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Memory in the iterative processes for nonlinear problems

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In this paper, we study different ways for introducing memory to a parametric family of optimal two-step iterative methods. We study the convergence and the stability, by means of real dynamics, of the methods obtained by introducing memory in order to compare them. We also perform several numerical experiments to see how the methods behave.

KEYWORDS

divided difference, dynamical planes, iterative methods, nonlinear equations, optimal scheme, real dynamics

MSC CLASSIFICATION

65H05

1 | INTRODUCTION

Iterative methods are one of the most widely used tools for solving problems with nonlinear equations $f(x) = 0$, where $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$. These methods obtain a sequence of approximations, which, under certain conditions, converge to the solution of the equation. One of the best known schemes is Newton's method, which has the iterative expression

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, \ldots . \tag{1}$$

Newton's method is well known for its efficiency and simplicity, as well as for its quadratic convergence and optimality in the sense of Kung-Traub conjecture.[1] When the derivative in (1) is replaced by the divided difference $f[x_k + f(x_k), x_k]$, we obtain the Steffensen's method,[2] which is a derivative-free and also optimal scheme. Many other optimal schemes appearing in the literature can be found, for example, in Petkovic[3] and Milovanovich and Cvetcovic,[4] and the references therein. The existence of derivatives in the iterative expression of a method can be a drawback when the function to be studied cannot be derived or its derivative is too costly to calculate. For this reason, derivative-free methods have arisen in the literature; see, for example, Chun and Neta[5] and Kumar et al[6] and the overviews about iterative methods.[7,8]

In order to increase the quadratic convergence, Traub[9] proposed the following scheme:

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = y_k - \dfrac{f(y_k)}{f'(x_k)}, \; k = 0, 1, \ldots . \end{cases} \tag{2}$$

This method has order of convergence 3, but it is not optimal in the sense of Kung-Traub conjecture.

In this paper, we design a derivative-free variant of Traub's method by replacing the derivatives by a divided difference with a parameter and a weight function (see this and other techniques in Behl et al.,[10] Chun & Neta,[11] Jarrat,[12] and King[13]). This yields the following parametric family, which as we shall see below is a class of optimal iterative methods of fourth order, that we denote by $M_{4,\beta}$.

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f[w_k, x_k]}, \; \text{where } w_k = x_k + \beta f(x_k) \; \forall \beta \in \mathbb{R} \setminus \{0\}, \\ x_{k+1} = y_k - H(\mu_k)\dfrac{f(y_k)}{f[y_k, x_k]}, \; \text{where } \mu_k = \dfrac{f(y_k)}{f(w_k)}, \; k = 0, 1, \ldots . \end{cases} \tag{3}$$

In addition, to designing such a family of optimal methods, we study several ways to introduce memory in it, by replacing the parameter with an expression that uses the previous iterates and their functional evaluations. In this way, we increase the order of the methods without more computational cost.

To prove the order of convergence of the methods with memory we use the following Ortega-Rheinboldt's Theorem, which can be found in Ortega and Rheinboldt[14]:

Theorem 1. *Let ϕ be an iterative method with memory that generates a sequence $\{x_k\}$ of approximations to the root α, and let this sequence converges to α. If there exist a nonzero constant η and positive numbers t_i, $i = 0, \ldots, m$ such that the inequality*

$$|e_{k+1}| \leq \eta \prod_{i=0}^{m} |e_{k-i}|^{t_i},$$

holds, then the R-order of convergence of the iterative method ϕ is at least p, where p is the unique positive root of the equation

$$p^{m+1} - \sum_{i=0}^{m} t_i p^{m-i} = 0.$$

The convergence of an iterative method is not the only thing to analyze, but it is also important to study its stability in terms of the set of initial approximations that generate convergence or give rise to chaotic behavior (see, e.g., Chicharro et al.,[15,16] and Sharma et al.[17]). This stability is analyzed using discrete real dynamics tools, which will allow us to differentiate family members with stable behavior from others with chaotic behavior. In this work, we study the real dynamics of the proposed memory methods on the polynomial $x^2 - c$, where c is an arbitrary positive real value.

The manuscript finishes with some numerical experiments to make a comparison between different elements of the family and the methods obtained by introducing memory.

2 | CONVERGENCE ANALYSIS

Let $f : \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in an open set $D \subset \mathbb{R}$ that contains a root α of $f(x) = 0$. Let us consider the expression

$$f[x+h, x] = \int_0^1 f'(x+th)dt, \tag{4}$$

obtained by Genochi-Hermite in Ortega and Rheinboldt.[14] Using the Taylor's expansion $f'(x+th)$ around x and integrating, we obtain the following development:

$$f[x+h, x] = f'(x) + \frac{1}{2}f''(x)h + \frac{1}{6}f'''(x)h^2 + O(h^3), \tag{5}$$

which we use to prove that the order of convergence of methods $M_{4,\beta}$, defined in (3), is 4 for any $\beta \in \mathbb{R} \setminus \{0\}$.

Theorem 2. *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in an open neighborhood D of α such that $f(\alpha) = 0$. We assume that $f'(\alpha) \neq 0$. Let $H(t)$ be a real function that verifies $H(0) = 1$, $H'(0) = 1$ and $|H''(0)| < \infty$. Then, taking an estimate x_0 sufficiently close to α, the sequence of iterates $\{x_k\}$ generated by the proposed family (3) converges to α with order 4, and its error equation is*

$$e_{k+1} = \frac{1}{2}C_2(1 + \beta f'(\alpha))(-2C_3(1 + \beta f'(\alpha)) + C_2^2(6 + 4\beta f'(\alpha) - H_2))e_k^4 + O(e_k^5), \tag{6}$$

where $C_j = \frac{1}{j}\frac{f^{(j)}(\alpha)}{f'(\alpha)}$ for $j = 2, 3, \dots$, $e_k = x_k - \alpha$ and $H_2 = H''(0)$.

Proof. Let us consider the Taylor expansion of $f(x_k)$ and $f(w_k)$ around α:

$$f(x_k) = f'(\alpha)\left(e_k + C_2e_k^2 + C_3e_k^3 + C_4e_k^4 + C_5e_k^5 + O(e_k^6)\right) \tag{7}$$

and

$$f(w_k) = f'(\alpha)\left(e_w + C_2e_w^2 + C_3e_w^3 + C_4e_w^4 + C_5e_w^5 + O(e_w^6)\right), \tag{8}$$

where $e_w = w_k - \alpha$.

Now, we calculate $f[w_k, x_k]$ using the above equations.

$$\begin{aligned} f[w_k, x_k] &= \frac{f(w_k) - f(x_k)}{w_k - x_k} = \frac{f(w_k) - f(x_k)}{w_k - \alpha + \alpha - x_k} = \frac{f(w_k) - f(x_k)}{e_w - e_k} \\ &= f'(\alpha)\left(1 + C_2(e_w + e_k) + C_3\frac{(e_w^3 - e_k^3)}{e_w - e_k} + C_4\frac{(e_w^4 - e_k^4)}{e_w - e_k} + O_4(e_k, e_w)\right). \end{aligned}$$

Since $w_k = x_k + \beta f(x_k)$, then it follows that

$$\begin{aligned} f[w_k, x_k] =& f'(\alpha)(1 + C_2(2 + \beta f'(\alpha))e_k + (\beta C_2^2 f'(\alpha) + C_3(3 + 3\beta f'(\alpha) + \beta^2 f'(\alpha)^2))e_k^2 \\ &+ (2 + \beta f'(\alpha))(2\beta C_2C_3 f'(\alpha) + C_4(2 + 2\beta f'(\alpha) + \beta^2 f'(\alpha)^2))e_k^3) + O(e_k^4). \end{aligned}$$

From this, we have

$$\begin{aligned} y_k - \alpha &= e_k - \frac{f(x_k)}{f[w_k, x_k]} \\ &= C_2(1 + \beta f'(\alpha))e_k^2 + (-C_2^2(2 + 2\beta f'(\alpha) + \beta^2 f'(\alpha)^2) + C_3(2 + 3\beta f'(\alpha) + \beta^2 f'(\alpha)^2))e_k^3 + O(e_k^4). \end{aligned}$$

Let us calculate e_{k+1}. Let consider the Taylor expansion of $f(y_k)$ around α:

$$f(y_k) = f'(\alpha)\left(e_y + C_2e_y^2 + C_3e_y^3 + C_4e_y^4 + C_5e_y^5 + O(e_y^6)\right), \tag{9}$$

where $e_y = y_k - \alpha$.

By using the previous equations, we obtain the following expression for $f[y_k, x_k]$

$$\begin{aligned} f[y_k, x_k] &= \frac{f(y_k) - f(x_k)}{y_k - x_k} = \frac{f(y_k) - f(x_k)}{y_k - \alpha + \alpha - x_k} = \frac{f(y_k) - f(x_k)}{e_y - e_k} \\ &= f'(\alpha)(1 + C_2e_k + (C_3 + C_2^2(1 + \beta f'(\alpha)))e_k^2 + O(e_k^3). \end{aligned}$$

Let us now calculate $\mu_k = \frac{f(y_k)}{f(w_k)}$.

$$\frac{f(y_k)}{f(w_k)} = C_2e_k + (C_3(2 + \beta f'(\alpha)) - C_2^2(3 + 2\beta f'(\alpha)))e_k^2 + O(e_k^3),$$

and, therefore,

$$H(\mu_k) = H_0 + H_1\mu_k + \frac{1}{2}H_2\mu_k^2 + O(\mu_k^3) = 1 + \mu_k + \frac{H_2}{2}\mu_k^2 + O(\mu_k^3)$$
$$= 1 + C_2 e_k + (C_3(2 + \beta f'(\alpha)) + \frac{1}{2}C_2^2(-6 - 4\beta f'(\alpha) + H_2))e_k^2 + O(e_k^3).$$

Then, we calculate $e_{k+1} = e_y - H(\mu_k)\frac{f(y_k)}{f[y_k,x_k]}$ using the above results.

$$e_{k+1} = \frac{1}{2}C_2(1 + \beta f'(\alpha))(-2C_3(1 + \beta f'(\alpha)) + C_2^2(6 + 4\beta f'(\alpha) - H_2))e_k^4 + O(e_k^5).$$

So it is proved that family (3) has order 4 under these conditions. □

According to the Kung-Traub conjecture, all the elements of family (3) are optimal iterative schemes.

From the error equation, we note that if $\beta = -\frac{1}{f'(\alpha)}$, then the order increase at least one unit. Since the value of α is unknown, we approximate the value of $f'(\alpha)$ in order to increase the order of the iterative scheme. In this way, we obtain a method with memory.

If we take the Newton interpolation polynomial of degree 1 at nodes x_k and x_{k-1}, that is, $N_1(t) = f(x_k) + f[x_k, x_{k-1}](t - x_k)$, then we approximate the derivative of f evaluated at the solution as

$$f'(\alpha) \approx N_1'(x_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}},$$

so we choose $\beta_k = -\frac{1}{N_1'(x_k)}$, and we obtain a method with memory, which we denote by M_4N_1.

Theorem 3. *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in a neighborhood of a simple root α of $f(x) = 0$. Let $H(t)$ be a real function that satisfies $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$. Then, taking an initial estimation x_0 sufficiently close to α, the sequence of iterates $\{x_k\}$ generated by method M_4N_1 converges to α with order* $2 + \sqrt{6} \approx 4.44948974$.

Proof. From the error equation (6) and taking $H_2 = H''(0) = 2$, we have

$$e_{k+1} \sim (1 + \beta f'(\alpha))^2 C_2(2C_2^2 - C_3)e_k^4 + O(e_k^5).$$

By using Taylor's series developments of $f(x_k)$ and $f(x_{k-1})$ around α in the same way as in the previous theorem, we obtain

$$\beta_k = -\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} = -\frac{1}{f'(\alpha)(1 + C_2(e_k + e_{k-1}) + O_2(e_{k-1}, e_k))}.$$

Therefore, $1 + \beta_k f'(\alpha) \sim C_2 e_{k-1}$.

From the error equation (6) and the above relation, it follows that

$$e_{k+1} \sim (C_2 e_{k-1})^2 C_2(2C_2^2 - C_3)e_k^4 \sim e_{k-1}^2 e_k^4. \tag{10}$$

On the other hand, we suppose that the R-order of the method is at least p. Therefore,

$$e_{k+1} \sim D_{k,p} e_k^p,$$

where $D_{k,p}$ tends to the asymptotic error constant, D_p, when $k \to \infty$.

Analogously,

$$e_k \sim D_{k-1,p} e_{k-1}^p.$$

Then,

$$e_{k+1} \sim D_{k,p}(D_{k-1,p} e_{k-1}^p)^p = D_{k,p} D_{k-1,p}^p e_{k-1}^{p^2}. \tag{11}$$

In the same way that relation (10) is obtained, it follows that

$$e_{k+1} \sim e_{k-1}^2 (D_{k-1,p} e_{k-1}^p)^4 = D_{k-1,p}^4 e_{k-1}^{4p+2}. \tag{12}$$

Then, by equating the exponents of e_{k-1} of (11) and (12), we obtain $p^2 = 4p + 2$, whose only positive solution is the order of convergence of M_4N_1 method, where $p \approx 4.44948974$. □

Other way to approximate the derivative of the function is by the Kurchatov's divided difference, which has the following expression:

$$f'(\alpha) \approx f[2x_k - x_{k-1}, x_{k-1}].$$

Then, if we take

$$\beta_k = -\frac{1}{f[2x_k - x_{k-1}, x_{k-1}]},$$

we obtain an iterative method with memory, denoted by M_4K, whose convergence is analyzed in the following result.

Theorem 4. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in an neighborhood of a simple root α of $f(x) = 0$. Let $H(t)$ be a real function that satisfies $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$. Then, taking an initial estimate x_0 sufficiently close to α, the sequence of iterates $\{x_k\}$ generated by method M_4K converges to α with order $p = 2 + 2\sqrt{2} \approx 4.82842713$.*

Proof. From the error equation (6) and taking $H_2 = H''(0) = 2$, we have

$$e_{k+1} \sim (1 + \beta f'(\alpha))^2 C_2 (2C_2^2 - C_3) e_k^4 + O(e_k^5).$$

By using Taylor's series developments in the same way as in Theorem 2 and by applying the Genocchi-Hermite formule, we obtain

$$\begin{aligned}[2x_k - x_{k-1}, x_{k-1}, f] &= f'(x_{k-1}) + \frac{1}{2} f''(x_{k-1})(2x_k - 2x_{k-1}) + \frac{1}{6} f'''(x_{k-1})(2x_k - 2x_{k-1})^2 + O_3(e_k, e_{k-1}) \\ &= f'(\alpha)\left(1 + 2C_2 e_k + 4C_3 e_k^2 + C_3 e_{k-1}^2 - 2C_3 e_k e_{k-1}\right) + O_3(e_k, e_{k-1}).\end{aligned}$$

Then, we get

$$1 + \beta_k f'(\alpha) \sim 2C_2 e_k + 4C_3 e_k^2 + C_3 e_{k-1}^2 - 2C_3 e_k e_{k-1}.$$

As e_k converges less quickly to 0 than e_k^2 and $e_k e_{k-1}$, the behavior of $1 + \beta_k f'(\alpha)$ is like that of e_k or like that of e_{k-1}^2. Suppose that the R-order of the method is at least p. Therefore,

$$e_{k+1} \sim D_{k,p} e_k^p,$$

where $D_{k,p}$ tends to the asymptotic error constant, D_p, when $k \to \infty$. Analogously,

$$\frac{e_k}{e_{k-1}^2} \sim D_{k-1,p} e_{k-1}^{p-2},$$

which means that $1 + \beta_k f'(\alpha) \sim e_{k-1}^2$ provided that $p > 2$.

Let us suppose $p > 2$. From the error equation (6) and the above relation, it follows that

$$e_{k+1} \sim (e_{k-1}^2)^2 e_k^4 \sim e_{k-1}^4 e_k^4. \tag{13}$$

On the other hand,

$$e_{k+1} \sim D_{k,p}(D_{k-1,p} e_{k-1}^p)^p = D_{k,p} D_{k-1,p}^p e_{k-1}^{p^2}. \tag{14}$$

In a similar way as relation (13) is obtained, it follows that

$$e_{k+1} \sim e_{k-1}^4 (D_{k-1,p} e_{k-1}^p)^4 = D_{k-1,p}^4 e_{k-1}^{4p+4}. \tag{15}$$

Then, by equating the exponents of e_{k-1} of (14) and (15), we obtain $p^2 = 4p + 4$, whose only positive solution is the order of convergence of the M_4N_1 method, where $p \approx 4.82842713$. □

We can use other approximations of $f'(\alpha)$ by means of Newton interpolation polynomials of higher degree.

If we define $N_2(t) = f(x_k) + f[x_k, x_{k-1}](t - x_k) + f[x_k, x_{k-1}, y_{k-1}](t - x_k)(t - x_{k-1})$, an approximation of the derivative is

$$f'(\alpha) \approx N_2'(x_k).$$

So we will choose $\beta_k = -\frac{1}{N_2'(x_k)}$, and so we obtain an iterative method with memory, denoted by M_4N_2, whose convergence is analyzed in the next result.

Theorem 5. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in a neighborhood of a simple root α of $f(x9 = 0$. Let $H(t)$ be a real function that satisfies $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$. Then, taking an initial estimation x_0 sufficiently close to α, the sequence of iterates $\{x_k\}$ generated by the method M_4N_2 converges to α with order $\frac{1}{2}\left(5 + \sqrt{33}\right) \approx 5.37228$.*

Proof. From the error equation (6) and knowing that $H_2 = H''(0) = 2$,

$$e_{k+1} \sim (1 + \beta f'(\alpha))^2 C_2 (2C_2^2 - C_3) e_k^4 + O(e_k^5).$$

Using Taylor's series of $f(x_k)$, $f(x_{k-1})$ and $f(y_{k-1})$ around α, we obtain

$$\begin{aligned} N_2'(x_k) &= f[x_k, x_{k-1}] + f[x_k, x_{k-1}, y_{k-1}](x_k - x_{k-1}) \\ &= f'(\alpha) + 2C_2 f'(\alpha) e_k + C_3 f'(\alpha) e_k e_y + C_3 f'(\alpha)(e_k - e_{y,k-1}) e_{k-1} + O(e_{k-1}^2) + O(e_k^2) + O(e_y^2) + O_3(e_{y,k-1}, e_k, e_{k-1}). \end{aligned}$$

This means that $1 + \beta_k f'(\alpha)$ will be able to behave as e_k, as $e_k e_{y,k-1}$, as $e_{k-1} e_k$, or as $e_{k-1} e_{y,k-1}$. It is clear that $e_k e_{y,k-1}$ tends faster to zero than e_k when $k \to \infty$ and that $e_{k-1} e_k$ tends to zero faster than $e_{k-1} e_{y,k-1}$. For this reason, we need to analyze if e_k converges faster to zero or does it $e_{k-1} e_{y,k-1}$.

Suppose the R-order of the method is at least p. Let us consider the sequence $\{y_k\}$ generated by the first step of the method, and let us assume that converges to R-order at least p_1. Therefore, it is satisfied

$$e_{k+1} \sim D_{k,p} e_k^p \qquad \text{and} \qquad e_{y,k} \sim D_{k,p_1} e_k^{p_1},$$

where $D_{k,p}$ tends to the asymptotic error constant, D_p, and where D_{k,p_1} tends to the asymptotic error constant, D_{p_1}, when $k \to \infty$.

As $e_k \sim D_{k-1,p} e_{k-1}^p$, then

$$\frac{e_k}{e_{k-1} e_{y,k-1}} \sim \frac{D_{k-1,p} e_{k-1}^p}{e_{k-1} e_{y,k-1}} \sim \frac{D_{k-1,p} e_{k-1}^p}{D_{k-1,p_1} e_{k-1} e_{k-1}^{p_1}}.$$

Then, if $p > p_1 + 1$, it follows that

$$1 + \beta_k f'(\alpha) \sim -C_3 e_{k-1} e_{y,k-1}. \tag{16}$$

From the error equation (6) and the above relation,

$$e_{k+1} \sim (-C_3 e_{k-1} e_{y,k-1})^2 C_2 (2C_2^2 - C_3) e_k^4 \sim e_{k-1}^2 e_{y,k-1}^2 e_k^4. \tag{17}$$

Assuming that the R-order of the method is at least p, we obtain the relation (11). If sequence $\{y_k\}$ converges to R-order at least p_1, we obtain the relation

$$e_{y,k} \sim D_{k,p_1} e_k^{p_1} \sim D_{k,p_1} (D_{k-1,p} e_{k-1}^p)^{p_1} \sim D_{k,p_1} D_{k-1,p}^{p_1} e_{k-1}^{pp_1}. \tag{18}$$

In the same way that relation (17) is obtained, it follows that

$$e_{k+1} \sim e_{k-1}^2 (D_{k-1,p_1} e_{k-1}^{p_1})^2 (D_{k-1,p} e_{k-1}^{p})^4 = D_{k-1,p_1}^2 D_{k-1,p}^4 e_{k-1}^{2p_1} e_{k-1}^{4p+2}. \tag{19}$$

On the other hand, we know that

$$e_{k,y} \sim (1+\beta_k f'(\alpha)) e_k^2 \sim e_{k-1} e_{y,k-1} e_k^2 \sim e_{k-1} (D_{k-1,p_1} e_{k-1}^{p_1})(D_{k-1,p} e_{k-1}^{p})^2 \sim e_{k-1}^{2p+1+p_1}. \tag{20}$$

Then, by equating the exponents of e_{k-1} of (11) and (19), and equating those of (18) and (20), it follows that

$$p^2 = 4p + 2 + 2p_1,$$
$$pp_1 = 2p + 1 + p_1,$$

whose only positive solution is the order of convergence of the method M_4N_2, being $p \approx 5.37228$ and $p_1 \approx 2.68614$.□

We can also approximate $f'(\alpha)$ by using the Newton interpolating polynomial of third-degree $N_3(t) = f(x_k) + f[x_k, x_{k-1}](t-x_k) + f[x_k, x_{k-1}, y_{k-1}](t-x_k)(t-x_{k-1}) + f[x_k, x_{k-1}, y_{k-1}, w_{k-1}](t-x_k)(t-x_{k-1})(t-y_{k-1})$. In this case,

$$f'(\alpha) \approx N_3'(x_k),$$

and by choosing $\beta_k = -\frac{1}{N_3'(x_k)}$, we design a new iterative method with memory, denoted by M_4N_3.

Theorem 6. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in a neighborhood of a simple root α of $f(x) = 0$. Let $H(t)$ be a real function which verifies that $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$. Then, taking an estimation x_0 close enough to α, the sequence of iterates $\{x_k\}$ generated by the M_4N_3 method converges to α with order $p \approx 6$.*

Proof. In the same way as in the previous results, it follows that

$$N_3'(x_k) = f[x_k, x_{k-1}] + f[x_k, x_{k-1}, y_{k-1}](x_k - x_{k-1}) + f[x_k, x_{k-1}, y_{k-1}, w_{k-1}](x_k - x_{k-1})(x_k - y_{k-1})$$

and

$$1 + \beta_k f'(\alpha) \sim 2C_2 e_k + C_4 e_{y,k-1} e_{k-1} e_{w,k-1}.$$

This means that $1 + \beta_k f'(\alpha)$ may behave as e_k or as $e_{k-1} e_{y,k-1} e_{w,k-1}$, as the other terms converge faster than these two. We now check that the behavior of $1 + \beta_k f'(\alpha)$ is like the behavior of $e_{k-1} e_{y,k-1} e_{w,k-1}$.

Suppose that the R-order of the method is at least p. Moreover, we assume that the sequence $\{y_k\}$ generated by the first step of the method and the sequence $\{w_k\}$ converge with R-order at least p_1 and at least p_2, respectively. Then,

$$\frac{e_k}{e_{k-1} e_{y,k-1} e_{w,k-1}} \sim \frac{D_{k-1,p} e_{k-1}^{p}}{D_{k-1,p_1} D_{k-1,p_2} e_{k-1} e_{k-1}^{p_1} e_{k-1}^{p_2}},$$

where D_{k,p_1} and D_{k,p_2} tend to the asymptotic error constants, D_{p_1} and D_{p_2}, respectively, when $k \to \infty$.

Then, if $p > p_1 + p_2 + 1$, it follows that

$$1 + \beta_k f'(\alpha) \sim C_4 e_{k-1} e_{y,k-1} e_{w,k-1}.$$

From Equation (6) and the above relation, we have

$$e_{k+1} \sim (C_4 e_{k-1} e_{y,k-1} e_{w,k-1})^2 C_2 (2C_2^2 - C_3) e_k^4 \sim e_{k-1}^2 e_{y,k-1}^2 e_{w,k-1}^2 e_k^4. \tag{21}$$

Assuming that the R-order of the method is at least p yields the relation (11). On the other hand, assuming that sequence $\{y_k\}$ and sequence $\{w_k\}$ converge with R-order at least p_1 and at least p_2, respectively, we obtain the relation defined in (18) and the following relation:

$$e_{w,k} \sim D_{k,p_2} e_k^{p_2} \sim D_{k,p_2}(D_{k-1,p} e_{k-1}^{p})^{p_2} \sim D_{k,p_2} D_{k-1,p}^{p_2} e_{k-1}^{pp_2}. \tag{22}$$

In the same way that relation (21) is obtained, it follows that

$$e_{k+1} \sim D_{k-1,p_1}^2 D_{k-1,p_2}^2 D_{k-1,p}^4 e_{k-1}^{2p_1+2p_2+4p+2}. \tag{23}$$

In addition, we know

$$e_{k,y} \sim (1+\beta_k f'(\alpha)) e_k^2 \sim e_{k-1}^{2p+1+p_1+p_2} \tag{24}$$

and

$$e_{w,y} \sim (1+\beta_k f'(\alpha)) e_k \sim e_{k-1}^{p+1+p_1+p_2}. \tag{25}$$

Then, by equating the exponents of e_{k-1} of (11) and (23), those of (18) and (24), and those of (22) and (25), it is obtained the nonlinear system

$$\begin{aligned} p^2 &= 4p + 2 + 2p_1 + 2p_2, \\ pp_1 &= 2p + 1 + p_1 + p_2, \\ pp_2 &= p + 1 + p_1 + p_2, \end{aligned}$$

whose only positive solution is the order of convergence of the method M_4N_3, being $p \approx 6$, $p_1 \approx 3$ and $p_2 \approx 2$. □

Finally, we can approximate the derivative of the equation using the following divided differences operators

- $f'(\alpha) \approx f[x_k, y_{k-1}]$,
- $f'(\alpha) \approx f[2x_k - y_{k-1}, y_{k-1}]$,

which allow us to design two new iterative methods with memory, denoted by M_4N_y and by M_4K_y, respectively. Let us note that these divided differences are of first order and with the last one we reach the maximum possible order of convergence by introducing memory in family (3).

Theorem 7. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in a neighborhood of a simple root α of $f(x) = 0$. Let $H(t)$ be a real function that verifies that $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$. Then, taking an estimation x_0 sufficiently close to α, sequence $\{x_k\}$ generated by method M_4N_y converges to α with order 5, and sequence generated by method M_4K_y converges to α with order 6.*

Proof. In the same way that the demonstration of the order of the two previous methods was done, it can be verified that in this case what is obtained is that the order is the positive root p of the following systems:

For method M_4N_y, we obtain

$$\begin{aligned} p^2 &= 4p + 2p_1, \\ pp_1 &= 2p + p_1, \end{aligned}$$

whose only positive solution is the order of convergence of the method M_4N_y, being $p = 5$.

On the other hand, for method M_4K_y, we obtain

$$\begin{aligned} p^2 &= 4p + 4p_1, \\ pp_1 &= 2p + 2p_1, \end{aligned}$$

whose only positive solution is the order of convergence of the method M_4K_y, being $p = 6$. □

3 | STABILITY OF THE METHODS WITH MEMORY

In this section, we analyze the stability of the methods with memory M_4N_1 and M_4N_y, by using some tools of real dynamics.

Polynomial weight functions have been used, satisfying the convergence conditions of the corresponding theorems.

The standard form of an iterative method with memory that uses only two previous iterations to calculate the next is

$$x_{k+1} = \phi(x_{k-1}, x_k),\ k \geq 1,$$

being x_0 and x_1 the initial estimations. A function defined from $\mathbb{R}^2$ to $\mathbb{R}$ cannot have fixed points. Therefore, an auxiliary vectorial function O is defined by means of $O(x_{k-1}, x_k) = (x_k, x_{k+1}) = (x_k, \phi(x_{k-1}, x_k))$, $k = 1, 2, \ldots$.

If (x_{k-1}, x_k) is a fixed point of O, then $O(x_{k-1}, x_k) = (x_{k-1}, x_k)$, and from the definition of O, we have that $(x_{k-1}, x_k) = (x_k, x_{k+1})$.

Thus, the discrete dynamical system $O : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as

$$O(\bar{x}) = O(z, x) = (x, \phi(z, x)),$$

where ϕ is the operator of the iterative scheme with memory.

Then, a point (z, x) is a fixed pint of O if $z = x$ and $x = \phi(z, x)$. If a fixed point (z, x) of the operator O does not verify that $f(x) = 0$, it is called strange fixed point.

In Robinson,[18] the stability of a fixed point is defined in the following result:

Theorem 8. *Let O from $\mathbb{R}^2$ to $\mathbb{R}^2$ be a sufficiently differentiable function. Assume that $\bar{x}$ is a fixed point. Let λ_1 and λ_2 be the eigenvalues of the Jacobian matrix of O evaluated at $\bar{x}$. Then,*

- *If all the eigenvalues satisfy $|\lambda_j| < 1$, then $\bar{x}$ is attracting.*
- *If one eigenvalue λ_i satisfy $|\lambda_i| > 1$, then $\bar{x}$ is unstable, that is, repelling or saddle.*
- *If all the eigenvalues satisfy $|\lambda_j| > 1$, then $\bar{x}$ is repelling.*

Moreover, if all the eigenvalues are equal to zero, the fixed point is superattracting.

A critical point $\bar{y}$ satisfies that the determinant of the Jacobian matrix evaluated at $\bar{y}$, its 0. All superattracting fixed points are critical points.

The basin of attraction of a fixed point x^* is defined as the set of pre-images of any order such that

$$\mathcal{A}(x^*) = \{y \in \mathbb{R}^n : O^m(y) \to x^*,\ m \to \infty\}.$$

We study the stability of the fixed points of the rational operator obtained when the methods is applied on the polynomial $p(x) = x^2 - c$, when c is a positive real value. It is easy to observe that both schemes give us the same rational polynomial.

In order to obtain the fixed points of the rational operator associated to a method with memory, we need to construct the auxiliary vectorial operator of two variables where $x_{k-1} = z$ and $x_k = x$. If we choose the weight function $H(\mu) = \mu^2 + \mu + 1$, the operator obtained is

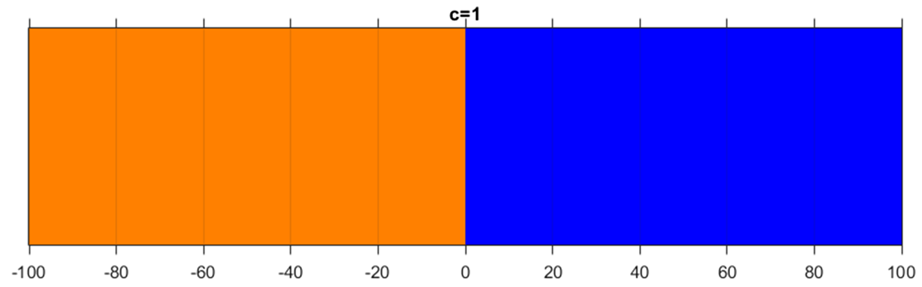
$$O_{M_4N_1}(z,x) = \left(x,\ \frac{\left(c-x^2\right)^3\left(c-z^2\right)(x+z)\left(\frac{\left(c-x^2\right)^2(x+z)^4}{(c+x(x+2z))^4} - \frac{\left(c-x^2\right)(x+z)^2}{(c+x(x+2z))^2} + 1\right)}{(c+x(x+2z))^3\left(\frac{\left(c(2x+z)+x^2z\right)^2}{(c+x(x+2z))^2} - x^2\right)} + \frac{2cx + cz + x^2z}{c + x^2 + 2xz} \right).$$

To calculate the fixed points, we will simultaneously do $z = x$ and $O_{M_4N_1}(z,x) = (x,x)$, which gives us the following operator:

$$O_{M_4N_1}(x,x) = \left(x,\ \frac{2x\left(\frac{16x^4\left(c-x^2\right)^2}{\left(c+3x^2\right)^4} - \frac{4x^2\left(c-x^2\right)}{\left(c+3x^2\right)^2} + 1\right)\left(c-x^2\right)^4}{\left(c+3x^2\right)^3\left(\frac{\left(3cx+x^3\right)^2}{\left(c+3x^2\right)^2} - x^2\right)} + \frac{3cx + x^3}{c + 3x^2} \right).$$

It is easy to prove the following result from which we conclude the good stability of the methods on quadratic polynomials.

FIGURE 1 Dynamic line M_4N_1 for $c = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]



Theorem 9. *The only fixed points of the operator* $O_{M_4N_1}(x,x)$ *are the roots of the polynomial* $p(x)$*, that is,* $\left(\sqrt{c},\sqrt{c}\right)$ *and* $\left(-\sqrt{c},-\sqrt{c}\right)$*, and both fixed points have superattractor character, taking into account the order of the method.*

For any value of $c > 0$*, operator* $O_{M_4N_1}$ *does not have critical points different of the roots of* $p(x)$*.*

Therefore, it is obtained that the methods M_4N_1 and M_4N_y have as fixed points only the roots of the polynomial $p(x)$ and do not have free critical points when $c > 0$, that is, when the roots of the polynomial $p(x)$ are real.

In Figure 1, we draw the dynamical line, see Chicharro et al,[19] when $c = 1$ so that we can compare it with the dynamical lines of the rest of the methods. The dynamical line represents the basins of attraction, plotting in different colors where the orbit of each initial estimation tends. We paint in blue the initial points that converge to the root $\{1, 1\}$ and we paint in orange the initial points that converge to $\{-1, -1\}$.

In Figure 2A, we show different real dynamical planes of these methods for the polynomial $p(x)$ varying the value of c. These planes have been generated with a mesh of 1,000 points, a tolerance of 10^{-3}, and a maximum number of 2,000 iterations. The tolerance must be less than the distance from the last iteration to one of the two roots. The dynamical planes represent the basins of attraction of each method. We choose a initial point from the mesh, and applying the iterative method, we study whether or not this initial point converges. We paint in orange the initial points that converge to the superattracting fixed point $\{1, 1\}$, and we paint in blue the initial points that converge to $\{-1, -1\}$. In black are painted the initial points that do not converge to any fixed point.

As can be seen in these dynamical planes, the convergence zones are similar, but the M_4N_y method obtains more convergence points since the approximation used to obtain the parameter is a little better than that of the M_4N_1 method; for this reason, there may be points that converge for the M_4N_y method in a much smaller number of iterations than in the case of the M_4N_1 method, which may diverge or take considerably longer.

3.1 | Real dynamics of other methods with memory

The study of the fixed and critical points of the other methods with memory is the same, since the same rational operator is obtained. For this reason, we only study the case of the method with memory M_4N_2.

As in the previous case, we choose the polynomial $p(x) = x^2 - c$, when c is a positive real value, and as weight function $H(\mu)$ the polynomial $\mu^2 + \mu + 1$.

The previous section discussed how the operator was obtained for two previous iterations. For the case of three previous iterations, the definitions and results seen for the previous case are developed in an equivalent way.

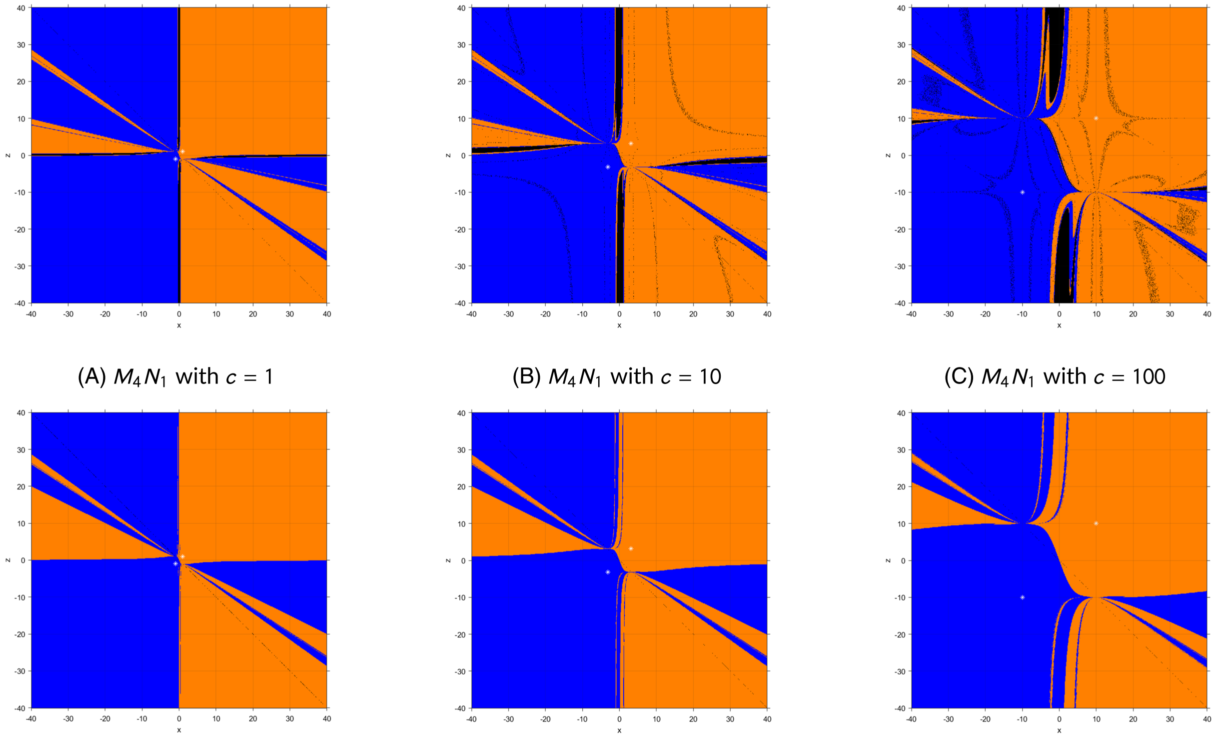
The vectorial operator obtained by applying the method on $p(x)$ is as follows, where $x_{k-1} = z$, $y_{k-1} = zy$, $x_k = x$ and $y_k = xy$,

$$O_{M_4N_2}(z,zy,x) = \left(x, xy, \frac{191x^{14} + 2063cx^{12} + 2659c^2x^{10} + 2291c^3x^8 + 781c^4x^6 + 189c^5x^4 + 17c^6x^2 + c^7}{4x\left(c+x^2\right)\left(c+3x^2\right)^5}\right).$$

To calculate the fixed points, we simultaneously do $z = x$, $zy = x$ and $O_{M_4N_2}(x,x,x) = (x,x,x)$, which gives us the following operator:

$$O_{M_4N_2}(x,x,x) = \left(x, x, \frac{191x^{14} + 2063cx^{12} + 2659c^2x^{10} + 2291c^3x^8 + 781c^4x^6 + 189c^5x^4 + 17c^6x^2 + c^7}{4x\left(c+x^2\right)\left(c+3x^2\right)^5}\right).$$

The next result establishes the stability of M_4N_2 on quadratic polynomials.



(A) M_4N_1 with $c = 1$ (B) M_4N_1 with $c = 10$ (C) M_4N_1 with $c = 100$

(D) M_4N_y with $c = 1$ (E) M_4N_y with $c = 10$ (F) M_4N_y with $c = 100$

FIGURE 2 Dynamic planes of the M_4N_1 and M_4N_y methods for different values of c [Colour figure can be viewed at wileyonlinelibrary.com]

Theorem 10. *The only fixed points of operator* $O_{M_4N_2}(x, x, x)$ *are the roots of polynomial* $p(x)$*, that is,* $\left(\sqrt{c}, \sqrt{c}, \sqrt{c}\right)$ *and* $\left(-\sqrt{c}, -\sqrt{c}, -\sqrt{c}\right)$*, and both fixed points have superattractor character, taking into account the order of convergence of method* M_4N_2*.*

For any value of $c > 0$*, operator* $O_{M_4N_2}$ *does not have critical points different of the roots of* $p(x)$*.*

In this case, the dynamical line when $c = 1$ is the same of method M_4N_1, as we see in Figure 3.

In this case, we can say that there are no strange fixed points (fixed points different of the roots) for any of the methods with memory obtained and that there are no free critical points (critical points different of the roots) either when $c > 0$.

Moreover, the two roots of the polynomial are superattractor fixed points, and as can be seen from the dynamical lines, they are stable methods when $c > 0$.

The dynamical lines were only made for one value of the parameter c since they were the same basins of attraction in all cases where $c > 0$.

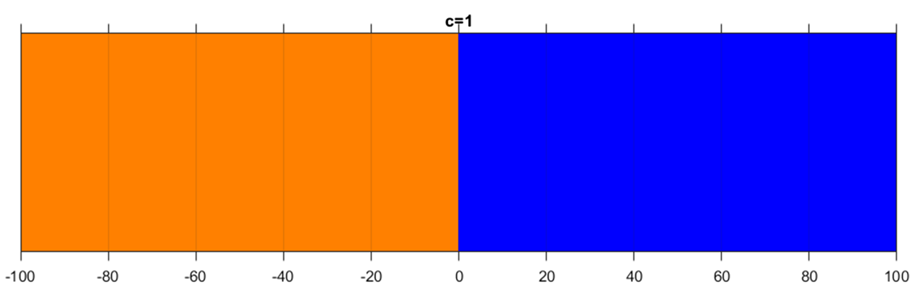


FIGURE 3 Dynamic line M_4N_2 for $c = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

4 | NUMERICAL EXPERIMENTS

In this section, we solve the nonlinear equations discussed below in order to make another comparison, in addition to that of the order of convergence, between the element of the parametric family M_4 corresponding to $\beta = -1$ and the memory methods obtained above.

We use Matlab R2020b, for the computational calculations, with arithmetic precision of 2,000 digits. We iterate from an initial estimate x_0 until it is verified that the distance between consecutive iterations plus the absolute value of the function evaluated in the last iteration is less than a tolerance of 10^{-100}. The items used to compare the methods in the examples are the approximation obtained, the norm of the equation evaluated in this approximation, the distance between the last two iterates, the number of iterations necessary to verify the tolerance, the computational time and the approximate computational convergence order (ACOC), defined by Cordero and Torregrosa,[20] which has the following expression:

$$p \approx ACOC = \frac{\ln(|x_{k+1} - x_k|/|x_k - x_{k-1}|)}{\ln(|x_k - x_{k-1}|/|x_{k-1} - x_{k-2}|)}.$$

The functions used are as follows:

- $f_1(x) = \cos(x) - x$, which has a zero at 0.73908513,
- If $f_2(x) = \arctan(x) - \frac{2x}{x^2+1} = 0$, it has a real root at -1.39175,
- $f_3(x) = (x-1)^3 - 1$, that has a zero at 2.

We use the quadratic polynomial $H(\mu) = \mu^2 + \mu + 1$ as the weight function for all methods. Table 1 lists the initial estimates that are used for each equation.

Let us observe the results obtained for the equation $\cos(x) - x = 0$ in Table 2. As it can be seen, all the methods converge to the solution, but there are differences between them.

The parametric family method takes one more iteration to reach the stopping criterion, but it is one of the methods that takes the lowest computational time.

As we can see in Table 2, the method that takes the highest time is method M_4N_1, and it is not the scheme that obtains the best approximation, so this would not be the ideal procedure in this case.

The next method taking more time is M_4K, but this one obtains a better approximation than the previous one in less time, although if we had to choose which method obtains a better approximation we would not choose this one or the previous one.

The best approximation is obtained by M_4K_y, followed by M_4N_3. By summarizing, method M_4K_y is the best one because of its low cpu-time, its great approximation and its ACOC, that fits the theoretical order of convergence.

Let us observe now the results obtained for the equation $f_2(x) = \arctan x - \frac{2x}{x^2+1} = 0$ in Table 3. As in the previous case, all methods converge to the solution, but there are some differences among them. The member of the parametric family and method M_4K take one more iteration to reach the stopping criterion, but the scheme needing more iterations is M_4N_1.

If we search the best approximations, we realize that they are obtained by M_4 and M_4K, although the method getting the best approximation with the lowest number of iterations is method M_4N_3.

TABLE 1 Initial estimations

Function	x_0	x_{-1}	y_{-1}	w_{-1}
$f_1(x)$	1	2	1.5	1.75
$f_2(x)$	−1	−0.25	−0.75	−0.5
$f_3(x)$	1.5	0	1.1	0.5

TABLE 2 Results for the equation $f_1(x) = 0$

Method	$\lvert x_{k+1} - x_k \rvert$	$\lvert f(x_k) \rvert$	Iteration	ACOC	Time
M_4 with $\beta = -1$	7.98384×10^{-225}	1.76651×10^{-897}	5	4	0.5437
M_4N_1	6.36668×10^{-125}	6.52553×10^{-555}	4	4.4822	0.7719
M_4K	3.88156×10^{-171}	3.96378×10^{-824}	4	4.95099	0.6625
M_4N_y	9.72756×10^{-165}	3.46507×10^{-823}	4	4.99744	0.5313
M_4K_y	6.84252×10^{-270}	$2.18741 \times 10^{-1618}$	4	6.00073	0.4969
M_4N_2	2.43352×10^{-190}	$8.10487 \times 10^{-1022}$	4	5.3755	0.5656
M_4N_3	8.22231×10^{-257}	$5.63335 \times 10^{-1540}$	4	5.99732	0.6219

TABLE 3 Results for the equation $f_2(x) = 0$

Method	$\|x_{k+1} - x_k\|$	$\|f(x_k)\|$	Iteration	ACOC	Time
M_4 with $\beta = -1$	3.83003×10^{-287}	$4.31925 \times 10^{-1149}$	5	4	0.5469
M_4N_1	4.49563×10^{-177}	2.0329×10^{-788}	6	4.4329	0.8000
M_4K	1.08803×10^{-451}	$2.69748 \times 10^{-2008}$	5	4.82737	0.8063
M_4N_y	3.16171×10^{-164}	7.17119×10^{-823}	4	4.95602	1.0938
M_4K_y	1.46333×10^{-220}	$6.63869 \times 10^{-1325}$	4	6.07599	0.6937
M_4N_2	1.69774×10^{-171}	6.23524×10^{-921}	4	5.38703	0.5469
M_4N_3	7.11925×10^{-222}	$9.37588 \times 10^{-1330}$	4	5.99633	0.7219

TABLE 4 Results for the equation $f_3(x) = 0$

Method	$\|x_{k+1} - x_k\|$	$\|f(x_k)\|$	Iteration	ACOC	Time
M_4 with $\beta = -1$	1.067×10^{-181}	2.59228×10^{-723}	6	4	0.3969
M_4N_1	2.03723×10^{-285}	$7.96453 \times 10^{-1267}$	6	4.44956	0.5000
M_4K	2.69898×10^{-346}	$4.85264 \times 10^{-1669}$	6	4.8361	0.7312
M_4N_y	4.33709×10^{-318}	$4.60379 \times 10^{-1587}$	6	5.00014	0.5531
M_4K_y	1.39248×10^{-181}	$7.13286 \times 10^{-1085}$	5	6.00027	0.5938
M_4N_2	1.5561×10^{-416}	$1.49505 \times 10^{-2234}$	6	5.4087	0.5750
M_4N_3	6.26407×10^{-151}	4.83317×10^{-901}	5	6	0.5938

Now, let us observe the results obtained for the equation $f_3(x) = (x-1)^3 - 1 = 0$ and presented in Table 4. In this case, all the methods need six iterations to converge, except M_4K_y and M_4N_3, which use five iterations. This table shows similar cpu-times, except in case of M_4 and M_4K, which are the methods taking the lowest and the highest time, respectively.

If we look at the approximations obtained, we notice that, among the methods needing five iterations, the best approximation is obtained by M_4K_y; among the methods taking six iterations, M_4N_2 stands out.

5 | CONCLUSIONS

In this manuscript, we design a family of two-step optimal iterative methods with convergence order 4. We introduce memory in several ways in this parametric family to increase the order without adding functional evaluations. Therefore, we increase the order up to two units, thus obtaining a method with memory with order 6, which is the highest order of convergence allowed by the error equation of its partner without memory.

Moreover, we study the stability of these schemes with memory for the sake of comparison. We conclude that, in general, the behavior of these methods is similar and that wide convergence zones are obtained for the function analyzed. Finally, we also perform numerical experiments, and it can be seen that the introduction of memory helps to obtain better results in general.

CONFLICT OF INTEREST

The authors declare no potential conflict of interest.

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