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NORMAL FUNCTIONALS ON LIPSCHITZ SPACES ARE WEAK* CONTINUOUS

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Abstract Let $\mathrm{Lip}_0(M)$ be the space of Lipschitz functions on a complete metric space M that vanish at a base point. We prove that every normal functional in $\mathrm{Lip}_0(M)^*$ is weak* continuous; that is, in order to verify weak* continuity it suffices to do so for bounded monotone nets of Lipschitz functions. This solves a problem posed by N. Weaver. As an auxiliary result, we show that the series decomposition developed by N. J. Kalton for functionals in the predual of $\mathrm{Lip}_0(M)$ can be partially extended to $\mathrm{Lip}_0(M)^*$.

Keywords and phrases: Lipschitz-free space, Lipschitz function, Lipschitz space, normal functional

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1. Introduction

Let (M,d) be a complete metric space with a selected base point, which we shall denote by 0. Then the space $\mathrm{Lip}_0(M)$ of all real-valued Lipschitz functions on M that vanish at 0 is a Banach space when endowed with the norm given by the Lipschitz constant

$$\|f\|_L = \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)} : x \neq y \in M\right\}$$

(the requirement that $f(0)=0$ gets rid of the constant functions; otherwise, $\|\cdot\|_L$ is merely a seminorm). Moreover, $\mathrm{Lip}_0(M)$ is a dual Banach space. Its canonical predual $\mathcal{F}(M)$, usually called *Lipschitz-free space* or *Arens-Eells space* over M, can be realised as the closed subspace $\mathcal{F}(M) = \overline{\mathrm{span}}\,\{\delta(x) : x \in M\}$ of $\mathrm{Lip}_0(M)^*$, where $\delta(x) \in \mathrm{Lip}_0(M)^*$ denotes the evaluation functional on $x \in M$. Note that δ is an isometric embedding of M into $\mathrm{Lip}_0(M)^*$, so $\mathcal{F}(M)$ contains a linearly dense and linearly independent isometric copy of M.

The Lipschitz spaces $\mathrm{Lip}_0(M)$ are in many ways the metric counterparts of the classical $C(K)$ spaces of real-valued continuous functions on Hausdorff compacts, so their study is interesting in its own right; for a detailed analysis of their properties, see the reference monograph [13] by Weaver. However, they currently attract a lot of attention due to their applications to the nonlinear geometry of Banach spaces. These usually involve the following extension property satisfied by Lipschitz-free spaces: *any Lipschitz mapping from M into a Banach space X can be extended to a linear operator from $\mathcal{F}(M)$ into X whose norm is the Lipschitz constant of the original mapping* (here, each $x \in M$ is identified with its associated evaluation functional $\delta(x) \in \mathcal{F}(M)$). In [6], Godefroy and Kalton famously used this to prove that the bounded approximation property of Banach spaces is stable under Lipschitz isomorphisms. Since then, numerous other applications to nonlinear functional analysis have been found; see, for example, the recent survey [5] by Godefroy.

The weak* topology induced by $\mathcal{F}(M)$ on $\mathrm{Lip}_0(M)$ coincides with the topology of pointwise convergence on norm-bounded subsets of $\mathrm{Lip}_0(M)$. Therefore, by a straightforward application of the Banach-Dieudonné theorem, a functional $\phi \in \mathrm{Lip}_0(M)^*$ is weak* continuous (i.e., it belongs to $\mathcal{F}(M)$) precisely when it satisfies the following condition: given any norm-bounded net (f_i) in $\mathrm{Lip}_0(M)$ that converges pointwise to $f \in \mathrm{Lip}_0(M)$, one has that $\langle f_i, \phi\rangle$ converges to $\langle f, \phi\rangle$.

In [12], Weaver considered the following weaker notion, by analogy with the corresponding notion for von Neumann algebras.

Definition 1. A functional $\phi \in \mathrm{Lip}_0(M)^*$ is *normal* when it satisfies the following: given any norm-bounded net (f_i) in $\mathrm{Lip}_0(M)$ that converges pointwise *and monotonically* to $f \in \mathrm{Lip}_0(M)$, one has that $\langle f_i, \phi\rangle$ converges to $\langle f, \phi\rangle$.

Equivalently, ϕ is normal if $\langle f_i, \phi\rangle \to 0$ for any net (f_i) of nonnegative functions in $B_{\mathrm{Lip}_0(M)}$ that decreases pointwise to 0.

By a well-known theorem, states on a von Neumann algebra are normal if and only if they belong to its predual (see, e.g., [10, Theorem 1.13.2]). In particular, because normality only depends on the order structure of the von Neumann algebra, this implies that von Neumann algebras have unique preduals [10, Corollary 1.13.3]. In our setting, any weak* continuous element of $\mathrm{Lip}_0(M)^*$ is obviously normal. Weaver asked in [12, Open problem on p. 37] whether the converse is also true. He first gave an affirmative answer for the very specific case of evaluation functionals on elements of the Stone-Čech compactification of M [12, Proposition 2.1.6] and for weak* limits of nets of elementary molecules [13, Theorem 3.43]. Later, he extended the result to all positive functionals [14, Theorem 2.3]; that is, those $\phi \in \mathrm{Lip}_0(M)^*$ such that $\langle f, \phi\rangle \geq 0$ for any nonnegative $f \in \mathrm{Lip}_0(M)$. This allowed him to show, similar to von Neumann algebras, that the Lipschitz-free space $\mathcal{F}(M)$ is in fact the unique predual of $\mathrm{Lip}_0(M)$ when M is bounded or geodesic [14]. It is currently an open problem whether this holds for all metric spaces M.

In this short note, we settle the question about normality in the general case.

Theorem 2. *Let M be a complete pointed metric space and $\phi \in \mathrm{Lip}_0(M)^*$. Then ϕ is normal if and only if it is weak* continuous.*

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Let us note that Theorem 2, besides being an analogue of the corresponding von Neumann algebra result, can also be considered as an abstract version of the Radon-Nikodým theorem for Lipschitz-free spaces; compare, for example, to [11, Theorem 8.7]. The classical Radon-Nikodým theorem implies that L_1 is 1-complemented in its bidual L_1^{**} (see, e.g., [1, Proposition 6.3.10]). Therefore, because $\mathcal{F}(\mathbb{R})$ is isometric to $L_1(\mathbb{R})$, we obtain that $\mathcal{F}(\mathbb{R})$ is complemented in its bidual $\mathrm{Lip}_0(\mathbb{R})^*$. In a deep paper [4], Cúth, Kalenda and Kaplický extended this result and proved that the Lipschitz-free space over any finite-dimensional Banach space is complemented in its bidual. This is, however, not true in general in the infinite-dimensional case; for instance, $\mathcal{F}(c_0)$ is not complemented in its bidual because it contains a complemented copy of c_0 by the lifting property [6, Theorem 3.1]. It remains an important open problem to decide for which metric spaces M the Lipschitz-free space $\mathcal{F}(M)$ is complemented in its bidual. Of particular interest is the case when $M = \ell_1$, because the complementability would imply that ℓ_1 is determined by its Lipschitz structure (see, e.g., [7, Problem 16]). Based on the similarity to the Radon-Nikodým theorem, one might try to investigate whether Theorem 2 could be helpful in addressing this problem.

In order to give the proof of Theorem 2 in Section 3, we first establish some auxiliary results concerning series decomposition of functionals on Lipschitz spaces in Section 2.

Let us now briefly introduce the notation used in this note. B_X will stand for the closed unit ball of a Banach space X. The closed ball with radius r around $x \in M$ will be denoted $B(x,r)$. We will use the notation

$$d(x,A) = \inf\{d(x,a) : a \in A\}$$
$$\mathrm{rad}(A) = \sup\{d(0,a) : a \in A\}$$

for $x \in M$ and $A \subset M$. $\mathrm{Lip}_0(M)^+$ will be the set of all nonnegative functions in $\mathrm{Lip}_0(M)$. The pointwise maximum and minimum of real-valued functions f and g will be written as $f \vee g$ and $f \wedge g$, respectively. We will also denote $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$. Note that $f = f^+ - f^-$. By the support of f we mean the set

$$\mathrm{supp}(f) = \{x \in M : f(x) \neq 0\},$$

and we will put $\|f\|_\infty = \sup\{|f(x)| : x \in M\}$, which can be infinite.

Let us recall that for any two Lipschitz functions f,g on M we have

$$\|fg\|_L \leq \|f\|_L \|g\|_\infty + \|g\|_L \|f\|_\infty .$$

It follows that for any Lipschitz function h on M with bounded support, the mapping

$$T_h\colon f \mapsto f \cdot h$$

is a linear operator on $\mathrm{Lip}_0(M)$ whose norm is bounded by

$$\|T_h\| \leq \|h\|_\infty + \mathrm{rad}(\mathrm{supp}(h))\,\|h\|_L . \tag{1}$$

Moreover T_h is weak*-weak*-continuous; that is, its adjoint $T_h{}^* : \phi \to \phi \circ T_h$ takes $\mathcal{F}(M)$ into $\mathcal{F}(M)$. See [2, Lemma 2.3] for the proof of these facts. We will be using these operators with weighting functions h such that $h = 1$ on some region of interest $A \subset M$ and $h = 0$ on

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some region $B \subset M$ that is to be ignored and takes intermediate values in some transition region. In particular, we will consider the functions Λ_n for $n \in \mathbb{Z}$ defined by

$$\Lambda_n(x) = \begin{cases} 0 & \text{if } d(x,0) \le 2^{n-1} \\ 2^{-(n-1)}d(x,0) - 1 & \text{if } 2^{n-1} \le d(x,0) \le 2^n \\ 2 - 2^{-n}d(x,0) & \text{if } 2^n \le d(x,0) \le 2^{n+1} \\ 0 & \text{if } 2^{n+1} \le d(x,0) \end{cases}$$

and Π_n for $n \in \mathbb{N}$, defined by

$$\Pi_n(x) = \begin{cases} 0 & \text{if } d(x,0) \le 2^{-(n+1)} \\ 2^{n+1}d(x,0) - 1 & \text{if } 2^{-(n+1)} \le d(x,0) \le 2^{-n} \\ 1 & \text{if } 2^{-n} \le d(x,0) \le 2^n \\ 2 - 2^{-n}d(x,0) & \text{if } 2^n \le d(x,0) \le 2^{n+1} \\ 0 & \text{if } 2^{n+1} \le d(x,0) \end{cases}$$

for $x \in M$. Notice that

$$\Pi_n = \sum_{k=-n}^{n} \Lambda_k \tag{2}$$

for any $n \in \mathbb{N}$. Moreover, $\|\Lambda_k\|_\infty, \|\Pi_n\|_\infty \le 1$, and we have

$$\begin{aligned} \operatorname{rad}(\operatorname{supp}(\Lambda_k)) \le 2^{k+1}, \quad \|\Lambda_k\|_L \le 2^{-(k-1)} \\ \operatorname{rad}(\operatorname{supp}(\Pi_n)) \le 2^{n+1}, \quad \|\Pi_n\|_L \le 2^{n+1} \end{aligned} \tag{3}$$

for every $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. In particular, (1) yields $\|T_{\Lambda_k}\| \le 5$.

2. Series decomposition in $\mathrm{Lip}_0(M)^*$

In Section 4 of [9], Kalton established that elements of $\mathcal{F}(M)$ admit a decomposition as a series with terms whose action is limited to annuli around the base point. Let us prove that this decomposition is also valid for normal functionals in $\mathrm{Lip}_0(M)^*$. We will use a slightly different version of the decomposition, based on the functions Λ_n instead of the original ones because they make computations easier.

Lemma 3. *For any* $\phi \in \mathrm{Lip}_0(M)^*$ *we have*

$$\sum_{n\in\mathbb{Z}} \|\phi \circ T_{\Lambda_n}\| \le 45\,\|\phi\|. \tag{4}$$

Hence,

$$\sum_{k=-n}^{n} \phi \circ T_{\Lambda_k} = \phi \circ T_{\Pi_n}$$

converges in norm as $n \to \infty$ *to a functional in* $\mathrm{Lip}_0(M)^*$.

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Proof. Fix $\varepsilon > 0$ and a finite set $F \subset \mathbb{Z}$. For $i = 0,1,2$, let F_i be the set of those $n \in F$ that are congruent with i modulo 3. We will show that

$$\sum_{n\in F_i} \|\phi \circ T_{\Lambda_n}\| < 15\,\|\phi\| + \varepsilon,$$

and this will be enough to prove (4). The second part of the statement is then obvious in view of (2).

Fix i, and for $n \in F_i$ choose $f_n \in B_{\mathrm{Lip}_0(M)}$ such that

$$\|\phi \circ T_{\Lambda_n}\| - \frac{\varepsilon}{|F_i|} < \langle f_n, \phi \circ T_{\Lambda_n}\rangle = \langle f_n \Lambda_n, \phi\rangle .$$

Notice that $\|f_n \Lambda_n\|_L \leq \|T_{\Lambda_n}\| \leq 5$ by (1) and (3). Now define $g = \sum_{n\in F_i} f_n \Lambda_n$ and let us estimate $\|g\|_L$. Fix $x \in \mathrm{supp}(g)$, then $x \in \mathrm{supp}(\Lambda_n)$ for some $n \in F_i$. If $y \in \mathrm{supp}(\Lambda_m)$ for $m \in F_i \setminus \{n\}$, assume $m > n$ without loss of generality, then $d(x,y) \geq d(x,0)$ and

$$\begin{aligned}|g(x) - g(y)| &\leq |f_n(x)\Lambda_n(x)| + |f_m(y)\Lambda_m(y)|\\ &\leq 5(d(x,0) + d(y,0))\\ &\leq 5(2d(x,0) + d(x,y)) \leq 15 d(x,y).\end{aligned}$$

Otherwise,

$$|g(x) - g(y)| = |f_n(x)\Lambda_n(x) - f_n(y)\Lambda_n(y)| \leq 5d(x,y).$$

So we get $\|g\|_L \leq 15$. Therefore,

$$\sum_{n\in F_i} \|\phi \circ T_{\Lambda_n}\| < \sum_{n\in F_i} \langle f_n\Lambda_n, \phi\rangle + \varepsilon = \langle g, \phi\rangle + \varepsilon \leq 15\,\|\phi\| + \varepsilon$$

as was claimed. □

Lemma 4. *If $\phi \in \mathrm{Lip}_0(M)^*$ is normal, then*

$$\phi = \sum_{n\in\mathbb{Z}} \phi \circ T_{\Lambda_n} = \lim_{n\to\infty} \phi \circ T_{\Pi_n} \tag{5}$$

with respect to the norm convergence in $\mathrm{Lip}_0(M)^*$.

Proof. It will suffice to show that $(\phi \circ T_{\Pi_n})$ converges weak* to ϕ, because Lemma 3 implies that the sequence converges in norm. That is, we need to show that $\langle f, \phi \circ T_{\Pi_n}\rangle \to \langle f, \phi\rangle$ for any $f \in \mathrm{Lip}_0(M)$; we may assume that $f \geq 0$, and the general case then follows by expressing $f = f^+ - f^-$.

So fix $f \in \mathrm{Lip}_0(M)^+$. For $n \in \mathbb{Z}$ define the function h_n by

$$h_n(x) = \begin{cases} 1 & \text{if } d(x,0) \leq 2^n \\ 2 - 2^{-n} d(x,0) & \text{if } 2^n \leq d(x,0) \leq 2^{n+1} \\ 0 & \text{if } 2^{n+1} \leq d(x,0) \end{cases}$$

for $x \in M$, which satisfies $\|T_{h_n}\| \leq 3$ by (1). Now notice that $\Pi_n = h_n(1 - h_{-(n+1)})$; hence, $T_{\Pi_n} = T_{h_n} \circ (I - T_{h_{-(n+1)}})$, where I is the identity operator on $\mathrm{Lip}_0(M)$, and

$$\|T_{\Pi_n}\| \leq \|T_{h_n}\| \left\|I - T_{h_{-(n+1)}}\right\| \leq 12$$

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for any $n \in \mathbb{N}$. Then $\|T_{\Pi_n}(f)\|_L \leq 12\|f\|_L$, and $T_{\Pi_n}(f)(x)$ converges pointwise and monotonically (increasing) to $f(x)$ for every $x \in M$. By the normality of ϕ we have

$$\lim_{n\to\infty} \langle f, \phi \circ T_{\Pi_n} \rangle = \lim_{n\to\infty} \langle T_{\Pi_n}(f), \phi \rangle = \langle f, \phi \rangle .$$

This ends the proof. □

Moreover, each term in the decomposition series and in the limit in (5) is also normal:

Lemma 5. *Let h be a nonnegative Lipschitz function on M with bounded support. If $\phi \in \mathrm{Lip}_0(M)^*$ is normal, then $\phi \circ T_h$ is normal.*

Proof. Let (f_i) be a bounded net in $\mathrm{Lip}_0(M)$ that decreases to 0 pointwise. Then $\|f_i h\|_L \leq \|T_h\| \|f_i\|_L$ is bounded by (1), so $(f_i h)$ is also a bounded net that decreases to 0 pointwise. Because ϕ is normal, we have

$$\lim_i \langle f_i, \phi \circ T_h \rangle = \lim_i \langle f_i h, \phi \rangle = 0.$$

It follows that $\phi \circ T_h$ is normal, too. □

3. Proof of Theorem 2

In addition to the above decomposition result, another essential ingredient for our proof is the following simple but powerful lemma from [3], which is itself based on a weaker version found in [8]. We include a short proof for the sake of completeness. In fact, the same argument yields a stronger statement than the one in [3]. Recall that a series $\sum_n x_n$ in a Banach space X is *weakly unconditionally Cauchy* if $\sum_n |\langle x_n, x^* \rangle| < \infty$ for every $x^* \in X^*$.

Lemma 6 ([3, Lemma 1.5])**.** *Let (f_n) be a bounded sequence in $\mathrm{Lip}_0(M)$. Suppose that the supports of the functions f_n are pairwise disjoint. Then $\sum_n f_n$ is a weakly unconditionally Cauchy series. In particular, (f_n) is weakly null.*

Proof. Let (f_n) be a sequence in $B_{\mathrm{Lip}_0(M)}$ with disjoint supports and let $(t_n) \in \ell_\infty$. Then

$$\sum_{n=1}^k t_n f_n = \sum_{n=1}^k (t_n f_n)^+ - \sum_{n=1}^k (t_n f_n)^- = \bigvee_{n=1}^k (t_n f_n)^+ - \bigvee_{n=1}^k (t_n f_n)^- .$$

Hence, $\left\| \sum_{n=1}^k t_n f_n \right\|_L \leq 2\|(t_n)\|_\infty$ for every $k \in \mathbb{N}$ and $\sum_{n=1}^\infty f_n$ is weakly unconditionally Cauchy by [15, Proposition II.D.4]. □

We can now finally prove our main result. The sufficiency part of Theorem 2 is obvious. To prove the necessity, let $\phi \in \mathrm{Lip}_0(M)^*$ be a normal functional. Lemma 4 says that $\phi = \lim_{n\to\infty} \phi \circ T_{\Pi_n}$ with respect to the norm convergence, so it suffices to show that $\phi \circ T_{\Pi_n}$, for any $n \in \mathbb{N}$, is weak* continuous. Moreover, by Lemma 5, such $\phi \circ T_{\Pi_n}$ for any $n \in \mathbb{N}$ is also normal. Therefore, for the rest of the proof we will assume that $\phi \in \mathrm{Lip}_0(M)^*$ is a normal functional with norm 1 and that there exist real numbers $0 < r < R$ such that $\langle f, \phi \rangle = 0$ whenever $f \in \mathrm{Lip}_0(M)$ equals 0 on the set

$$K = \{x \in M : r \leq d(x,0) \leq R\} .$$

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We will repeatedly make use of the function

$$e(x) = \left(1 - \frac{4}{r}d(x,K)\right) \vee 0 \quad \text{for all } x \in M,$$

the support of which is contained in

$$K' = \left\{x \in M : \frac{3}{4}r \le d(x,0) \le R + \frac{r}{4}\right\}$$

and which equals 1 on K, and the function

$$e'(x) = \left(1 - \frac{4}{r}d(x,K')\right) \vee 0 \quad \text{for all } x \in M,$$

the support of which is contained in

$$K'' = \left\{x \in M : \frac{r}{2} \le d(x,0) \le R + \frac{r}{2}\right\}$$

and which equals 1 on K'. (We think of them as the 'unit on K', which will be used to restrict functions, and the 'unit on K'', which will be used to translate functions, respectively.) Note that $e, e' \in \mathrm{Lip}_0(M)^+$ with $\|e\|_L, \|e'\|_L \le \frac{4}{r}$; in particular,

$$|\langle e', \phi\rangle| \le \frac{4}{r}. \tag{6}$$

For brevity, denote

$$\alpha = 2 + (R+1)\frac{4}{r}.$$

We will proceed by contradiction. Suppose that $\phi \notin \mathcal{F}(M)$. By the Hahn-Banach theorem, there exists $\psi \in B_{\mathrm{Lip}_0(M)^{**}}$ such that $\langle \phi, \psi\rangle = c > 0$ and that $\langle \mu, \psi\rangle = 0$ for every $\mu \in \mathcal{F}(M)$. Our argument relies on a construction presented in the following claim:

Claim 1. *With the notation as above, for a given nonempty finite set $A \subset K'$ and an $\varepsilon \in \left(0, \min\{1, \frac{rc}{48}\}\right)$, there exists a function $g : M \to \mathbb{R}$ satisfying the following:*

(i) $g \in \mathrm{Lip}_0(M)^+$ *with* $\|g\|_L \le \alpha$.

(ii) $g(x) \le 2\varepsilon$ *for every* $x \in A$.

(iii) $g(x) \ge \varepsilon$ *for every* $x \in K'$.

(iv) $g(x) = \varepsilon e'(x)$ *for every* $x \in M \setminus K'$*; in particular,* $\mathrm{supp}(g) \subset K''$ *and therefore* $\|g\|_\infty \le \alpha(R + \frac{r}{2})$.

(v) $|\langle g, \phi\rangle| \ge \frac{c}{4}$.

Proof. Consider the weak* neighbourhood U of ψ in $\mathrm{Lip}_0(M)^{**}$ given by

$$U = \left\{\varrho \in \mathrm{Lip}_0(M)^{**} : |\langle \phi, \varrho - \psi\rangle| < \frac{c}{3} \text{ and } |\langle \delta(x), \varrho\rangle| < \varepsilon \text{ for all } x \in A\right\}$$

(notice that $\langle \delta(x), \varrho - \psi\rangle = \langle \delta(x), \varrho\rangle$). Thanks to the weak* density of $B_{\mathrm{Lip}_0(M)}$ in $B_{\mathrm{Lip}_0(M)^{**}}$, we may find an $f \in B_{\mathrm{Lip}_0(M)} \cap U$, which means that $\langle f, \phi\rangle > \frac{2}{3}c$ and

$|f(x)| = |\langle \delta(x), f\rangle| < \varepsilon$ for every $x \in A$. By replacing f with f^+ or f^-, we obtain $f \in B_{\mathrm{Lip}_0(M)} \cap \mathrm{Lip}_0(M)^+$ such that

$$|\langle f, \phi\rangle| > \frac{c}{3} \tag{7}$$

and $f(x) < \varepsilon$ for every $x \in A$.

Now, put

$$g = T_e(f) + \varepsilon e'.$$

Then $g \in \mathrm{Lip}_0(M)^+$ and by (1) we have

$$\|g\|_L \leq 1 + \left(R + \frac{r}{4}\right)\frac{4}{r} + \varepsilon\frac{4}{r} \leq \alpha,$$

so g satisfies (i). Moreover, $\mathrm{supp}(T_e(f)) \subset \mathrm{supp}(e) \subset K'$, which establishes (iv). In particular, the bound on $\|g\|_\infty$ then follows from (i) and the definition of K''. Properties (ii) and (iii) are straightforward to verify. Finally, because the evaluation of ϕ only depends on the restriction of a function to the set K and because $g\restriction_K = (f + \varepsilon e')\restriction_K$, we get by (7) and (6) that

$$|\langle g, \phi\rangle| = |\langle f + \varepsilon e', \phi\rangle| \geq |\langle f, \phi\rangle| - \varepsilon|\langle e', \phi\rangle| > \frac{c}{3} - \frac{c}{12} = \frac{c}{4};$$

thus, (v) also holds. □

To proceed with the main proof, fix a decreasing sequence $(\varepsilon_n)_{n=1}^\infty \subset \left(0, \min\left\{1, \frac{cr}{48}, \frac{r}{2}\right\}\right)$ such that $\varepsilon_n \to 0$ and

$$(2 + \alpha)\varepsilon_{n+1} < \varepsilon_n \tag{8}$$

for every $n \in \mathbb{N}$.

Let $\mathfrak{F}$ be the family of all nonempty finite subsets of K', and for every $A \in \mathfrak{F}$ let $\mathfrak{F}_A = \{B \in \mathfrak{F} : A \subset B\}$. Note that the sets $\mathfrak{F}$ and $\mathfrak{F}_A$ are directed by inclusion. We will now construct a net $(g_A)_{A\in\mathfrak{F}}$ in $\mathrm{Lip}_0(M)$ that satisfies conditions (i)–(iv) above with $\varepsilon = \varepsilon_{|A|}$ (where $|A|$ denotes the cardinality of A), and also these two:

(vi) $|\langle g_A, \phi\rangle| \geq \frac{c}{8}$.

(vii) If $E \subset A$ then $g_A(x) \leq g_E(x)$ for every $x \in M$.

This will be enough to end the proof. Indeed, $(g_A)_{A\in\mathfrak{F}}$ decreases pointwise to 0 because $g_A(x) \leq 2\varepsilon_n$ whenever $|A| \geq n$ and either $x \in A$ or $x \in M \setminus K'$ by (ii) and (iv), respectively, but $|\langle g_A, \phi\rangle| \geq \frac{c}{8}$ for every $A \in \mathfrak{F}$, contradicting the normality of ϕ.

We proceed by induction on $n = |A|$. For $n = 1$ – that is, singletons $A = \{x\}$ with $x \in K'$ – let g_A be the function g given by Claim 1 for $\varepsilon = \varepsilon_1$. It clearly satisfies (i)–(vi) and also (vii) by vacuity. Now let $n > 1$, assume that the functions g_A have been constructed for all nonempty subsets $A \subset K'$ with fewer than n elements and fix $A \subset K'$ with $|A| = n$. To complete the induction, it suffices to prove that there exists g_A satisfying (i)–(iv) and (vi)–(vii) with $\varepsilon = \varepsilon_n$.

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To this end, denote $h = \bigwedge_{E \subsetneq A} g_E$, which satisfies conditions (i)–(iv) with A and $\varepsilon = \varepsilon_{n-1}$. Next, for any $B \in \mathfrak{F}_A$ let $\mathfrak{g}_B$ be the function given by Claim 1 for the set B and $\varepsilon = \varepsilon_n$. Notice that the function $\mathfrak{g}_B \wedge h$ satisfies conditions (i)–(iv) for $\varepsilon = \varepsilon_n$ and set A because it is bounded by $\mathfrak{g}_B$, and also condition (vii) because it is bounded by h. We will show that g_A can be found among the functions $\mathfrak{g}_B \wedge h$; that is, at least one of the functions $\mathfrak{g}_B \wedge h$ satisfies also condition (vi). The proof will proceed by contradiction, and we will need the following claim.

Claim 2. *With the notation as above, if $|\langle \mathfrak{g}_B \wedge h, \phi \rangle| < \frac{c}{8}$ for every $B \in \mathfrak{F}_A$, then there is a constant $\beta > 0$ with the following property: for any $B \in \mathfrak{F}_A$, there exist $E \in \mathfrak{F}_B$ and $f \in \mathrm{Lip}_0(M)^+$ such that*

(a) $\|f\|_L \leq \beta$.

(b) $\mathrm{supp}(f) \subset \left(\bigcup_{x \in E} B(x, \varepsilon_n) \right) \setminus \left(\bigcup_{x \in B} B(x, \varepsilon_n) \right)$.

(c) $|\langle f, \phi \rangle| \geq \frac{c}{16}$.

Proof. Fix $B \in \mathfrak{F}_A$ and define $f = T_e(\mathfrak{g}_B - (\mathfrak{g}_B \wedge h))$. Clearly, $f \geq 0$ and, moreover, $\|f\|_L \leq 2\alpha\left(2 + \frac{4}{r}R\right)$ by (1). Suppose that $x \in B(b, \varepsilon_n)$ for some $b \in B$. If $x \notin K'$, then $e(x) = 0$, and if $x \in K'$, then by (8) we have

$$\mathfrak{g}_B(x) \leq \mathfrak{g}_B(b) + |\mathfrak{g}_B(x) - \mathfrak{g}_B(b)| \leq 2\varepsilon_n + \alpha\varepsilon_n < \varepsilon_{n-1},$$

whereas $h(x) \geq \varepsilon_{n-1}$, so $\mathfrak{g}_B(x) \leq h(x)$. In any case $f(x) = 0$ for all $x \in \bigcup_{b \in B} B(b, \varepsilon_n)$. Moreover,

$$|\langle f, \phi \rangle| = |\langle \mathfrak{g}_B - (\mathfrak{g}_B \wedge h), \phi \rangle| \geq |\langle \mathfrak{g}_B, \phi \rangle| - |\langle \mathfrak{g}_B \wedge h, \phi \rangle| > \frac{c}{4} - \frac{c}{8} = \frac{c}{8}.$$

Similar to functions e and e' introduced above, for a given $E \in \mathfrak{F}_B$ define the function

$$e_E(x) = \left(1 - \frac{1}{\varepsilon_n} d(x, E)\right) \vee 0 \quad \text{for all } x \in M,$$

which clearly satisfies that $\mathrm{supp}(e_E) \subset \bigcup_{x \in E} B(x, \varepsilon_n)$. Then the net $(T_{e_E}(f))_{E \in \mathfrak{F}_B}$ is a norm-bounded increasing net in $\mathrm{Lip}_0(M)^+$ converging pointwise to f. Indeed, by (1) we have $\|T_{e_E}(f)\|_L \leq \beta$, where

$$\beta = \left(1 + \frac{1}{\varepsilon_n}\left(R + \frac{r}{4}\right)\right) \cdot 2\alpha\left(2 + \frac{4}{r}R\right)$$

does not depend on B or E, and the rest is immediate from the definition. Hence, the normality of ϕ implies that $\langle T_{e_E}(f), \phi \rangle$ converges to $\langle f, \phi \rangle$, and in particular there exists $E \in \mathfrak{F}_B$ such that

$$|\langle f, \phi \rangle - \langle T_{e_E}(f), \phi \rangle| < \frac{c}{16}.$$

The function $T_{e_E}(f)$ satisfies the requirements of the claim. Indeed, we have already verified (a), (b) follows from $\mathrm{supp}(T_{e_E}(f)) \subset \mathrm{supp}(f) \cap \mathrm{supp}(e_E)$ and we get (c) from

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$$|\langle T_{e_E}(f),\phi\rangle| \geq |\langle f,\phi\rangle| - |\langle f,\phi\rangle - \langle T_{e_E}(f),\phi\rangle| > \frac{c}{8} - \frac{c}{16} = \frac{c}{16}.$$

This ends the proof of Claim 2. □

To conclude our main argument, suppose that $|\langle \mathfrak{g}_B \wedge h,\phi\rangle| < \frac{c}{8}$ for every $B \in \mathfrak{F}_A$. We then construct sequences $(B_n) \subset \mathfrak{F}_A$ and $(f_n) \subset \mathrm{Lip}_0(M)^+$ as follows: Take $B_0 = A$, and for any $n \in \mathbb{N}$ let B_n and f_n be the set E and function f, respectively, given by Claim 2 for $B = B_{n-1}$. Then the sequence (f_n) is norm-bounded by (a) and has pairwise disjoint supports by (b). However, it is not weakly null due to (c), which is in contradiction with Lemma 6. This ends the proof of Theorem 2.

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