Fundamentals of transient heat transfer

Surnames, name	Castelló Gómez, Marisa (mcasgo@upv.es) Fito Suñer, Pedro (pedfisu@tal.upv.es) Tarrazó Morell, José (jtarrazo@tal.upv.es)
Department	Department of Food Technology
Centre	Universitat Politècnica de València

1 Abstract

In this article we are going to present the basic characteristics that must be taken into account when dealing with situations in which there are **temperature changes** in a system **as a function of the treatment time** to which it is subjected. To do so, we will make use of Fourier's second law, we will see what are the basic geometries, also what are the four dimensionless modules that are handled in these cases and, finally, we will work with Newman's rule when there are several transfer directions.

2 Introduction

Transient heat transfer is a very frequent unit operation, both at domestic and industrial levels. We all know how long it takes to fry French fries to reach the optimal conditions for consumption or that it does not take the same time to defrost food at 4ºC as it does at room temperature. At the industrial level, establishing precise process times for heat treatments helps to optimize energy resources and to standardize the quality of products, which can be foodstuffs, pharmaceuticals, fermentation broths, etc. Figure 1 shows some common examples where heat transfer takes place.

Therefore, in this article we are going to deal with **situations in which the temperature of a body is modified** until it reaches a certain value **at a specific point** of the body **for a specific time**. Therefore, we will not only take into account the time variable but also the position, since the temperature reached at the surface is not the same as the temperature at the centre of the product subjected to a heating or cooling stage for the same length of time.

Figure 1. Example of some of the applications of transient heat transfer

3 Objectives

Once you read this document carefully, you will be able to:

-**Predict the temperature** reached **at a** given **point** of a product when subjected to a heating or cooling process **for a given time**.
-**Establish the "system dynamics**" in continuous processes in order to find out the "response" to an external modification of the conditions that separates them from the steady state.
- Recommend the appropriate product **size** as a function of the heat treatment applied
- Estimate the **thermal diffusivity** (α) of a product

4 Development

In these cases, we will resort to the **differential equation of the conservation of energy**, which gives us a relationship between temperature, time and position in space. This relationship will depend on the physical properties of the solid and we must bear in mind the following considerations:

- There is no internal generation in the solid
- There is no phase change
- Density and specific heat are constant
- Thermal conductivity of the solid does not depend on T

This equation will be as follows:

$$\rho \cdot c_p \cdot \frac{\partial T}{\partial t} = k \cdot \nabla^2 \cdot T$$

Considering a control volume, in a transient state, the temperature value depends not only on the position, but also on the time. Therefore, it is necessary to consider the term of ACCUMULATION and it is necessary to work with partial derivative equations (Figure 2).

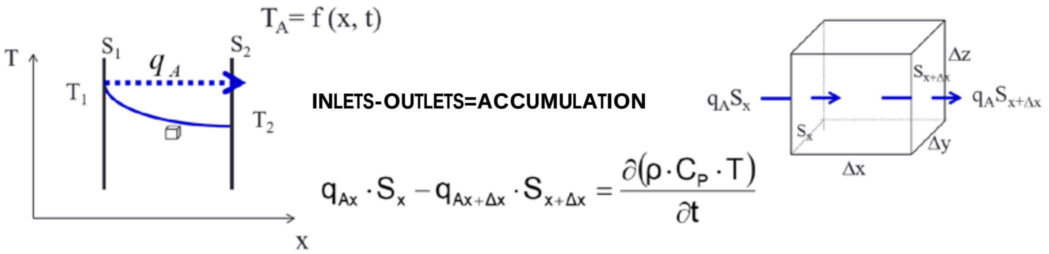


Figure 2. Example of the application of energy balances to a control volume

This gives rise to **Fourier's 2nd Law**, which describes in differential form the relationship between temperature and time at a given point in the system and is expressed as follows:

$$\frac{\partial T}{\partial t} = -\alpha \left[\frac{\partial^2 T}{\partial^2 x} + \frac{\partial^2 T}{\partial^2 y} + \frac{\partial^2 T}{\partial^2 z}\right]$$

This law has a simple solution in situations where there is only one direction of transfer. This implies that one has to work with **basic geometries**, which in this context are the infinite sheet (li), the infinite cylinder (ci) and the sphere (e) (Figure 3).

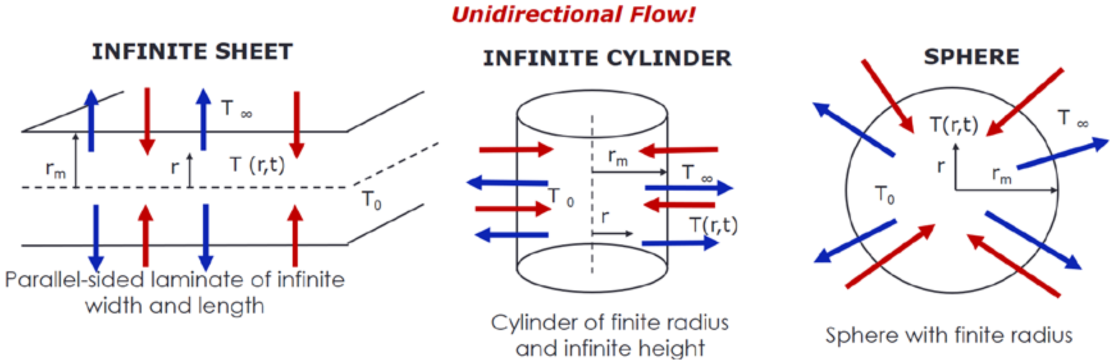


Figure 3. Basic geometries for transient heat transfer

In Figure 3, the blue arrows refer to the cases in which the product is cooled and the red arrows refer to the cases in which it is heated. In addition, the nomenclature included refers to the following:

- $\mathbf{r_m}$: characteristic dimension. Distance from the point at which it is most difficult to reach the conditions of the medium to the point at which it is least difficult.
- **r**: distance from the point where it is most difficult to reach the conditions of the medium to the specific point where we want to find out the temperature or where the temperature data is given to us
- $\mathbf{T_\infty}$: temperature of the medium surrounding the product being heated or cooled and that must be kept constant throughout the process time
- $\mathbf{T_0}$: initial body temperature, which should be uniform
- **T(r,t)**: temperature at a specific point of the product after a certain time has elapsed

4.1 Dimensionless Modules

To solve Fourier's 2nd law in each of the basic geometries (BG), we must work with four dimensionless parameters which are the driving force (Y), the reduced time or Fourier number (X), the relative position (n) and the ratio between the external resistance to heat transfer and the internal resistance (m), whose formulas are as follows:

$$Y = \frac{T_\infty - T_{(r,t)}}{T_\infty - T_0} \quad [1,0] \qquad X = \frac{\alpha}{r_m^2} t \qquad n = \frac{r}{r_m} \quad [0,1] \qquad m = \frac{1}{\mathrm{Bi}} = \frac{k}{h r_m}$$

Where:

- α is the thermal diffusivity (m^2/s), obtained by: $\alpha = k/(\rho \cdot c_p)$, where **k** is the thermal conductivity (W/mK), ρ is the density (kg/m^3) and $\mathbf{c_p}$ is the specific heat (J/kgºC) of the product under study.
- **t**: is the time (s)
- **Bi**: is the Biot number
- **h** is the individual convective heat transfer coefficient (W/m^2K)

Knowing 3 of these 4 dimensionless modules, we will be able to deduce the fourth and, therefore, to find out the unknown parameter of the problem.

4.2 Analytical, graphical or Excel file method of resolution

In order to solve the cases in which the transient heat transfer is estimated to occur in a single direction, different methodologies can be used that are based on the development of Fourier's 2nd law for each case.

Let's start with the analytical resolution. For the three BG, solving Fourier's 2nd law when the dimensionless parameter **m=0** gives rise to the following equations which are convergent series:

$$Y_{li} = \frac{4}{\pi}\left[\begin{array}{c} exp\left(-\frac{\pi^2}{4}X\right) cos\left(\frac{\pi}{2}n\right) - \frac{1}{3}\ exp\left(-\frac{9\pi^2}{4}X\right) cos\left(\frac{3\pi}{2}n\right) \\ +\frac{1}{5}\ exp\left(-\frac{25\pi^2}{4}X\right) cos\left(\frac{5\pi}{2}n\right) + \frac{1}{7}\ exp\left(-\frac{49\pi^2}{4}X\right) cos\left(\frac{7\pi}{2}n\right)\ldots \end{array}\right]$$

$$\begin{aligned} Y_{ci} = {} & 1.602\ \exp(\text{-}5.784X)\ J_0(2.405 \cdot n) - 1.066\ \exp(\text{-}30.47X)\ J_0(5.52 \cdot n) \\ & + 0.853\ \exp(\text{-}74.892X)\ J_0(8.654 \cdot n) - (0.731)\exp(139.051X) J_o(11.792n) + \ldots \end{aligned}$$

$$Y_e = 2\left[\begin{array}{c} (exp(-\pi^2X)\frac{sen\pi n}{\pi n} - exp(-4\pi^2X)\frac{sen2\pi n}{2\pi n} + \\ + exp(-16\pi^2X)\frac{sen4\pi n}{4\pi n}) - exp(-16\pi^2X)\frac{sen(4\pi n)}{4\pi n} + \ldots \end{array}\right]$$

If, in addition, we assume that <u>the induction period has been exceeded</u> (generally for values of X>0.5), <u>only the first term</u> of the above equations <u>will be necessary</u> to obtain the value of the driving force, since the rest will not change their value. In this case:

$$Y_{li} = \frac{4}{\pi}\ exp\left(-\frac{\pi^2}{4}X\right) cos\left(\frac{\pi}{2}n\right)$$

$$Y_{ci} = 1.602\ \exp(\text{-}5.784 \cdot X)\ J_0(2.405 \cdot n) \qquad sabiendo\ que\ J_0(0) = 1$$

$$Y_e = 2\ \ exp(-\pi^2X)\frac{sen\pi n}{\pi n} \qquad sabiendo\ que\ \lim_{n\to 0}\left(\frac{sen\pi n}{\pi n}\right) = 1$$

For the ***graphical resolution***, we will make use of abacuses that represent several solutions of Fourier's 2nd law, but only when the induction period has been exceeded, so they will not be suitable when the driving force is very close to 1, that is, for moments close to the beginning of the heating or cooling periods. An example of the abacus for each GB is shown in Figure 4. As can be seen, in the upper left part of the 3 abacuses there is a region where the straight-line bundles of the different values of m and n do not reach because in that area the Y is close to 1. It should also be borne in mind that, in these cases, there is a logarithmic scale on the ordinate axis and, therefore, this must be taken into account when handling the abacuses.

Shall we put ourselves to the test? *Knowing that a product has a spherical geometry, whose value of X is 2.5, m=2 and n=0, how much is Y worth?*

Solution: Y=0.04

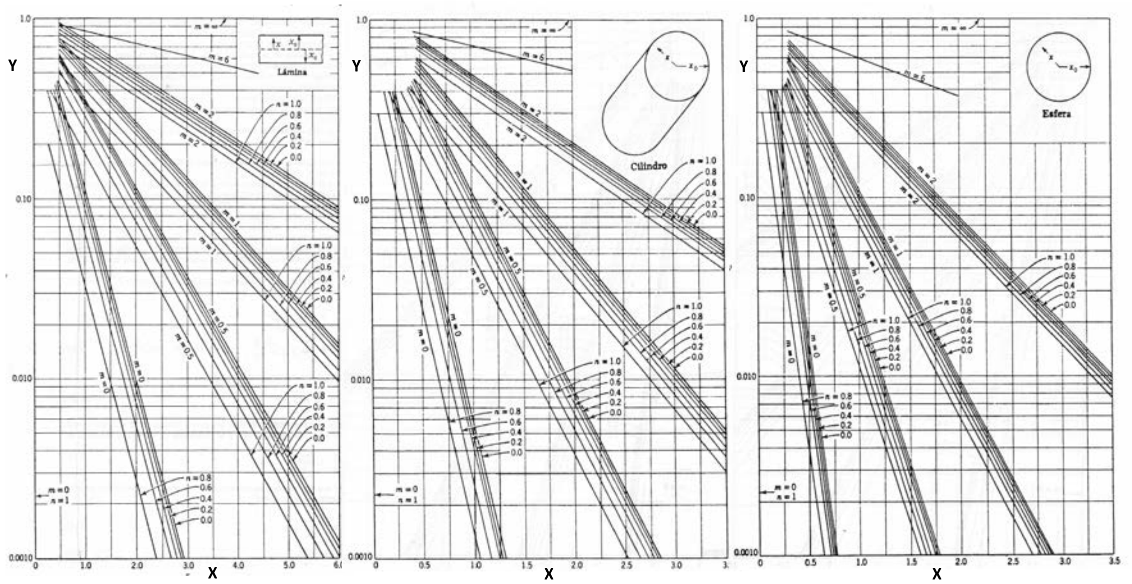


Figure 4. Example of abacuses for graphical resolution in basic geometries (adapted from Costa et al., 1991).

As you can imagine, it is difficult for these two procedures to be able to cover all the cases, forcing us to go to arduous and difficult integration programs. An alternative and versatile solution is to resort to a tool programmed in **Excel macros**, by the professors who wrote the article, which calculates 2000 terms of the solved series for each BG, giving a dimensionless parameter from the other three. Below is a screenshot of the interface of this file to give you an idea of its versatility (Figure 5).

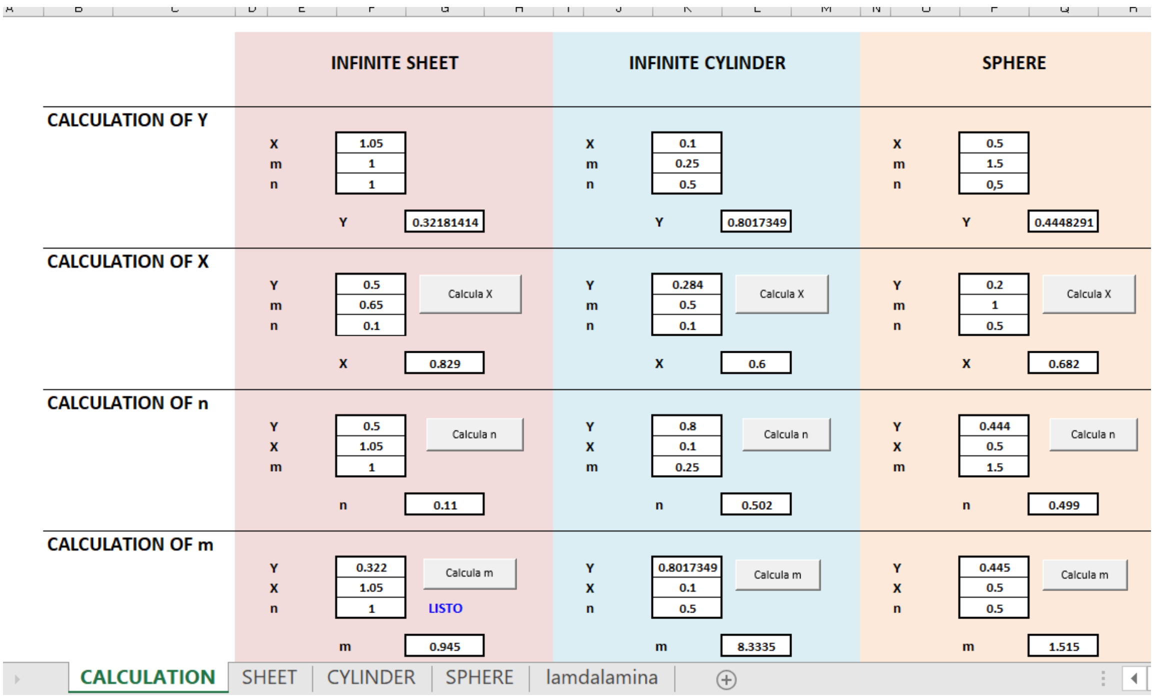


Figure 5. Screenshot of the Excel file in which all possible solutions of Fourier's 2nd Law have been incorporated

4.3. Geometries with several transfer directions. Newman's rule

As you can imagine, in real life, products have several directions in which they are heated or cooled, so we have to resort to **Newman's rule** which states that *the driving force of a body with several directions of transfer is given by the product of the driving forces corresponding to the basic figures that have generated it*.

In general, we usually find parallelepipeds with finite dimensions, or cylinders in which both diameter and height are finite. In these cases, the driving force will be obtained as follows:

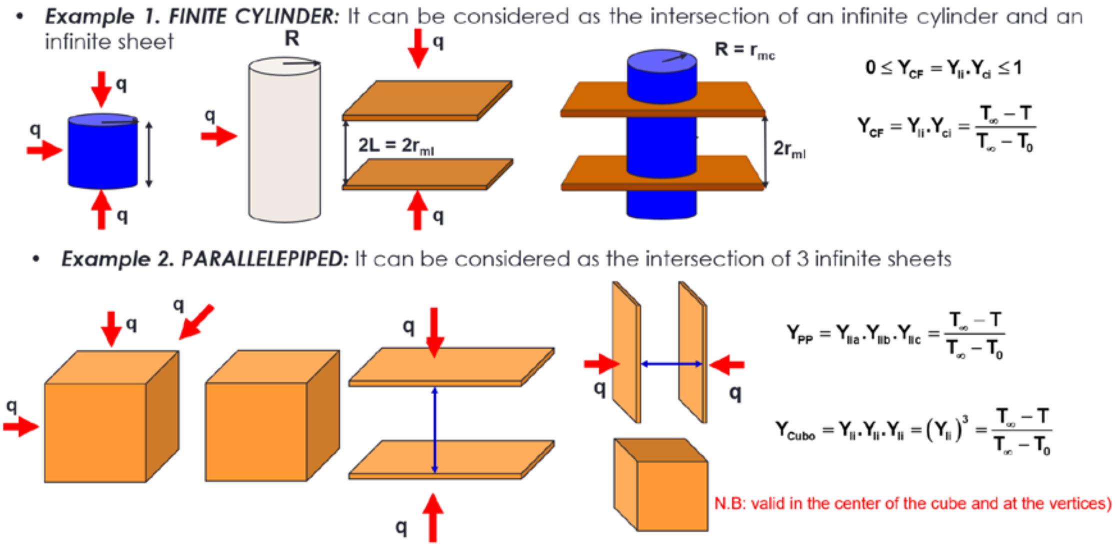
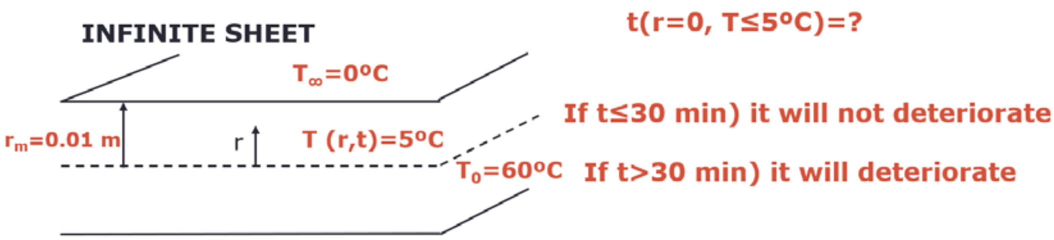


Figure 6. Examples of the obtaining of the driving force of a finite cylinder and a parallelepiped by applying Newman's rule.

4.4 Examples

The best way to know if we are clear about the methodology is to try it out, so let's solve a couple of examples involving transient heat transfer.

Case 1*. Biological material in the form of flat sheets, 2 cm thick, freshly processed and at 60 °C, is to be cooled rapidly to avoid deterioration. The material has a density of 1.07 gcm^{-3}, a specific heat of 3 kJkg^{-1}K^{-1} and a thermal conductivity of 0.5 Wm^{-1}K^{-1}. It has been estimated that the solid cannot be above 5 ºC for more than 30 minutes. The sheets are placed in a refrigerator at 0 ºC and with an individual heat transfer coefficient of 50 Wm^{-2}K^{-1}. Predict whether the material will deteriorate.*



Using graphical resolution

$$Y = \frac{(T_\infty - T_t)}{(T_\infty - T_o)} = \frac{0-5}{0-60} = 0{,}083$$

$$n = \frac{r}{r_m} = \frac{0}{0{,}01} = 0$$

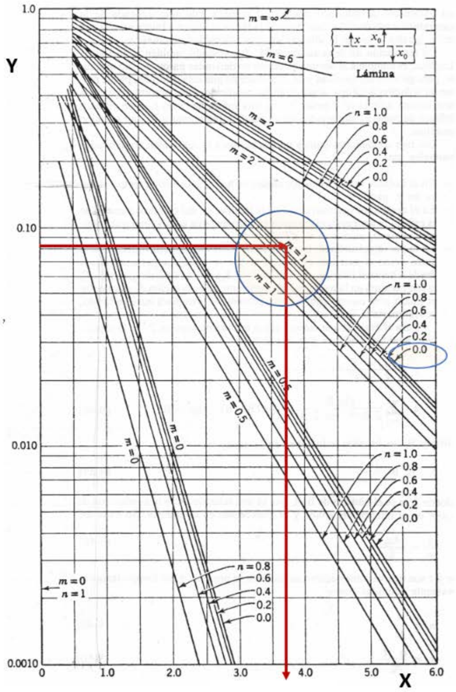
$$m = \frac{k}{h \cdot r_m} = \frac{0{,}50}{50.0{,}01} = 1$$

$$X = \frac{k}{\rho \cdot c_p} \frac{t}{r_m^2} = \frac{0{,}5}{3000 \cdot 1070} \cdot \frac{t}{(0{,}01)^2} = 1{,}56.10^{-3}.t$$

$$3{,}76 = 1{,}56.10^{-3}.t$$

$$t = 21414 \text{ s} = 40 \text{ min}$$

The biological material will deteriorate

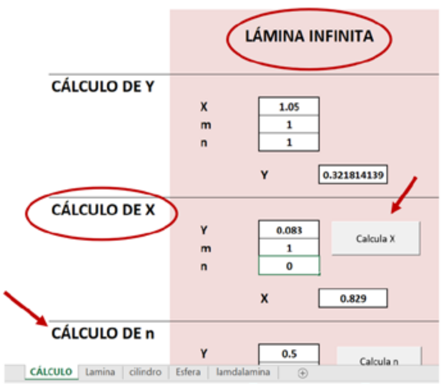


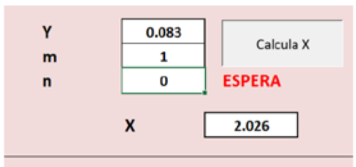
X=3.76

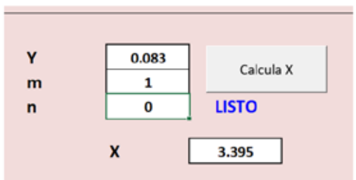
Using the Excel Fourier file

2) Position yourself on the part of the spreadsheet of the corresponding basic geometry

1) Knowing that **Y=0.083**, **m=1** and **n=0**, we shall introduce these numerical values into the corresponding cells of the Excel, where the calculation of the unknown variables is indicated:







$$t = \frac{X \cdot r_m^2}{\alpha} = 2180 \text{ s}$$

T=36 min>30 min

The biological material will deteriorate

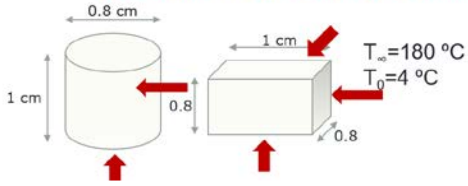
This first case was simple because the product had only one transfer direction. Let's look at another case with several transfer directions.

Case 2. *We want to know the frying time of two pieces of potato (initially refrigerated at 4 ºC) cut into parallelepiped and cylinder shapes. The potato pieces are immersed in sunflower oil at a temperature of 180 ºC (h= 20 W/m²K). Knowing that the gelatinization of potato starches occurs at 85 ºC and the Maillard reactions of coloring at 165 ºC:*

a. How long does it take for each piece of potato to be fully gelatinized?
b. How long will it take for the surface to be colored?

Data: K: 0.4 W/mK, ρ: 960 kg/m³, c_p: 3.9 kJ/kgK

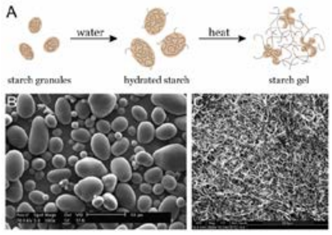
Transient heat transfer with several transfer directions. Fourier's 2nd law. Newman's rule. Finite cylinder: $Y_{CF}=Y_{CI}\cdot Y_{LI}$, Parallelepiped $Y_{PP}=Y_{LIa}\cdot Y_{LIb}\cdot Y_{LIc}$



a) t(r=0, T=85 °C)=?

b) t(r=r$_m$, T=165 °C)=?

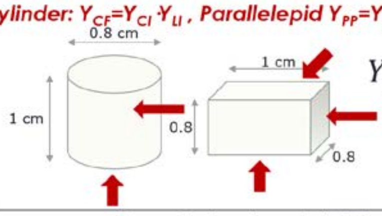
In these cases, it is advisable to build a table to enter the values of each GB:



a) t(r=0, T=85 °C)=?

a) t(r=0, T=85 °C)=?

Finite cylinder: $Y_{CF}=Y_{CI}\cdot Y_{LI}$, Parallelepid $Y_{PP}=Y_{LIa}\cdot Y_{LIb}\cdot Y_{LIc}$



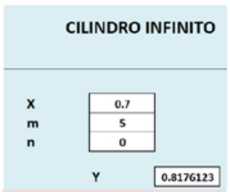
$$Y_{potato}=\frac{180-85}{180-4}=0,54$$

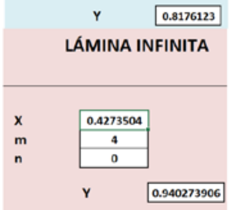
	Finite cylinder (CF)				Parallelepiped (PP)			
	CI	LI	CF		LI_a	LI_b	LI_c	PP
r_m			X	r_m				X
r				r				
n				n				
m				m				
x				x				
Y			0.540	Y				0.540

How can we obtain the driving force values to apply Newman's rule? By iterating. To do this, we fill in the iteration table considering different times. In the range of driving force values that, in this case, considers the value of 0.540, we iterate as shown below*:*

Itaration table

$t_{supuesto}$ (s)	X_{CI}	X_{LI}	Y_{CI}	Y_{LI}	Y_{CF}	$X_{LIa}=X_{LIb}$	X_{LIc}	$Y_{LIa}=Y_{LIb}$	Y_{LIc}	Y_{PP}
100	0.668	0.427	0.818	0.940	0.769	0.668	0.427	0.913	0.940	0.783
400	2.671	1.709	0.389	0.700	0.273	2.671	1.709	0.635	0.700	0.282
300	2.003	1.282	0.499	0.773	0.385	2.003	1.282	0.716	0.773	0.397
200	1.335	0.855	0.639	0.853	0.544	1.335	0.855	0.809	0.853	0.558





Linear interpolation

Potato with finite cylinder geometry

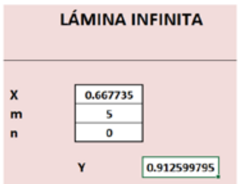
(300-200)=100 s ⇨ 0.544-0.385 β = 97 s

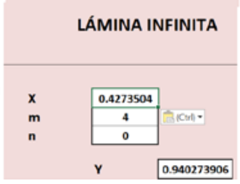
β ⇨ 0.54-0.385 $t_{real}=300-\beta$ = 203 s=3.4 min

Potato with parallelepiped geometry

(300-200)=100 s ⇨ 0.558-0.397 β' = 97 s

β' ⇨ 0.54-0.397 $t'_{real}=300-\beta$ = 211 s=3.5 min



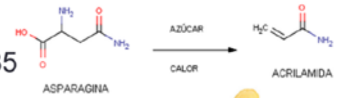


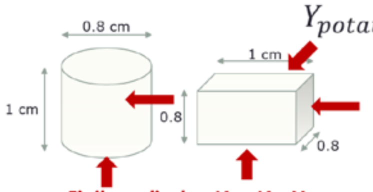
For the next section, we shall proceed in the same way, but now taking into account the new driving force value.

b) t(r=0. T=165 ºC)=?

b) t($r=r_m$. T=165 ºC)=?

$$Y_{potato}=\frac{180-165}{180-4}=0.085$$





Finite cylinder: $Y_{CF}=Y_{CI}\cdot Y_{LI}$ Parallelepiped $Y_{PP}=Y_{LIa}\cdot Y_{LIb}\cdot Y_{LIc}$

	Finite cylinder (CF)				Parallelepiped (PP)			
	CI	LI	CF		LI_a	LI_b	LI_c	PP
r_m	0.004	0.005	X	r_m	0.004	0.004	0.005	X
r	0.004	0.005		r	0.004	0.004	0.005	
n	1	1		n	1	1	1	
m	5	4		m	5	5	4	
x	?	?		x	?	?	?	
Y	?	?	0.085	Y	?	?	?	0.085

$Y_{cf}=Y_{pp}=$ 0.085

Iteration table

$t^{supuesto}$ (s)	X_{CI}	X_{LI}	Y_{CI}	Y_{LI}	Y_{CF}	$X_{LIa}=X_{LIb}$	X_{LIc}	$Y_{LIa}=Y_{LIb}$	Y_{LIc}	Y_{PP}
100	0.668	0.427	0.744	0.835	0.621	0.668	0.427	0.831	0.835	0.577
400	2.671	1.709	0.354	0.621	0.220	2.671	1.709	0.578	0.621	0.208
600	4.006	2.564	0.216	0.510	0.110	4.006	2.564	0.454	0.510	0.105
800	5.342	3.419	0.132	0.419	0.055	5.342	3.419	0.356	0.419	0.053

CILINDRO INFINITO

x	0.7
m	5
n	1
Y	0.743658

LÁMINA INFINITA

x	0.4273504
m	4
n	1
Y	0.835286266

Linear interpolation

Potato with cylinder geometry

(800-600)=200 s ⇨ 0.110-0.055

β' ⇨ 0.110-0.085

β' = 91 s

$t_{real}=600+\beta'$ = 691 s=11.5 min

Potato with parallelepiped geometry

(800-600)=200 s ⇨ 0.105-0.053

β' ⇨ 0.105-0.085

β' = 77 s

$t'_{real}=600+\beta'$ = 677 s=11.3 min

LÁMINA INFINITA

x	0.667735
m	5
n	1
Y	0.831192455

LÁMINA INFINITA

x	0.4273504
m	4
n	1
Y	0.835286266

5 Closing Statement

Throughout this learning object we have seen how to work with the mathematical tools that allow us to solve situations involving transient heat transfer. Best of luck with their application!

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