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Theoretical and computational analysis of a new formulation for the Rural Postman Problem and the General Routing Problem

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ABSTRACT

The Rural Postman Problem (RPP) is one of the most well-known problems in arc routing. Given an undirected graph, the RPP consists of finding a closed walk traversing and servicing a given subset of edges with minimum total cost. In the General Routing Problem (GRP), there is also a subset of vertices that must be visited. Both problems were introduced by Orloff and proved to be NP-hard. In this paper, we propose a new formulation for the RPP and the GRP using two sets of binary variables representing the first and second traversal, respectively, of each edge. We present several families of valid inequalities that induce facets of the polyhedron of solutions under mild conditions. Using this formulation and these families of inequalities, we propose a branch-and-cut algorithm, test it on a large set of benchmark instances, and compare its performance against the exact procedure that, as far as we know, produced the best results. The results obtained show that the proposed formulation is useful for solving undirected RPP and GRP instances of very large size.

1. Introduction

Arc routing problems (ARPs) refer to problems where one or more vehicles must fulfill the demand of customers, represented as edges or arcs on a graph, to optimize a given objective. These customers can be streets, highways, or contour lines defining shapes, among others, requiring services such as cleaning, snow removal, or contour cutting. The objective may involve minimizing the total distance traveled, balancing the duration of routes, or maximizing the benefit derived from serving customers. For more information on models, applications, and solution procedures for ARPs, readers can refer to the cited Dror (2000), Corberán and Prins (2009), Corberán and Laporte (2015), Mourão and Pinto (2017), and Corberán et al. (2021).

This paper addresses two important problems: the Rural Postman Problem (RPP) and the General Routing Problem (GRP). The RPP is an extension of the well-known Chinese Postman Problem (Guan, 1962), while the GRP further extends the complexity of the RPP. Orloff (1974) initially introduced both the RPP and the GRP, which were later proven to be NP-hard by Lenstra and Rinnooy Kan (1976). Notably, Orloff highlighted that the difficulty of the problem increases with the number of connected components in the graph formed by the required edges and vertices. We will refer to these components as R-connected components and their corresponding sets of vertices as R-sets.

The RPP and the GRP are defined as follows. Consider an undirected graph $G = (V, E)$ consisting of a set of vertices V and a set of edges E. Let E_R be a subset of E that represents the required edges, and let $E_{NR} = E \setminus E_R$ be the set of non-required edges. Each edge $e \in E$ has a deadheading cost $c_e \geq 0$, while each required edge $e \in E_R$ incurs a service cost $c_e^s \geq c_e$, which applies only to its first traversal. The objective of the RPP is to find a tour, a closed walk, with the minimum total cost that traverses and services all the required edges. In the case of the GRP, where we further consider a set of required vertices $V_R \subseteq V$, the objective is to find a tour with the minimum total cost that traverses all the required edges and visits all the required vertices at least once.

It is important to point out that when $V_R = \emptyset$, the General Routing Problem simplifies to the Rural Postman Problem. Furthermore, if in addition $E_R = E$, the problem becomes the Chinese Postman Problem. On the other hand, when $E_R = \emptyset$, we encounter the Steiner Graphical Traveling Salesman Problem, as discussed by Cornuéjols et al. (1985), which is also referred to as the Road Traveling Salesman Problem by Fleischmann (1985). Finally, if both $E_R = \emptyset$ and $V_R = V$, we obtain the Graphical Traveling Salesman Problem (GTSP), as studied in Cornuéjols et al. (1985).

To simplify the structure and formulation of the problem, it is common practice to convert the original graph $G = (V, E)$ into a simplified graph $G^T = (V_R, E^T)$ using the transformation method proposed

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by Christofides et al. (1981) for the RPP. This transformation involves the elimination of non-required vertices, which generally facilitates both problem formulation and its solution. However, it is important to keep in mind that this transformed graph may have more edges than the original graph and, in certain cases, polynomial instances could become non-polynomial. For this reason, some researchers choose to formulate the problem directly in the original graph to avoid these complexities. Although in this paper we will do the theoretical study on general graphs, we have carried out the computational experiments on RPP and GRP benchmark instances, which are all defined on simplified graphs.

The initial formulation of the Rural Postman Problem (RPP) was introduced by Christofides et al. (1981). This formulation is based on the simplified graph G^T and uses variables x_e, which represent the number of times edge e is traversed without servicing it (in dead-heading), as well as variables z_i, which, multiplied by 2, represent the degree of vertex i. In their approach, they proposed a lower bound by incorporating constraints that impose the even degree of each vertex within the objective function in a Lagrangian fashion. To evaluate the performance of their formulation, the authors solved twenty-four randomly generated instances, satisfying the conditions $|V| \leq 84, |E| \leq 184$ and up to 8 R-sets, using a branch-and-bound algorithm. All instances were solved optimally.

Corberán and Sanchis proposed formulations for the RPP (Corberán and Sanchis, 1994) and the GRP (Corberán and Sanchis, 1998) on the original graph G using the same variables x_e introduced in Christofides et al. (1981). In their paper, they showed that, apart from the trivial ones, a facet-inducing inequality for the RPP and GRP polyhedra can be obtained from every facet-inducing inequality for the GTSP polyhedron. In addition, they introduced new families of facet-inducing inequalities, including K-C and Honeycomb inequalities, as well as the so-called R-odd inequalities,

$$\sum_{e\in\delta(S)} x_e \geq 1, \quad \text{for all } S \subset V, \quad |S \cap V_R| \text{ is odd.} \tag{1}$$

In their paper, Corberán and Sanchis (1994) presented a cutting plane algorithm for the RPP, where the separation problems were solved visually. The algorithm proved to be efficient, successfully solving 23 out of the 24 instances initially introduced by Christofides et al. (1981). Furthermore, the algorithm was able to solve two bigger instances derived from the street network of Albaida, Spain.

Letchford (1997) introduced a generalization of K-C inequalities, called path-bridge inequalities. The author gave a polynomial-time exact separation routine for a simple subset of the path-bridge inequalities but did not report computational results. In Corberán et al. (2001), a cutting plane algorithm was proposed for both the RPP and the GRP. The algorithm incorporated connectivity constraints, R-odd cut inequalities, K-C inequalities, path-bridge inequalities, and honeycomb inequalities. Computational experiments were conducted using a SUN Sparc 20 workstation, which had performance comparable to that of a 66 MHz Pentium processor. The algorithm yielded optimal solutions for the RPP instances presented in Christofides et al. (1981), as well as for those proposed in Hertz et al. (1999). In addition, it achieved the optimal solution in 34 out of 40 GRP instances, with up to 196 vertices, 316 edges, and 111 R-sets. Furthermore, the algorithm solved 6 out of the 7 Graphical Traveling Salesman Problem instances obtained from the TSPLIB (Reinelt, 1991), with up to 200 vertices.

Ghiani and Laporte (2000) proposed a slightly modified formulation for the RPP compared to the formulation presented by Corberán and Sanchis (1994). Their formulation differs in that it exclusively employs binary variables. This formulation marks the first case of using 0/1 variables for the RPP (which can easily be extended to the GRP). The authors based their approach on a significant observation: there always exists an optimal solution for the RPP in which, at most, the variables associated with the edges of a minimum cost tree spanning the R-connected components are equal to 2. Recognizing that all other variables are binary, it becomes possible to express the RPP using solely 0/1 variables by duplicating those associated with the minimum spanning tree. Then, Barahona and Grötschel's cocircuit inequalities (Barahona and Grötschel, 1986) can be adapted to the RPP thus formulated as:

$$x\big(\delta(S) \setminus F\big) \geq x(F) - |F| + 1, \tag{2}$$

for each set $S \subset V$ and each $F \subset \delta(S)$ such that $|F| + |\delta(S) \cap E_R|$ is odd. In their paper, Ghiani and Laporte presented a branch-and-cut algorithm for the RPP. This algorithm uses a combination of connectivity inequalities, R-odd inequalities, and the subset of cocircuit inequalities with $|F| = 1$, known as R-even inequalities. This algorithm was able to solve instances with up to 350 vertices optimally.

Garfinkel and Webb (1999) proposed a distinct formulation for the RPP, which was later enhanced by Fernández et al. (2003). Their approach involves computing matchings connecting the R-connected components on a transformed graph. Expanding upon this formulation, Fernández et al. (2003) developed an exact algorithm that successfully solved 145 out of 158 RPP and GRP instances from Corberán et al. (2001). Additionally, they addressed 15 new, larger RPP instances, ranging up to 284 vertices and 31 R-sets. Remarkably, even for the instances that remained unsolved, the optimality gap observed was consistently below 1%.

As far as we are aware, the best exact algorithm for the RPP and GRP is the one proposed by Corberán et al. (2007) for the Windy General Routing Problem (WGRP). The WGRP is a generalization of the GRP in which the problem is defined on a windy graph, an undirected graph with asymmetric costs in the edges. In Corberán et al. (2007) it is reported that this algorithm was able to solve to optimality instances involving up to 1000 vertices, 3080 edges, and 304 R-sets. The performance of the algorithm proposed in our work will be compared against the results obtained by that algorithm.

The Maximum Benefit Chinese Postman Problem (MBCPP) was studied in Corberán et al. (2013). In the MBCPP, each edge is associated with a set of benefits that are collected in a specific order during each traversal of the edge, and the objective is to find a tour that maximizes the total net benefit obtained. The formulation of the MBCPP involves the utilization of two binary variables, x_e and y_e, for each edge $e \in E$. These variables represent the first and second traversals, respectively, of the edge. Encouraged by the favorable computational results observed in that study, we have been motivated to explore a similar formulation using these types of variables for both the Rural Postman Problem and the General Routing Problem. This is the purpose of this work.

The paper is structured as follows. Section 2 presents a new formulation for the Rural Postman Problem and the General Routing Problem. Moving on to Section 3, we define the polytope associated with the feasible solutions of these problems. Within this section, we also determine its dimension and provide proof for the facet-inducing property of certain inequalities. Section 3.1 examines parity inequalities in detail. Sections 3.2 and 3.3 focus on the exploration of p-connectivity and K-C inequalities, respectively. The branch-and-cut algorithm is detailed in Section 4, while Section 5 presents the computational experiments conducted. Section 6 summarizes the conclusions drawn from the study and the Appendix includes the detailed computational results on each instance and with each algorithm.

2. The RPP and GRP formulation

In the RPP and the GRP, it is commonly assumed that all vertices are either required or incident with a required edge. This assumption is made to simplify the problem, as any graph can be transformed into a new one that satisfies this condition. However, in our approach, we choose not to perform this transformation and instead allow the graph to contain non-required vertices that are not connected to any required edges.

The GRP is defined on an undirected graph $G = (V, E)$ with edge set $E = E_R \cup E_{NR}$ and vertex set $V = V_R \cup V_{NR}$. Here, E_R represents the required edges and V_R represents the required vertices. Each edge $e \in E$ has a non-negative deadheading cost c_e, and each required edge $e \in E_R$ has a service cost $c_e^s \geq c_e$ corresponding to the first traversal of that edge. The objective of the GRP is to find a tour (closed walk) with the minimum total cost, which traverses all the required edges and visits all the required vertices.

It is important to note that the Rural Postman Problem (RPP) is a special case of the GRP where there are no required vertices. Therefore, throughout this paper, we will primarily refer to the GRP, while recognizing that all the results discussed are also applicable to the RPP.

Note that if vertex $i \in V$ is incident with any required edge $e \in E_R$, the requirement for the tour to traverse edge e inherently includes the condition of visiting vertex i. Therefore, for the sake of simplicity, we will assume that V_R contains the set of vertices that are incident with the required edges.

Consider the (generally disconnected) subgraph (V_R, E_R) of G. Let Q be the number of its connected components and V^1, V^2, …, V^Q their corresponding sets of vertices that we call R-sets. The induced subgraphs $G(V^i)$ in G will be referred to as R-connected components of G. Note that an R-connected component (R-set) may consist only of a single (required) vertex and that $\cup_i V^i = V_R$.

The following notation is used. Given two subsets of vertices $S, S' \subseteq V$, $(S : S')$ denotes the edge set with one endpoint in S and the other one in S'. Given a subset $S \subseteq V$, let us denote $\delta(S) = (S : V \setminus S)$ and $E(S) = (S : S)$. For any subset $F \subseteq E$, we will denote $F_R = F \cap E_R$ and $F_{NR} = F \cap E_{NR}$. Similarly, For any subset $W \subseteq V$, we will denote $W_R = W \cap V_R$ and $W_{NR} = W \cap V_{NR}$. Given a vector x indexed on the set E of edges, for each subset $F \subseteq E$, we denote $x(F) = \sum_{e \in F} x_e$.

To formulate the GRP we define the following variables: x_e, y_e, $\forall e \in E$, representing the first and second traversal, respectively, of the edge e. The GRP can be formulated as follows:

$$\text{Minimize} \quad \sum_{e \in E_R} \left(c_e^s x_e + c_e y_e\right) + \sum_{e \in E_{NR}} c_e \left(x_e + y_e\right)$$

$$\sum_{e \in \delta(i)} \left(x_e + y_e\right) \equiv 0 \pmod 2, \ \forall i \in V \tag{3}$$

$$\sum_{e \in \delta(S)} \left(x_e + y_e\right) \geq 2, \ \forall S = \left(\bigcup_{i \in T} V^i\right) \cup W, \quad \emptyset \neq T \subsetneq \{1, \dots, Q\}, W \subset V_{NR} \tag{4}$$

$$\sum_{e \in \delta(S)} \left(x_e + y_e\right) \geq 2x_f, \ \forall S \subset V \text{ such that } (V \setminus S)_R \neq \emptyset, \quad \forall f \in E(S), \tag{5}$$

$$x_e = 1, \ \forall e \in E_R \tag{6}$$

$$x_e \geq y_e, \ \forall e \in E_{NR} \tag{7}$$

$$x_e, y_e \in \{0, 1\}, \ \forall e \in E. \tag{8}$$

The constraints (3) enforce that each vertex in the route is visited an even number of times, which could be zero as well. The conditions (4) ensure that the route connects all the R-sets, thereby visiting all the "isolated" required vertices. Furthermore, conditions (5) prevent the inclusion of "solutions" that consist of isolated cycles formed by non-required edges and vertices. To ensure the traversal of all required edges, we have constraints (6). Constraints (7) guarantee that a second traversal of a non-required edge can only occur after it has been traversed previously. Finally, constraints (8) define the binary conditions for the variables.

Obviously, this formulation is also valid for the RPP. The only distinction is that in the RPP, there are no subsets V^i that consist of a single vertex.

Furthermore, in any GRP tour, it is always true that $x_e = 1$ for all $e \in E_R$. As a result, these variables can be eliminated from the formulation. Additionally, for non-required edges connecting vertices within the same set V^i, it is possible to fix all corresponding y_e variables to zero, as demonstrated by Corberán and Sanchis (1994). Moreover, based on the dominance relations described in Chapter 5 of Corberán and Laporte (2015), other y_e variables can also be fixed to zero. It is important to note that in our formulation, we will retain all these variables as they assist in expressing certain inequalities and facilitate the polyhedral study.

Let us call GRP tour to a closed walk on graph G traversing all the required edges and visiting all the required vertices. Associated with each GRP tour we can consider:

(a) An incidence vector $(x, y) \in \mathbb{Z}^{2|E|}$, where variables x_e take the value 1 if edge e is traversed once, variables y_e take the value 1 if edge e is traversed twice, and

(b) a support graph $(V, E^{(x,y)})$, where $E^{(x,y)}$ contains one copy of edge $e \in E$ for each variable $x_e = 1$ or $y_e = 1$.

Note that the support graphs are even and connected. Conversely, any even and connected subgraph of G corresponds to a tour on G. In fact, it is important to highlight that an incidence vector or a subgraph may correspond to several different closed walks, but all of them have the same cost and can be easily computed (with the Hierholzer algorithm, Hierholzer and Wiener (1873), for example). Hence, and for the sake of simplicity, we will refer to the closed walk, its incidence vector, and its corresponding support graph as a GRP tour on G.

3. The RPP and GRP polytope

The polyhedron *GRP(G)* is defined as the convex hull of all GRP tours in graph G. To analyze this polyhedron, we rely on certain results presented in Corberán et al. (2013) for the MBCPP. In Corberán et al. (2013), the MBCPP is initially defined in a general setting, considering multiple benefits associated with each edge. However, it is later simplified as follows. Given an undirected connected graph $G = (V, E)$, where vertex 1 represents the depot, and each edge $e \in E$ has two associated benefits: one for the first traversal and another for the second traversal. The objective of the MBCPP is to find a tour that starts from the depot, traverses a subset of edges in E at most twice, and returns to the depot, while maximizing the total benefit. As mentioned earlier, the MBCPP formulation in Corberán et al. (2013) employs two binary variables, x_e and y_e, for each edge $e \in E$, representing the first and second traversal of e, respectively. It is shown that the convex hull of all MBCPP tours, i.e., the vectors (x, y) satisfying

$$\sum_{e \in \delta(i)} \left(x_e + y_e\right) \equiv 0 \pmod 2, \ \forall i \in V \tag{9}$$

$$\sum_{e \in \delta(S)} \left(x_e + y_e\right) \geq 2x_f, \ \forall S \subset V \setminus \{1\}, \quad \forall f \in E(S) \tag{10}$$

$$x_e \geq y_e, \ \forall e \in E \tag{11}$$

$$x_e, y_e \in \{0, 1\}, \ \forall e \in E, \tag{12}$$

is a full dimensional polytope and several families of valid and facet-inducing inequalities are described.

Let $v_0 \in V_R$ and consider the MBCPP defined on graph G, with the vertex v_0 serving as the depot. Every GRP tour $(x, y) \in \mathbb{Z}^{(2|E|)}$ is a closed walk starting and ending at the depot and is also a MBCPP tour. Hence, we can state the following theorem:

Theorem 1. *Let $f(x, y) \geq \alpha$ be a valid inequality for the MBCPP on graph G with depot $v_0 \in V_R$. The corresponding inequality $f(x, y) \geq \alpha$ is valid for the GRP on G.*

For example, from inequalities (10) we obtain inequalities (5). Furthermore, from several families of valid inequalities for the MBCPP, namely parity, p-connectivity, and K-C inequalities, we will obtain valid inequalities for the GRP (see Sections 3.1–3.3).

In the following, we will obtain the dimension of GRP(G) and will study conditions under which some of the above constraints, as well as other valid inequalities, define facets of the polyhedron. To conduct this study, we will construct various GRP tours in graph G. One such tour is formed by taking two copies of each edge in E. This basic tour is used in proving Theorem 2. As we proceed with the proofs of other theorems, we will develop more specific and detailed GRP tours. To do this, we need some additional definitions.

Consider a vertex subset $V^o \subseteq V$, with $|V^o|$ even. A subset of edges $M \subseteq E$ is a T-join if, in the subgraph (V, M), the degree of v is odd if and only if $v \in V^o$. It is well-known that for a connected graph G, there exists a T-join for every set $V^o \subseteq V$ with $|V^o|$ even (see Nemhauser and Wolsey (1988) for further details).

Given $G = (V, E) = (V, E_R \cup E_{NR})$, let $V_R^o \subseteq V_R$ be the set of R-odd vertices, i.e., the vertices incident with an odd number of required edges. Let $M \subseteq E$ be any T-join corresponding to V_R^o. The graph $(V_R, M \cup E_R)$, where $M \cup E_R$ contains two copies of each required edge in M, is an even graph, although it may not be connected. By incorporating the edges from a closed walk that visits at least one node in each connected component of this graph, we obtain a GRP tour.

Theorem 2. *dim(GRP(G))=* $2|E| - |E_R|$ *if and only if G is a 3-edge connected graph.*

Proof. GRP(G) is a polytope in $\mathbb{R}^{2|E|}$. Since all its points satisfy Eq. (6), which are linearly independent, we have dim(GRP(G))≤ $2|E| - |E_R|$.

If G is not 3-edge connected, there exists a cut-set $\delta(S)$ with at most 2 edges. If $\delta(S)$ contains exactly two edges, namely e and f, all GRP tours satisfy the equation $x_e - y_e = x_f - y_f$. Similarly, if $\delta(S) = \{e\}$, all GRP tours satisfy the equation $x_e = y_e$. Given that these equations are linearly independent from Eq. (6), we conclude that dim(DRPP(G))< $2|E| - |E_R|$. On the other hand, let us now suppose that graph G is 3-edge connected. We will prove that dim(GRP(G))≥ $2|E| - |E_R|$. Let $ax + by = c$, i.e.,

$$\sum_{e\in E} a_e x_e + \sum_{e\in E} b_e y_e = c \tag{13}$$

be an equation satisfied by all the GRP tours. We have to prove that this equation is a linear combination of Eq. (6), i.e., to prove that

$$a_e = 0, \quad \forall e \in E_{NR},$$
$$b_e = 0, \quad \forall e \in E,$$
$$c = \sum_{e\in E_R} a_e.$$

Consider the vector $T = (x, y)$ formed with two copies of each edge in E. In other words, T is a vector where all its entries are equal to 1. Since T is a GRP tour on G, it satisfies (13) and we have

$$\sum_{e\in E} a_e + \sum_{e\in E} b_e = c. \tag{14}$$

Let $f \in E_{NR}$. Since G is a 3-connected graph, we can remove the two copies of edge f from the vector T, resulting in a new GRP tour denoted as T^{-2f}. Consequently, it satisfies (13) and we have

$$\sum_{e\in E\setminus\{f\}} a_e + \sum_{e\in E\setminus\{f\}} b_e = c.$$

By subtracting this equation from (14), we obtain $a_f + b_f = 0$, for all $f \in E_{NR}$.

Let C be an arbitrary cycle on G. By removing one copy of each edge in C from the vector T, we obtain another GRP tour denoted $T - C$. This new tour also satisfies (13). Substituting $T - C$ into (13) and subtracting the resulting equation from (14), we obtain $b(C) = 0$, where recall that $b(C) = \sum_{e\in C} b_e = 0$.

Consider an arbitrary edge $f = (i, j) \in E$. As graph G is a 3-edge connected graph, there exist two edge-disjoint paths $\mathcal{P}_1$ and $\mathcal{P}_2$, that join vertices i and j using edges other than f. Now, let us consider three cycles: $\mathcal{P}_1 \cup \{f\}$, $\mathcal{P}_2 \cup \{f\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$. It holds that $b(C) = 0$ for each of these cycles. Consequently, we can deduce that
$b(\mathcal{P}_1) + b_f = 0$, $\;b(\mathcal{P}_2) + b_f = 0$, and $\;b(\mathcal{P}_1) + b(\mathcal{P}_2) = 0$.

From these equations, we can conclude that $b_f = 0$ for each $f \in E$. Given that $a_f + b_f = 0$ holds for each $f \in E_{NR}$, we have $a_f = 0$ for all $f \in E_{NR}$. By substituting these values in (14), we obtain $\sum_{e\in E_R} a_e = c$. This concludes the proof. ♦

In what follows, to prove that some inequalities are facet-defining for GRP(G), we will assume that G is a 3-edge connected graph.

Theorem 3. *The inequality* $y_e \geq 0$, *for each edge* $e \in E_{NR}$, *is facet-inducing for GRP(G). The inequality* $y_e \geq 0$, *for each edge* $e \in E_R$, *is facet-inducing for GRP(G) if the graph* $G \setminus \{e\}$ *is 3-edge connected.*

Proof. Let $ax + by \geq c$, i.e., $\sum_{f\in E} a_f x_f + \sum_{f\in E} b_f y_f \geq c$, be a valid inequality such that
$\{ (x, y) \in \text{GRP(G)} : \;\; y_e = 0 \} \subseteq \{ (x, y) \in \text{GRP(G)} : \;\; ax + by = c \}$.

To establish that this inequality is a linear combination of the equalities (6) and $y_e \geq 0$, we need to prove the following conditions:

$$a_f = 0, \quad \forall f \in E_{NR},$$
$$b_f = 0, \quad \forall f \in E,\ f \neq e,$$
$$c = \sum_{f\in E_R} a_f.$$

(a) We will first prove it for $e \in E_{NR}$. Let T be the GRP tour formed with two copies of each edge in E and T^{-2e} the GRP tour obtained by removing the two copies of e from T. Since T^{-2e} satisfies $y_e = 0$, it also satisfies $ax + by = c$, and we have

$$\sum_{f\in E\setminus\{e\}} a_f + \sum_{f\in E\setminus\{e\}} b_f = c. \tag{15}$$

Let $g \in E_{NR}$, $g \neq e$. Since graph G is 3-connected, the vector T^{-2e-2g}, obtained by removing the two copies of g from T^{-2e}, is also a GRP tour satisfying $y_e = 0$. Hence, it also satisfies $ax + by = c$ and, by subtracting this equation from (15), we obtain $a_g + b_g = 0$, $\forall g \in E_{NR}$, $g \neq e$.

Let us consider a cycle C on G that does not contain edge e. The tour $T^{-2e} - C$ is obtained by removing one copy of each edge in C from T^{-2e}. Since it does not include edge e, it remains a valid GRP tour and satisfies $y_e = 0$. By substituting its incidence vector in the equation $ax + by = c$, we have:

$$\sum_{f\in E\setminus\{e\}} a_f + \sum_{f\in E\setminus\{e\cup C\}} b_f = c. \tag{16}$$

By subtracting Eq. (16) from (15), we obtain that $b(C) = 0$.

Now, let us consider a cycle C on G that contains edge e. The tour $T - C$ is obtained by removing one copy of each edge in C from T. It is a GRP tour and satisfies $y_e = 0$. By substituting its incidence vector in the equation $ax + by = c$, we have:

$$\sum_{f\in E} a_f + \sum_{f\in E\setminus\{C\}} b_f = c. \tag{17}$$

By subtracting Eq. (17) from (15), we obtain $-a_e + b(C \setminus \{e\}) = 0$ and, hence, $b(C) = a_e + b_e$.

Consider the edge e. Since G is a 3-edge connected graph, there exist two edge-disjoint paths, $\mathcal{P}_1$, $\mathcal{P}_2$, joining the end-nodes of e using edges other than e. Now, let us consider the three cycles $\mathcal{P}_1 \cup \{e\}$, $\mathcal{P}_2 \cup \{e\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$. We have that
$b(\mathcal{P}_1) + b_e = a_e + b_e$, $\;b(\mathcal{P}_2) + b_e = a_e + b_e$, and $\;b(\mathcal{P}_1) + b(\mathcal{P}_2) = 0$.

From these equations we obtain $a_e = 0$.

Let $f \in E \setminus \{e\}$ be any other edge. Since G is a 3-edge connected graph, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining the end-nodes of f with edges different from f. Again, consider the three cycles $\mathcal{P}_1 \cup \{f\}$, $\mathcal{P}_2 \cup \{f\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$. We will distinguish two cases. Assume first that $e \notin \mathcal{P}_1 \cup \mathcal{P}_2$. In this case,
$b(\mathcal{P}_1) + b_f = 0$, $\;b(\mathcal{P}_2) + b_f = 0$, and $\;b(\mathcal{P}_1) + b(\mathcal{P}_2) = 0$.

Assume now that, for example, $e \in \mathcal{P}_1$. In this case,
$b(\mathcal{P}_1) + b_f = a_e + b_e$, $\;b(\mathcal{P}_2) + b_f = 0$, and $\;b(\mathcal{P}_1) + b(\mathcal{P}_2) = a_e + b_e$.

In both cases, we obtain $b_f = 0$, for each $f \in E \setminus \{e\}$. Given that $a_f + b_f = 0$ holds for each $f \in E_{NR} \setminus \{e\}$, we have $a_f = 0$ for all $f \in E_{NR} \setminus \{e\}$. By substituting these values in (15), we obtain $\sum_{f \in E_R} a_f = c$ and we are done.

(b) We will now prove it for the case $e \in E_R$ and the graph $G \setminus \{e\}$ is 3-edge connected. Since G is a 3-edge connected graph, there exist two edge-disjoint paths, $\mathcal{P}_1$, $\mathcal{P}_2$, that join the end-nodes of e using edges different from e. Let T^1 and T^2 be the GRP tours formed with two copies of each edge in E except the edges on $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. The edge e is traversed once in each of the tours.

Consider any edge $f \in E_{NR}$. If $f \notin \mathcal{P}_1$, we consider the tours T^1 and T^{1-2f}, which satisfy $y_e = 0$, and by subtracting their corresponding equalities $ax + by = c$ we obtain that $a_f + b_f = 0$. The case $f \notin \mathcal{P}_2$ can be handled similarly with the tours T^2 and T^{2-2f}. Hence, $a_f + b_f = 0$ for all $f \in E_{NR}$.

Let C be a cycle in G that does not contain the edge e. Suppose, for example, that $f \notin \mathcal{P}_1$. Given the GRP tour T^1 above, we can define a new GRP tour as follows: $T^* = T^1 \setminus C + 2(C \cap \mathcal{P}_1)$. This tour also satisfies $y_e = 0$. By subtracting the equalities $ax + by = c$ corresponding to T^1 and T^*, we obtain that $b(C \setminus \mathcal{P}_1) = b(C \cap \mathcal{P}_1)$ and, hence, $b(C) = 2b(C \cap \mathcal{P}_1)$ for all cycles C in G that do not contain the edge e.

Let $f \in E$, $f \neq e$. Since $G \setminus \{e\}$ is 3-edge connected, there exist two edge-disjoint paths, $\bar{\mathcal{P}}_1$ and $\bar{\mathcal{P}}_2$, joining the end-nodes of f using edges other than f and e. We consider three cycles: $\bar{\mathcal{P}}_1 \cup \{f\}$, $\bar{\mathcal{P}}_2 \cup \{f\}$, and $\bar{\mathcal{P}}_1 \cup \bar{\mathcal{P}}_2$. Given that $e \notin \bar{\mathcal{P}}_1$, $e \notin \bar{\mathcal{P}}_2$, and $f \notin \mathcal{P}_1$, we have

$b(\bar{\mathcal{P}}_1) + b_f = 2b(\bar{\mathcal{P}}_1 \cap \mathcal{P}_1)$,

$b(\bar{\mathcal{P}}_2) + b_f = 2b(\bar{\mathcal{P}}_2 \cap \mathcal{P}_1)$, and

$b(\bar{\mathcal{P}}_1) + b(\bar{\mathcal{P}}_2) = 2b(\bar{\mathcal{P}}_1 \cap \mathcal{P}_1) + 2b(\bar{\mathcal{P}}_2 \cap \mathcal{P}_1)$,

and we obtain that $b_f = 0$ for all $f \in E$, $f \neq e$. Finally, since $a_f + b_f = 0$ for every $f \in E_{NR}$, we conclude that $a_f = 0$ for all $f \in E_{NR}$. This completes the proof. ⧫

Theorem 4. *Inequality $x_e \leq 1$, for each edge $e \in E_{NR}$, is facet-inducing for GRP(G).*

Proof. Let $ax + by \leq c$ be a valid inequality such that $\{ (x, y) \in \text{GRP(G)} : x_e = 1 \} \subseteq \{ (x, y) \in \text{GRP(G)} : ax + by = c \}$. We need to prove that this inequality is a linear combination of the equalities (6) and $x_e \leq 1$. This implies proving that

$$\begin{aligned} a_f &= 0, \quad \forall f \in E_{NR},\ f \neq e, \\ b_f &= 0, \quad \forall f \in E, \\ c &= \sum_{f \in E_R} a_f + a_e. \end{aligned}$$

Let T be the GRP tour formed by taking two copies of each edge in E. This tour satisfies $x_e = 1$, and therefore it also satisfies $ax + by = c$, and we have that

$$\sum_{f \in E} a_f + \sum_{f \in E} b_f = c. \tag{18}$$

Consider any edge $f \in E_{NR}$, $f \neq e$. Since G is a 3-connected graph, we can construct a GRP tour T^{-2f} that also satisfies $x_e = 1$. By substituting this tour into $ax + by = c$ and subtracting the resulting equality from (18), we obtain that $a_f + b_f = 0$, $\forall f \in E_{NR}$, $f \neq e$.

For any cycle C in G, the tour $T - C$ is also a GRP tour that satisfies $x_e = 1$. Substituting its incidence vector into the equation $ax + by = c$, we have

$$\sum_{f \in E} a_f + \sum_{f \in E \setminus \{C\}} b_f = c. \tag{19}$$

Subtracting Eq. (19) from (18), we obtain: $b(C) = 0$.

Let $f \in E$ be any edge (perhaps e). Since G is a 3-edge connected graph, there exist two edge-disjoint paths, $\mathcal{P}_1$ and $\mathcal{P}_2$, that join the end-nodes of f with distinct edges of f. Now, let us consider the three cycles: $\mathcal{P}_1 \cup \{f\}$, $\mathcal{P}_2 \cup \{f\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$. From these cycles, we have $b(\mathcal{P}_1) + b_f = 0$, $b(\mathcal{P}_2) + b_f = 0$, and $b(\mathcal{P}_1) + b(\mathcal{P}_2) = 0$, and we obtain $b_f = 0$ for each $f \in E$. Furthermore, since $a_f + b_f = 0$ holds for each $f \in E_{NR} \setminus \{e\}$, we conclude that $a_f = 0$ for all $f \in E_{NR} \setminus \{e\}$. By substituting these values into Eq. (18), we obtain $\sum_{f \in E_R} a_f + a_e = c$, which completes the proof. ⧫

Theorem 5. *The inequalities (7), $x_e \geq y_e$ for each edge $e \in E_{NR}$, are facet-inducing for GRP(G) if graph $G \setminus \{e\}$ is 3-edge connected.*

Proof. Let $ax + by \geq c$ be a valid inequality such that $\{ (x, y) \in \text{GRP(G)} : x_e = y_e \} \subseteq \{ (x, y) \in \text{GRP(G)} : ax + by = c \}$.

We have to prove that this inequality is a linear combination of the equalities (6) and $x_e - y_e \geq 0$. Note that this means proving that

$$\begin{aligned} a_f &= 0, \quad \forall f \in E_{NR},\ f \neq e, \\ b_f &= 0, \quad \forall f \in E,\ f \neq e, \\ a_e &= -b_e, \\ c &= \sum_{f \in E_R} a_f. \end{aligned}$$

Let T be the GRP tour formed with two copies of each edge in E, which satisfies $x_e = y_e$ and, therefore, also satisfies $ax + by = c$ and we have

$$\sum_{f \in E} a_f + \sum_{f \in E} b_f = c. \tag{20}$$

Consider any edge $f \in E_{NR}$. Since G is a 3-edge connected graph, we can construct a GRP tour T^{-2f} that also satisfies $x_e = y_e$. By substituting it into $ax + by = c$ and subtracting the resulting equality from Eq. (20), we obtain that $a_f + b_f = 0$, $\forall f \in E_{NR}$. Note that this equality also holds for edge e, giving $a_e + b_e = 0$.

For every cycle C in G that does not contain edge e, the tour $T - C$ is also a GRP tour satisfying $x_e = y_e$ and, therefore, $ax + by = c$ and we have

$$\sum_{f \in E} a_f + \sum_{f \in E \setminus \{C\}} b_f = c. \tag{21}$$

Subtracting (21) from (20) we obtain that $b(C) = 0$.

Let $f \in E$ be any edge, $f \neq e$. Since $G \setminus \{e\}$ is a 3-edge connected graph, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ that join the end-nodes of f with distinct edges of f. Consider the three cycles $\mathcal{P}_1 \cup \{f\}$, $\mathcal{P}_2 \cup \{f\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$. Therefore, we have $b(\mathcal{P}_1) + b_f = 0$, $b(\mathcal{P}_2) + b_f = 0$, and $b(\mathcal{P}_1) + b(\mathcal{P}_2) = 0$, and we obtain $b_f = 0$, for all $f \neq e$. Since $a_f + b_f = 0$, $\forall f \in E_{NR}$, we obtain that $a_f = 0$ for all $f \in E_{NR} \setminus \{e\}$. Substituting these values for a, b into (20) we get $\sum_{f \in E_R} a_f = c$ and we are done. ⧫

Note 1. The inequalities $x_e \geq 0$ and $y_e \leq 1$, for $e \in E_{NR}$, are not facet-inducing because they are dominated by the inequality $x_e \geq y_e$.

In the following, we will describe the conditions under which the connectivity inequalities (4) and (5) are facet inducing. It is important to note that, since $x_e = 1$ holds for all $e \in E_R$, if $\delta_R(S) \neq \emptyset$, then inequalities (4) and (5) are trivially satisfied. Therefore, we will assume $\delta_R(S) = \emptyset$.

Theorem 6. *Let $S = \left(\bigcup_{i \in T} V^i\right) \cup W$ with $T \subsetneq \{1, \dots, p\}$, $T \neq \emptyset$, and $W \subset V_{NR}$. The connectivity inequality (4):*

$$(x + y)(\delta(S)) \geq 2,$$

is facet-inducing for GRP(G) if $\delta_R(S) = \emptyset$ and the subgraphs $G(S)$ and $G(V \setminus S)$ are both 3-edge connected.

Proof. Let $ax + by \geq c$ be a valid inequality such that $\{ (x, y) \in \text{GRP(G)} : (x + y)(\delta(S)) = 2 \} \subseteq \{ (x, y) \in \text{GRP(G)} : ax + by = c \}$.

We need to prove that this inequality is a linear combination of the equalities (6) and $(x + y)(\delta(S)) \geq 2$. This can be expressed as:

$\sum_{e\in E_R} a_e x_e + \sum_{e\in\delta(S)} \left(\lambda x_e + \lambda y_e\right) \geq \sum_{e\in E_R} a_e + 2\lambda$. Therefore, we have to prove that

$$\begin{aligned} a_e = b_e = 0, &\quad \forall e \in E_{NR}(S) \cup E_{NR}(V \setminus S),\\ b_e = 0, &\quad \forall e \in E_R,\\ a_e = b_e = \lambda, &\quad \forall e \in \delta(S),\\ c = \sum_{e\in E_R} a_e + 2\lambda. & \end{aligned}$$

Let $T = (x, y)$ here be the GRP tour formed with two copies of each edge in $E(S) \cup E(V \setminus S)$ plus two copies of a given edge $f \in \delta(S)$. Since this tour satisfies $(x + y)(\delta(S)) = 2$, it also satisfies $ax + by = c$, and we have

$$\sum_{e\in E\setminus\delta(S)} a_e + \sum_{e\in E\setminus\delta(S)} b_e + a_f + b_f = c. \tag{22}$$

For each edge $e \in E_{NR} \setminus \delta(S)$, T^{-2e} is a GRPP tour because $G(S)$ and $G(V \setminus S)$ are 3-edge connected graphs, satisfying $(x + y)(\delta(S)) = 2$. By substituting T^{-2e} in the equation $ax + by = c$ and subtracting the resulting equality from (22), we obtain $a_e + b_e = 0$ for each edge $e \in E_{NR} \setminus \delta(S)$.

For each cycle C either in $G(V \setminus S)$ or in $G(S)$, the vector $T - C$ is also a GRP tour satisfying $(x + y)(\delta(S)) = 2$. Substituting its incidence vector into $ax + by = c$ and subtracting the resulting equality from (22) we obtain that $b(C) = 0$.

Let $\mathcal{P}_1$ be any path in $G(S)$ or in $G(V \setminus S)$ that joins two vertices i, j. Since graphs $G(S)$ and $G(V \setminus S)$ are 3-edge connected, there are two other edge-disjoint paths $\mathcal{P}_2$ and $\mathcal{P}_3$ joining i and j. Considering the three cycles $\mathcal{P}_1 \cup \mathcal{P}_2$, $\mathcal{P}_2 \cup \mathcal{P}_3$, and $\mathcal{P}_2 \cup \mathcal{P}_3$, for which $b(C) = 0$ holds, we obtain that $b(\mathcal{P}) = 0$ for each path either in $G(S)$ or in $G(V \setminus S)$. In particular, $b_e = 0$ for each edge $e \in E\setminus\delta(S)$. Moreover, if $e \in E_{NR}\setminus\delta(S)$, since $a_e + b_e = 0$, we have $a_e = b_e = 0$ for all $e \in E_{NR} \setminus \delta(S)$.

Let us denote the edges in $\delta(S)$ as $e_1, \dots, e_p$, where $p \geq 3$ since graph G is 3-edge connected. Consider two edges $f, e \in \delta(S)$. Let T be the GRP tour described above, and let T^* be the tour obtained from T by removing the second traversal of f and one copy of each edge on the two paths $\mathcal{P}_1, \mathcal{P}_2$ joining the endpoints of f and e, and adding the first traversal of e. Both tours T and T^* satisfy $(x + y)(\delta(S)) = 2$. By subtracting the corresponding equalities, we obtain $b(\mathcal{P}_1) + b(\mathcal{P}_2) + b_f - a_e = 0$ and, hence, $b_f = a_e$. By interchanging the roles of the edges f and e, we obtain $b_e = a_f$. Proceeding in this way for all pairs of edges in $\delta(S)$, we conclude that $a_{e_i} = b_{e_j}$ for all $i \neq j \in \{1, \dots, p\}$. This implies that $a_{e_i} = a_{e_j} = b_{e_i} = b_{e_j} = \lambda$, for a certain constant value λ, for all i, j (since $p \geq 3$ holds).

Substituting $a_e = b_e = 0$ for each $e \in E_{NR} \setminus \delta(S)$, $b_e = 0$ for each edge $e \in E_R$, and $a_e = b_e = \lambda$ for each $e \in \delta(S)$ in Eq. (22), we get that $\sum_{e\in E_R} a_e + 2\lambda = c$, and we are done. ◆

Theorem 7. *Let $S \subseteq V$ be such that $V_R \subseteq (V \setminus S)$ and let $f \in E(S)$. The connectivity inequality (5),*

$$(x + y)(\delta(S)) \geq 2x_f,$$

is facet inducing for GRP(G) if the subgraphs $G(S)$ and $G(V \setminus S)$ are 3-edge connected.

Proof. Note that $V_R \subseteq (V \setminus S)$ implies that $S_R = \emptyset$ and $E_R(S) = \delta_R(S) = \emptyset$. Let $ax + by \geq c$ be a valid inequality such that
$\{\ (x, y) \in \text{GRP(G)} :\quad (x + y)(\delta(S)) - 2x_f = 0\ \} \subseteq \{\ (x, y) \in \text{GRP(G)} : ax + by = c\ \}$.

We have to prove that this inequality is a linear combination of the equalities (6) and $(x + y)(\delta(S)) - 2x_f \geq 0$. Note that such a linear combination is $\sum_{e\in E_R} a_e x_e + \sum_{e\in\delta(S)} \left(\lambda x_e + \lambda y_e\right) \geq \sum_{e\in E_R} a_e + 2\lambda$, and we have to prove that

$$\begin{aligned} a_e = b_e = 0, &\quad \forall e \in E_{NR}(S) \cup E_{NR}(V \setminus S), \quad e \neq f\\ a_f = -2\lambda,\ b_f = 0, &\\ b_e = 0 &\quad \forall e \in E_R,\\ a_e = b_e = \lambda, &\quad \forall e \in \delta(S),\\ c = \sum_{e\in E_R} a_e. & \end{aligned}$$

Let T be the GRP tour formed with two copies of each edge in $E(S) \cup E(V \setminus S)$ along with two copies of a given edge $g \in \delta(S)$. Since $\delta(S) \subseteq E_{NR}$, T satisfies $(x + y)(\delta(S)) = 2x_f = 2$. Therefore, T also satisfies $ax + by = c$, and we have

$$\sum_{e\in E\setminus\delta(S)} a_e + \sum_{e\in E\setminus\delta(S)} b_e + a_g + b_g = c. \tag{23}$$

For each edge $e \in E_{NR}\setminus\delta(S)$, with $e \neq f$, T^{-2e} is a GRP tour because both subgraphs $G(S)$ and $G(V\setminus S)$ are 3-edge connected. Moreover, T^{-2e} satisfies $(x + y)(\delta(S)) = 2x_f$. By substituting T^{-2e} into $ax + by = c$ and subtracting the resulting equality from (23), we obtain $a_e + b_e = 0$ for each edge $e \in E_{NR} \setminus \delta(S)$, $e \neq f$. For each cycle C either in $G(V \setminus S)$ or in $G(S)$, the vector $T - C$ is also a GRP tour satisfying $(x+y)(\delta(S)) = 2x_f$. By substituting its incidence vector into $ax + by = c$ and subtracting the resulting equality from (23), we obtain $b(C) = 0$.

Let $\mathcal{P}_1$ be any path in either $G(S)$ or $G(V \setminus S)$ joining two vertices i and j. Since $G(S)$ and $G(V \setminus S)$ are 3-edge connected, there exist two more edge-disjoint paths $\mathcal{P}_2$ and $\mathcal{P}_3$ joining i and j. Considering the three cycles $\mathcal{P}_1 \cup \mathcal{P}_2$, $\mathcal{P}_2 \cup \mathcal{P}_3$, and $\mathcal{P}_2 \cup \mathcal{P}_3$, for which $b(C) = 0$ holds, we obtain that $b(\mathcal{P}) = 0$ for each path in either $G(S)$ or $G(V \setminus S)$. In particular, $b_e = 0$ for each edge $e \in E \setminus \delta(S)$ (including $b_f = 0$). Furthermore, if $e \in E_{NR} \setminus \delta(S)$, $e \neq f$, since $a_e + b_e = 0$, we have $a_e = b_e = 0$ for all $e \in E_{NR} \setminus \delta(S)$, $e \neq f$.

Let us denote the edges in $\delta(S)$ as $e_1, \dots, e_p$, where $p \geq 3$ since G is 3-edge connected. Now consider two edges $g, e \in \delta(S)$. Consider the GRP tour T described earlier, and let T^* be the tour obtained from T after removing the second traversal of g and one copy of each edge on two paths $\mathcal{P}_1, \mathcal{P}_2$ joining the endpoints of g and e, and adding the first traversal of e. Both tours satisfy $(x+y)(\delta(S)) = 2x_f$ and, after subtracting the corresponding equalities, we obtain $b(\mathcal{P}_1) + b(\mathcal{P}_2) + b_g - a_e = 0$, which implies $b_g = a_e$. If we interchange the roles of the edges g and e, we obtain $b_e = a_g$. Proceeding in this way with all pairs of edges in $\delta(S)$, we obtain $a_{e_i} = b_{e_j}$ for all $i \neq j \in \{1, \dots, p\}$, and then $a_{e_i} = a_{e_j} = b_{e_i} = b_{e_j} = \lambda$, where λ is a certain constant value, for all i, j (because $p \geq 3$ holds).

Let $e = (i, j) \in \delta(S)$ with $j \in S$. Consider the GRP tour T^* formed with two copies of each edge in $E(V \setminus S)$, two copies of e, two copies of each edge in a given path $\mathcal{P}$ in $G(S)$ joining j to an end node of f, and two copies of edge f. T^* is a GRP tour because all the required edges and vertices are in $G(V \setminus S)$ and satisfies $(x + y)(\delta(S)) = 2x_f = 2$. By comparing this tour with the GRP tour obtained after removing the two copies of the edges in $\mathcal{P}$ and the two copies of e and f, which satisfies $(x + y)(\delta(S)) = 2x_f = 0$, we obtain that $a_e + b_e + a_f + b_f = 0$. Given that $b_f = 0$ and $a_e = b_e$, we have $a_f = -2a_e$ for any edge $e \in \delta(S)$.

By substituting $a_e = b_e = 0$ for each $e \in E_{NR} \setminus \delta(S)$, $e \neq f$, $b_e = 0$ for each edge $e \in E_R$, $a_e = b_e = \lambda$ for each $e \in \delta(S)$, and $a_f = -2\lambda$ into Eq. (23), we obtain $\sum_{e\in E_R} a_e = c$, and the proof is finished. ◆

Note that if condition $V_R \subseteq (V \setminus S)$ is not satisfied, then there exists an associated connectivity inequality (4) $(x+y)(\delta(S)) \geq 2$. Consequently, inequality (5) is not facet-inducing in this case.

In the remainder of the paper, we introduce several new families of valid inequalities for the GRP, including parity inequalities, p-connectivity inequalities, and K-C inequalities.

3.1. Parity inequalities

In Corberán et al. (2013), the following constraints, which generalize the well-known co-circuit inequalities (Barahona and Grötschel, 1986), were proposed for the MBCPP. They are called parity inequalities and, from Theorem 1, are also valid for GRP(G):

$$(x-y)(\delta(S)\setminus F) \geq (x-y)(F)-|F|+1, \quad \forall S \subset V,\ \forall F \subseteq \delta(S) \text{ with } |F| \text{ odd.} \tag{24}$$

These parity inequalities (24) cut (infeasible) solutions in which there is a cut-set with an odd number of edges traversed exactly once (these edges define the set F) and the remaining edges are traversed twice or not at all.

Note 2. Before proving that some parity inequalities (24) induce facets of GRP(G), we will describe two types of GRP tours that satisfy them with equality. Recall that $|F|$ is odd, and consider a cut-set $\delta(S)$ such that the graphs $G(S)$ and $G(V\setminus S)$ are connected. GRP tours satisfying (24) with equality traverse the cut-set $\delta(S)$ in the following two ways:
Type 1: All edges in F plus one edge of $\delta(S)\setminus F$ are traversed once ($x_e = 1,\ y_e = 0$) while all other edges of $\delta(S)$ are either traversed twice ($x_e = y_e = 1$) or not traversed ($x_e = y_e = 0$).
Type 2: All edges in F except one of them are traversed once, while all other edges in $\delta(S)$ are traversed twice or not at all.

Given a "traversal" of the cut-set $\delta(S)$ as described above, let $V^o \subset V\setminus S$ be the set of vertices incident with an odd number of the edges traversed once. Since the number of such edges is even, $|V^o|$ is also even, and there is a T-join in $G(V\setminus S)$ associated with V^o. This same process takes place in $G(S)$. Consider two copies of each edge in $G(V\setminus S)$ and in $G(S)$ not belonging to the T-joins. All these edges plus the two T-joins, plus the "traversal" of $\delta(S)$, define a GRP tour satisfying (24) with equality.

Theorem 8. *Parity inequalities (24), for all $S\subset V$ and for all $F\subseteq\delta(S)$ with $|F|$ odd, are facet-inducing for GRP(G) if subgraph $G(S)$ and $G(V\setminus S)$ are 3-edge connected.*

Proof. Inequalities (24) can be written as:

$$(x-y)(\delta(S)\setminus F)-(x-y)(F) \geq 1-|F|. \tag{25}$$

After substituting $x_e = 1$ for all $e\in E_R$ in (25) we obtain the following equivalent inequality:

$$(x-y)(\delta_{NR}(S)\setminus F)-(x-y)(F_{NR})-y(\delta_R(S)\setminus F)+y(F_R) \geq 1-|F_{NR}|-|\delta_R(S)\setminus F|. \tag{26}$$

Let us suppose there is another valid inequality $ax+by\geq c$ such that,

$$\left\{(x,y)\in \mathrm{GRP}(G):\ (x-y)(\delta(S)\setminus F)-(x-y)(F)=1-|F|\right\} \subseteq$$
$$\subseteq \left\{(x,y)\in \mathrm{GRP}(G):\ \ ax+by=c\right\}.$$

We have to prove that inequality $ax+by\geq c$ is a linear combination of equalities (6) and (26). It can be seen (by adding the equalities (6), each one multiplied by $a_e\in\mathbb{R}$, and inequality (6) multiplied by $\lambda\in\mathbb{R}$, and then comparing with $ax+by\geq c$) that this is equivalent to proving that

$$\begin{aligned}
b_e = 0, &\quad \forall e\in E_R\setminus\delta(S),\\
a_e = b_e = 0, &\quad \forall e\in E_{NR}\setminus\delta(S),\\
b_e = \lambda, &\quad \forall e\in F_R,\\
a_e = -\lambda,\ b_e = \lambda, &\quad \forall e\in F_{NR},\\
b_e = -\lambda, &\quad \forall e\in \delta_R(S)\setminus F,\\
a_e = \lambda,\ b_e = -\lambda, &\quad \forall e\in \delta_{NR}(S)\setminus F,\\
c = \sum_{e\in E_R} a_e + \lambda\left(1-|F_{NR}|-|\delta_R(S)\setminus F|\right).
\end{aligned}$$

Let $e\in E_{NR}\setminus\delta(S)$. Given that graphs $G(S)$ and $G(V\setminus S)$ are 3-edge connected, they remain connected after deleting edge e. Moreover, there exists a GRP tour T in $G\setminus\{e\}$ that satisfies (24) with equality (see Note 2). The GRP tour T^{+2e} also satisfies (24) with equality. By subtracting the equations $ax+by=c$ corresponding to both tours, we obtain $a_e+b_e=0 \quad \forall e\in E_{NR}\setminus\delta(S)$.

Let T be any GRP tour constructed as described in Note 2, which traverses all the edges in $G(V\setminus S)$ and $G(S)$. Let $\mathcal{T}$ denote the set of required edges in the T-join associated with T. For each cycle C in either $G(V\setminus S)$ or $G(S)$, the tour $T-C$ obtained by removing one copy of each edge in C from T is also even and connected. However, it may not traverse the required edges in $\mathcal{T}\cap C$. Let T^* be the tour obtained from $T-C$ by adding two copies of each edge in $\mathcal{T}\cap C$. T^* is a GRP tour that satisfies (24) with equality. By subtracting the equations $ax+by=c$ corresponding to T and T^*, we find that

$$b(C\setminus(\mathcal{T}\cap C))-b((\mathcal{T}\cap C))=0.$$

Adding $2b(\mathcal{T}\cap C)$ to both terms yields

$$b(C)=2b((\mathcal{T}\cap C)),$$

for each cycle C in either $G(V\setminus S)$ or $G(S)$.

Let $e\in E(V\setminus S)$. Since $G(V\setminus S)$ is a 3-connected graph, there exist two edge-disjoint paths $\mathcal{P}_1,\mathcal{P}_2$ that join the end-nodes of e using edges other than e. Consider the three cycles $\mathcal{P}_1\cup\{e\}$, $\mathcal{P}_2\cup\{e\}$, and $\mathcal{P}_1\cup\mathcal{P}_2$, for which the equation $b(C)=2b(\mathcal{T}\cap C)$ holds. If $e\notin\mathcal{T}$ we have:
$b(\mathcal{P}_1)+b_e = 2b(\mathcal{T}\cap\mathcal{P}_1)$,
$b(\mathcal{P}_2)+b_e = 2b(\mathcal{T}\cap\mathcal{P}_2)$, and
$b(\mathcal{P}_1)+b(\mathcal{P}_2) = 2b(\mathcal{T}\cap\mathcal{P}_1)+2b(\mathcal{T}\cap\mathcal{P}_2)$,
which implies $b_e=0$. If $e\in\mathcal{T}$, we have:
$b(\mathcal{P}_1)+b_e = 2b(\mathcal{T}\cap\mathcal{P}_1)+2b_e$, $b(\mathcal{P}_2)+b_e = 2b(\mathcal{T}\cap\mathcal{P}_2)+2b_e$ and $b(\mathcal{P}_1)+b(\mathcal{P}_2) = 2b(\mathcal{T}\cap\mathcal{P}_1)+2b(\mathcal{T}\cap\mathcal{P}_2)$,
which again leads to $b_e=0$. Hence, we have $b_e=0$ for all $e\in E(V\setminus S)$. Similarly, we can show that $b_e=0$ for each edge $e\in E(S)$. Furthermore, since $a_e+b_e=0$ holds for all $e\in E_{NR}\setminus\delta(S)$, we conclude that $a_e=b_e=0$ for all $e\in E_{NR}\setminus\delta(S)$.

Let $e\in\delta_{NR}(S)$. If $e\in F_{NR}$ or $e\in\delta_{NR}(S)\setminus F$, there exists a GRP tour T that does not traverse e and satisfies (24) with equality. The tour T^{+2e} also satisfies (24) with equality. By subtracting the equations $ax+by=c$ corresponding to both tours, we obtain $a_e+b_e=0$ for all $e\in\delta_{NR}(S)$.

Now, let us suppose there exists $e_1,e_2\in F$. Consider T^1, the GRP tour of Type 2 in Note 2 that traverses once all the edges in F except e_1, which is traversed twice. Similarly, consider T^2 for edge e_2. Both tours satisfy (24) with equality. By subtracting the equalities $ax+by=c$ corresponding to T^1 and T^2, and considering that $a_e=b_e=0$ for all edges $e\in E_{NR}\setminus\delta(S)$ and $b_e=0$ for all $e\in E_R\setminus\delta(S)$, we obtain that $\sum_{e\in E_R\setminus\delta(S)} a_e+b_{e_1} = \sum_{e\in E_R\setminus\delta(S)} a_e+b_{e_2}$. Hence, $b_{e_1}=b_{e_2}$. By iterating this argument, we obtain $b_e=\lambda$ for all $e\in F$, where λ is a certain constant value. Furthermore, since $a_e+b_e=0$ for each $e\in\delta_{NR}(S)$, we have $a_e=-\lambda$ for all $e\in F_{NR}$. Note that this is obviously true if F contains only one edge.

Let us suppose there are $e_1,e_2\in\delta(S)\setminus F$. Let T^1 be the GRP tour of Type 1 in Note 2 that traverses once the edges in $F\cup\{e_1\}$ and twice the edge e_2. Similarly, let T^2 be the GRP tour that traverses once the edges in $F\cup\{e_2\}$ and twice e_1. Both tours satisfy (24) with equality. By subtracting the equalities $ax+by=c$ corresponding to T^1 and T^2, we obtain $\sum_{e\in E_R\setminus\delta(S)} a_e+b_{e_2} = \sum_{e\in E_R\setminus\delta(S)} a_e+b_{e_1}$. Hence, we have $b_{e_1}=b_{e_2}$. By iterating this argument, we obtain $b_e=\mu$ for all $e\in\delta(S)\setminus F$ and, if $e\in\delta_{NR}(S)$, $a_e=-\mu$, where μ is a certain constant value. Again, this is obviously true if $\delta(S)\setminus F$ contains only one edge.

Now, let us consider the case where $|F|\geq 1$. Let $e_1\in F$. If there exists $e_2\in\delta(S)\setminus F$, we can subtract the equalities $ax+by=c$ corresponding to the two GRP tours: T^1, which traverses once the edges in $F\cup\{e_2\}$, and T^2, which traverses once the edges in $F\setminus\{e_1\}$ and twice e_1,e_2. This subtraction leads to $\sum_{e\in E_R\setminus\delta(S)} a_e = \sum_{e\in E_R\setminus\delta(S)} a_e+b_{e_1}+b_{e_2}$, and hence, $b_{e_1}+b_{e_2}=0=\lambda+\mu$ and we find that $\mu=-\lambda$. Note that if $\delta(S)\setminus F=\emptyset$, there is no parameter μ.

Furthermore, if we replace the (x, y) values corresponding to any of the previous tours T and the values for a_e, b_e obtained above into the equation $ax + by = c$, we have

$$\sum_{e\in E_R} a_e + \alpha\left(1 - |F_{NR}| - |\delta_R(S)\setminus F|\right) = c,$$

and we have completed the proof. ⧫

Note 3. Theorem 8 also applies when one of the two shores S or $V \setminus S$ consists of only one vertex.

Theorem 9. *The following is a complete formulation for the GRP:*

$$\left.\begin{array}{rl} x_e = 1, & \forall e \in E_R \\ x_e \geq y_e, & \forall e \in E_{NR} \\ \textit{Connectivity inequalities} & (4) + (5) \\ \textit{Parity inequalities} & (24) \\ x_e \in \{0,1\}, & \forall e \in E \\ y_e \in \{0,1\}, & \forall e \in E_{NR} \end{array}\right\} \tag{27}$$

Proof. We need to prove that any solution (x^*, y^*) of (27) is a feasible solution for the GRP. (x^*, y^*) is a binary vector representing a graph on the edges of G that satisfies the following conditions: the second copy of an edge e, $y^*_e = 1$, exists only if the first copy exists ($x^*_e \geq y^*_e$), and each required edge must be traversed ($x^*_e = 1$, $\forall e \in E_R$). Now we need to prove that the graph represented by (x^*, y^*) is Eulerian. Suppose that the graph represented by (x^*, y^*) is not connected. Then there exist a cut-set $\delta(S)$ that is not traversed, i.e., $(x^* + y^*)(\delta(S)) = 0$. Since at least one of the two shores S or $V \setminus S$ contains required edges, let us suppose for example $E(V \setminus S)$, and some edge in $E(S)$ is traversed, say $x^*_f = 1$ for any $f \in E(S)$. Then, the corresponding connectivity inequality (4) (if $f \in E_R$) or (5) (if $f \in E_{NR}$) is not satisfied by (x^*, y^*).

Let us now suppose that the graph represented by (x^*, y^*) is not even. Then, there exist, at least, a cut-set $\delta(S)$ such that $(x^* + y^*)(\delta(S))$ is an odd number. Let $F \subseteq \delta(S)$ be the set of edges $e \in \delta(S)$ satisfying $x^*_e = 1$ and $y^*_e = 0$. Note that $|F|$ is odd. Note also that the edges in the set $\delta(S) \setminus F$ satisfy $x^*_e = y^*_e = 1$ or $x^*_e = y^*_e = 0$, i.e., $x^*_e - y^*_e = 0$. Then, (x^*, y^*) does not satisfy the parity inequality (24) corresponding to $\delta(S)$ and set F because
$(x^* - y^*)(\delta(S) \setminus F) = 0$ and $(x^* - y^*)(F) - |F| + 1 = 1$. ⧫

3.2. *p*-connectivity inequalities

The constraints described in this section are an extension of those introduced in Corberán et al. (2013) for the MBCPP to cut off fractional solutions similar to the one described as follows. Consider the GRP instance shown in Fig. 1(a), in which vertex 1 is required, each thick line represents a required edge, and each thin line represents a non-required one. Consider the fractional solution (x^*, y^*) with values $x^*_{(1,2)} = y^*_{(1,2)} = x^*_{(1,4)} = y^*_{(1,4)} = x^*_{(2,4)} = y^*_{(2,4)} = 0.5$, and $x^*_{(2,3)} = y^*_{(2,3)} = x^*_{(4,5)} = y^*_{(4,5)} = 1$, and the remaining variables equal to zero. It can be seen that this fractional solution satisfies all the inequalities presented in previous sections but it is cut off with one of the p-connectivity inequalities we present in what follows.

Let $\{S_0, \dots, S_p\}$ be a partition of V such that $\delta(S_i) \cap E_R = \emptyset$ for all i. Assume we divide the set $\{0, 1, \dots, p\} = \mathcal{R} \cup \mathcal{N}$ (from 'Required' and 'Non-required') in such a way that $i \in \mathcal{R}$ if $(S_i)_R \neq \emptyset$ (i.e. if $G(S_i)$ contains isolated required vertices or required edges) and $i \in \mathcal{N}$ otherwise. Select one edge $e_i \in E(S_i)$ for every $i \in \mathcal{N}$. Note that $e_i \in E_{NR}$ and that $1 \leq |\mathcal{R}| \leq p + 1$, $0 \leq |\mathcal{N}| \leq p$, and $|\mathcal{R}| + |\mathcal{N}| = p + 1$. Note that all subsets in $\mathcal{R}$ have to be visited by all the solutions, while those in $\mathcal{N}$ will be visited necessarily by the solutions that traverse an edge inside them. The following inequality

$$(x + y)(\delta(S_0)) + 2\sum_{1\leq r<t\leq p} x(S_r : S_t) \geq 2\sum_{i\in\mathcal{N}} x_{e_i} + 2\,(|\mathcal{R}| - 1) \tag{28}$$

will be referred to as a p-connectivity inequality. The special case of inequalities (28) when all the subsets $G(S_i)$ with $i \in \mathcal{R}$ contain required edges are valid for the GRP from Theorem 1 because they are obtained from the corresponding p-connectivity inequality for the MBCPP,

$$(x + y)(\delta(S_0)) + 2\sum_{1\leq r<t\leq p} x(S_r : S_t) \geq 2\sum_{i=0, i\neq d}^{p} x_{e_i}, \tag{29}$$

after replacing the equalities $x_{e_i} = 1$ for all $i \in \mathcal{R}$. It can be seen that the general case of inequalities (28) are also valid for the GRP with a proof similar to that in Corberán et al. (2013) for the MBCPP.

This inequality with $p = 2$ and $|\mathcal{N}| = 1$ is represented in Fig. 1(b) and (c), where for each pair (a, b) associated with an edge e, a and b represent the coefficients of x_e and y_e, respectively.

Theorem 10. *p-connectivity inequalities (28) are facet-inducing for GRP(G) if subgraphs $G(S_i)$, $i = 0, \dots, p$, are 3-edge connected, $|(S_0 : S_i)| \geq 2$, $\forall\ i = 1, \dots, p$, and the graph induced by $V \setminus S_0$ is connected.*

Proof. Inequality (28) can be written as:

$$(x + y)(\delta(S_0)) + 2\sum_{1\leq r<t\leq p} x(S_r : S_t) - 2\sum_{i\in\mathcal{N}} x_{e_i} \geq 2|\mathcal{R}| - 2. \tag{30}$$

Let us suppose there is another valid inequality $ax + by \geq c$ such that

$$\Big\{ (x, y) \in \mathrm{GRP}(G) : \quad (x + y)(\delta(S_0)) + 2\sum_{1\leq r<t\leq p} x(S_r : S_t)$$
$$-2\sum_{i\in\mathcal{N}} x_{e_i} = 2|\mathcal{R}| - 2 \ \Big\} \subseteq$$
$$\subseteq \big\{(x, y) \in \mathrm{GRP}(G) : \quad ax + by = c\big\}.$$

We have to prove that inequality $ax + by \geq c$ is a linear combination of equalities (6) and inequality (30). It can be seen (by adding the equalities (6), each one multiplied by an $a_e \in \mathbb{R}$, and inequality (30) multiplied by a $\lambda \in \mathbb{R}$, and then equating it to $ax + by \geq c$) that this is equivalent to prove that

$$\begin{aligned} a_e = b_e &= \lambda, \quad \forall e \in \delta(S_0), \\ a_e = 2\lambda,\ b_e &= 0, \quad \forall e \in (S_r : S_t), 1 \leq r < t \leq p, \\ a_{e_i} = -2\lambda,\ b_{e_i} &= 0, \quad \forall i \in \mathcal{N}, \\ a_e = b_e &= 0, \quad \forall e \in E_{NR}(S_i),\ i = 0, \dots, p,\ e \neq e_i, \\ b_e &= 0, \quad \forall e \in E_R(S_i),\ i = 0, \dots, p, \\ c &= \sum_{e\in E_R} a_e + 2\lambda\,(|\mathcal{R}| - 1)\,. \end{aligned}$$

In the GRP tours used in this proof we will not describe how the edges in each set $E(S_i)$, $i = 0, \dots, p$ are traversed. It can be seen that all these tours can be completed by using T-joins in $E(S_i)$ plus two copies of each edge in $E(S_i)$ not belonging to the T-joins, as described in Note 2 for the parity inequalities.

Similar arguments to those used in the proof of Theorem 8 lead to prove that $a_e + b_e = 0$, for each $e \in E_{NR}(S_i)$, $i \in \mathcal{R}$, and for each $e \in E_{NR}(S_i) \setminus \{e_i\}$, $i \in \mathcal{N}$. Furthermore, using the 3-edge connectivity of each graph $G(S_i)$ (hence, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining the end-nodes of an edge $e \in E(S_i)$ with edges different from e), we obtain that $b_e = 0$ for each edge $e \in E(S_i)$. Hence, we have $a_e = b_e = 0$ for all $e \in E_{NR}(S_i)$, $i \in \mathcal{R}$, and for all $e \in E_{NR}(S_i) \setminus \{e_i\}$, $i \in \mathcal{N}$.

Let S_i and S_j, with $i, j \neq 0$, be two sets such that there is an edge $e \in (S_i : S_j)$. Note that $e \in E_{NR}$. For the sake of simplicity, let us assume $i \in \mathcal{R}$, $j \in \mathcal{N}$ (with the other possibilities we would proceed similarly). Since all the sets $(S_0 : S_k)$ are non-empty, and subgraph $G(S_j)$ is 3-edge connected, we can construct the GRP tour that traverses twice an edge $f \in (S_0 : S_j)$, traverses once the edge e_j, traverses all the required edges, and visits all the sets S_i, $i \in \mathcal{R}$ (see Fig. 2(a), where we assume $\mathcal{R} = \{0, \dots, |\mathcal{R}|-1\}$ and $\mathcal{N} = \{|\mathcal{R}|, \dots, p\}$). This tour satisfies inequality (30) as an equality. If we subtract the equality $ax + by = c$

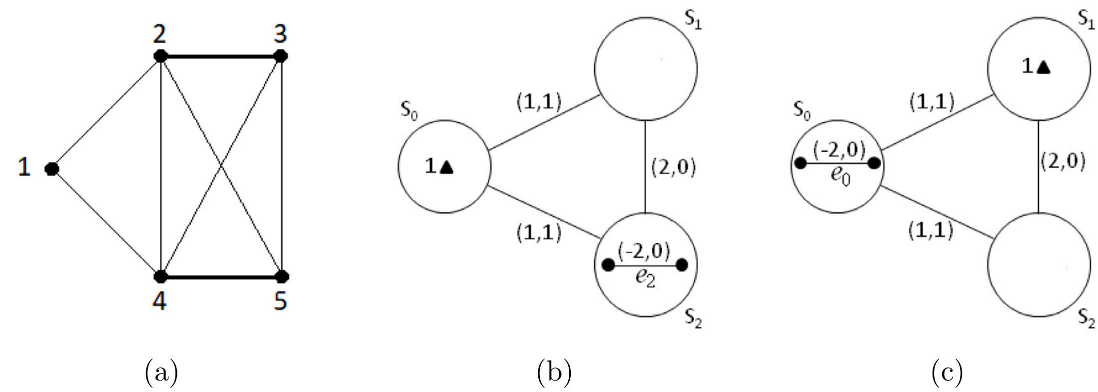

Fig. 1. 2-connectivity inequalities.

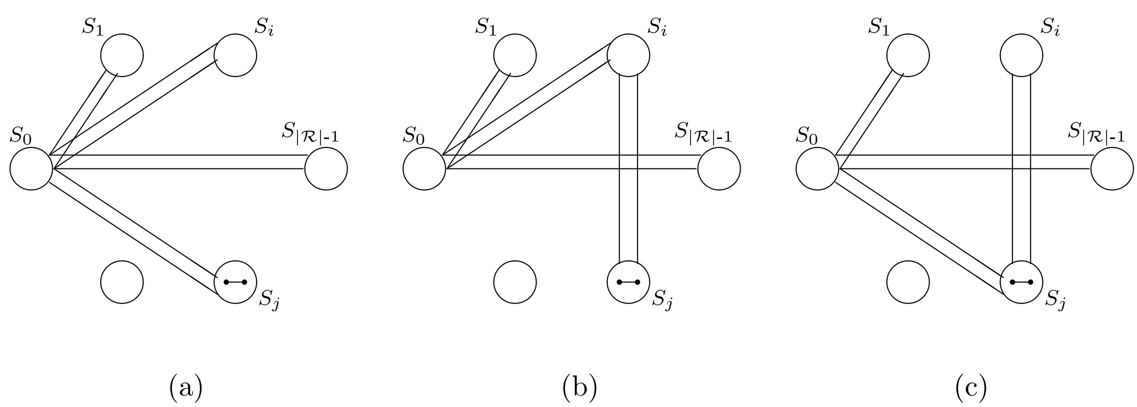

Fig. 2. GRP tours satisfying (30) with equality.

corresponding to this tour from the one corresponding to the GRP tour obtained after removing the two traversals of f and all the traversals of edges in $E(S_j)$, which also satisfies inequality (30) as equality, we get $a_f + b_f + a_{e_j} = 0$. We construct two more GRP tours satisfying (30) with equality such as those depicted in Fig. 2(b) and (c). By comparing (i.e., by subtracting their corresponding equalities $ax + by = c$) the GRP tours (a) and (b), we obtain $a_{0j} + b_{0j} = a_{ij} + b_{ij} = -a_{e_j}$, and by comparing (a) and (c) we obtain $a_{0i} + b_{0i} = a_{ij} + b_{ij} = -a_{e_j}$, where a_{kl} (b_{kl}) represents the coefficient of the variable x (y) corresponding to any edge in $(S_k : S_l)$. Given that the graph induced by $V \setminus S_0$ is connected, we can iterate this argument to conclude that $a_e + b_e = 2\lambda$ for every edge $e \in (S_i : S_j)$ (including $(S_0 : S_i)$), and $a_{e_i} = -2\lambda$ for each e_i, $i \in \mathcal{N}$, where λ is a certain constant value. Given that graph $G(S_i)$ is 3-edge connected and $b_e = 0$ for all edge $e \in E(S_i) \setminus \{e_i\}$, by comparing a GRP tour traversing e_i twice and the tour obtained by replacing the second traversal of e_i by the traversal of a path joining its end-vertices, we obtain $b_{e_i} = 0$ for each e_i, $i \in \mathcal{N}$.

For each $i \in \{1, 2, \dots, p\}$, let e_1, e_2 be two edges in $(S_0 : S_i)$ (recall that $|(S_0 : S_i)| \geq 2$ holds). We have already proved that $a_{e_1} + b_{e_1} = a_{e_2} + b_{e_2} = 2\lambda$. It can be seen that we can construct four GRP tours satisfying inequality (30) as an equality as follows. One tour traverses e_1 once and does not traverses e_2. Another tour traverses e_2 once and does not traverse e_1. By comparing these tours we obtain $a_{e_1} = a_{e_2}$ and, hence, $b_{e_1} = b_{e_2}$. The third tour traverses both e_1 and e_2 once, and the fourth one traverses e_1 twice and does not traverse e_2. By comparing them, we obtain $a_{e_2} = b_{e_1}$ and, hence, also $a_{e_1} = b_{e_2}$, and $a_{e_1} = b_{e_1} = a_{e_2} = b_{e_2} = \lambda$. Therefore, $a_e = b_e = \lambda$ for each edge $e \in (S_0 : S_i)$, $i = 1, \dots, p$, i.e., for each edge $e \in \delta(S_0)$.

As above, let S_i and S_j, with $i, j \neq 0$, be two sets such that there is an edge $e = (u, v) \in (S_i : S_j)$ (again with $i \in \mathcal{R}$, $j \in \mathcal{N}$, for example). There is a GRP tour T that traverses once edge e, an edge $a_i \in (S_0 : S_i)$, and an edge $a_j \in (S_0 : S_j)$ and satisfies inequality (30) as an equality. If we remove from T the traversal of $e = (u, v)$ and add the traversal of the edges in a path joining u and v formed with edges a_i and a_j, plus some edges in $G(S_0)$, $G(S_i)$ and $G(S_j) \setminus \{e_j\}$ (if any of these last edges is traversed three times, two copies would be removed), we obtain another GRP tour satisfying (30) as equality. By comparing both tours we obtain $a_e = b_{e_i} + b_{e_j} = 2\lambda$, which implies $b_e = 0$ (recall that $a_e + b_e = 2\lambda$). Hence, $a_e = 2\lambda, b_e = 0$, for each edge $e \in (S_i : S_j)$, $i \neq j$.

Furthermore, if we replace in $ax+by = c$ the (x, y) values corresponding to any of the previous tours T and the values for a_e, b_e obtained above, we have

$$\sum_{e \in E_R} a_e + 2\lambda (|\mathcal{R}| - 1) = c,$$

and we are done. ♦

3.3. K-C inequalities

K-C inequalities were introduced and proved to be facet-inducing for the undirected Rural Postman Problem (RPP) in Corberán and Sanchis (1994). In this section, we describe a new version of these inequalities and prove they are valid and facet-inducing for the GRP.

Consider the GRP instance shown in Fig. 3, in which each thick line represents a required edge, each thin line represents a non-required one, and each large circle represents an arbitrary subgraph containing at least a required edge or vertex.

Let (x^*, y^*) be the fractional solution with $x^*_e = 1$, $y^*_e = 0$ for the required edges, $x^*_{(2,4)} = y^*_{(2,4)} = x^*_{(3,5)} = y^*_{(3,5)} = 0.5$ and $x^*_{(6,7)} = 1, y^*_{(6,7)} = 0$. This solution is "connected" but is not "even" at vertex 2 nor at vertex 3. Furthermore, it cannot be cut off with parity inequalities: For example, associated with the cut-set $\delta(\{2\})$ and $F = \{(1,2), (2,3), (2,4)\}$ we have the following parity inequality (24)

$$0 \geq x_{(1,2)} - y_{(1,2)} + x_{(2,3)} - y_{(2,3)} + x_{(2,4)} - y_{(2,4)} - 3 + 1,$$

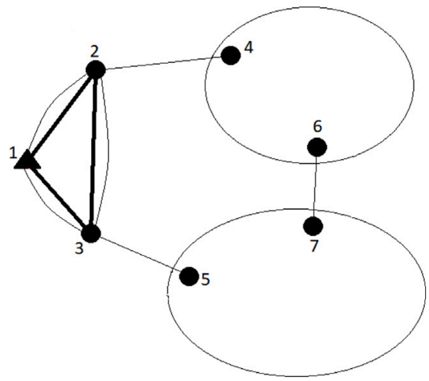

Fig. 3. A GRP instance to illustrate K-C inequalities.

which is not violated by (x^*, y^*) (as $0 \geq 0$ holds). Note that the fractional solution similar to (x^*, y^*) except for $x^*_{(2,4)} = x^*_{(3,5)} = 1$, $y^*_{(2,4)} = y^*_{(3,5)} = 0$, is indeed cut off by the above parity inequality. It can also be seen that (x^*, y^*) satisfies all the p-connectivity inequalities (28). However, the fractional solution similar to (x^*, y^*) except for $x^*_{(6,7)} = y^*_{(6,7)} = 0.5$, is indeed cut off by a p-connectivity inequality. We will see that (x^*, y^*) is cut off with the inequalities presented in this section.

Let $\{S_0, \dots, S_K\}$, with $K \geq 3$, be a partition of V such that $\delta(S_i) \cap E_R = \emptyset$ for all $i = 1, 2, \dots, K-1$. Assume we divide the set $\{1, \dots, K-1\} = \mathcal{R} \cup \mathcal{N}$ (from 'Required' and 'Non-required') in such a way that $i \in \mathcal{R}$ if $(S_i)_R \neq \emptyset$ and $i \in \mathcal{N}$ otherwise, and select one edge $e_i \in E(S_i)$ for every $i \in \mathcal{N}$. Note that $e_i \in E_{NR}$, $0 \leq |\mathcal{R}| \leq K-1$ and $0 \leq |\mathcal{N}| \leq K-1$, and $|\mathcal{R}| + |\mathcal{N}| = K-1$. As for the p-connectivity inequalities, note that all subsets in $\mathcal{R}$ have to be visited by all the solutions, while those in $\mathcal{N}$ will be visited necessarily by the solutions that traverse an edge inside them. Let $F \subseteq (S_0 : S_K)$ be a set of edges, with $|F| \geq 2$ and even. The K-C inequalities for the GRP are defined as:

$$\begin{aligned}&(K-2)(x-y)\Big((S_0 : S_K) \setminus F\Big) - (K-2)(x-y)(F) + \\ &+ \sum_{\substack{0 \leq i < j \leq K \\ (i,j) \neq (0,K)}} \Big((j-i)x(S_i : S_j) + (2-j+i)y(S_i : S_j)\Big) \\ &\geq 2\sum_{i \in \mathcal{N}} x_{e_i} + 2|\mathcal{R}| - (K-2)|F|. \end{aligned} \tag{31}$$

The coefficients and structure of the K-C inequalities are shown in Fig. 4, where, for the sake of simplicity, we assume $\mathcal{R} = \{1, \dots, |\mathcal{R}|\}$ and $\mathcal{N} = \{|\mathcal{R}|+1, \dots, K-1\}$. Edges in F are represented by thick lines. For each pair (a, b) associated with an edge e, a and b represent the coefficients in the inequality of x_e and y_e, respectively.

The special case of the K-C inequalities (31) when all the subsets $G(S_i)$ with $i \in \mathcal{R}$ contain required edges are valid for the GRP from Theorem 1 because they are obtained from the corresponding K-C inequality for the MBCPP,

$$\begin{aligned}&(K-2)(x-y)\Big((S_0 : S_K) \setminus F\Big) - (K-2)(x-y)(F) + \\ &+ \sum_{\substack{0 \leq i < j \leq K \\ (i,j) \neq (0,K)}} \Big((j-i)x(S_i : S_j) + (2-j+i)y(S_i : S_j)\Big) \geq 2\sum_{i=1}^{K-1} x_{e_i} - (K-2)|F|,\end{aligned}$$

after replacing the equalities $x_{e_i} = 1$ for all $i \in \mathcal{R}$. In general, it can be seen that K-C inequalities (31) are valid for the GRP with a proof similar to that in Corberán et al. (2013).

It is easy to see that, when $K = 2$, the K-C inequality (31) reduces to a connectivity inequality (4) when $(S_0 \cup S_2)_R \neq \emptyset$ and $(S_1)_R \neq \emptyset$, and to a connectivity inequality (5) when $(S_1)_R = \emptyset$.

Regarding the fractional solution (x^*, y^*) described above for the instance represented in Fig. 3, note that the K-C inequality (31) with $K = 3$ and $F = \{(1,2), (2,3)\}$,

$$-x_{(1,2)} + y_{(1,2)} - x_{(2,3)} + y_{(2,3)} + x_{(2,4)} + y_{(2,4)} + x_{(6,7)} + y_{(6,7)} + x_{(3,5)} + y_{(3,5)} \geq 4-2,$$

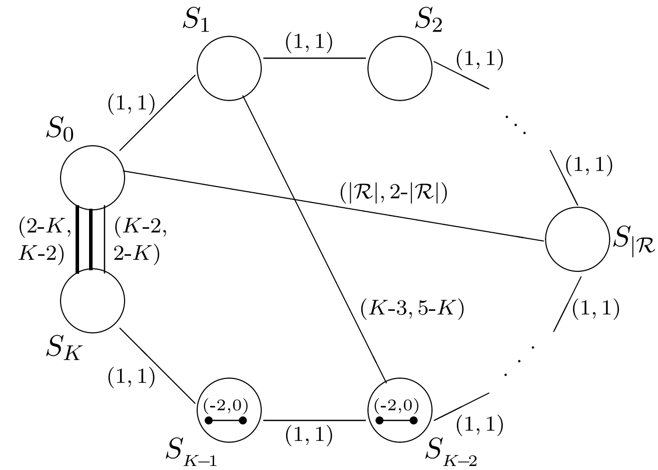

Fig. 4. Coefficients of the K-C inequality.

is violated by (x^*, y^*) as $-2 + 3 < 2$ holds.

Note 4. Les us describe several types of GRP tours that satisfy the K-C inequality (31) with equality that will be used in the proof of Theorem 11. We do not detail how the edges in each set $E(S_i)$ are traversed. Note that if subgraphs $G(S_i)$, $i = 0, \dots, K$, are 3-edge connected, all these tours can be completed by using T-joins as described in Note 2 for the parity inequalities.

The GRP tours represented in Fig. 5(a), (b) and (c) traverse each edge $e \in F$ (represented in bold lines in the figures) exactly once ($x_e = 1$, $y_e = 0$) and each edge not in F either twice ($x_e = y_e = 1$) or none ($x_e = y_e = 0$) and, therefore, these tours traverse the set $(S_0 : S_K)$ an even number of times. These GRP tours traverse each edge e_i twice, for all $i \in \mathcal{N}$, and connect all the sets S_j, $j = 0, 1, 2, \dots, K-1$ in an "even" way. It can be seen that these tours satisfy (31) with equality.

The GRP tours represented in Fig. 5(d) are similar to those represented in Fig. 5(a) except that they do not traverse a given edge e_p, $p \in \mathcal{N}$ nor reaches the corresponding set S_p. These tours also satisfy (31) with equality. The GRP tours represented in Fig. 5(e) and (f) traverse the set $(S_0 : S_K)$ an odd number of times, and they connect the sets S_i with a "path" from S_0 to S_K, traversing each edge e_i, $i \in \mathcal{N}$ once. The tours in (e) traverse exactly once each edge in F and one more edge in $(S_0 : S_K)$ while the remaining edges in $(S_0 : S_K)$ are either traversed twice or not traversed. The tours in (f) traverse exactly once each edge in F except one of them while the remaining edges in $(S_0 : S_K)$ either twice or none. All these tours satisfy (31) with equality.

Theorem 11. *K-C inequalities (31) are facet-inducing for GRP(G) if subgraphs $G(S_i)$, $i = 0, \dots, K$, are 3-edge connected and $|(S_i : S_{i+1})| \geq 2$ for $i = 0, \dots, K-1$.*

Proof. After substituting in (31) $x_e = 1$ for each edge in $(S_0 : S_K)_R$ and in F_R, we obtain the following equivalent inequality:

$$\begin{aligned}&(K-2)(x-y)\Big((S_0 : S_K)_{NR} \setminus F\Big) - (K-2)(x-y)(F_{NR}) - \\ &-(K-2)y\Big((S_0 : S_K)_R \setminus F\Big) + (K-2)y(F_R) + \\ &+ \sum_{\substack{0 \leq i < j \leq K \\ (i,j) \neq (0,K)}} \Big((j-i)x(S_i : S_j) + (2-j+i)y(S_i : S_j)\Big) - 2\sum_{i \in \mathcal{N}} x_{e_i} \geq \\ &\geq 2|\mathcal{R}| - (K-2)|F_{NR}| - (K-2)|(S_0 : S_K)_R \setminus F|. \end{aligned} \tag{32}$$

Let us suppose there is another valid inequality $ax + by \geq c$ such that

$$\Big\{(x, y) \in \mathrm{GRP}(G) : \ (x, y) \text{ satisfies (31) with equality}\Big\}$$

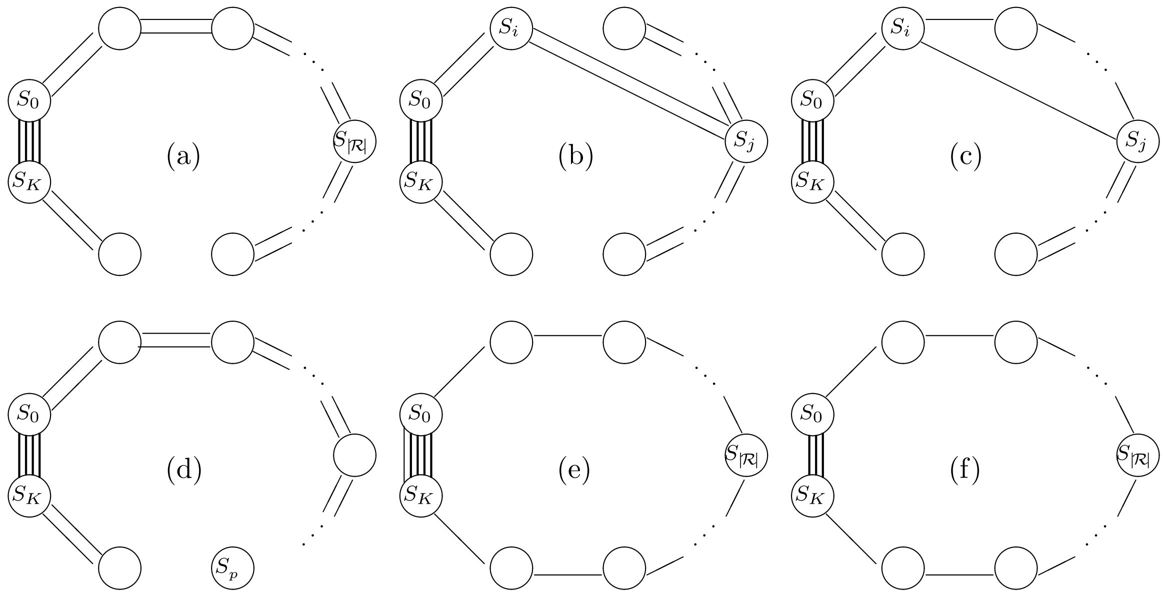

Fig. 5. GRP tours described in Note 4 and used in the proof of Theorem 11.

$$\subseteq \left\{(x,y) \in \mathrm{GRP}(G): \;\; ax + by = c\right\}.$$

We have to prove that inequality $ax + by \geq c$ is a linear combination of the equalities (6) and inequality (32).

It can be seen (by adding the equalities (6), each one multiplied by an $a_e \in \mathbb{R}$, and inequality (32) multiplied by a $\lambda \in \mathbb{R}$, and then equating it to $ax + by \geq c$) that this is equivalent to prove that

$$\begin{aligned}
a_e = \lambda(K-2),\; b_e &= -\lambda(K-2), && \forall e \in (S_0 : S_K)_{NR} \setminus F,\\
b_e &= -\lambda(K-2), && \forall e \in (S_0 : S_K)_{R} \setminus F,\\
a_e = -\lambda(K-2),\; b_e &= \lambda(K-2), && \forall e \in F_{NR},\\
b_e &= \lambda(K-2), && \forall e \in F_{R},\\
a_e = \lambda(j-i),\; b_e &= -\lambda(2-j+i), && \forall e \in (S_i : S_j),\;\; 0 \leq i < j \leq K,\;\; (i,j) \neq (0,K),\\
a_e = -2\lambda,\; b_e &= 0, && \forall e_i,\; i \in \mathcal{N},\\
a_e = b_e &= 0, && \forall e \in E(S_i)_{NR},\; i = 0,1,\ldots,K,\; e \neq e_i,\\
b_e &= 0, && \forall e \in E(S_i)_{R},\; i = 0,1,\ldots,K,\\
c &= \sum_{e \in E_R} a_e + 2\lambda|\mathcal{R}| - \lambda(K-2)|F_{NR}| \\
&\quad -\lambda(K-2)|(S_0 : S_K)_R \setminus F|.
\end{aligned}$$

Similar arguments to those used in the proof of Theorems 8 and 10 using the 3-edge connectivity of graphs $G(S_i)$ lead to prove that $a_e = b_e = 0$, for each $e \in E_{NR}(S_i)$, $i \in \mathcal{R}$, and for each $e \in E_{NR}(S_i) \setminus \{e_i\}$, $i \in \mathcal{N}$, and, also that $b_e = 0$ for each edge $e \in E(S_i)$.

For each $i \in \mathcal{N}$, let T^1 be the GRP tour of type (a) in Note 4 traversing twice an edge in each set $(S_j : S_{j+1})$, $j \neq i$, and let T^2 be the GRP tour of type (d) traversing twice the same edge in each set $(S_j : S_{j+1})$, $j \neq i-1, i$. By subtracting the equations $ax + by = c$ corresponding to both tours, we obtain that $a_e + b_e + a_{e_i} + b_{e_i} = 0$ for all $e \in (S_{i-1} : S_i)$. If we consider the GRP tour T^3 of type (a) traversing twice an edge in each set $(S_j : S_{j+1})$, $j \neq i-1$, by subtracting the equations corresponding to T^2 and T^3, we conclude $a_e + b_e + a_{e_i} + b_{e_i} = 0$, for all $e \in (S_i : S_{i+1})$. For each $i \in \mathcal{R}$, let T^1 and T^3 two GRP tours of type (a) defined as above. By comparing them (i.e., by subtracting their corresponding equalities $ax + by = c$) we conclude that $a_e + b_e = a_f + b_f$ for all $e \in (S_{i-1} : S_i)$ and $f \in (S_i : S_{i+1})$. By iterating this argument, we obtain that $a_e + b_e = 2\lambda$ for all $e \in (S_i : S_{i+1})$, $i = 1, \ldots, K-1$, and $a_{e_i} + b_{e_i} = -2\lambda$ for all $i \in \mathcal{N}$, where λ is a certain constant value.

For each $i \in \mathcal{N}$, let T^1 be the GRP tour of type (e) in Note 4 traversing edge $e_i = (u, v)$ once. Given that $G(S_i)$ is a 3-edge connected graph, we can find a path connecting u and v that does not traverse e_i. If we add this path plus one copy of e_i to T^1, we obtain a GRP tour T^2 also satisfying (32) with equality. By comparing both tours, and given that $a_e = b_e = 0$ for all $e \in E(S_i) \setminus \{e_i\}$, we obtain $b_{e_i} = 0$ and, therefore, $a_{e_i} = -2\lambda$.

For each $i \in \{0, 1, 2, \ldots, K-1\}$, let e, f be two edges in $E(S_i : S_{i+1})$ (there exist because $|(S_i : S_{i+1})| \geq 2$ holds). There are two GRP tours T^1 and T^2 of type (e) in Note 4 traversing edges e and f once respectively. By comparing both tours, we get $a_e = a_f$. Since we have proved that $a_e + b_e = 2\lambda = a_f + b_f$, we have $b_e = b_f$. Furthermore, let T^3 be a tour of type (a) traversing edge e twice and T^4 a similar tour traversing e and f once. By comparing these tours, we obtain $b_e = a_f$ and, since $a_f = a_e$, we get $a_e = b_e$. Therefore $a_e = b_e = \lambda$ for each edge $e \in E(S_i : S_{i+1})$, for all $i \in \{0, 1, 2, \ldots, K-1\}$.

Let $e \in (S_0 : S_K)_{NR}$ and let T be a GRP tour of type (f) in Note 4 that does not traverse edge e. The GRP tour T^{+2e} also satisfies (32) with equality, since $x_e = y_e = 1$ and the sum of the coefficients of both variables in (32) is zero. By comparing both tours, we obtain that $a_e + b_e = 0$, for all $e \in (S_0 : S_K)_{NR}$.

Let $e \in F_{NR}$. By comparing the GRP tour of type (a) traversing once all the edges in F and the GRP tour of type (f) traversing once all the edges in F, except edge e that is not traversed, we obtain that $a_e + \lambda(K-1) - \lambda = 0$. Hence, $a_e = -\lambda(K-2)$ and $b_e = \lambda(K-2)$.

Let $e \in (S_0 : S_K)_{NR} \setminus F$. By comparing the GRP tour of type (a) traversing once all the edges in F and the GRP tour of type (e) traversing once all the edges in $F \cup \{e\}$, we obtain that $\lambda(K-1) - \lambda = a_e$ and, hence, $b_e = -\lambda(K-2)$.

Let $e \in F_R$. By comparing the GRP tour of type (a) traversing once all the edges in F and the GRP tour of type (e) traversing once all the edges in F except edge e that is traversed twice, we obtain that $\lambda(K-1) = b_e + \lambda = 0$. Hence, $b_e = \lambda(K-2)$.

Let $e \in (S_0 : S_K)_R \setminus F$. By comparing the GRP tour of type (a) traversing once all the edges in F and twice edge e and the GRP tour of type (e) traversing once all the edges in $F \cup \{e\}$, we obtain that $a_e + b_e + \lambda(K-1) = a_e + \lambda$ and, hence, $b_e = -\lambda(K-2)$.

For each edge $e \in E(S_i : S_j)$, with $|i-j| > 1$, by comparing tours of type (a) and (c) in Fig. 5, we obtain $a_e = \lambda|i-j|$, and by comparing tours of type (b) and (c), we obtain that $b_e + \lambda\big(|i-j| - 1\big) = \lambda$ and, therefore, $b_e = \lambda\big(2 - |i-j|\big)$.

Finally, if we replace in $ax + by = c$ the (x, y) values corresponding to any of the previous tours T and the values for a_e, b_e obtained above, we have

$$\sum_{e\in E_R} a_e + 2\lambda|\mathcal{R}| - \lambda(K-2)|F_{NR}| - \lambda(K-2)|(S_0 : S_K)_R \setminus F| = c,$$

and we are done. ♦

4. A branch-and-cut algorithm

We describe in this section a branch-and-cut algorithm whose cutting-plane procedure uses inequalities from the polyhedral description presented in the previous sections.

The initial LP relaxation contains all the inequalities (6) and (7), the bounds 0,1 on the variables, a connectivity inequality (4) for each R-set V^i, and a parity inequality (24) with $S = \{v\}$ for each R-odd degree vertex v and $F = \delta_R(v)$. At each iteration of the cutting-plane algorithm, we use several separation procedures to find valid inequalities that are violated by the current LP solution (x^*, y^*).

As mentioned previously, the RPP and GRP benchmark instances are all simplified. Therefore all the computational experiments have been conducted on instances where $V_{NR} = \emptyset$ and $\cup V^i = V$. This implies that the inequalities presented before become slightly simpler.

4.1. Separation of connectivity inequalities

Since $V_{NR} = \emptyset$, connectivity inequalities of the type (5) are not needed and inequalities (4) are defined only for all $S = \left(\cup_{i\in T} V^i\right)$, with $T \subset \{1,\dots,Q\}$. These inequalities (4) can be separated exactly in polynomial time with the well-known Gomory and Hu algorithm (Gomory and Hu, 1961). This is done by finding a minimum weight cut in the weighted shrunk graph $G_s = (V_s, E_s, \bar{x}^* + \bar{y}^*)$ obtained by shrinking each R-set into a single vertex. A minimum weight cut in an undirected graph can be found by using the more efficient algorithms in Padberg and Rinaldi (1990) and Nagamochi et al. (1994).

Although polynomial, the above exact algorithm is quite time-consuming and we use it exceptionally. Instead, we use two faster heuristic algorithms. The first one is based on the computation of the connected components of the subgraph of G_s induced by the edges $e \in E_s$ with $\bar{x}^*_e + \bar{y}^*_e > \varepsilon$, where ε is a given parameter. For each set of vertices S in the original graph corresponding to a connected component such that $(x^* + y^*)(\delta(S)) < 2$, a violated connectivity inequality (4) is obtained. Note that, for integral solutions, this heuristic algorithm with $\varepsilon = 0$ finds any violated connectivity inequality if it exists.

A second heuristic consists of iteratively shrinking in G_s two components among which $\bar{x}^* + \bar{y}^* \geq 1$ into a single one, as long as possible. Next, we calculate the minimum cut from a given resulting component to all other ones. Again, if this minimum cut is less than 2 we have a violated connectivity inequality (4).

4.2. Separation of parity inequalities

Parity inequalities (24) can be separated in polynomial time. Note that if we change $x - y$ for x in

$$(x-y)(\delta(S)\setminus F) \geq (x-y)(F) - |F| + 1, \quad \forall S \subset V, \quad \forall F \subset \delta(S) \text{ with } |F| \text{ odd},$$

we obtain the cocircuit inequalities presented in Ghiani and Laporte (2000). These inequalities can be separated exactly in polynomial time with an algorithm (see Benavent et al. (2000), for example) based on the computation of odd minimum cuts on an auxiliary graph $\bar{G}^*$ in which each edge $e = (i,j)$ of G^* has been split into two edges: (i, u_e) with weight $x^*_e - y^*_e$ and (u_e, j) with weight $1-(x^*_e - y^*_e)$, and where some parity labels on the vertices are considered. Any odd cut-set (with respect to those labels) in the graph $\bar{G}^*$ with weight less than 1 provides a violated parity inequality (24). The minimal odd cut-sets can be computed with the classical Padberg–Rao procedure (Padberg and Rao, 1982) or with the improved one of Letchford et al. (2008).

For parity inequalities with $S = \{v\}$, $v \in V_R$, one can apply an exact and simple procedure (Ghiani and Laporte, 2000) to obtain the set F of edges corresponding to the maximally violated inequality associated with the cut-set $\delta(v)$, if it exists. The same procedure can be applied to find the set F corresponding to any subset $S \subset V$. In particular, we do so for all R-sets.

A heuristic algorithm based on the computation of the connected components of the support graph G^* induced by the edges with $\varepsilon < x^*_e - y^*_e < 1-\varepsilon$, where ε is a given parameter, is also used. For the set S of vertices of each connected component of G^*, the corresponding set F is found by applying the above procedure.

4.3. Separation of p-connectivity inequalities

When $V_{NR} = \emptyset$, the p-connectivity inequality (28) becomes simpler and reduces to

$$(x+y)(\delta(S_0)) + 2\sum_{1\leq r<t\leq p} x(S_r : S_t) \geq 2p \tag{33}$$

To separate these inequalities, we have devised the following heuristic algorithm. Let (x^*, y^*) be the current LP solution. Starting with $C^i = V^i$, $i = 1,\dots,Q$ (the R-sets of G), we iteratively shrink C^i and C^j if $(x^* + y^*)(C^i : C^j) \geq 2$, while possible. For each resulting set of vertices C^i we proceed as follows. We choose $S_0 = C^i$ and denote the remaining sets C^j as $S_1,\dots,S_p$. While $p > 2$, we try to merge pairs of sets S_i and S_j according to the following rules:

- If $i, j \neq 0$ and $x^*(S_i : S_j) \geq 1$, we merge S_i and S_j. Notice that if we shrink S_i and S_j, the LHS of the inequality (33) decreases by $2x^*(S_i : S_j)$ while the RHS decreases by 2.
- If $i = 0$ (or $j = 0$) and $(x^* + y^*)(S_0 : S_i) + \sum_{r\neq 0,i}(x^* - y^*)(S_i : S_r) \geq 2$, we make $S_0 := S_0 \cup S_i$. Note that if we shrink S_0 and S_i, the LHS of the inequality (33) decreases by $(x^* + y^*)(S_0 : S_i)$ since these variables disappear, and by $\sum_{r\neq 0,i}(x^* - y^*)(S_i : S_r)$ since each $2x^*(S_i : S_r)$ becomes $(x^* + y^*)(S_i : S_r)$, while the RHS is decreased by 2.

The p-connectivity inequality (33) associated with the resulting sets $S_0, S_1, \dots, S_p$ is checked for violation.

4.4. Separation of K-C inequalities

Since $V_{NR} = \emptyset$, in the K-C inequalities (31), we have $\mathcal{N} = \emptyset$ and $|\mathcal{R}| = K - 1$. Therefore these inequalities can be separated with an algorithm similar to the one proposed in Corberán et al. (2001) for the General Routing Problem (GRP).

4.5. The cutting-plane algorithm

At each iteration of the cutting plane algorithm the separation procedures are called in the following order:

1. The first heuristic algorithm for separating connectivity inequalities is applied with $\varepsilon = 0$. If no violated inequalities are found, it is called again with $\varepsilon = 0.25, 0.5$. If this fails, the second heuristic algorithm is run.
2. Parity inequalities are separated exactly for all subsets of vertices consisting of a single required vertex or an R-set.
3. The heuristic procedure for parity inequalities is applied with values of $\varepsilon = 0, 0.2, 0.4$, while it does not find any violated inequality.
4. The heuristic algorithm for separating the p-connectivity inequalities is run at every node whose depth is less than or equal to 6.

Table 1
Characteristics of the instances.

Set	Type	Vertices			Edges			R-sets		
		Aver.	Min	Max	Aver.	Min	Max	Aver.	Min	Max
UR500	RPP	446.0	298	499	1128.9	597	1526	35.3	1	99
UR750	RPP	665.7	452	749	1698.4	915	2314	55.7	1	140
UR1000	RPP	886.2	605	1000	2289.9	1122	3083	74.8	1	204
UR40 × 25	RPP	1000	1000	1000	2562	2551	2576	182.3	136	232
UR40 × 50	RPP	2000	2000	2000	5201.1	5159	5248	339.3	219	458
UR60 × 50	RPP	3000	3000	3000	7827.2	7796	7846	523	344	686
UG500	GRP	500	500	500	1213.1	858	1571	92.2	1	259
UG750	GRP	750	750	750	1826.6	1346	2288	135.0	1	412
UG1000	GRP	1000	1000	1000	2437.2	1765	3105	176.8	1	573
GTSP	GTSP	181.7	150	225	329.9	296	392	181.7	150	225

5. Only at the root node, if no violated connectivity inequalities have been found and the number of R-sets is not greater than 250, the heuristic algorithm for separating K-C inequalities is executed.
6. The exact procedure for parity inequalities is applied with different strategies:
    - For instances with up to 1000 vertices, if no violated parity inequalities are found and the depth of the node is no greater than 3.
    - For instances with $1000 < |V| \leq 2000$, if no more than two violated inequalities of any kind are found and the depth of the node is no greater than 3.
    - Finally, if $|V| > 2000$, only at the root node and when no violated inequalities of any kind are found.
7. The exact separation algorithm for connectivity inequalities is applied only at the root node if the previous heuristics fail and $|V| \leq 1000$.

Moreover, a tailing-off strategy is applied when the lower bound increases less than 0,0015% in the last 20 iterations. If $|V| \leq 2000$, the strong branching strategy with a maximum of 10 candidates is applied, otherwise, the pseudo reduced costs strategy is used to select the variable for branching. Best bound is the strategy chosen for selecting the node to study and Cplex cuts are turned off, as well as its heuristic.

5. Computational experiments

In this section, we present the computational experiments carried out to assess the performance of the new algorithm, denoted from here on **NewB&C**. The results are compared with those obtained by the exact procedure proposed in Corberán et al. (2007), denoted **B&C**, which, as far as we know, is the one producing the best results known for the RPP and GRP. First, we describe the instances that have been used for the comparison and then we show the results on these instances. Both the instances and computational results are available at https://www.uv.es/plani/instancias.htm.

5.1. The instances

Both algorithms were tested on the 79 instances defined on undirected graphs described in Corberán et al. (2007), 36 of them corresponding to RPP instances, 36 to GRP, and 7 to GTSP. The 36 RPP instances were generated as follows. First, $|V| \in \{500, 750, 1000\}$ points are randomly selected on a 1000×1000 grid. Then, for each vertex v, $d \in \{3, 4, 5, 6\}$ edges connecting v to its d nearest neighbors are added. If the resulting graph is not connected, the edges of five trees spanning the connected components are added. The costs of the edges are given by rounded Euclidean distances, and an edge is defined as required with a given probability $p_r \in \{0.25, 0.50, 0.75\}$. Since the graphs associated with these instances could contain vertices not incident with the required edges, the authors applied a simplification procedure similar to the one described in Christofides et al. (1981). Therefore, the number of vertices of the simplified instances may be less than the initial value set for $|V|$. The RPP instances are grouped into the sets denoted by UR500, UR750, and UR1000 in Table 1. The 36 GRP instances were generated as above, except that the graph was not simplified and all the vertices were considered required. These instances are grouped in three sets UG500, UG750, and UG1000, consisting of 12 instances each. Finally, the set named GTSP contains seven GTSP instances generated from well-known TSPLIB instances.

Since all the RPP instances were solved optimally by **B&C**, to compare its behavior with that of the new exact algorithm **NewB&C** we have generated three new sets of RPP instances with 1000, 2000, and 3000 vertices, respectively. Ten instances have been generated on a 40×25 grid, ten on a 40×50 grid, and ten on a 60×50 grid. All horizontal and vertical edges are included, while diagonal edges are included with a probability of 2/3. An edge is selected as required with a given probability $p_r \in \{0.2, 0.3\}$. If a vertex is not incident with required edges, one of its incident edges is randomly selected and declared as required. Fig. 6 shows the required edges of the RPP-GRID-40-25-3-4 instance, which is the fourth instance generated on a 40×25 grid with $p_r = 0.3$. Five instances are generated with each value of p_r and each grid for a total of $5{\times}2{\times}3 = 30$ very large instances. These three sets are named UR40 × 25, UR40 × 50, and UR50 × 60, respectively, and their characteristics, as well as those of the 36 RPP, 36 GRP, and 7 GTSP instances described above, are presented in Table 1. Note that the largest instances have 3000 vertices, 7846 edges, and 686 R-sets.

5.2. Computational results

Here we present the results obtained after running both algorithms on the same machine, an Intel(R) Core(TM) i7-7700HQ @2.80 GHz CPU and 16 GB RAM, using CPLEX Studio 12.10, on the sets of instances presented above with a time limit of one hour. Table 2 reports, for each set of instances and each algorithm, the number of instances solved to optimality, the average time in seconds, the average number of nodes of the search tree explored, and the average percentage gap between the final upper and lower bounds for the unsolved instances. Note that, in some of these unsolved instances, no feasible solution was found and, therefore, this gap could not be calculated. The number in brackets after the gap indicates the number of unsolved instances for which a feasible solution was found. The complete detailed results can be found in Appendix.

Regarding computing times, we can observe in Table 2 that **B&C** is, in general, faster than **NewB&C**. Note that, in some sets, the average time reported is considerably lower for **NewB&C**. However, if we did not take into account the unsolved instances, which account for 3600 s each, the average times would be similar or lower for **B&C**. This behavior can be observed in Fig. 7, that depicts the number of instances solved optimally against the computing time consumed. Looking at the number of optimal solutions found, **NewB&C**, which solves 96 out of 109 instances to optimality, clearly outperforms **B&C**, which is capable of solving only 86. Moreover, **NewB&C** is able to find feasible solutions

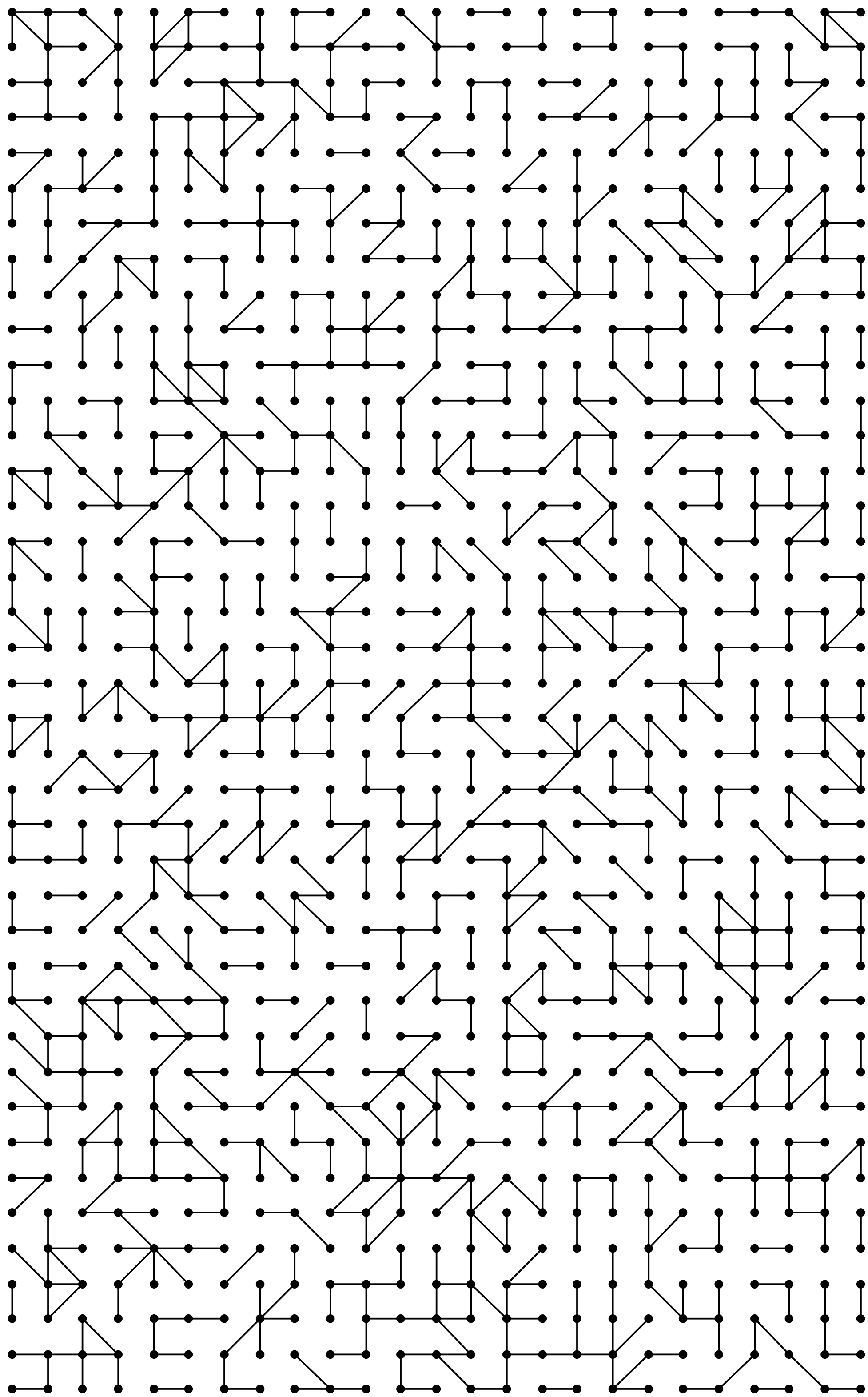

Fig. 6. Instance RPP-GRID-40-25-3-4.

for all the instances, while **B&C** fails to do so in 13 cases. A remarkable result is the one obtained for the GTSP instance TS225G, based on a TSPLIB instance that was deliberately designed to be especially difficult for cutting-plane algorithms. The new algorithm **NewB&C** is able to

Table 2
Results with the two branch-and-cut algorithms.

Set	B&C				NewB&C			
	Optima	Time	Nodes	Gap (%)	Optima	Time	Nodes	Gap (%)
UR500	12/12	2.9	7.1		12/12	40.8	2.0	
UR750	12/12	31.9	21.2		12/12	160.6	115.2	
UR1000	12/12	88.0	33.2		12/12	422.6	210.3	
UR40 × 25	9/10	975.6	158.4	0.50 (1)	10/10	347.6	53.8	
UR40 × 50	3/10	2803.7	125	0.19 (6)	9/10	2169.9	139.8	1.09 (1)
UR60 × 50	0/10	3600	14.2	0.14 (1)	0/10	3600	581.1	1.01 (10)
UG500	11/12	430.1	72.3	–	12/12	159.4	8.9	
UG750	10/12	772.7	53.8	0.84 (2)	12/12	963.4	214.5	
UG1000	9/12	1219.1	72.7	0.28 (1)	10/12	1240.6	286.7	0.50 (2)
GTSP	6/7	515.5	6424.6	2.60 (1)	7/7	9.5	1.0	

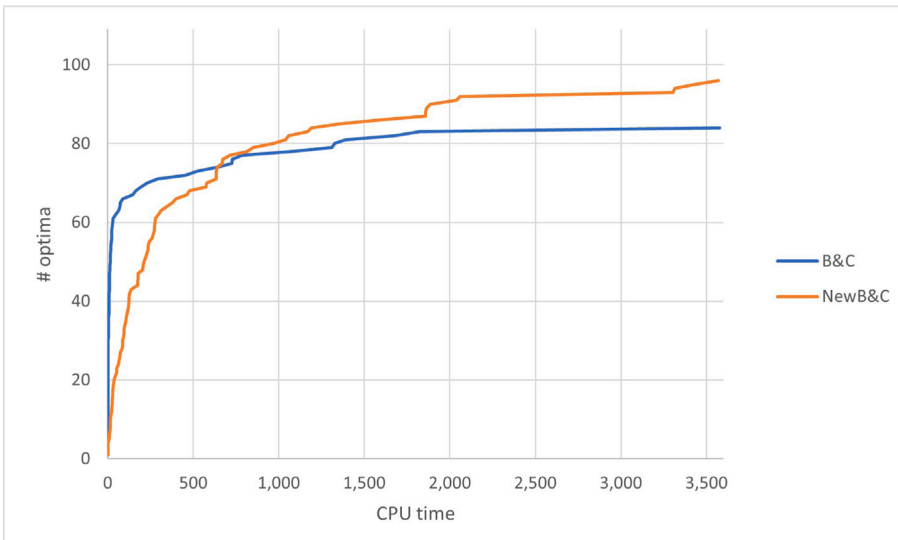

Fig. 7. Number of optimal solutions found vs computing time.

solve this instance to optimality in the root node in 18.8 s, while **B&C** could not solve it in one hour after exploring more than 44,000 nodes.

Overall, both algorithms are competent in solving large RPP and GRP instances. The **B&C** algorithm is a very sophisticated one that seems to be a better option for medium-sized instances, while **NewB&C** would be preferable for the very large ones.

6. Conclusions

In this work we have introduced a new formulation for the Rural Postman Problem (RPP) and the General Routing Problem (GRP). This formulation, used formerly in Corberán et al. (2013), uses two binary variables for each edge, representing its first and second traversal, respectively. We have studied the polytope associated with the feasible solutions and shown that several families of inequalities induce facets of it. With these inequalities, we have designed a branch-and-cut algorithm that has been tested on several sets of instances. After comparing its performance with the exact procedure proposed in Corberán et al. (2007), we can conclude that the formulation proposed is efficient for solving RPP and GRP instances of large size and seems promising for other arc routing problems on undirected graphs.

CRediT authorship contribution statement

Ángel Corberán: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing – original draft, Writing – review & editing. **Isaac Plana:** Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing – original draft, Writing – review & editing. **José M. Sanchis:** Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing – original draft, Writing – review & editing. **Paula Segura:** Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing – original draft, Writing – review & editing.

Data availability

The data instances and computational results are available at https://www.uv.es/plani/instancias.htm.

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Appendix. Detailed computational results

Here we report the complete computational results for each instance. Tables 3 to 6 show for each instance and algorithm the lower bound at the end of the root node (LB0), the final lower (LB) and upper (UB) bounds (if any has been found), the number of nodes in the search tree and the computing time in seconds, as well as whether the instance has been solved to optimality or not. The number of R-sets of each instance is also given in the first column.

Table 3
RPP instances in sets UR500, UR750, and UR1000.

	R-sets	B&C						NewB&C					
		LB0	Optimal	LB	UB	Nodes	Time	LB0	Optimal	LB	UB	Nodes	Time
UR532	99	17 235.6	Yes	17 277	17 277	62	10.6	17 172.8	Yes	17 277	17 277	15	15.5
UR535	58	23 635	Yes	23 635	23 635	0	2.9	23 635	Yes	23 635	23 635	0	24.5
UR537	19	30 080.5	Yes	30 098	30 098	3	3.1	30 098	Yes	30 098	30 098	0	25.3
UR542	85	17 824	Yes	17 830	17 830	0	4.3	17 800.5	Yes	17 830	17 830	4	16.2
UR545	19	29 646.1	Yes	29 648	29 648	0	1.0	29 648	Yes	29 648	29 648	0	22.8
UR547	2	38 673.1	Yes	38 692	38 692	3	2.5	38 692	Yes	38 692	38 692	0	86.0
UR552	80	20 097	Yes	20 097	20 097	3	2.0	20 085	Yes	20 097	20 097	5	14.9
UR555	5	34 473.9	Yes	34 488	34 488	4	2.3	34 488	Yes	34 488	34 488	0	62.9
UR557	1	48 292.2	Yes	48 307	48 307	4	3.4	48 307	Yes	48 307	48 307	0	84.9
UR562	53	24 556	Yes	24 556	24 556	0	0.7	24 556	Yes	24 556	24 556	0	12.3
UR565	2	42 821.1	Yes	42 828	42 828	3	1.3	42 828	Yes	42 828	42 828	0	71.1
UR567	1	58 958.6	Yes	58 971	58 971	3	0.8	58 971	Yes	58 971	58 971	0	52.8
UR732	140	21 066.4	Yes	21 114	21 114	175	87.3	21 057	Yes	21 114	21 114	12	26.4
UR735	100	28 663	Yes	28 663	28 663	0	9.0	28 663	Yes	28 663	28 663	0	52.5
UR737	16	36 579.2	Yes	36 588	36 588	3	2.5	36 588	Yes	36 588	36 588	0	73.2
UR742	122	22 555.9	Yes	22 557	22 557	2	23.2	22 547.8	Yes	22 557	22 557	4	174.0
UR745	57	32 476.3	Yes	32 493	32 493	0	16.1	32 493	Yes	32 493	32 493	0	94.2
UR747	3	47 763	Yes	47 764	47 764	2	3.2	47 454	Yes	47 764	47 764	23	107.2
UR752	108	25 103.4	Yes	25 131	25 131	57	193.3	25 088.2	Yes	25 131	25 131	75	280.0
UR755	15	41 774	Yes	41 774	41 774	2	2.8	41 774	Yes	41 774	41 774	0	96.4
UR757	1	58 412.4	Yes	58 416	58 416	4	3.4	58 193.5	Yes	58 416	58 416	5	125.2
UR762	103	27 876.1	Yes	27 880	27 880	6	28.8	27 876	Yes	27 880	27 880	6	123.0
UR765	2	50 492	Yes	50 492	50 492	0	9.8	50 177	Yes	50 492	50 492	1015	463.2
UR767	1	72 949.8	Yes	72 950	72 950	2	3.3	72 694	Yes	72 950	72 950	242	311.4
UR132	204	23 861.6	Yes	23 913	23 913	133	230.5	23 853.4	Yes	23 913	23 913	21	128.5
UR135	124	33 087.6	Yes	33 088	33 088	7	30.8	33 064.8	Yes	33 088	33 088	7	109.4
UR137	24	42 796.3	Yes	42 797	42 797	2	3.3	42 447.9	Yes	42 797	42 797	7	177.3
UR142	167	25 547.6	Yes	25 548	25 548	3	48.4	25 537.9	Yes	25 548	25 548	8	272.5
UR145	71	39 007.6	Yes	39 008	39 008	3	24.0	38 992.2	Yes	39 008	39 008	5	124.2
UR147	4	55 949	Yes	55 959	55 959	3	7.7	55 387.5	Yes	55 959	55 959	1044	969.8
UR152	149	28 971.2	Yes	28 975	28 975	21	144.9	28 973	Yes	28 975	28 975	3	174.3
UR155	5	49 155.6	Yes	49 156	49 156	2	6.2	48 914.5	Yes	49 156	49 156	46	275.7
UR157	2	70 229.1	Yes	70 231	70 231	4	7.5	69 849.5	Yes	70 231	70 231	548	714.6
UR162	138	32 320.2	Yes	32 341	32 341	207	522.8	32 317.9	Yes	32 341	32 341	101	850.0
UR165	9	58 790.8	Yes	58 800	58 800	4	16.5	58 514.8	Yes	58 800	58 800	514	641.1
UR167	1	82 473.2	Yes	82 481	82 481	9	13.8	82 216	Yes	82 481	82 481	219	634.1

Table 4
GRP instances.

	R-sets	B&C						NewB&C					
		LB0	Optimal	LB	UB	Nodes	Time	LB0	Optimal	LB	UB	Nodes	Time
UG532	259	20 970.2	Yes	21 023	21 023	554	1068.9	20 969	Yes	21 023	21 023	36	236.8
UG535	113	25 004	Yes	25 004	25 004	0	8.9	25 004	Yes	25 004	25 004	0	34.5
UG537	25	30 068	Yes	30 068	30 068	0	2.0	30 068	Yes	30 068	30 068	0	9.3
UG542	242	21 274.3	No	21 282.1	–	107	3600.0	21 255.2	Yes	21 324	21 324	62	574.2
UG545	60	28 119	Yes	28 119	28 119	0	2.5	28 119	Yes	28 119	28 119	0	29.5
UG547	2	39 598	Yes	39 598	39 598	2	1.5	39 598	Yes	39 598	39 598	0	32.2
UG552	186	23 926.2	Yes	23 958	23 958	198	452.9	23 926.3	Yes	23 958	23 958	9	343.9
UG555	27	35 397	Yes	35 397	35 397	0	1.3	35 397	Yes	35 397	35 397	0	228.7
UG557	3	47 696.7	Yes	47 710	47 710	3	1.8	47 710	Yes	47 710	47 710	0	87.9
UG562	175	24 382	Yes	24 382	24 382	0	17.2	24 382	Yes	24 382	24 382	0	177.5
UG565	13	41 886	Yes	41 886	41 886	0	1.5	41 886	Yes	41 886	41 886	0	43.9
UG567	1	56 731.9	Yes	56 740	56 740	3	2.5	56 740	Yes	56 740	56 740	0	114.3
UG732	412	25 836.9	No	25 862.9	26 246	135	3614.9	25 809.2	Yes	25 978	25 978	1038	3421.3
UG735	154	30 987.8	Yes	30 989	30 989	3	21.4	30 989	Yes	30 989	30 989	0	119.4
UG737	27	37 574.7	Yes	37 580	37 580	3	2.9	37 580	Yes	37 580	37 580	0	37.5
UG742	348	26 719.7	No	26 737.7	26 797	284	3605.7	26 686.2	Yes	26 771	26 771	615	3316.0
UG745	96	35 335	Yes	35 335	35 335	0	5.5	35 335	Yes	35 335	35 335	0	137.5
UG747	5	48 237.6	Yes	48 241	48 241	3	3.8	47 856	Yes	48 241	48 241	268	668.5
UG752	304	28 069.2	Yes	28 087	28 087	177	1823.3	28 024.1	Yes	28 087	28 087	83	2038.9
UG755	36	43 062.3	Yes	43 071	43 071	3	8.9	43 046.7	Yes	43 071	43 071	7	298.4
UG757	1	59 591.9	Yes	59 594	59 594	4	4.4	59 359	Yes	59 594	59 594	81	237.2
UG762	222	31 629.5	Yes	31 644	31 644	19	165.1	31 636.9	Yes	31 644	31 644	5	814.2
UG765	13	50 869	Yes	50 875	50 875	3	8.3	50 690	Yes	50 875	50 875	172	207.3
UG767	2	71 555.2	Yes	71 562	71 562	11	8.7	71 428	Yes	71 562	71 562	305	264.5

(continued on next page)

Table 4 (*continued*).

	R-sets	B&C						NewB&C					
		LB0	Optimal	LB	UB	Nodes	Time	LB0	Optimal	LB	UB	Nodes	Time
UG132	573	29 528.1	No	29 528.1	–	2	3616.5	29 353.1	No	29 614	29 863	69	3600.0
UG135	217	34 088.7	Yes	34 095	34 095	4	72.4	34 092.9	Yes	34 095	34 095	3	379.7
UG137	35	42 310.9	Yes	42 323	42 323	3	17.1	42 232	Yes	42 323	42 323	7	213.2
UG142	464	30 095.2	No	30 095.7	–	42	3610.3	30 079.3	No	30 097.6	30 148	48	3600.0
UG145	97	40 834.5	Yes	40 839	40 839	9	27.3	40 834.1	Yes	40 839	40 839	0	220.9
UG147	5	57 255.1	Yes	57 263	57 263	3	12.5	56 799.8	Yes	57 263	57 263	690	578.8
UG152	365	33 548	No	33 550.8	33 645	109	3613.3	33 521	Yes	33 560	33 560	201	1885.3
UG155	48	47 818	Yes	47 818	47 818	0	4.5	47 774.2	Yes	47 818	47 818	3	100.9
UG157	5	69 437.2	Yes	69 442	69 442	3	8.9	69 148.5	Yes	69 442	69 442	422	1037.9
UG162	294	36 789.1	Yes	36 826	36 826	626	3576.2	36 789.1	Yes	36 826	36 826	120	1603.3
UG165	18	60 278.6	Yes	60 283	60 283	4	7.9	60 209.5	Yes	60 283	60 283	305	477.1
UG167	1	84 824.4	Yes	84 833	84 833	67	62.6	84 544.5	Yes	84 833	84 833	1573	1189.8

Table 5
RPP instances in sets UR40 × 25, UR40 × 50, and UR60 × 50.

	R-sets	B&C						NewB&C					
		LB0	Optimal	LB	UB	Nodes	Time	LB0	Optimal	LB	UB	Nodes	Time
UR40 × 25_2_1	222	31 024.7	Yes	31 042	31 042	159	728.5	31 021.1	Yes	31 042	31 042	12	241.2
UR40 × 25_2_2	232	30 598.2	Yes	30 607	30 607	34	292.2	30 570.2	Yes	30 607	30 607	69	672.6
UR40 × 25_2_3	217	30 884	No	30 900.2	31 054	305	3600.0	30 876.7	Yes	30 930	30 930	272	635.7
UR40 × 25_2_4	217	31 179.1	Yes	31 190	31 190	199	1388.9	31 175.8	Yes	31 190	31 190	6	203.6
UR40 × 25_2_5	227	30 514.1	Yes	30 533	30 533	94	644.5	30 510.1	Yes	30 533	30 533	35	633.9
UR40 × 25_3_1	147	34 222.6	Yes	34 228	34 228	8	15.2	34 220.6	Yes	34 228	34 228	3	65.8
UR40 × 25_3_2	149	33 569.9	Yes	33 582	33 582	298	1309.8	33 553.9	Yes	33 582	33 582	8	259.1
UR40 × 25_3_3	136	33 557.3	Yes	33 559	33 559	10	23.3	33 554.2	Yes	33 559	33 559	3	92.5
UR40 × 25_3_4	140	33 962.7	Yes	33 982	33 982	55	73.5	33 962.4	Yes	33 982	33 982	22	273.0
UR40 × 25_3_5	136	33 753.7	Yes	33 773	33 773	422	1680.4	33 747	Yes	33 773	33 773	108	398.7
UR40 × 50_2_1	458	48 560.7	No	48 571	48 626	45	3600.0	48 569.1	Yes	48 595	48 595	179	3304.9
UR40 × 50_2_2	442	49 124.5	No	49 128.8	–	24	3600.0	49 126.1	No	49 135.3	49 679	271	3600.0
UR40 × 50_2_3	433	49 168.5	No	49 181.4	49 195	57	3600.0	49 176.5	Yes	49 188	49 188	82	2059.5
UR40 × 50_2_4	417	49 155.7	No	49 156.1	49 376	82	3600.0	49 142.9	Yes	49 180	49 180	714	3570.1
UR40 × 50_2_5	432	48 901.9	No	48 915	48 918	427	3600.0	48 894.6	Yes	48 918	48 918	88	1858.8
UR40 × 50_3_1	276	53 125.4	Yes	53 137	53 137	42	725.2	53 129.2	Yes	53 137	53 137	11	1355.0
UR40 × 50_3_2	231	53 855.7	No	53 875.3	54 164	182	3600.0	53 855.1	Yes	53 884	53 884	18	1858.9
UR40 × 50_3_3	226	54 636.6	Yes	54 651	54 651	56	1326.2	54 640.3	Yes	54 651	54 651	27	1861.9
UR40 × 50_3_4	259	54 086.2	Yes	54 091	54 091	30	786.4	54 084.7	Yes	54 091	54 091	4	1170.5
UR40 × 50_3_5	219	54 007	No	54 014.4	54 028	305	3600.0	53 983	Yes	54 016	54 016	4	1059.0
UR60 × 50_2_1	662	60 640.5	No	60 640.5	–	0	3600.0	60 632.6	No	60 648.2	61 149	409	3600.0
UR60 × 50_2_2	686	59 857.4	No	59 857.4	–	1	3600.0	59 838.6	No	59 864.3	60 787	392	3600.0
UR60 × 50_2_3	675	60 330.5	No	60 330.5	–	0	3600.0	60 302.8	No	60 335.9	61 067	590	3600.0
UR60 × 50_2_4	679	60 578	No	60 578	–	0	3600.0	60 492.5	No	60 494.6	61 273	383	3600.0
UR60 × 50_2_5	675	60 510.2	No	60 510.2	–	0	3600.0	60 409.7	No	60 419.3	60 971	584	3600.0
UR60 × 50_3_1	379	66 934	No	66 934	–	1	3600.0	66 908.5	No	66 936.2	68 270	685	3600.0
UR60 × 50_3_2	386	66 287	No	66 287.2	–	15	3600.0	66 248.1	No	66 274.2	66 780	700	3600.0
UR60 × 50_3_3	366	67 259.8	No	67 259.8	67 354	50	3600.0	67 222.8	No	67 260.5	67 285	854	3600.0
UR60 × 50_3_4	378	66 763.6	No	66 770.5	–	19	3600.0	66 734.6	No	66 768.2	66 885	746	3600.0
UR60 × 50_3_5	344	67 622.4	No	67 622.9	–	56	3600.0	67 592.2	No	67 613.4	68 634	468	3600.0

Table 6
GTSP instances.

	R-sets	B&C						NewB&C					
		LB0	Optimal	LB	UB	Nodes	Time	LB0	Optimal	LB	UB	Nodes	Time
KROA150G	150	26 428.2	Yes	26 524	26 524	108	2.1	26 524	Yes	26 524	26 524	0	8.5
KROB150G	150	26 070.8	Yes	26 130	26 130	31	0.9	26 090.5	Yes	26 130	26 130	4	3.2
KROA200G	200	29 357.5	Yes	29 368	29 368	9	1.2	29 368	Yes	29 368	29 368	0	2.9
KROB200G	200	29 437	Yes	29 437	29 437	0	0.9	29 437	Yes	29 437	29 437	0	3.4
PR152G	152	73 682	Yes	73 682	73 682	0	0.2	73 682	Yes	73 682	73 682	0	1.2
RAT195G	195	2311.25	Yes	2323	2323	92	3.0	2322.67	Yes	2323	2323	3	28.7
TS225G	225	122 284	No	123 407	127 533	44 732	3600.0	126 643	Yes	126 643	126 643	0	18.7

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