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Regular Articles

Asplunds, yet again ☆

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A B S T R A C T

We present some partially new characterizations of Asplund spaces via Baire class one selectors. Sufficient conditions for Asplundness in terms of properties of duality mappings are also considered.

Our motivation for this article was the realization that there are still some less known, if possibly even new, characterizations of Asplund spaces. We present them below, thus hoping that the Asplund property can be yet better understood. By the very definition, a Banach space X is called an *Asplund space* provided every convex continuous function defined on an open convex subset D of X is Fréchet differentiable at the points of a dense (necessarily G_δ) subset of D. This concept comes from E. Asplund's paper [1]. The name "Asplund space" was articulated a few years later by I. Namioka and R.R. Phelps in their paper [18].

Sufficient conditions for Asplundness in the setting of smoothness (i.e., Gateaux differentiability) of the norm were provided from the very beginning of the theory (Asplund [1] used local uniform rotundity of the dual, Ekeland and Lebourg [8] profited from the existence of a Fréchet differentiable bump, and Diestel and Faires [7] considered Gateaux differentiability of the norm combined with norm-to-weak continuity of its derivative —the so-called "very smoothness"). In absence of smoothness, it is natural to consider continuity properties (upper, lower semicontinuity) of the duality mapping, see Remark 5 below.

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For a rich supply of information about Asplund spaces we refer, e.g., to [19], [6, Section I.5], or [10, Chapter 11].

Here we concentrate on the existence of selectors of the first Baire class both for the duality mapping on X and for the argmin function on the Banach space $\mathcal{C}((B_{X^*}, w^*))$ (abbreviated $\mathcal{C}(B_{X^*})$) of all continuous functions on the compact topological space (B_{X^*}, w^*) (i.e., the closed unit ball of the dual X^*, endowed with the restriction of the w^*-topology on X^*), equipped with the supremum norm $\|\cdot\|$ (for the definition of the argmin function, see equation (1) below), something related —but not equivalent— to upper semicontinuity properties of the duality mapping. The reader may find two proofs of one of the crucial implications of our main result: The first one, in the spirit of Godefroy [15], is circumscribed to the duality mapping, and is based on Simons' inequality [20], see also [10, Lemma 3.123] (for the reader's convenience, we shall record its precise formulation in Lemma 2 below). The other, in a sense simpler, follows a clever argument due to Stegall done on the space $\mathcal{C}(B_{X^*})$; thus avoiding the need of Simons' Lemma 2.

Let $(X, \|\cdot\|)$ be a Banach space with the dual space X^*. For $x \in X$ and $x^* \in X^*$ we put $\langle x^*, x\rangle := x^*(x)$. On X^*, we can consider three topologies: "norm", "weak" (w, for short) and "weak*" (w^*, for short). The *duality mapping*, that is, the Moreau-Rockafellar subdifferential $\partial\|\cdot\| : X \longrightarrow 2^{B_{X^*}}$ of the norm $\|\cdot\|$, is defined by

$$\partial\|\cdot\|(x) := \{x^* \in B_{X^*} :\ \langle x^*, x\rangle = \|x\|\}, \quad x \in X.$$

Alaoglu's theorem asserts that the closed unit ball B_{X^*} in X^* provided with the w^* topology is a compact space.

Consider the set-valued mapping

$$\mathcal{C}(B_{X^*}) \ni f \longmapsto \big\{x^* \in B_{X^*} :\ f(x^*) = \min f(B_{X^*})\big\} =: \operatorname{argmin}(f) \subset B_{X^*}; \tag{1}$$

thus argmin: $\mathcal{C}(B_{X^*}) \longrightarrow 2^{B_{X^*}}$. The mapping argmin is widely used in variational analysis. Notice that for every $f \in \mathcal{C}(B_{X^*})$ we have agmin $(f) = B_{X^*} \cap \partial\|\cdot\|(-f)$ where $\|\cdot\|$ means the "maximum" norm on $\mathcal{C}(B_{X^*})$ (and we assume that B_{X^*} is a subset of $\mathcal{C}(B_{X^*})^*$).

For a compact space K, the mapping argmin: $C(K) \to 2^K$ satisfies $\operatorname{argmin}(f) = \partial\|\cdot\|(-f) \cap K$, where $f \in C(K)$ and $\|\cdot\|$ is the maximum norm on $C(K)$. The mapping $\kappa : X \longrightarrow \mathcal{C}(B_{X^*})$ given by

$$\kappa(x)(x^*) := \langle x^*, x\rangle, \quad x \in X,\ x^* \in B_{X^*} \tag{2}$$

is a linear isometry into, and $\operatorname{argmin}(\kappa(x)) = \partial\|\cdot\|(-x) \cap B_{X^*}$. Throughout this note we shall always have in mind its action on the broader scope of the continuous functions on the compact space (B_{X^*}, w^*). Remarks 7 and 8 below help to stress this point.

We shall use several times in this note the following well-known fact. For the reader's convenience, we include the proof:

Lemma 1. *Every w-separable subset M of a Banach space is norm-separable.*

Proof. If M is nonempty, find a countable set $\{s_n :\ n \in \mathbb{N}\}$ in M such that its weak closure contains M. Let Q be the family of all (infinite) sequences $(q_1, q_2, \ldots)$ with non-negative rational entries, all of them zero but a finite number, and such that $q_1 + q_2 + \cdots = 1$. Clearly, Q is a countable set. Thus, the set $\{q_1 s_1 + q_2 s_2 + \cdots :\ (q_1, q_2, \ldots) \in Q\}$ is also countable. Moreover, the latter set is obviously w-dense in $\operatorname{co} M$. Hence $\operatorname{co} M$ is w-separable. Now, Mazur's theorem (see, e.g., [10, Theorem 3.45]) guarantees that $\operatorname{co} M$ (and so M) is norm-separable. $\square$

Theorem 3 below is the main result of this note. The proof of (iii-w)⇒(i) depends, at a certain stage, of Simons' inequality. As promised, we reproduce here its statement as a lemma:

Lemma 2 *(Simons). Let* Γ *be a nonempty set. Let* (g_k) *be a sequence in* $\ell_\infty(\Gamma)$, *and let* Δ *be a subset of* Γ *with the following property: Whenever* $\lambda_1, \lambda_2, \ldots$ *are nonnegative numbers such that* $\sum_{n=1}^\infty \lambda_n = 1$, *there exists* $\gamma \in \Delta$ *with* $(\sum_{n=1}^\infty \lambda_n g_n)(\gamma) = \| \sum_{n=1}^\infty \lambda_n g_n\|_\infty$. *Then,*

$$\sup\left\{\limsup_n g_n(\gamma): \ \gamma \in \Delta\right\} \geq \inf\left\{\|g\|_\infty : \ g \in \mathrm{co}\left\{g_n : \ n \in \mathbb{N}\right\}\right\}.$$

Theorem 3. *For a Banach space* $(X, \|\cdot\|)$ *the following are equivalent:*

(i) X *is an Asplund space.*

(ii) The set-valued mapping $\mathrm{argmin}\colon \mathcal{C}(B_{X^*}, w^*) \longrightarrow 2^{B_{X^*}}$ *admits a selector which is of the norm-to-norm first Baire class.*

(ii-w) The set-valued mapping $\mathrm{argmin}\colon \mathcal{C}(B_{X^*}, w^*) \longrightarrow 2^{B_{X^*}}$ *admits a selector which is of the norm-to-weak first Baire class.*

(iii) The duality mapping $\partial\|\cdot\| : X \longrightarrow 2^{B_{X^*}}$ *admits a selector which is of the norm-to-norm first Baire class.*

(iii-w) The duality mapping $\partial\|\cdot\| : X \longrightarrow 2^{B_{X^*}}$ *admits a selector which is of the norm-to-weak first Baire class.*

Proof. (i)⇒(ii). Clearly, for every $f \in \mathcal{C}(B_{X^*})$ the set $\mathrm{argmin}(f)$ is non-empty and w^*-compact. The mapping argmin is norm-to-w^*-upper semicontinuous. To check this, fix any $f \in \mathcal{C}(B_{X^*})$ and any w^*-relatively open subset W of B_{X^*} containing argmin (f). Then for sure $\min f(B_{X^*} \setminus W) > \min f(B_{X^*})$. Hence, if $g \in \mathcal{C}(B_{X^*})$ and $\|g - f\|$ is small enough, we have that $\min g(B_{X^*} \setminus W) > \min g(B_{X^*})$, and so argmin (g) must be a subset of W. This proves the norm-to-w^*-upper semi-continuity of the mapping argmin.

Now, once X is an Asplund space, the dual space X^* is w^*-dentable by [19, Theorem 2.32]; hence, the Jayne-Rogers selection theorem, see [17, Theorem 8] or [9, Theorem 8.1.2], provides the desired selector $\varphi_0 : \mathcal{C}(B_{X^*}) \longrightarrow B_{X^*}$ for the set-valued mapping argmin. In more detail, it gives a sequence $(\varphi_j)_{j=1}^\infty$, where each $\varphi_j : \mathcal{C}(B_{X^*}) \longrightarrow B_{X^*}$ is a norm-to-norm continuous mapping, such that for every $f \in \mathcal{C}(B_{X^*})$ we have $\varphi_j(f) \longrightarrow \varphi_0(f) \in \mathrm{argmin}(f)$ as $j \to \infty$.

(ii)⇒(ii-w) and (iii)⇒(iii-w) are trivial.

(ii)⇒(iii) and (ii-w)⇒(iii-w) are based on the observation after equation (2) above.

(iii-w)⇒(i). We will profit from Godefroy's technology of working with the so called James boundaries, see [13], [11], [15], and [10, Sections 3.11.8.2 and 3.11.8.3]. Its definition and main features are made explicit in Remark 2 below. Here we provide the complete argument for the reader's convenience: From (iii-w) we find a sequence $(\varphi_j)_{j=1}^\infty$, where each $\varphi_j : X \longrightarrow B_{X^*}$ is norm-to-w-continuous and such that, for every $x \in X$, we have $\varphi_j(x) \longrightarrow \varphi_0(x) \in \partial\|\cdot\|(x)$ weakly as $j \to \infty$. Put

$$\Phi(x) := \{\varphi_1(x), \varphi_2(x), \ldots\} \ (\subset B_{X^*}), \quad x \in X.$$

Take any separable subspace Y of X. We will show that

$$\overline{\mathrm{sp}\, \Phi(Y)\restriction_Y}^{\|\cdot\|} = Y^*. \tag{3}$$

(Here the closure is computed in the canonical dual norm of Y^*, and $x^*\restriction_Y$ means the restriction of $x^* \in X^*$ to Y; accordingly, we put $M\restriction_Y := \left\{x^*\restriction_Y : \ x^* \in M\right\}$ for any $M \subset X^*$.) Observe that

$$\Phi(Y) \subset \varphi_1(Y) \cup \varphi_2(Y) \cup \cdots \tag{4}$$

Since the φ_j's are norm-to-w-continuous, the union in (4) is w-separable (thus, norm-separable, see Lemma 1 above). This shows that $\mathrm{sp}\, \Phi(Y)$ is also norm-separable in X^* (and so $\mathrm{sp}\, \Phi(Y)\restriction_Y$ is norm-separable in Y^*).

Thus (3) will guarantee the separability of Y^*, and hence the whole space X will be Asplund by [19, Corollary 2.15].

Suppose that (3) is false. Then the Hahn-Banach separation theorem yields a $y_0^{**} \in S_{Y^{**}}$ and a $y_0^* \in B_{Y^*}$ such that

$$\langle y_0^{**}, y_0^* \rangle > 0 = \langle y_0^{**}, y^* \rangle \quad \text{for every} \quad y^* \in \overline{\operatorname{sp} \Phi(Y)\restriction_Y}^{\|\cdot\|} \ \big(= \overline{\operatorname{sp} \Phi(Y)\restriction_Y}^{w} \big). \tag{5}$$

Put $\Delta := \overline{\Phi(Y)}^{\,w}\restriction_Y \ (\subset \overline{\varphi_1(Y) \cup \varphi_2(Y) \cup \cdots}^{\,w}\restriction_Y \subset B_{Y^*})$ (see (4) above). The arguments around (4) show that Δ is norm-separable. We can easily verify that for every $y \in Y$ there is a $\delta \in \Delta$ such that $\|y\| = \langle \delta, y \rangle$; thus Δ is a James boundary of B_{Y^*}; see [10, page 132] and Remark 2 below. We know that (the canonical embedding of) the unit ball B_Y is w^*-dense in the double dual unit ball $B_{Y^{**}}$. Thus, the norm-separability and boundedness of Δ yield a sequence $(y_k)_{k=1}^{\infty}$ in B_Y which converges to y_0^{**} pointwise on each element of $\Delta \cup \{y_0^*\}$. By omitting a few elements of the sequence, and then relabeling it, we can and do assume that

$$\langle y_0^*, y_k \rangle > \frac{1}{2} \langle y_0^{**}, y_0^* \rangle \ (> 0) \quad \text{for all} \quad k \in \mathbb{N}. \tag{6}$$

Now, we will apply Simons' Lemma 2, with $\Gamma := B_{Y^*}$, with our Δ, and with $g_k := y_k \restriction_{B_{Y^*}}$, $k \in \mathbb{N}$. From the definition of Φ we can easily deduce that the premise of Lemma 2 is satisfied. In fact, consider any non-negative numbers $\lambda_1, \lambda_2, \ldots$ satisfying $\lambda_1 + \lambda_2 + \cdots = 1$ and put $y := \lambda_1 y_1 + \lambda_2 y_2 + \cdots$. Clearly y is well defined and belongs to Y, and we noticed above that Δ is a James boundary of B_{Y^*}.

Observe that $u(y^*) := \limsup_k g_k(y^*) = 0$ for $y^* \in \Delta$ (see equation (5) above). Thus,

$$\sup\{u(y^*) : \ y^* \in \Delta\} = 0. \tag{7}$$

However, if $g \in \operatorname{co}\{g_k : \ k \in \mathbb{N}\}$, we have $\|g\| > (1/2)\langle y_0^{**}, y_0^* \rangle \ (> 0)$ (see equation (6) above), hence

$$\inf\{\|g\| : \ g \in \operatorname{co}\{g_k : \ k \in \mathbb{N}\}\} \geq \frac{1}{2}\langle y_0^{**}, y_0^* \rangle \ (> 0). \tag{8}$$

Finally, (7) and (8) together contradict Simons' inequality. □

It is worth to mention that the implication (ii-w)⇒(i) has a simpler, more direct proof, not needing Simons' lemma. (No wonder because the set-valued mapping argmin is defined on $\mathcal{C}(B_{X^*})$, which is much bigger/richer than its subspace X, the domain of the set-valued mapping $\partial\|\cdot\|$.) We will imitate ideas of Ch. Stegall, see [21] and [4]. Assume that there are norm-to-w-continuous mappings $\varphi_j : \mathcal{C}(B_{X^*}) \longrightarrow B_{X^*}$, $j \in \mathbb{N}$, such that $\varphi_j(f) \longrightarrow \varphi_0(f) \in \operatorname{argmin}(f)$ weakly for every $f \in C(B_{X^*})$.

In order to show that X is Asplund, we must prove that $(Y^*, \|\cdot\|)$ is separable for every separable subspace Y of X (see [19, Theorem 2.19]). So, fix a separable subspace Y of X. Let $\{s_n : \ n \in \mathbb{N}\}$ be a norm-dense subset of the unit ball of Y. By $\mathcal{L}$ we denote the family of all functions of the form

$$B_{X^*} \ni x^* \longmapsto \sum_{n=1}^{k} 2^{-n} \big| a_n - \langle x^*, s_n \rangle \big|, \tag{9}$$

where $k \in \mathbb{N}$ and $a_1, a_2, \ldots, a_k$ are rational numbers in $[-1, 1]$. Clearly, each element in $\mathcal{L}$ is w^*-continuous. Thus $\mathcal{L}$ is a countable subset of $\mathcal{C}(B_{X^*})$. Further put

$$\Lambda := \big\{\varphi_j(g) : \ j \in \mathbb{N}, \ g \in \mathcal{L}\big\} \ (\subset B_{X^*});$$

This set is also countable.

Pick any $y^* \in B_{Y^*}$. Put

$$f(x^*) := \sum_{n=1}^{\infty} 2^{-n} \big| \langle y^*, s_n \rangle - \langle x^*, s_n \rangle \big|, \text{ for } x^* \in B_{X^*}.$$

Clearly, f belongs to $\overline{\mathcal{L}}$, the norm-closure of $\mathcal{L}$. Notice that, for $x^* \in B_{X^*}$,

$$f(x^*) = \begin{cases} 0 & \text{if } x^*\restriction_Y = y^*, \\ > 0 & \text{otherwise.} \end{cases}$$

Thus, $\operatorname{argmin}(f)\restriction_Y = \{y^*\}$ and, since $\varphi_0(f) \in \operatorname{argmin}(f)$, we have that $\varphi_0(f)\restriction_Y = \{y^*\}$.

Fix any w-open neighborhood W of the origin in X^*. Recalling that $\varphi_j(f) \rightharpoonup \varphi_0(f)$ (weakly) as $j \to \infty$, there is $j \in \mathbb{N}$ so that $\varphi_j(f) \in \varphi_0(f) - \frac{1}{2}W$. As $f \in \overline{\mathcal{L}}$ and φ_j is continuous, there is $g \in \mathcal{L}$ so that $\varphi_j(g) \in \varphi_j(f) - \frac{1}{2}W$. Adding the latter two inclusions we get that $\varphi_0(f) \in \varphi_j(\mathcal{L}) + W \subset \Lambda + W$. This happens for every w-open neighborhood W of 0; so $\varphi_0(f) \in \overline{\Lambda}^w$. Then finally, $(\{y^*\} =)\ \ \varphi_0(f)\restriction_Y \subset \overline{\Lambda}^w\restriction_Y$. But the latter set here is w-separable, hence also norm-separable (see again Lemma 1 above). And y^* being arbitrary, we get that B_{Y^*} lies in the norm-separable set $\overline{\Lambda}^w\restriction_Y$. Therefore $(Y^*, \|\cdot\|)$ is separable. We thus have got (i).

Remarks. 1. The equivalence (i)⇔(iii) can be found in [6, page 32].

2. The key step in the proof of (iii-w)⇒(i) follows from the statement: If JB is any norm-separable James boundary of B_{X^*}, then $\overline{\text{co}}^{\|\cdot\|}(JB) = B_{X^*}$; see [13], and also [10, Section 3.11.8.3, in particular Theorem 3.122] (a *James boundary of* B_{X^*} is any subset JB of B_{X^*} such that every $x \in X$ attains its norm at a point of JB). Assume that $\partial\|\cdot\|$ has a selector that is the pointwise-weak limit of a sequence $(\varphi_n)_{n=1}^{\infty}$ of norm-to-weak continuous mappings from X into X^*. Let Y be an arbitrary separable subspace of X. Then $JB := \overline{\bigcup_{n=1}^{\infty} \varphi_n(Y)}^{\,w}$ is a separable James boundary for Y (see the discussion in the proof of (iii-w)⇒(i), Theorem 3 above). Hence Y^* is separable, and so X is Asplund by [19, Corollary 2.15].

3. From Theorem 3 (ii-w) we immediately get: *If the duality mapping $\partial\|\cdot\|$ admits a norm-to-norm, or just norm-to-weak, continuous selection on $X \setminus \{0\}$, then X is Asplund.* (We avoid here the origin 0 because there is no chance for the lower semicontinuity of $\partial\|\cdot\|$ there —look at the norm $\mathbb{R} \ni x \mapsto |x|$.) In particular, if the norm $\|\cdot\|$ on X is Gateaux smooth, with norm-to-weak continuous derivative on $X \setminus \{0\}$, then X is Asplund. This is a result of Diestel and Faires quoted in the introduction.

4. What if $\partial\|\cdot\|$ is norm-to-norm, or norm-to-weak, or just norm-to-weak* lower semicontinuous off 0? Then we can rather easily squeeze that $\partial\|\cdot\|$ must be a singleton off 0, that is, the norm is Gateaux smooth. We actually have the following equivalence: Given a convex function $\varphi : X \longrightarrow \mathbb{R}$, continuous at some $x \in X$, then $\partial\varphi(x)$ is a singleton if and only if the subdifferential $\partial\varphi$ is norm-to-weak* lower semicontinuous at x. Proof. Assume that $|\partial\varphi(x)| = 1$. By [19, Proposition 2.5], $\partial\varphi$ is norm-to-weak* upper semicontinuous at x; this is a general fact. Hence, using the very definitions, $\partial\varphi$ is also norm-to-weak* lower semicontinuous at x. Further assume that $|\partial\varphi(x)| > 1$. Then for sure there is an $h \in X$ such that $|\partial\varphi(x)h| > 1$, which, by the convexity of the set $\partial\varphi(x)h$, means that this set is a non-degenerate linear segment, $[a, b]$ say. Find $\xi \in \partial\varphi(x)$ such that $\langle \xi, h \rangle = a$. Put $U := \{x^* \in X^* :\ \langle x^*, h \rangle > \frac{1}{2}(a + b)\}$. This is a weak* open set and $U \cap \partial\varphi(x) \neq \emptyset$. But $U \cap \partial\varphi(x - th) = \emptyset$ for all $t > 0$ small enough. Indeed, if $\eta \in \partial\varphi(x - th)$, then from the monotonicity of $\partial\varphi$ we get

$$0 \leq \langle \xi - \eta, x - (x - th) \rangle = t(a - \langle \eta, h \rangle);$$

thus $\langle \eta, h \rangle \leq a$ $(< \frac{1}{2}(a + b))$ and so $\eta \notin U$. We disproved the norm-to-weak* lower semicontinuity of φ at x.

5. Sufficient conditions for Asplundness in the presence of smoothness were considered in the introduction. See also Remark 1 above for a particular case. In absence of smoothness, and besides looking at the existence of selectors of the multivalued duality mapping with particular continuity properties (see again Remark 1 above), it is natural to rely on continuity properties (upper semicontinuity, to be precise; for lower semicontinuity see Remark 3 above) of this mapping in order to obtain Asplundness, as it was already mentioned in the introduction. It is well known that the duality mapping is always norm-to-weak* upper semicontinuous (Cudia [3], [19, Proposition 2.5]). Hu and Lin [16] proved that its norm-to-weak upper semicontinuity implies Asplundness, and Giles, Gregory, and Sims [12] got the same conclusion under a weaker form of the condition. Other important contributions were provided by Godefroy [13] (strong subdifferentiability —see below in this same item— implies Asplundness), and, by using the same technology (Simons' inequality, to be precise), Contreras and Payá [2] proved that every quite smooth Banach space is Asplund. A Banach space X is said to be quite smooth if for every non-zero $x \in X$ and for every weak neighborhood V of 0 in X^* there exists $\delta > 0$ such that $\partial\|\cdot\|(x) \subset \partial\|\cdot\|(x_0) + V$ whenever $0 \neq x \in X$ and $\|x - x_0\| < \delta$. Clearly, norm-to-weak upper semicontinuity of the duality mapping implies quite smoothness. If, instead of V being a weak neighborhood of 0 in the definition of quite smoothness we let V be a norm neighborhood, we get the so-called strong subdifferentiability of the norm. In [14] this concept was used for getting a characterization of Asplundness in the setting of separable Banach spaces: A separable space X whose dual is not separable has an equivalent norm that is nowhere strongly subdifferentiable except at the origin. Apparently, the nonseparable case is still open.

6. Theorem 3 is useful when deriving structural results on Asplund spaces, see [11], [6, Section VI.3], [4], and [5].

7. In the conditions (ii) and (ii-w), it is possible to consider the set-valued mapping argmin only on the family

$$\Big\{ \sum_{n=1}^{\infty} 2^{-n} |a_n - \langle \cdot, s_n \rangle| : \ a_1, a_2, \ldots \in [-1, 1], \ s_1, s_2, \ldots \in B_X \Big\} \ \Big(\subset \mathcal{C}(B_{X^*}) \Big).$$

Note that this family is disjoint from $\kappa(X)$ (see equation (2) above), and that Simons' lemma is still not needed.

8. What are the novelties in this note? The use of $\mathcal{C}(B_{X^*})$, of the set-valued mapping argmin, considering the weak topology instead of the norm topology in X^*, and finally a rather careful account of what happens if the duality mapping $\partial\|\cdot\|$ has various (semi) continuity properties.

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