
		0.004	70	2	46	2	84	2	40	2	98	2	36	2
		0.006	73	2	47	2	88	2	41	2	103	2	37	2

TABLE III
Optimal reliability test plans (n^*, k^*) for lognormal $LN(\mu,\sigma)$ distributions when $\alpha_0 = 0.05$ and $\alpha_1 = 0.10$.

			$\sigma=1$				$\sigma=2$				$\sigma=3$			
			$c=0.8$		$c=1.2$		$c=0.8$		$c=1.2$		$c=0.8$		$c=1.2$	
R_0	R_1	d	n^*	k^*	n^*	k^*	n^*	k^*	n^*	k^*	n^*	k^*	n^*	k^*
0.98	0.88	0.000	64	2	40	3	53	2	47	3	49	2	49	3
		0.002	85	3	41	3	68	3	48	3	64	3	50	3
		0.004	88	3	50	4	71	3	49	3	66	3	52	3
		0.006	111	4	52	4	88	4	60	4	82	4	64	4
	0.90	0.000	100	3	57	4	80	3	67	4	75	3	59	3
		0.002	105	3	59	4	101	4	69	4	94	4	73	4
		0.004	134	4	70	5	105	4	83	5	98	4	88	5
		0.006	189	6	72	5	148	6	86	5	119	5	91	5
0.99	0.88	0.000	47	1	32	2	38	1	27	1	36	1	28	1
		0.002	49	1	33	2	39	1	27	1	37	1	29	1
		0.004	50	1	33	2	40	1	39	2	38	1	41	2
		0.006	52	1	34	2	58	2	40	2	54	2	42	2
	0.90	0.000	58	1	38	2	47	1	44	2	43	1	47	2
		0.002	83	2	39	2	66	2	45	2	62	2	48	2
		0.004	87	2	40	2	69	2	47	2	64	2	49	2
		0.006	91	2	41	2	72	2	48	2	66	2	51	2

k^* tend to increase as the dispersion grows in the Weibull, gamma and lognormal cases.

Fig. 3 depicts the optimal sample size n^* as a function of the censoring factor $c = t/v$ when the dispersion parameter is $d = 0.001, 0.003, 0.005$, the lifetime distribution is Weibull $W(2,\lambda)$, $R_0 = 0.99$, $R_1 = 0.90$, $\alpha_0 = 0.05$ and $\alpha_1 = 0.10$.

In accordance with Fig. 3, n^* is rapidly reduced as the prefixed test duration t increases. Moreover, n^* becomes larger when d grows. Note also that the influence of the dispersion level on the required sample size is reduced when the test time (or, equivalently, the censoring factor) increases.

Based on the results obtained above, in addition to considering historical data and expert opinions, it would be advisable to select a preliminary random sample and perform goodness-of-fit tests to choose the most appropriate lifetime model.

V. RDT plans with optimal test times

The experimental duration t is not prefixed in this section. Hence, we have also to find the optimal test time. An RDT plan is now characterized by a triple (n, k, t), where n and k are integers such that $0 \le k < n$ and $t > 0$. The optimal RDT plan should minimize the incurred cost for lot sentencing in addition to providing the required protections to producers and consumers.

TABLE IV
OPTIMAL RELIABILITY TEST PLANS (n^*, k^*) FOR WEIBULL $W(s, \lambda)$, GAMMA $G(r, \theta)$ AND LOGNORMAL $LN(\mu, \sigma)$ DISTRIBUTIONS WITH IDENTICAL COEFFICIENTS OF VARIATION WHEN $s = 2$, $\alpha_0 = 0.05$ AND $\alpha_1 = 0.10$.

			Weibull case				Gamma case				Lognormal case			
			$c = 0.8$		$c = 1.2$		$c = 0.8$		$c = 1.2$		$c = 0.8$		$c = 1.2$	
R_0	R_1	d	n^*	k^*	n^*	k^*	n^*	k^*	n^*	k^*	n^*	k^*	n^*	k^*
0.98	0.88	0.000	84	3	38	3	78	2	35	3	102	2	30	3
		0.002	87	3	39	3	103	3	35	3	108	2	37	4
		0.004	110	4	40	3	108	3	36	3	147	3	38	4
		0.006	135	5	41	3	138	4	44	4	190	4	38	4
	0.90	0.000	101	3	46	3	119	3	50	4	128	2	43	4
		0.002	128	4	57	4	153	4	51	4	175	3	50	5
		0.004	159	5	58	4	192	5	52	4	270	5	52	5
		0.006	242	8	70	5	325	9	62	5	471	9	60	6
0.99	0.88	0.000	49	1	22	1	57	1	20	1	74	1	24	2
		0.002	50	1	22	1	59	1	28	2	78	1	24	2
		0.004	72	2	23	1	85	2	29	2	83	1	25	2
		0.006	75	2	32	2	89	2	29	2	88	1	25	2
	0.90	0.000	80	2	37	2	69	1	33	2	93	1	28	2
		0.002	84	2	37	2	101	2	34	2	100	1	29	2
		0.004	89	2	38	2	107	2	34	2	149	2	29	2
		0.006	119	3	39	2	113	2	35	2	161	2	37	3

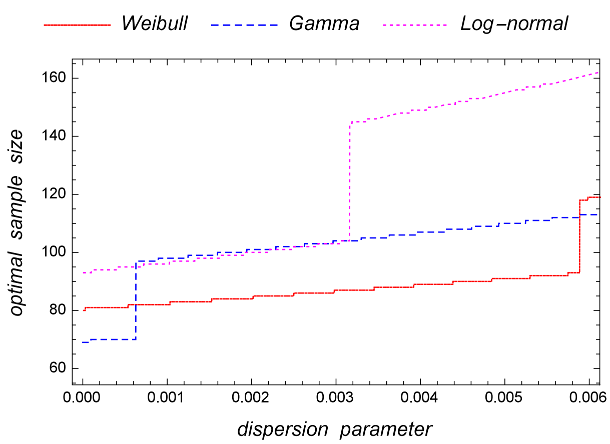

Fig. 1. Optimal sample size n^* versus the dispersion parameter d for Weibull $W(s, \lambda)$, gamma $G(r, \theta)$ and lognormal $LN(\mu, \sigma)$ distributions with identical coefficients of variation when $s = 2$, $R_0 = 0.99$, $R_1 = 0.90$, $c = 0.8$, $\alpha_0 = 0.05$ and $\alpha_1 = 0.10$.

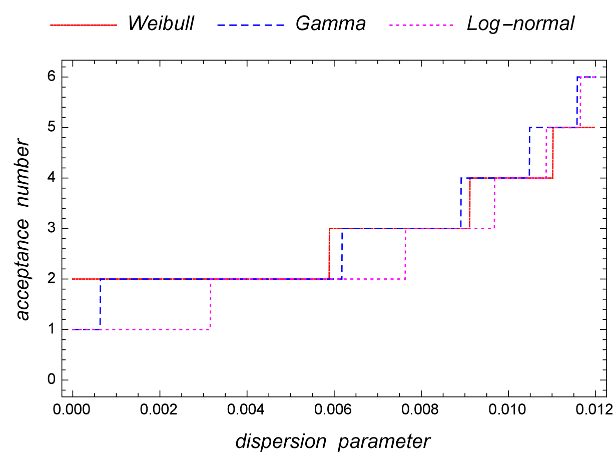

Fig. 2. Optimal acceptance number k^* versus the dispersion parameter d for Weibull $W(s, \lambda)$, gamma $G(r, \theta)$ and lognormal $LN(\mu, \sigma)$ distributions with identical coefficients of variation when $s = 2$, $R_0 = 0.99$, $R_1 = 0.90$, $c = 0.8$, $\alpha_0 = 0.05$ and $\alpha_1 = 0.10$.

In our situation, a suitable cost function must be increasing in both n and t, and also represents an appropriate trade-off between test duration and sample size. A linear combination of n and t is the natural and simplest choice. Given the positive constants c_0, c_1, and c_1, the cost function is defined by

$$C[n, k, t] = c_0 + c_1 n + c_2 t, \tag{8}$$

where c_0 is the initial cost of the life experiment, c_1 is the cost per sampled item, and c_2 is the cost per test time unit.

The constrained optimization problem to determine the minimum number of devices to test, n^*, the maximum tolerable number of failures, k^*, to accept the submitted lot and the best test duration, t^*, is a mixed integer nonlinear programming problem, which can be formulated as follows:

$$\begin{aligned} \text{Minimize} \quad & c_0 + c_1 n + c_2 t \\ \text{Subject to} \quad & A(R_0; n, k, t) \geq 1 - \alpha_0, \\ & A(R_1; n, k, t) \leq \alpha_1, \\ & n, k \in \mathbb{N}_0, \quad n > k, \quad t > 0. \end{aligned} \tag{9}$$

This minimization problem can be stated more compactly as

$$\min\{C[n, k, t] : (n, k, t) \in D\},$$

where $D = \{(n, k, t) : (n, k) \in D_t\}$ denotes the feasible region associated with (9).

The RDT plan with minimal cost that simultaneously satisfies the risk requirements (3), i.e. the global solution of (9), would be denoted by $S^* = (n^*, k^*, t^*)$.

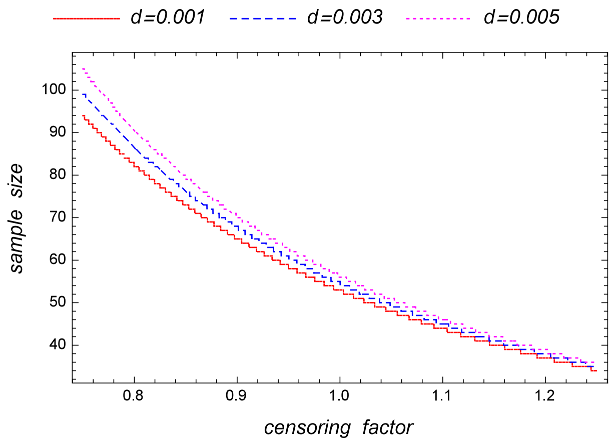

Fig. 3. Optimal sample size n^* versus the censoring factor $c = t/v$ when the dispersion parameter is $d = 0.001, 0.003, 0.005$, the lifetime distribution is Weibull $W(2, \lambda)$, $R_0 = 0.99$, $R_1 = 0.90$, $\alpha_0 = 0.05$ and $\alpha_1 = 0.10$.

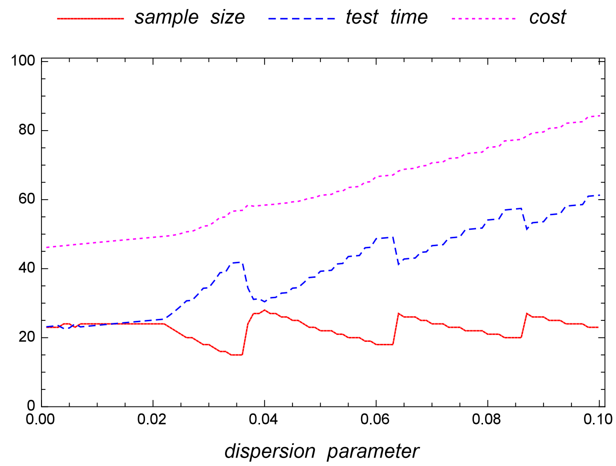

Fig. 4. Optimal sample size, test time and cost versus the dispersion parameter d when the lifetime distribution is Weibull $W(1, \lambda)$, $R_0 = 0.99$, $R_1 = 0.90$, $\alpha_0 = 0.05$, $\alpha_1 = 0.10$, $v = 10$ and $C[n, k, t] = n + t$.

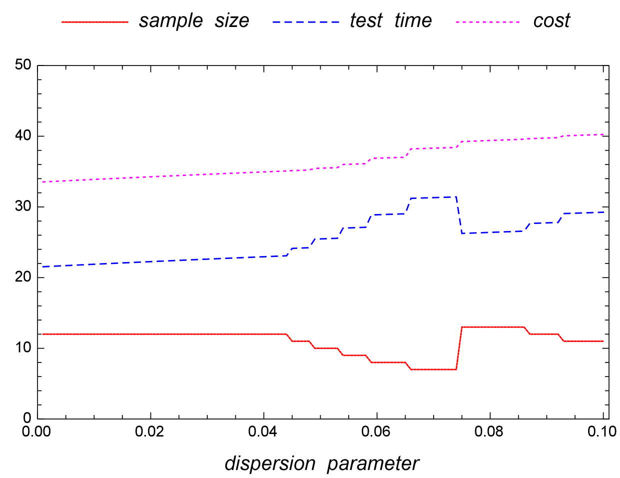

Fig. 5. Optimal sample size, test time and cost versus the dispersion parameter d when the lifetime distribution is Weibull $W(2, \lambda)$, $R_0 = 0.99$, $R_1 = 0.90$, $\alpha_0 = 0.05$, $\alpha_1 = 0.10$, $v = 10$ and $C[n, k, t] = n + t$.

Suppose that k_0 denotes the smallest feasible value of k. Assuming that $S_i = (n_i, k_i, t_i)$ is the optimal RDT plan when $k_i = k_0 + i$ is the acceptance number for $i \in \mathbb{N}_0$, it then follows that

$$C[S^*] = \min\{C[S_i] : i \in \mathbb{N}_0\}.$$

is the minimum cost. In addition, n_i is the feasible value of n that minimizes $C[n, k, t_{n,k}]$ when $k = k_i$, where

$$t_{n,k} = \min\{t > 0 : (n, k, t) \in D\}.$$

Clearly, $t_i = t_{n_i,k_i}$ is the optimal test duration when $k = k_i$. It is also evident that, if $C[S^*] = C[S_{i^*}]$, then the best plan S^* is $S_{i^*} = (n_{i^*}, k_{i^*}, t_{i^*})$, which implies that $n^* = n_{i^*}$, $k^* = k_{i^*}$ and $t^* = t_{i^*}$.

In order to solve the optimization problem (9) for Weibull, gamma and lognormal lifetime distributions, we refer to the corresponding computational methods proposed by Fernández [25], [51], [58] in the binomial case.

Several tables and figures are now presented with the aim of studying the influence of the dispersion parameter d on the optimal sample size, acceptance number and test time. Table V reports the optimal-duration RDT plan $S^* = (n^*, k^*, t^*)$ and minimal cost C^* for Weibull $W(s, \lambda)$ lifetime distributions when $\alpha_0 = 0.05$, $\alpha_1 = 0.10$, the conforming lifetime is $v = 10$ and the cost function is defined by $C[n, k, t] = n + 5t$. The corresponding values when $C[n, k, t] = 5n + t$ are provided in Table VI.

In light of Tables V and VI, the influence of the overdispersion on reliability test planning with optimal test duration is less significant. That is, $S^* = (n^*, k^*, t^*)$ is reasonably insensitive to slight changes in d. For example, if $C[n, k, t] = n + 5t$, $R_0 = 0.98$, $R_1 = 0.90$, $s = 1$ and $c = 0.8$, then it follows from Table V that n^* is 58, 64, 64 and 59 and k^* is 3, 4, 4 and 4 when d is 0.000, 0.002, 0.004 and 0.006, respectively. According to Table VI, if the cost function is $C[n, k, t] = 5n + t$, the corresponding sample sizes are 13, 13, 13 and 15, while the respective acceptance numbers are 3, 3, 3 and 4.

Assuming that the lifetime distribution is Weibull $W(s, \lambda)$, $R_0 = 0.99$, $R_1 = 0.90$, $\alpha_0 = 0.05$, $\alpha_1 = 0.10$, $v = 10$ and $C[n, k, t] = n + t$, Figs. 4, 5 and 6 shows the optimal sample size, test time and cost versus the dispersion parameter d when the shape parameter s is 1, 2 and 3.

From Figs. 4, 5 and 6 it is clear that the effects of the dispersion parameter d on the optimal sample size, test time and cost are not crucial in most cases. Generally, optimal-duration RDT plans S^* are much more robust than fixed-duration RDT plans S_t^* to small variations in the dispersion level.

TABLE V
OPTIMAL RELIABILITY TEST PLAN (n^*, k^*, t^*) AND MINIMAL COST C^* FOR WEIBULL $W(s, \lambda)$ DISTRIBUTIONS WHEN $\alpha_0 = 0.05$, $\alpha_1 = 0.10$, $v = 10$ AND $C[n, k, t] = n + 5t$.

			$s=1$				$s=2$				$s=3$			
R_0	R_1	d	n^*	k^*	t^*	C^*	n^*	k^*	t^*	C^*	n^*	k^*	t^*	C^*
0.98	0.88	0.000	53	3	10.153	103.77	34	3	12.690	97.448	24	3	13.256	90.281
		0.002	53	3	10.445	105.22	33	3	13.005	98.027	24	3	13.314	90.569
		0.004	53	3	10.726	106.63	33	3	13.117	98.587	24	3	13.370	90.851
		0.006	53	3	10.999	108.00	33	3	13.226	99.130	23	3	13.625	91.123
	0.90	0.000	58	3	11.228	114.14	36	3	13.565	103.83	25	3	13.934	94.670
		0.002	64	4	12.627	127.14	36	3	13.698	104.49	25	3	13.997	94.986
		0.004	64	4	12.994	128.97	38	4	14.796	111.98	26	4	14.820	100.10
		0.006	59	4	14.432	131.16	38	4	14.923	112.62	26	4	14.880	100.40
0.99	0.88	0.000	40	1	7.7047	78.524	27	1	10.718	80.589	20	1	11.602	78.009
		0.002	40	1	7.9312	79.656	27	1	10.824	81.120	20	1	11.659	78.294
		0.004	31	1	10.431	83.157	27	1	10.928	81.638	20	1	11.715	78.573
		0.006	47	2	9.8176	96.088	20	1	12.768	83.839	20	1	11.769	78.846
	0.90	0.000	51	2	10.106	101.53	33	2	12.569	95.845	23	2	13.200	88.999
		0.002	51	2	10.423	103.11	32	2	12.896	96.481	23	2	13.262	89.311
		0.004	51	2	10.729	104.65	32	2	13.018	97.091	23	2	13.323	89.616
		0.006	51	2	11.027	106.14	32	2	13.137	97.684	23	2	13.383	89.914

TABLE VI
OPTIMAL RELIABILITY TEST PLAN (n^*, k^*, t^*) AND MINIMAL COST C^* FOR WEIBULL $W(s, \lambda)$ DISTRIBUTIONS WHEN $\alpha_0 = 0.05$, $\alpha_1 = 0.10$, $v = 10$ AND $C[n, k, t] = 5n + t$.

			$s=1$				$s=2$				$s=3$			
R_0	R_1	d	n^*	k^*	t^*	C^*	n^*	k^*	t^*	C^*	n^*	k^*	t^*	C^*
0.98	0.88	0.000	12	3	50.446	110.45	6	3	35.433	65.433	4	3	30.564	50.564
		0.002	12	3	50.776	110.78	6	3	35.487	65.487	4	3	30.556	50.556
		0.004	12	3	51.104	111.10	6	3	35.541	65.541	4	3	30.547	50.547
		0.006	12	3	51.430	111.43	6	3	35.595	65.595	4	3	30.539	50.539
	0.90	0.000	13	3	55.757	120.76	6	3	39.029	69.029	6	3	24.789	54.789
		0.002	13	3	56.154	121.15	6	3	39.089	69.089	6	3	24.814	54.814
		0.004	13	3	56.546	121.55	6	3	39.148	69.148	6	3	24.839	54.839
		0.006	15	4	60.489	135.49	7	3	35.020	70.020	7	3	23.061	58.061
0.99	0.88	0.000	8	1	40.779	80.779	4	1	29.837	49.837	3	1	23.365	38.365
		0.002	8	1	41.021	81.021	4	1	29.879	49.879	3	1	23.380	38.380
		0.004	8	1	41.262	81.262	4	1	29.921	49.921	3	1	23.395	38.395
		0.006	8	1	41.502	81.502	4	1	29.963	49.963	3	1	23.410	38.410
	0.90	0.000	11	2	50.912	105.91	5	2	36.450	61.450	4	2	26.443	46.443
		0.002	11	2	51.260	106.26	5	2	36.504	61.504	4	2	26.461	46.461
		0.004	11	2	51.605	106.61	5	2	36.557	61.557	4	2	26.479	46.479
		0.006	11	2	51.948	106.95	5	2	36.609	61.609	4	2	26.496	46.496

VI. ILLUSTRATIVE EXAMPLES

Several practical applications are discussed in this section in order to illustrate the conclusions outlined above.

A. *Checking of microelectronic chips*

Suppose that a reliability engineer wishes to decide the acceptability of a submitted lot of microelectronic chips using an RDT plan based on failure count data. The selected chips are tested at specified high levels of temperature and voltage in order to accelerate the occurrence of wear-out failures. Weibull, gamma and lognormal distributions are adopted to describe the random behavior of the time-to-failure variable in hours, T.

The analyst deems that the minimum lifetime of a conforming chip under the extreme conditions assumed is $v = 5$ hours. Moreover, the minimal acceptable and maximal rejectable reliability levels at v are $R_0 = 0.98$ and $R_1 = 0.88$, respectively. That is, a chip is acceptable if its reliability at v, denoted as R, is at least $R_0 = 0.98$, and it is rejectable if R is not greater than $R_1 = 0.88$. In addition, the corresponding maximum tolerable producer and consumer risks are $\alpha_0 = 0.05$ and $\alpha_1 = 0.10$.

A random sample of n chips is chosen from the submitted lot. These chips are then simultaneously place on life test for a certain time t. Assuming the above quality and risk requirements, the reliability engineer wants to judge the acceptability of the lot using the number of observed failures by time t, denoted as $X_{n,t}$.

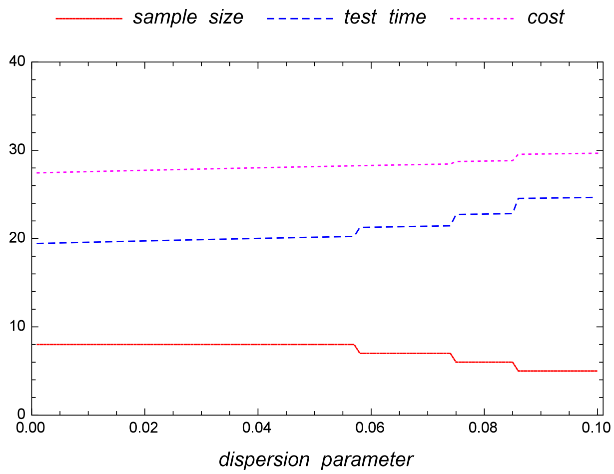

Fig. 6. Optimal sample size, test time and cost versus the dispersion parameter d when the lifetime distribution is Weibull $W(3, \lambda)$, $R_0 = 0.99$, $R_1 = 0.90$, $\alpha_0 = 0.05$, $\alpha_1 = 0.10$, $v = 10$ and $C[n, k, t] = n + t$.

The engineer considers that the dispersion parameter is $d = 0$ and also that the lifetime variable T follows a Weibull distribution with shape parameter $s = 2$. Thus, as the failure rate is proportional to the test time when $s = 2$, the engineer intrinsically assumes that the wear-out of the chips is linear. According to Fernández [25] the choice of the Weibull shape parameter s is usually based on previous data, expert opinions, and available knowledge of the underlying failure mechanism. In general, the RDT plans are quite robust to slight changes in s.

In view of Table IV, the best t-duration Weibull $W(s, \lambda)$ RDT plan with $t = 4$ hours (i.e., $c = 0.8$) is $S_t^* = (84, 3)$. Hence, the analyst has to test 84 chips randomly selected from the submitted lot during $t = 4$ hours under the assumed extreme conditions, and the entire lot is accepted iff there are at most three failures. The corresponding optimal inspection schemes in the gamma and lognormal cases with identical coefficients of variation are $S_t^* = (78, 2)$ and $S_t^* = (102, 2)$. The best RDT plans when $d = 0.006$ would be $(135, 5)$, $(138, 4)$ and $(190, 4)$ when T has Weibull, gamma and lognormal distributions. Clearly, the impact of the dispersion on the best fixed-duration schemes is substantial.

According to Table IV, the optimal Weibull, gamma and lognormal t-duration RDT plans when the test time is $t = 6$ (i.e., $c = 1.2$) and the dispersion parameter is $d = 0$ are $(38, 3)$, $(35, 3)$ and $(30, 3)$, respectively. The corresponding best inspection schemes when $d = 0.006$ would be $(41, 3)$, $(44, 4)$ and $(38, 4)$. The influence of the dispersion parameter d on the optimal RDT plans is now less drastic.

Suppose now that a reliability engineer wants to also determine the optimal test duration assuming that the time-to-failure of a microelectronic chip follows a Weibull $W(2, \lambda)$ distribution. When the cost function is defined as $C[n, k, t] = n + 10t$, it can be deduced from Table V that the optimal-duration RDT plan is $S^* = (34, 3, 12.690/2)$ if $d = 0$ and $S^* = (33, 3, 13.226/2)$ if $d = 0.006$. The minimum costs are $C^* = 97.448$ and $C^* = 99.130$, respectively. In the latter case, the submitted lot is accepted iff no more than three failed chips occur in a random sample of 33 chips, which are tested independently during 13.226/2 hours. Note that in Tables V and VI, as the conforming lifetime v is 10/2, the cost parameter c_2 must be multiplied by 2, and one has to divide the test times by 2 to find the optimal durations. In light of Table VI, the corresponding best schemes would be $S^* = (6, 3, 35.433/2)$ and $S^* = (6, 3, 35.595/2)$ when $C[n, k, t] = 5n + 2t$. Thus, the minimum costs would be $C^* = 65.433$ and $C^* = 65.595$, respectively.

The robustness of the optimal-duration RDT plans against slight modifications in the dispersion parameter d is therefore evident in the above situations.

B. *Testing of semiconductor lasers*

Assume that a large lot of a certain type of semiconductor lasers has been submitted for sampling inspection. Semiconductor lasers often fail due to degradation processes, such as diffusion, migration or corrosion. Testing high-reliability semiconductors under normal conditions is usually too costly. Accelerated aging is practically necessary to reduce both sample size and test time, in addition to the experimental cost.

Suppose that T denotes the time-to-failure in hours of a semiconductor device under the specified extreme stress degree. Moreover, $v = 10$ hours is the minimal lifetime of a conforming laser, and the corresponding acceptable and rejectable reliability levels at v are $R_0 = 0.99$ and $R_1 = 0.90$. In addition, the maximum risks allowed by the producer and the consumer are 5% and 10%, respectively; i.e., $\alpha_0 = 0.05$ and $\alpha_1 = 0.10$. Furthermore, a quality manager considers that the lifetime variable T follows a Weibull $W(s, \lambda)$ distribution. In this situation, the goal of the manager is to find the optimal RDT plan based on failure count data in order to determine whether the submitted lot of semiconductors is admissible.

Assuming that the Weibull shape parameter s is 2, it is seen in Table IV that the optimal (minimum sample size) RDT plan with fixed test time $t = 8$ hours (i.e., $c = 0.8$) is $S_t^* = (80, 2)$ when $d = 0$, whereas $S_t^* = (119, 2)$ if $d = 0.006$. The impact of the dispersion parameter on the required sample size is quite high in this case. The reliability test is successful if no more than two failures occur, but the manager has to select 119 lasers when $d = 0.006$ and only 80 lasers when $d = 0$. If the fixed test duration was $t = 12$ hours (i.e., $c = 1.2$), the best schemes would be $S_t^* = (37, 2)$ and $S_t^* = (39, 2)$ when the values of d are 0 and 0.006, respectively. The effect of d on the optimal plan S_t^* is now much lower.

The cost function defined by $C[n, k, t] = n + 5t$ is adopted by the quality manager. This implies that a test duration of one hour is five times as important as a single laser; i.e., a reliability test of 12 minutes is equivalent to a single semiconductor laser. The quality manager then wishes to determine the optimal test duration, as well as required sample size the maximum number of failures allowed to pass the test, under the above cost function. According to Table V, the optimal (minimum cost) RDT plans when $d = 0$ are $S^* = (51, 2, 10.106)$, $S^* = (33, 2, 12.569)$ and $S^* = (23, 2, 13.200)$ if the Weibull shape parameters are 1, 2 and 3, respectively. The corresponding best inspection schemes when $d = 0.006$ are $S^* = (51, 2, 11.027)$, $S^* = (33, 2, 13.137)$ and $S^* = (23, 2, 13.383)$.

Clearly, the optimal-duration RDT plans S^* are quite insensitive to small changes in the dispersion degree. For example, if the shape parameter is $s = 2$, then 33 randomly selected lasers from the lot must simultaneously put on test for 12.569 and 13.137 hours when $d = 0$ and $d = 0.006$, respectively, and the submitted lot is accepted iff the number of failures is at most two.

The quality manager now deems that a tested laser is as costly as a five-hour experiment; i.e., the cost function is defined by $C[n, c, t] = n + 5t$. In view of Table VI, the best designs with $d = 0$ would be $S^* = (11, 2, 50.912)$, $S^* = (5, 2, 36.450)$ and $S^* = (4, 2, 26.443)$ when the values of s were 1, 2 and 3, respectively. The corresponding optimal-duration plans would be $S^* = (11, 2, 51.948)$ $S^* = (5, 2, 36.609)$ and $S^* = (4, 2, 26.496)$ if $d = 0.006$. The robustness of the optimal-duration plans to slight variations of d is again evident.

VII. Concluding remarks

Quality assurance managers often have to demonstrate that the manufactured devices have achieved the desired reliability standards. The number of failed devices does not adjust faithfully to a binomial distribution in many reliability tests because the failure probability p cannot be assumed constant.

In this paper, the conditional distribution of the number of observed failures in the experiment given p is binomial, and the random behavior of the failure probability is modeled by a beta distribution. Moreover, decision criteria to judge the acceptability of submitted lots and production processes are based on uniformly most powerful beta-binomial reliability tests. In addition, producer and consumer risks are limited in advance, i.e. the probability of accepting/rejecting bad/good batches is sufficiently low.

Assuming Weibull, gamma and lognormal lifetime distributions, the best RDT plans with fixed and optimal test times have been found by solving pure and mixed integer nonlinear programming problems. The fixed-duration inspection scheme minimizes the sample size, whereas the optimal-duration plan minimizes a cost function selected by the decision maker according to the objective and subjective information available. In essence, the cost function to be minimized is a linear combination of sample size and test time.

The influence of the overdispersion on the best plan with fixed test time t, $S_t^* = (n^*, k^*)$, is usually noticeable. The optimal t-duration RDT plan S_t^* is often quite sensitive to small variations in the dispersion parameter d. Both the optimal sample size, n^*, and acceptance number, k^*, tend to increase as d grows. In most situations, k^* is fairly stable but n^* increases quickly, especially when the prefixed test time t is relatively small. In contrast, the effect of the overdispersion on the inspection scheme with optimal duration, $S^* = (n^*, k^*, t^*)$, is much smaller.

Clearly, optimal-duration test plans are superior to fixed-duration schemes in terms of costs. Moreover, the RDT plans with optimal test times are more robust against overdispersion than the RDT plans with fixed durations. Generally, the optimal-duration test plans are rather insensitive to slight changes in the dispersion level. This novel approach allows the practitioners to reduce the impact of overdispersion on the optimal test time and numbers of test devices and failures tolerated. Consequently, the use of optimal-duration RDT plans is highly recommended in practical applications where overdispersion is possible.

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References

[1] R. R. L. Kantam, G. Srinivasa Rao, and B. Sriram, "An economic reliability test plan: Log-logistic distribution," *Journal of Applied Statistics,* vol. 33, pp. 291–296, 2006.

[2] S. Balamurali and C.-H. Jun, "Multiple dependent state sampling plans for lot acceptance based on measurement data," *European Journal of Operational Research,* vol. 180, pp. 1221–1230, 2007.

[3] H. Guo and H. Liao, "Methods of reliability demonstration testing and their relationships," *IEEE Transactions on Reliability,* vol. 61, pp. 231–237, 2012.

[4] C.-C. Hsieh and Y.-T. Lu, "Risk-embedded Bayesian acceptance sampling plans via conditional value-at-risk with Type II censoring," *Computers & Industrial Engineering,* vol. 65, pp. 551–560, 2013.

[5] D.-C. Li and L.-S. Lin, "A new approach to assess product lifetime performance for small data sets," *European Journal of Operational Research,* vol. 230, pp. 290–298, 2013.

[6] G. Yang, "Heuristic degradation test plans for reliability demonstration," *IEEE Transactions on Reliability,* vol. 62, pp. 305–311, 2013.

[7] A. J. Fernández, "Optimal defects-per-unit acceptance sampling plans using truncated prior distributions," *IEEE Transactions on Reliability,* vol. 63, pp. 634–645, 2014.

[8] S.-J. Wu and S.-R. Huang, "Planning progressive Type-I interval censoring life tests with competing risks," *IEEE Transactions on Reliability,* vol. 63, pp. 511–522, 2014.

[9] H.-W. Wang, K.-N. Teng, and Y. Zhou, "Design an optimal accelerated-stress reliability acceptance test plan based on acceleration factor," *IEEE Transactions on Reliability,* vol. 67, pp. 1008–1018, 2018.

[10] M. Aslam, C.-H. Yen, C.-H. Chang, A. H. Al-Marshadi, and C.-H. Jun, "A multiple dependent state repetitive sampling plan based on performance index for lifetime data with Type II censoring," *IEEE Access,* vol. 7, pp. 49377–49391, 2019.

[11] H. Lewitschnig and A. Fanzott, "Fractional failures samplingAn alternative to zero failures sampling plans," IEEE Transactions on Reliability, vol. 68, pp. 609–619, 2019.

[12] S. Chen, L. Lu, Q. Zhang, and M. Li, "Optimal binomial reliability demonstration tests design under acceptance decision uncertainty," *Quality Engineering,* vol. 32, pp. 492–508, 2020.

[13] C.-W. Wu, M. H. Shu, and Y. N. Chang, "Variable-sampling plans based on lifetime-performance index under exponential distribution with censoring and its extensions," *Applied Mathematical Modelling,* vol. 55, pp. 81–93, 2018.

[14] S.-J. Wu, C.-C. Hsu, and S.-R. Huang, "Optimal designs and reliability sampling plans for one-shot devices with cost considerations," *Reliability Engineering & System Safety,* vol. 197, 106795, 2020.

[15] H. Guo, T. Jin, and A. Mettas, "Designing reliability demonstration tests for one-shot systems under zero component failures," *IEEE Transactions on Reliability,* vol. 60, pp. 286-294, 2011.

[16] F. Yalcin and S. Eryilmaz, "Start-up demonstration test based on total successes and total failures with dependent start-ups," *IEEE Transactions on Reliability,* vol. 61, pp. 227-230, 2012.

[17] R. Qin, E. A. Cudney, and Z. Hamzic, "An optimal plan of zero-defect single-sampling by attributes for incoming inspections in assembly lines," *European Journal of Operational Research,* vol. 246, pp. 907–915, 2015.

[18] L. Lu, M. Li, and C. M. Anderson-Cook, "Multiple objective optimization in reliability demonstration tests," *Journal of Quality Technology,* vol. 48, pp. 326–342, 2016.

[19] M. Li, W. Zhang, Q. Hu, H. Guo, and J. Liu, "Design and risk evaluation of reliability demonstration test for hierarchical systems with multilevel information aggregation," *IEEE Transactions on Reliability,* vol. 66, pp. 135–147, 2017.

[20] B. Bouslah, A. Gharbi, and R. Pellerin, "Joint production, quality and maintenance control of a two-machine line subject to operation-dependent and quality-dependent failures," *International Journal of Production Economics,* vol. 195, pp. 210–226, 2018.

[21] Q. Ge, H. Peng, G. J. van Houtum, and I. Adan, "Reliability optimization for series systems under uncertain component failure rates in the design phase," *International Journal of Production Economics,* vol. 196, pp. 163–175, 2018.

[22] C. J. Pérez-González, A. J. Fernández, A. Kohansal, and A. Asgharzadeh, "Optimal truncated repetitive lot inspection with defect rates," *Applied Mathematical Modelling,* vol. 75, pp. 223–235, 2019.

[23] C. J. Pérez-González, A. J. Fernández, and A. Kohansal, "Eficient truncated repetitive lot inspection using Poisson defect counts and prior information," *European Journal of Operational Research,* vol. 287, pp. 964–974, 2020.

[24] Q. Q. Zhao and W. Y. Yun, "A sampling plan for one-shot systems considering destructive inspection," *Communications in Statistics - Theory and Methods,* vol. 49, pp. 3744–3760, 2020.

[25] A. J. Fernández, "Optimal durations of Weibull reliability tests based on failure counts," *Computers & Industrial Engineering,* vol. 156, 107247, 2021.

[26] C.-H. Jun, S. Balamurali, and S.-H. Lee, "Variables sampling plans for Weibull distributed lifetimes under sudden death testing," *IEEE Transactions on Reliability,* vol. 55, pp. 53–58, 2006.

[27] J.-W. Chen, K.-H. Li, and Y. Lam, "Bayesian single and double variable sampling plans for the Weibull distribution with censoring," *European Journal of Operational Research,* vol. 177, pp. 1062–1073, 2007.

[28] I. Arizono, Y. Kawamura, and Y. Takemoto, "Reliability tests for Weibull distribution with variational shape parameter based on sudden death lifetime data," *European Journal of Operational Research,* vol. 189, pp. 570–574, 2008.

[29] T.-R. Tsai, Y.-T. Lu, and S.-J. Wu, "Reliability sampling plans for Weibull distribution with limited capacity of test facility," *Computers Industrial & Engineering,* vol. 55, pp. 721–728, 2008.

[30] J. H. Seo, M. Jung, and C. M. Kim, "Design of accelerated life test sampling plans with a nonconstant shape parameter," *European Journal of Operational Research,* vol. 197, pp. 659–666, 2009.

[31] M. Aslam, C.-H. Jun, A. J. Fernández, M. Ahmad, and M. Rasool, "Repetitive group sampling plan based on truncated tests for Weibull models," *Research Journal of Applied Sciences, Engineering and Technology,* vol. 7, pp. 1917–1924, 2014.

[32] A. J. Fernández, "Optimum attributes component test plans for k-out-of-n:F Weibull systems using prior information," *European Journal of Operational Research,* vol. 688–696, 2015.

[33] A. J. Fernández, "Planning reliability demonstration tests with limited expected risks," *Computers & Industrial Engineering,* vol. 165, 107918, 2022.

[34] S. Roy, "Bayesian accelerated life test plans for series systems with Weibull component lifetimes," *Applied Mathematical Modelling,* vol. 62, pp. 383–403, 2018.

[35] L. Gao, Z.-S. Ye, W. Chen, P. Qian, and J. Pan, "Accelerated Life Test Planning for Minimizing Misclassification Risks," *IEEE Transactions on Reliability,* vol. 70, pp. 459–471, 2021.

[36] M. Aslam, "Design of sampling plan for exponential distribution under neutrosophic statistical interval method," *IEEE Access,* vol. 6, pp. 64153-64158, 2018.

[37] O. Dey and D. Chakraborty, "A fuzzy random periodic review system with variable lead-time and negative exponential crashing cost," *Applied Mathematical Modelling,* vol. 36, pp. 6312–6322, 2012.

[38] A. J. Fernández, "Weibull inference using trimmed samples and prior information," *Statistical Papers,* vol. 50, pp. 119–136, 2009.

[39] A. J. Fernández, "Smallest Pareto confidence regions and applications," *Computational Statistics & Data Analysis,* vol. 62, pp. 11–25, 2013.

[40] A. J. Fernández, "Computing optimal confidence sets for Pareto models under progressive censoring," *Journal of Computational and Applied Mathematics,* vol. 258, pp. 168-180, 2014.

[41] W.-C. Lee, J.-W. Wu, and C.-W. Hong, "Assessing the lifetime performance index of products with the exponential distribution under progressively type II right censored samples," *Journal of Computational and Applied Mathematics,* vol. 231, pp. 648–656, 2009.

[42] W.-C. Lee, J.-W. Wu, and C. L. Lei, "Evaluating the lifetime performance index for the exponential lifetime products," *Applied Mathematical Modelling,* vol. 34, pp. 1217–1224, 2010.

[43] A. A. Soliman and F. M. Al-Aboud, "Bayesian inference using record values from Rayleigh model with application," *European Journal of Operational Research,* vol. 185, 659–672, 2008.

[44] W.-C. Lee, J.-W. Wu, M.-L. Hong, L.-S. Lin, and R.-L. Chan, "Assessing the lifetime performance index of Rayleigh products based on the Bayesian estimation under progressive type II right censored samples," *Journal of Computational and Applied Mathematics,* vol. 235, 1676–1688, 2011.

[45] S. Dey and T. Dey, "Statistical inference for the Rayleigh distribution under progressively type-II censoring with binomial removal," *Applied Mathematical Modeling,* vol. 38, 974–982, 2014.

[46] S. MirMostafaee, M. N. Qomi, and A. J. Fernández, "Tolerance limits for minimal repair times of a series system with Rayleigh distributed component lifetimes," *Applied Mathematical Modelling,* vol. 40, pp. 3153–3163, 2016.

[47] A. Asgharzadeh, A. J. Fernández, and M. Abdi, "Confidence sets for the two-parameter Rayleigh distribution under progressive censoring," *Applied Mathematical Modelling,* vol. 47, 656–667, 2017

[48] W. Lu and T.-R. Tsai, "Interval censored sampling plans for the gamma lifetime model," *European Journal of Operational Research,* vol. 192, pp. 116–124, 2009.

[49] S.-T. Tseng, N. Balakrishnan, and C.-C. Tsai, "Optimal step-stress accelerated degradation test plan for gamma degradation processes," *IEEE Transactions on Reliability,* vol. 58, pp. 611–618, 2009.

[50] A. J. Fernández, "Optimal reliability demonstration test plans for k-out-of-n systems of gamma distributed components," *IEEE Transactions on Reliability,* vol. 60, pp. 833–844, 2011.

[51] A. J. Fernández, "Gamma reliability test times with minimal costs and limited risks," *IEEE Transactions on Reliability,* 2021, doi: 10.1109/TR.2021.3125544.

[52] A. J. Fernández, C. J. Pérez-González, and A. Kohansal, "Improved time-censored reliability test plans for k-out-of-n gamma systems," *Journal of Computational and Applied Mathematics,* vol. 361, pp. 42-54, 2019.

[53] S. S. Gupta, "Life test sampling plan for normal and lognormal distributions," *Technometrics,* vol. 4, pp. 151–175, 1962.

[54] H. Schneider, "Failure-censored variables-sampling plans for lognormal and Weibull distributions," *Technometrics,* vol. 31, pp. 199–206, 1989.

[55] A. A. Alhadeed and S. S Yang, "Optimum simple step-stress plan for cumulative exposure model using log-normal distribution," *IEEE Transactions on Reliability,* vol. 50, pp. 64–68, 2005.

[56] Q. Wu and J. Lu, "Sampling inspection of reliability in (log) normal case with type I censoring," *Acta Mathematica Scientia,* vol. 26, pp. 331–343, 2006.

[57] I. B. Naqvi and M. Aslam, "Time truncated testing strategy using multiple testers: lognormal distributed lifetime," *Research Journal of Applied Sciences, Engineering and Technology,* vol. 7, pp. 4745–4748, 2014.

[58] A. J. Fernández, "Optimum lot inspection based on lognormal reliability tests," *International Journal of Production Research,* 2022, doi: 10.1080/00207543.2022.2036852.

[59] C. J. Pérez-González, A. J. Fernández, V. Giner-Bosch, and A. Carrión-García, "Optimal repetitive reliability inspection of manufactured lots for lifetime models using prior information," *International Journal of Production Research,* 2022, doi: 10.1080/00207543.2022.2068163.