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Solving Nonlinear Transcendental Equations by Iterative Methods with Conformable Derivatives: A General Approach

Giro Candelario [1], Alicia Cordero [2,*], Juan R. Torregrosa [2] and María P. Vassileva [1]

[1] Área de Ciencias Básicas y Ambientales, Instituto Tecnológico de Santo Domingo (INTEC), Av. Los Procéres, Gala, Santo Domingo 10602, Dominican Republic; giro.candelario@intec.edu.do (G.C.); maria.penkova@intec.edu.do (M.P.V.)

[2] Institute for Multidisciplinary Mathematics, Universitat Politècnica de València, 46022 València, Spain; jrtorre@mat.upv.es

* Correspondence: acordero@mat.upv.es

Abstract: In recent years, some Newton-type schemes with noninteger derivatives have been proposed for solving nonlinear transcendental equations by using fractional derivatives (Caputo and Riemann–Liouville) and conformable derivatives. It has also been shown that the methods with conformable derivatives improve the performance of classical schemes. In this manuscript, we design point-to-point higher-order conformable Newton-type and multipoint procedures for solving nonlinear equations and propose a general technique to deduce the conformable version of any classical iterative method with integer derivatives. A convergence analysis is given and the expected orders of convergence are obtained. As far as we know, these are the first optimal conformable schemes, beyond the conformable Newton procedure, that have been developed. The numerical results support the theory and show that the new schemes improve the performance of the original methods in some aspects. Additionally, the dependence on initial guesses is analyzed, and these schemes show good stability properties.

Keywords: nonlinear equations; conformable derivative; higher-order Newton's procedure; multipoint methods; optimal schemes; stability

MSC: 65H05

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1. Introduction

Fractional calculus dates back to shortly after the appearance of the classical one, when Leibniz and l'Hôpital came up with the concept of the half-derivative in 1695. Since then, some definitions of fractional and non fractional derivatives which preserve many properties of classical calculus have been introduced, and many real problems can be modeled by using these derivatives [1–3].

In recent papers, some fractional Newton-type iterative procedures with Caputo and Riemann–Liouville derivatives (see [4–9]), and fractal Newton-type iterative methods (see [10]), have been designed in order to find the solution $\bar{x} \in \mathbb{R}$ of a nonlinear function $f(x)$, where $f{:}I \subseteq \mathbb{R} \to \mathbb{R}$ is a continuous function in $I \subseteq \mathbb{R}$, but the theoretical order of convergence is neither preserved nor held in practice. Additionally, optimal conformable Newton-type schemes were proposed in [11,12] (in scalar and vectorial versions, respectively) by using the conformable derivative/Jacobian, and the order of convergence was obtained in theory and in practice too. As far as we know, there are no optimal conformable multipoint iterative methods in the literature. In using the conformable derivative in this work, we have three targets: obtain high-order optimal conformable Newton-type procedures (in the sense of Kung–Traub's conjecture [13]) based on the method in [11], devise a general technique to obtain the conformable version of any classical scheme, and compare each one of these procedures with its classical version. With the proposed technique, we are able to generate optimal conformable multipoint methods with an arbitrary order of convergence starting with any non-conformable optimal iterative scheme.

First, let us introduce some concepts related to conformable calculus. The left conformable derivative of a function $f : [a, \infty) \to \mathbb{R}$ starting from a of order $\alpha \in (0, 1]$, $\alpha, a, x \in \mathbb{R}$, $a < x$, is defined as [14,15]

$$(T^a_\alpha f)(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon(x-a)^{1-\alpha}) - f(x)}{\varepsilon}. \tag{1}$$

If the limit in (1) exists, f is said to be α-differentiable. If f is differentiable, $(T^a_\alpha f)(x) = (x - a)^{1-\alpha} f'(x)$. If f is α-differentiable in (a, b), for some $b \in \mathbb{R}$, $(T^a_\alpha f)(a) = \lim_{x \to a^+} (T^a_\alpha f)(x)$.

The left conformable derivative holds the property of the nonfractional derivative, $T^a_\alpha K = 0$, where K is a constant. This derivative does not require the evaluation of special functions, as Gamma or Mittag-Leffler functions do, as it uses fractional derivatives, as in Riemann–Liouville or Caputo functions.

The next result provides a suitable Taylor power series of a function $f(x)$, where the conformal derivatives start from a point a, distinct from another point a_1 where they are evaluated.

Theorem 1 (Theorem 4.1, [16])**.** *Let $f(x)$ be an infinitely α-differentiable function ($0 < \alpha \leq 1$), at the neighborhood of a_1, whose conformable derivative starts from a. The conformable power series for $f(x)$ is*

$$f(x) = f(a_1) + \frac{(T^a_\alpha f)(a_1)\delta_1}{\alpha} + \frac{(T^a_\alpha f)^{(2)}(a_1)\delta_2}{2\alpha^2} + R_2(x, a_1, a), \tag{2}$$

where $\delta_1 = H^\alpha - L^\alpha$, $\delta_2 = H^{2\alpha} - L^{2\alpha} - 2L^\alpha \delta_1, \dots$, and $H = x - a$, $L = a_1 - a$.

It can be shown that $\delta_2 = \delta_1^2$, $\delta_3 = \delta_1^3$, etc. Now, the conformable Taylor power series (2) can be written as

$$f(x) = f(a_1) + \frac{(T^a_\alpha f)(a_1)\delta_1}{\alpha} + \frac{(T^a_\alpha f)^{(2)}(a_1)\delta_1^2}{2\alpha^2} + R_2(x, a_1, a). \tag{3}$$

The following result states the quadratic order of convergence obtained in [11] by using the Taylor power series (3) for simple roots (multiplicity $m = 1$).

Theorem 2 (see [11])**.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function in the interval $I \in \mathbb{R}$ containing the zero $\bar{x}$ of $f(x)$. Let $(T^a_\alpha f)(x)$ be the conformable derivative of $f(x)$ starting from a, of order α, $\alpha \in (0, 1]$. Let us suppose that $(T^a_\alpha f)(x)$ is continuous and not null at $\bar{x}$. If an initial approximation x_0 is sufficiently close to $\bar{x}$, then the local order of convergence of the conformable Newton-type method*

$$x_{k+1} = a + \left((x_k - a)^\alpha - \alpha \frac{f(x_k)}{(T^a_\alpha f)(x_k)} \right)^{1/\alpha}, \quad k = 0, 1, 2, \dots, \tag{4}$$

is at least 2, where $0 < \alpha \leq 1$, and the error equation is

$$e_{k+1} = \alpha(\bar{x} - a)^{\alpha-1} C_2^{CO} e_k^2 + O\left(e_k^3\right), \tag{5}$$

being $C_j^{CO} = \dfrac{1}{j!\alpha^{j-1}} \dfrac{(T^a_\alpha f)^{(j)}(\bar{x})}{(T^a_\alpha f)(\bar{x})}$, $j = 2, 3, 4, \dots$

In [12], it is shown that the asymptotic error constant of the error equation in (5) may be expressed as

$$\alpha(\bar{x} - a)^{\alpha-1} C_2^{CO} = C_2 + \frac{1}{2} \frac{1-\alpha}{\bar{x} - a}, \tag{6}$$

where $C_j = \frac{1}{j!}\frac{f^{(j)}(\bar{x})}{f'(\bar{x})}$, $j = 2,3,4,\ldots$, with this being the classical asymptotic error constant.

Morover, it is also shown in [12] that the error equation of scheme (4), by using the classical Taylor series, is

$$e_{k+1} = \left(C_2 + \frac{1}{2}\frac{1-\alpha}{\bar{x}-a}\right)e_k^2 + O\left(e_k^3\right), \tag{7}$$

concluding that error equations in (5) and (7) are equivalents, no matter if the Taylor power series (3) or the classical one are used in convergence analysis. So, we are going to use classical Taylor expansions in convergence analysis later. Moreover, when $\alpha = 1$, the classical Newton procedure is obtained in (4), as well as its error equation in (5) and (7).

Knowing that the asymptotic error constant C of an iterative method $\phi(x)$ of order p is defined as (see [17])

$$C = \lim_{x\to\bar{x}} \frac{\phi(x)-\bar{x}}{(x-\bar{x})^p}, \tag{8}$$

the following result allows us to calculate the asymptotic error constant of an iterative scheme of order p if we know the asymptotic error constant of another iterative procedure of order p.

Theorem 3 (Theorem 2-8, [17])**.** *Let $\phi_1(x)$, $\phi_2(x)$ be of order p whose solution $\bar{x}$ is of multiplicity $m \geq 1$. Let*

$$G(x) = \frac{\phi_2(x)-\phi_1(x)}{(x-\bar{x})^p}, \quad x \neq \bar{x}. \tag{9}$$

Let C_1, C_2 be the asymptotic error constants of ϕ_1 and ϕ_2, respectively. Then,

$$C_2 = C_1 + \lim_{x\to\bar{x}} G(x). \tag{10}$$

We also confirm Theorem 3 for the conformable version of multipoint methods given later. In all our methods, we consider multiplicity $m = 1$.

In the next section, we derive three higher-order Newton-type schemes. We also provide a technique to obtain the conformable version of any classical method, and it is applied to derive some conformable multipoint schemes. The convergence analysis of all the proposed conformable methods is presented in Section 3; the numerical results and numerical stability are discussed in Section 4, showing good numerical performance and improving the original methods in some aspects of both convergence and stability; and some conclusions are given in Section 5.

2. From One-Point to Multipoint Conformable Methods

In order to deduce higher-order schemes by means of a modification of the conformable Newton scheme (4), we need its error equation in (7), but up to order four. Let us remark that this expression depends on a and α,

$$\begin{aligned} e_{k+1} = {} & \left(C_2 + \frac{1-\alpha}{2(\bar{x}-a)}\right)e_k^2 + \left(2C_3 - 2C_2^2 + \frac{(\alpha-1)C_2}{\bar{x}-a} + \frac{(1-\alpha)(\alpha-2)}{3(\bar{x}-a)^2}\right)e_k^3 \\ & + \left(3C_3 - 7C_2C_3 + 4C_2^3 + \frac{(1-\alpha)(5C_2^2-C_3)}{2(\bar{x}-a)} + \frac{(2\alpha^2-5\alpha+3)C_2}{2(\bar{x}-a)^2} + \frac{(1-\alpha)(2\alpha^2-7\alpha+7)}{8(\bar{x}-a)^3}\right)e_k^4 \\ & + O\left(e_k^5\right). \end{aligned} \tag{11}$$

In a first approximation, we want to find a value of α that nulls the quadratic term in (11), so

$$\begin{aligned}\alpha &= 1+2(\bar{x}-a)C_2\\ &= 1+(\bar{x}-a)\frac{f''(\bar{x})}{f'(\bar{x})}.\end{aligned}$$

By considering α_k and iterate x_k as approximations of α and the solution $\bar{x}$, respectively,

$$\alpha_k = 1+(x_k-a)\frac{f''(x_k)}{f'(x_k)}, \quad k=0,1,2,\ldots \tag{12}$$

Therefore, we can define a new iterative scheme that can improve the performance of the conformable Newton scheme, which we denote by NeL3:

$$x_{k+1} = a+\left((x_k-a)^{\alpha_k}-\alpha_k\frac{f(x_k)}{(T^a_{\alpha_k}f)(x_k)}\right)^{1/\alpha_k}, \quad k=0,1,2,\ldots, \tag{13}$$

where α_k is described by (12).

In a second approximation, we use a to improve the order of convergence, so

$$\begin{aligned}a &= \bar{x}+\frac{1-\alpha}{2C_2}\\ &= \bar{x}+(1-\alpha)\frac{f'(\bar{x})}{f''(\bar{x})}.\end{aligned}$$

Then, an estimation of a is obtained by means of the expression

$$a_k = x_k+(1-\alpha)\frac{f'(x_k)}{f''(x_k)}, \quad k=0,1,2,\ldots \tag{14}$$

Replacing a in (4) by (14) we obtain a new procedure, and we denote it as NeA3:

$$x_{k+1} = a_k+\left((x_k-a_k)^{\alpha}-\alpha\frac{f(x_k)}{(T^{a_k}_{\alpha}f)(x_k)}\right)^{1/\alpha}, \quad k=0,1,2,\ldots \tag{15}$$

where a_k is described by (14).

However, we can consider simultaneously how to define a and α in order to cancel not only the quadratic but also the cubic term of the error equation of the conformable Newton scheme. Solving the resulting nonlinear system generated, for α and a as unknowns, we obtain

$$\begin{aligned}\alpha &= 1+\frac{2C_2^2}{2C_2^2-3C_3}\\ &= 1+\frac{f''(\bar{x})^2}{f''(\bar{x})^2-f'(\bar{x})f'''(\bar{x})}.\end{aligned}$$

Considering α_k and iterate x_k as approximations of α and the solution $\bar{x}$, respectively, α_k can be estimated at each iteration by

$$\alpha_k = 1+\frac{f''(x_k)^2}{f''(x_k)^2-f'(x_k)f'''(x_k)}, \quad k=0,1,2,\ldots \tag{16}$$

Moreover, a is

$$\begin{aligned} a &= \bar{x} + \frac{C_2}{3C_3 - 2C_2^2} \\ &= \bar{x} + \frac{f'(\bar{x})f''(\bar{x})}{f'(\bar{x})f'''(\bar{x}) - f''(\bar{x})^2}. \end{aligned}$$

So, using x_k as an estimation of $\bar{x}$, an iterative estimation of a can be calculated as

$$a_k = x_k + \frac{f'(x_k)f''(x_k)}{f'(x_k)f'''(x_k) - f''(x_k)^2}, \quad k = 0, 1, 2, \ldots \tag{17}$$

Replacing α and a in (4) by (16) and (17), respectively, we obtain a new scheme, which we denote by NeLA4:

$$x_{k+1} = a_k + \left((x_k - a_k)^{\alpha_k} - \alpha_k \frac{f(x_k)}{(T_{\alpha_k}^{a_k} f)(x_k)} \right)^{1/\alpha_k}, \quad k = 0, 1, 2, \ldots, \tag{18}$$

where α_k and a_k correspond to (16) and (17), respectively.

So, we have designed one-point procedures that improve the order of convergence of (4), although they use derivatives of the nonlinear function up to order four.

In [17], Traub proves that any one-point method of order p depends on the first $p-1$ derivatives of f. So, in addition to the computational complexity of NeL3, NeA3, and NeLA4, another drawback is the evaluation of higher-order derivatives in their iterative schemes; this is reason why multipoint procedures are widely used in the literature (see [17,18]).

So, in order to devise a general technique to obtain the conformable version of any method, we start from the assumption that any fixed-point function coming from an iterative scheme not using conformable derivatives can be written as (see [17])

$$\phi(x) = x - f(x)g(x), \tag{19}$$

where $g(\bar{x})$ is finite and not null, and $\bar{x}$ is a fixed point of ϕ. If $g(x)$ in (19) is $1/f'(x)$, the classical Newton procedure is obtained.

From (19) we can obtain a general representation of classical Taylor series of f about $\bar{x}$ up to order one, as

$$f(x) \approx \frac{1}{g(\bar{x})}(x - \phi(\bar{x})). \tag{20}$$

Considering the conformable Taylor power series (3), we can obtain the conformable version of (20) as

$$f(x) \approx \frac{1}{\alpha g_\alpha(\bar{x})}[(x-a)^\alpha - (\phi(\bar{x}) - a)^\alpha]. \tag{21}$$

If the analytical expression of $g(x)$ includes classical derivatives of $f(x)$, then the analytical expression of $g_\alpha(x)$ includes conformable derivatives of $f(x)$. Now, isolating $\phi(\bar{x})$ from (21), we have

$$\phi(\bar{x}) \approx a + ((x-a)^\alpha - \alpha f(x) g_\alpha(\bar{x}))^{1/\alpha}. \tag{22}$$

As x is considered an estimation of $\bar{x}$, we finally obtain the conformable version of (19) as

$$\phi(x) = a + ((x-a)^\alpha - \alpha f(x) g_\alpha(x))^{1/\alpha}. \tag{23}$$

In (23), if $g_\alpha(x) = 1/(T_\alpha^a f)(x)$, then it corresponds to the fixed point of the conformable Newton scheme (4). So, by using (23), we can obtain the conformable version of any procedure.

Therefore, to transfer an iterative scheme to its conformable version, it is necessary to identify its analytical expression of $g(x)$ in the given classical method. In [11], we can see that the theoretical order of convergence of the classical version of Newton's procedure is held when its conformable version is analyzed. Furthermore, sometimes it presents some numerical advantages versus the classical version. By using the conformable version of Newton's scheme in [11], we have observed that

- We can find the solution when classical method fails (with $\alpha \neq 1$).
- Sometimes, the conformable method requires fewer iterations than the classical scheme, and the computational order of convergence can be slightly greater.
- It is possible to obtain a different root by choosing distinct values of index α, with the same initial estimation.
- Complex roots can be found by starting from real initial estimates.

Now, we use the procedure defined in (23) to derive the conformable version of some classical multipoint schemes. Let us consider the third-order Traub scheme [17,18]

$$\psi_1(x) = \phi_1(x) - \frac{f[\phi_1(x)]}{f'(x)}, \tag{24}$$

where the predictor step is

$$\phi_1(x) = x - \frac{f(x)}{f'(x)}. \tag{25}$$

In this case, from (24), we deduce that $g(x) = \dfrac{1}{f'(x)}$. Hence,

$$g_\alpha(x) = \frac{1}{(T_\alpha^a f)(x)}.$$

As the conformable version of predictor (25) is (4), the conformable version of Traub's procedure is

$$\psi_2(x) = a + \left((\phi_2(x) - a)^\alpha - \alpha \frac{f[\phi_2(x)]}{(T_\alpha^a f)(x)} \right)^{1/\alpha}, \tag{26}$$

being

$$\phi_2(x) = a + \left((x-a)^\alpha - \alpha \frac{f(x)}{(T_\alpha^a f)(x)} \right)^{1/\alpha}, \tag{27}$$

which we denote by TeCO.

On the other hand, let us consider now Chun–Kim's third-order method [18,19]

$$\psi_3(x) = x - \frac{1}{2}\left[3 - \frac{f'[\phi_1(x)]}{f'(x)}\right] \frac{f(x)}{f'(x)}, \tag{28}$$

where $\phi_1(x) = x - \dfrac{f(x)}{f'(x)}$. From (28), we deduce that $g(x) = \dfrac{1}{2}\left[3 - \dfrac{f'[\phi(x)]}{f'(x)}\right] \dfrac{1}{f'(x)}$. So,

$$g_\alpha(x) = \frac{1}{2}\left[3 - \frac{(T_\alpha^a f)[\phi(x)]}{(T_\alpha^a f)(x)}\right] \frac{1}{(T_\alpha^a f)(x)},$$

and the conformable version of Chun–Kim's scheme denoted by CKeCO is

$$\psi_4(x) = a + \left((x-a)^\alpha - \frac{\alpha}{2}\left[3 - \frac{(T_\alpha^a f)[\phi_2(x)]}{(T_\alpha^a f)(x)}\right] \frac{f(x)}{(T_\alpha^a f)(x)} \right)^{1/\alpha}, \tag{29}$$

being $\phi_2(x)$ as in (27).

Now, let us consider Ostrowski's optimal fourth-order procedure [17,18] (according to Kung–Traub's conjecture [13]),

$$\psi_5(x) = \phi_1(x) - \left[\frac{f(x)}{f(x) - 2f[\phi_1(x)]}\right]\frac{f[\phi_1(x)]}{f'(x)}, \tag{30}$$

where $\phi_1(x)$ is the fixed-point function of the classical Newton method. In this case, we deduce from (30) that

$$g(x) = \left[\frac{f(x)}{f(x) - 2f[\phi(x)]}\right]\frac{1}{f'(x)}.$$

Therefore,

$$g_\alpha(x) = \left[\frac{f(x)}{f(x) - 2f[\phi(x)]}\right]\frac{1}{(T^a_\alpha f)(x)},$$

and the conformable version of Ostrowski's scheme, denoted by OeCO, is

$$\psi_6(x) = a + \left((\phi_2(x) - a)^\alpha - \alpha\left[\frac{f(x)}{f(x) - 2f[\phi_2(x)]}\right]\frac{f[\phi_2(x)]}{(T^a_\alpha f)(x)}\right)^{1/\alpha}, \tag{31}$$

being $\phi_2(x)$ described by (27).

Finally, let us consider another fourth-order optimal procedure, Chun's method designed in [18],

$$\psi_7(x) = \phi_1(x) - \left[\frac{f(x) + 2f[\phi_1(x)]}{f(x)}\right]\frac{f[\phi_1(x)]}{f'(x)}, \tag{32}$$

where $\phi_1(x)$ is the iteration function of the classical Newton scheme. In this case, we deduce from (32),

$$g(x) = \left[\frac{f(x) + 2f[\phi(x)]}{f(x)}\right]\frac{1}{f'(x)},$$

hence,

$$g_\alpha(x) = \left[\frac{f(x) + 2f[\phi(x)]}{f(x)}\right]\frac{1}{(T^a_\alpha f)(x)},$$

and the conformable version of Chun's scheme, denoted by CeCO, is described by

$$\psi_8(x) = a + \left((\phi_2(x) - a)^\alpha - \alpha\left[\frac{f(x) + 2f[\phi_2(x)]}{f(x)}\right]\frac{f[\phi_2(x)]}{(T^a_\alpha f)(x)}\right)^{1/\alpha}, \tag{33}$$

being $\phi_2(x)$ defined by (27).

In the next section, the convergence analysis of these procedures, both point-to-point and multipoint, is performed.

3. Convergence Analysis

First, let us remember that we are going to use classical Taylor expansions in convergence analysis, due to the direct relation between classical and conformable expansions. The requirements guaranteeing the convergence of point-to-point scheme NeL3 are stated in the following result.

Theorem 4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function in the interval I containing the zero $\bar{x}$ of $f(x)$. Let $(T^a_{\alpha_k} f)(x)$ be the conformable derivative of $f(x)$ starting from a, with order α_k. Let us suppose that $(T^a_{\alpha_k} f)(x)$ is continuous and not null at $\bar{x}$. If an initial approximation x_0 is sufficiently close to $\bar{x}$, then the local order of convergence of the conformable Newton-type scheme (NeL3)*

$$x_{k+1} = a + \left((x_k - a)^{\alpha_k} - \alpha_k\frac{f(x_k)}{(T^a_{\alpha_k} f)(x_k)}\right)^{1/\alpha_k}, \quad k = 0, 1, 2, \ldots,$$

where

$$\alpha_k = 1 + (x_k - a)\frac{f''(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots,$$

is at least 3*, and its error equation is*

$$e_{k+1} = \frac{1}{3}\left(2C_2^2 - 3C_3 - \frac{C_2}{\bar{x} - a}\right)e_k^3 + O\left(e_k^4\right),$$

being $C_j = \dfrac{f^{(j)}(\bar{x})}{j!f'(\bar{x})}$ *for* $j \geq 2$*, such that* $a < x_k$*,* $\forall k$*.*

Proof. This scheme has been designed by replacing α in the conformable Newton scheme by α_k. Then, the error equation of NeL3 can be obtained by just replacing α in the error equation given in (11) by the Taylor expansion of α_k, defined in (12).

The Taylor expansions of $f'(x_k)$ and $f''(x_k)$ around $\bar{x}$ can be expressed as

$$f'(x_k) = f'(\bar{x})\left[1 + 2C_2e_k + 3C_3e_k^2 + 4C_4e_k^3\right] + O\left(e_k^4\right),$$

and

$$f''(x_k) = f'(\bar{x})\left[2C_2 + 6C_3e_k + 12C_4e_k^2\right] + O\left(e_k^3\right),$$

respectively, where $C_j = \dfrac{f^{(j)}(\bar{x})}{j!f'(\bar{x})}$, for $j \geq 2$. The expansion of the quotient $\dfrac{f''(x_k)}{f'(x_k)}$ is then calculated as

$$\frac{f''(x_k)}{f'(x_k)} = 2C_2 + 2\left(3C_3 - 2C_2^2\right)e_k + 2\left(4C_2^3 - 9C_2C_3 + 6C_4\right)e_k^2 + O\left(e_k^3\right).$$

The expansion of the product $(x_k - a)\dfrac{f''(x_k)}{f'(x_k)}$ results in

$$\begin{aligned}(x_k - a)\frac{f''(x_k)}{f'(x_k)} &= = (\bar{x} - a + e_k)\frac{f''(x_k)}{f'(x_k)} \\ &= 2(\bar{x} - a)C_2 + 2\left((\bar{x} - a)\left(3C_3 - 2C_2^2\right) + C_2\right)e_k \\ &\quad +2\left((\bar{x} - a)\left(4C_2^3 - 9C_2C_3 + 6C_4\right) + 3C_3 - 2C_2^2\right)e_k^2 + O\left(e_k^3\right).\end{aligned}$$

Hence,

$$\begin{aligned}\alpha_k &= 1 + (x_k - a)\frac{f''(x_k)}{f'(x_k)} = 1 + 2(\bar{x} - a)C_2 + 2\left((\bar{x} - a)\left(3C_3 - 2C_2^2\right) + C_2\right)e_k \\ &\quad +2\left((\bar{x} - a)\left(4C_2^3 - 9C_2C_3 + 6C_4\right) + 3C_3 - 2C_2^2\right)e_k^2 + O\left(e_k^3\right).\end{aligned}$$

Replacing α in (11) by the Taylor expansion of α_k, we finally have

$$e_{k+1} = \frac{1}{3}\left(2C_2^2 - 3C_3 - \frac{C_2}{\bar{x} - a}\right)e_k^3 + O\left(e_k^4\right).$$

This completes the proof. □

Now, let us prove the third-order convergence of NeA3 under the conditions stated in the next result.

Theorem 5. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a continuous function in the interval* I *containing the zero* $\bar{x}$ *of* $f(x)$*. Let* $(T_\alpha^{a_k} f)(x)$ *be the conformable derivative of* $f(x)$ *starting from* a_k*, with order* α*, for any* $\alpha \in (0, 1)$*. Let us suppose that* $(T_\alpha^{a_k} f)(x)$ *is continuous and not null at* $\bar{x}$*. If an initial* *approximation* x_0 *is sufficiently close to* $\bar{x}$*, then the local order of convergence of the conformable Newton-type scheme (NeA3)*

$$x_{k+1} = a_k + \left((x_k - a_k)^\alpha - \alpha \frac{f(x_k)}{(T_\alpha^{a_k} f)(x_k)} \right)^{1/\alpha}, \quad k = 0, 1, 2, \ldots,$$

where

$$a_k = x_k + (1-\alpha) \frac{f'(x_k)}{f''(x_k)}, \quad k = 0, 1, 2, \ldots,$$

is at least 3 *for* $0 < \alpha < 1$*, and the error equation is*

$$e_{k+1} = \left(\frac{2}{3} \frac{(2-\alpha)C_2^2}{1-\alpha} - C_3 \right) e_k^3 + O\left(e_k^4\right),$$

being $C_j = \dfrac{f^{(j)}(\bar{x})}{j! f'(\bar{x})}$ *for* $j \geq 2$.

Proof. In a similar way as in Theorem 4, the expansion of the quotient $\dfrac{f'(x_k)}{f''(x_k)}$ is calculated as

$$\frac{f'(x_k)}{f''(x_k)} = \frac{1}{2C_2} + \left(1 - \frac{3}{2}\frac{C_3}{C_2^2}\right) e_k + \frac{3}{2}\left(\frac{3C_3^2 - C_2^2 C_3 - 2C_2 C_4}{C_2^3}\right) e_k^2 + O\left(e_k^3\right).$$

Then,

$$(1-\alpha)\frac{f'(x_k)}{f''(x_k)} = \frac{1-\alpha}{2C_2} + (1-\alpha)\left(1 - \frac{3}{2}\frac{C_3}{C_2^2}\right) e_k + \frac{3}{2}(1-\alpha)\left(\frac{3C_3^2 - C_2^2 C_3 - 2C_2 C_4}{C_2^3}\right) e_k^2 + O\left(e_k^3\right).$$

So, the expansion of a_k can be expressed as

$$\begin{aligned} a_k &= x_k + (1-\alpha)\frac{f'(x_k)}{f''(x_k)} \\ &= \bar{x} + e_k + (1-\alpha)\frac{f'(x_k)}{f''(x_k)} = \bar{x} + \frac{1-\alpha}{2C_2} + \left(2 - \alpha - \frac{3}{2}\frac{(1-\alpha)C_3}{C_2^2}\right) e_k \\ &\quad + \frac{3}{2}(1-\alpha)\left(\frac{3C_3^2 - C_2^2 C_3 - 2C_2 C_4}{C_2^3}\right) e_k^2 + O\left(e_k^3\right). \end{aligned}$$

Replacing a in (11) by the Taylor expansion of a_k, we have

$$e_{k+1} = \left(\frac{2}{3} \frac{(2-\alpha)C_2^2}{1-\alpha} - C_3 \right) e_k^3 + O\left(e_k^4\right),$$

where $\alpha \neq 1$. This completes the proof. □

Let us remark that when $\alpha = 1$ in NeA3, we obtain the classical Newton scheme. Analogously, the convergence hypothesis of NeLA4 are stated in the following result.

Theorem 6. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a continuous function in the interval* I *containing the zero* $\bar{x}$ *of* $f(x)$*. Let* $(T_{\alpha_k}^{a_k} f)(x)$ *be the conformable derivative of* $f(x)$ *starting from* a_k*, with order* α_k*. Let us* *suppose that* $(T^{a_k}_{\alpha_k} f)(x)$ *is continuous and not null at* $\bar{x}$*. If an initial approximation* x_0 *is sufficiently close to* $\bar{x}$*, then the local order of convergence of the conformable Newton-type procedure (NeLA4)*

$$x_{k+1} = a_k + \left((x_k - a_k)^{\alpha_k} - \alpha_k \frac{f(x_k)}{(T^{a_k}_{\alpha_k} f)(x_k)} \right)^{1/\alpha_k}, \qquad k = 0, 1, 2, \ldots,$$

where

$$\alpha_k = 1 + \frac{f''(x_k)^2}{f''(x_k)^2 - f'(x_k) f'''(x_k)}, \quad k = 0, 1, 2, \ldots,$$

and

$$a_k = x_k + \frac{f'(x_k) f''(x_k)}{f'(x_k) f'''(x_k) - f''(x_k)^2}, \quad k = 0, 1, 2, \ldots,$$

is at least 4*, and the error equation is*

$$e_{k+1} = 2\left(C_2 C_3 - 3\frac{C_3^2}{C_2} + 2C_4 \right) e_k^4 + O\left(e_k^5\right),$$

being $C_j = \dfrac{f^{(j)}(\bar{x})}{j! f'(\bar{x})}$ *for* $j \geq 2$*.*

Proof. The Taylor expansions of $f'(x_k)$, $f''(x_k)$ and $f'''(x_k)$ around the root $\bar{x}$ allow us to calculate the expansion of the following products:

$$f'(x_k) f''(x_k) = f'(\bar{x})^2 \Big[2C_2 + 2(2C_2^2 + 3C_3) e_k + 6(3C_2C_3 + 2C_4) e_k^2 + 2(9C_3^2 + 16C_2C_4 + 10C_5) e_k^3 \Big] + O\left(e_k^4\right),$$

$$f'(x_k) f'''(x_k) = f'(\bar{x})^2 \Big[6C_3 + 12(C_2C_3 + 2C_4) e_k + 6(3C_3^2 + 8C_2C_4 + 10C_5) e_k^2 \Big] + O\left(e_k^3\right)$$

and

$$f''(x_k)^2 = f'(\bar{x})^2 \Big[4C_2^2 + 24C_2C_3 e_k + 12(3C_3^2 + 4C_2C_4) e_k^2 + 16(9C_3C_4 + 5C_2C_5) e_k^3 \Big] + O\left(e_k^4\right).$$

Hence, for α_k it results that

$$\begin{aligned} \alpha_k &= 1 + \frac{f''(x_k)^2}{f''(x_k)^2 - f'(x_k) f'''(x_k)} = 1 + \frac{2C_2^2}{2C_2^2 - C_3} + \left(\frac{12C_2 \left(C_2^2 C_3 - 3C_3^2 + 2C_2C_4\right)}{\left(2C_2^2 - 3C_3\right)^2} \right) e_k \\ &+ \left(\frac{6\left(27C_3^4 + 16C_2^5 C_4 - 48C_2^3 C_3 C_4 - 36C_2 C_3^2 C_4 + C_2^4 \left(20C_5 - 6C_3^2\right) + C_2^2 \left(9C_3^3 + 48C_4^2 - 30C_3C_5\right)\right)}{\left(2C_2^2 - 3C_3\right)^3} \right) e_k^2 \\ &+ O\left(e_k^3\right), \end{aligned}$$

and for a_k we have

$$\begin{aligned} a_k &= x_k + \frac{f'(x_k) f''(x_k)}{f'(x_k) f'''(x_k) - f''(x_k)^2} = \bar{x} + \frac{C_2}{3C_3 - 2C_2^2} + \left(\frac{6C_2^2 C_3 - 4C_2^4 + 9C_3^2 - 12C_2C_4}{\left(2C_2^2 - 3C_3\right)^2} \right) e_k \\ &+ \left(\frac{6\left(2C_2^5 C_3 + 12C_2^4 C_4 - 36C_2^2 C_3 C_4 - 9C_3^2 C_4 - 5C_2^3 \left(3C_3^2 - 2C_5\right) + 3C_2 \left(9C_3^3 + 8C_4^2 - 5C_3C_5\right)\right)}{\left(2C_2^2 - 3C_3\right)^3} \right) e_k^2 \\ &+ O\left(e_k^3\right). \end{aligned}$$

Replacing α and a in (11) by the Taylor expansions of α_k and a_k, respectively, we have

$$e_{k+1} = 2\left(C_2C_3 - 3\frac{C_3^2}{C_2} + 2C_4\right)e_k^4 + O\left(e_k^5\right).$$

This completes the proof of the fourth-order of convergence. □

Let us notice that, although NeLA4 has an order of convergence of four, it uses four functional evaluations per iteration, so it is not an optimal method. We proceed now with the convergence analysis of those multipoint conformable methods defined in the previous section.

Theorem 7. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function in the interval I containing the zero $\bar{x}$ of $f(x)$. Let $(T_\alpha^a f)(x)$ be the conformable derivative of $f(x)$ starting from a, with order α, for any $\alpha \in (0,1]$. Let us suppose that $(T_\alpha^a f)(x)$ is continuous and not null at $\bar{x}$. If an initial approximation x_0 is sufficiently close to $\bar{x}$, then the local order of convergence of the conformable Traub-type procedure (TeCO)*

$$x_{k+1} = a + \left((y_k - a)^\alpha - \alpha\frac{f(y_k)}{(T_\alpha^a f)(x_k)}\right)^{1/\alpha}, \quad k = 0,1,2,\ldots,$$

where

$$y_k = a + \left((x_k - a)^\alpha - \alpha\frac{f(x_k)}{(T_\alpha^a f)(x_k)}\right)^{1/\alpha}, \quad k = 0,1,2,\ldots,$$

is at least 3, and the error equation is

$$e_{k+1} = \left(2C_2^2 + 2\frac{(1-\alpha)C_2}{\bar{x} - a} + \frac{1}{2}\frac{(1-\alpha)^2}{(\bar{x} - a)^2}\right)e_k^3 + O\left(e_k^4\right),$$

being $C_j = \dfrac{f^{(j)}(\bar{x})}{j!f'(\bar{x})}$, for $j \geq 2$, such that $a < x_k$, $\forall k$.

Proof. The generalized binomial Theorem for noninteger powers is given by [20]

$$(x+y)^r = \sum_{k=0}^{\infty}\binom{r}{k}x^{r-k}y^k, \quad k \in \{0\} \cup \mathbb{N},$$

being the generalized binomial coefficient [21]

$$\binom{r}{k} = \frac{\Gamma(r+1)}{k!\Gamma(r-k+1)}, \quad k \in \{0\} \cup \mathbb{N},$$

where $\Gamma(\cdot)$ is the Gamma function. Thus, the Taylor expansion of $(T_\alpha^a f)(x_k)$ is

$$\begin{aligned}(T_\alpha^a f)(x_k) &= (x_k - a)^{1-\alpha}f'(x_k) = f'(\bar{x})\Big[(\bar{x} - a)^{1-\alpha} + (\bar{x} - a)^{-\alpha}(1 - \alpha + 2(\bar{x} - a)C_2)e_k \\ &\quad + \frac{1}{2}(\bar{x} - a)^{-1-\alpha}((\alpha - 1)\alpha + 4(1-\alpha)(\bar{x} - a)C_2 + 6(\bar{x} - a)^2C_3)e_k^2 \\ &\quad + \frac{1}{6}(\bar{x} - a)^{-2-\alpha}(\alpha^3 - \alpha + 6(1-\alpha)\alpha(\bar{x} - a)C_2 - 18(1-\alpha)(\bar{x} - a)^2C_3 \\ &\quad + 24(a^3 - 3a^2\bar{x} + 3a\bar{x}^2 - \bar{x}^3)C_4)e_k^3\Big] + O\left(e_k^4\right).\end{aligned}$$

The Taylor expansion of y_k up to order three is given by the error equation in (11) as

$$y_k - \bar{x} = \left(C_2 + \frac{1-\alpha}{2(\bar{x} - a)}\right)e_k^2 + \left(2C_3 - 2C_2^2 + \frac{(\alpha - 1)C_2}{\bar{x} - a} + \frac{(1-\alpha)(\alpha - 2)}{3(\bar{x} - a)^2}\right)e_k^3 + O\left(e_k^4\right).$$

So, the expansion of $f(y_k)$ is

$$f(y_k) = f'(\bar{x})\left[\left(C_2 + \frac{1-\alpha}{2(\bar{x}-a)}\right)e_k^2 + \left(2C_3 - 2C_2^2 + \frac{(\alpha-1)C_2}{\bar{x}-a} + \frac{(1-\alpha)(\alpha-2)}{3(\bar{x}-a)^2}\right)e_k^3\right] + O\left(e_k^4\right).$$

Then,

$$\begin{aligned}\alpha\frac{f(y_k)}{(T_\alpha^a f)(x_k)} &= \frac{\alpha}{2}(\bar{x}-a)^{\alpha-2}(1-\alpha-2(\bar{x}-a)C_2)e_k^2 \\ &+\frac{\alpha}{6}(\bar{x}-a)^{\alpha-3}((\alpha-1)(5\alpha-7)+18(1-\alpha)(\bar{x}-a)C_2+24(\bar{x}-a)^2C_2^2-12(\bar{x}-a)^2C_3)e_k^3 \\ &+O\left(e_k^4\right).\end{aligned}$$

By using again the generalized binomial theorem, the expansion of $(y_k-a)^\alpha$ results in

$$\begin{aligned}(y_k-a)^\alpha &= (\bar{x}-a)^\alpha + \frac{\alpha}{2}(\bar{x}-a)^{\alpha-2}(1-\alpha-2(\bar{x}-a)C_2)e_k^2 \\ &+\frac{\alpha}{3}(\bar{x}-a)^{\alpha-3}((\alpha-1)(\alpha-2)+3(1-\alpha)(\bar{x}-a)C_2+6(\bar{x}-a)^2C_2^2-6(\bar{x}-a)^2C_3)e_k^3+O\left(e_k^4\right),\end{aligned}$$

therefore,

$$a + \left((y_k-a)^\alpha - \alpha\frac{f(y_k)}{(T_\alpha^a f)(x_k)}\right)^{1/\alpha} = \bar{x} + \left(2C_2^2 + 2\frac{(1-\alpha)C_2}{\bar{x}-a} + \frac{1}{2}\frac{(1-\alpha)^2}{(\bar{x}-a)^2}\right)e_k^3 + O\left(e_k^4\right).$$

Finally, as $x_{k+1} = \bar{x} + e_{k+1}$, the error equation is

$$e_{k+1} = \left(2C_2^2 + 2\frac{(1-\alpha)C_2}{\bar{x}-a} + \frac{1}{2}\frac{(1-\alpha)^2}{(\bar{x}-a)^2}\right)e_k^3 + O\left(e_k^4\right).$$

This completes the proof. □

Let us now prove the convergence condition for the CKeCO scheme.

Theorem 8. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function in the interval I containing the zero $\bar{x}$ of $f(x)$. Let $(T_\alpha^a f)(x)$ be the conformable derivative of $f(x)$ starting from a, with order α, for any $\alpha \in (0,1]$. Let us suppose that $(T_\alpha^a f)(x)$ is continuous and not null at $\bar{x}$. If an initial approximation x_0 is sufficiently close to $\bar{x}$, then the local order of convergence of the conformable Chun–Kim-type scheme (CKeCO)*

$$x_{k+1} = a + \left((x_k-a)^\alpha - \frac{\alpha}{2}\left[3 - \frac{(T_\alpha^a f)(y_k)}{(T_\alpha^a f)(x_k)}\right]\frac{f(x_k)}{(T_\alpha^a f)(x_k)}\right)^{1/\alpha}, \quad k = 0,1,2,\ldots,$$

where

$$y_k = a + \left((x_k-a)^\alpha - \alpha\frac{f(x_k)}{(T_\alpha^a f)(x_k)}\right)^{1/\alpha}, \quad k = 0,1,2,\ldots,$$

is at least 3, and the error equation is

$$e_{k+1} = \left(2C_2^2 + \frac{1}{2}C_3 + \frac{5}{2}\frac{(1-\alpha)C_2}{\bar{x}-a} + \frac{1}{12}\frac{(1-\alpha)(7-8\alpha)}{(\bar{x}-a)^2}\right)e_k^3 + O\left(e_k^4\right),$$

being $C_j = \frac{f^{(j)}(\bar{x})}{j!f'(\bar{x})}$, for $j \geq 2$, such that $a < x_k$, $\forall k$.

Proof. Let us use the Taylor expansions calculated in the proof of Theorem 7. So, the expansion of $(T_\alpha^a f)(y_k)$ is

$$
\begin{aligned}
(T_\alpha^a f)(y_k) &= (y_k-a)^{1-\alpha} f'(y_k) = f'(\bar{x})\Big[(\bar{x}-a)^{1-\alpha} + \frac{1}{2}(\bar{x}-a)^{-1-\alpha}(\alpha-1-2(\bar{x}-a)C_2)^2 e_k^2 \\
&\quad + \frac{1}{3}(\bar{x}-a)^{-2-\alpha}(\alpha-1-2(\bar{x}-a)C_2)((\alpha-1)(\alpha-2)-3(\bar{x}-a)(C_2(\alpha-1-2(\bar{x}-a)C_2) \\
&\quad + 2(\bar{x}-a)C_3))e_k^3\Big] + O\Big(e_k^4\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{f(x_k)}{(T_\alpha^a f)(x_k)} &= (\bar{x}-a)^{\alpha-1}e_k - (\bar{x}-a)^{\alpha-2}(1-\alpha+(\bar{x}-a)C_2)e_k^2 \\
&\quad + \frac{1}{2}(\bar{x}-a)^{\alpha-3}((\alpha-1)(\alpha-2)-2(\bar{x}-a)(C_2(\alpha-1-2(\bar{x}-a)C_2)+2(\bar{x}-a)C_3))e_k^3 \\
&\quad + O\Big(e_k^4\Big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\alpha}{2}\left[3-\frac{(T_\alpha^a f)(y_k)}{(T_\alpha^a f)(x_k)}\right] &= \alpha\Bigg[1+\left(C_2+\frac{1}{2}\frac{1-\alpha}{\bar{x}-a}\right)e_k \\
&\quad + \left(\frac{\alpha(5-2\alpha)-3}{4(\bar{x}-a)^2}+\frac{2(\alpha-1)C_2}{\bar{x}-a}-3C_2^2+\frac{3}{2}C_3\right)e_k^2 \\
&\quad + \frac{1}{12}\Bigg(\frac{84(1-\alpha)C_2^2}{\bar{x}-a}+96C_2^3+2C_2\left(\frac{(\alpha-1)(17\alpha-22)}{(\bar{x}-a)^2}-48C_3\right) \\
&\quad + \frac{(1-\alpha)(\alpha(6\alpha-17)+13-30(\bar{x}-a)^2C_3)}{(\bar{x}-a)^3}+24C_4\Bigg)e_k^3\Bigg] \\
&\quad + O\Big(e_k^4\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\alpha}{2}\left[3-\frac{(T_\alpha^a f)(y_k)}{(T_\alpha^a f)(x_k)}\right]\frac{f(x_k)}{(T_\alpha^a f)(x_k)} &= \alpha\Bigg[(\bar{x}-a)^{\alpha-1}e_k+\frac{\alpha(\alpha-1)}{2}(\bar{x}-a)^{\alpha-2}e_k^2 \\
&\quad - \frac{1}{4}((\bar{x}-a)^{\alpha-3}((\alpha-1)(2\alpha-1) \\
&\quad - 2(\bar{x}-a)(5(\alpha-1)C_2-4(\bar{x}-a)C_2^2-(\bar{x}-a)C_3)))e_k^3\Bigg] \\
&\quad + O\Big(e_k^4\Big),
\end{aligned}
$$

Using the generalized binomial Theorem again,

$$
(x_k-a)^\alpha = (\bar{x}-a)^\alpha + \alpha(\bar{x}-a)^{\alpha-1}e_k + \frac{1}{2}\alpha(\alpha-1)(\bar{x}-a)^{\alpha-2}e_k^2 + \frac{1}{6}\alpha(\alpha-1)(\alpha-2)(\bar{x}-a)^{\alpha-3}e_k^3 + O\Big(e_k^4\Big),
$$

and

$$
\begin{aligned}
a+\left((x_k-a)^\alpha-\frac{\alpha}{2}\left[3-\frac{(T_\alpha^a f)(y_k)}{(T_\alpha^a f)(x_k)}\right]\frac{f(x_k)}{(T_\alpha^a f)(x_k)}\right)^{1/\alpha} &= \bar{x} \\
&\quad + \left(2C_2^2+\frac{1}{2}C_3+\frac{5}{2}\frac{(1-\alpha)C_2}{\bar{x}-a}+\frac{1}{12}\frac{(1-\alpha)(7-8\alpha)}{(\bar{x}-a)^2}\right)e_k^3 \\
&\quad + O\Big(e_k^4\Big).
\end{aligned}
$$

Finally, the error equation is

$$e_{k+1} = \left(2C_2^2 + \frac{1}{2}C_3 + \frac{5}{2}\frac{(1-\alpha)C_2}{\bar{x}-a} + \frac{1}{12}\frac{(1-\alpha)(7-8\alpha)}{(\bar{x}-a)^2}\right)e_k^3 + O\left(e_k^4\right),$$

and this completes the proof. □

Since the error equation of the classical Traub and Chun–Kim procedures are, respectively,

$$e_{k+1} = 2C_2^2 e_k^3 + O\left(e_k^4\right),$$

and

$$e_{k+1} = \left(2C_2^2 + \frac{1}{2}C_3\right)e_k^3 + O\left(e_k^4\right),$$

Theorem 3 is confirmed for conformable iterative schemes, and our procedure's ability to transform these non-optimal integer procedures to conformable ones is also supported.

Theorem 9. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function in the interval I containing the zero $\bar{x}$ of $f(x)$. Let $(T_\alpha^a f)(x)$ be the conformable derivative of $f(x)$ starting from a, with order α, for any $\alpha \in (0,1]$. Let us suppose that $(T_\alpha^a f)(x)$ is continuous and not null at $\bar{x}$. If an initial approximation x_0 is sufficiently close to $\bar{x}$, then the local order of convergence of the conformable Ostrowski-type method (OeCO)*

$$x_{k+1} = a + \left((y_k - a)^\alpha - \alpha\left[\frac{f(x_k)}{f(x_k) - 2f(y_k)}\right]\frac{f(y_k)}{(T_\alpha^a f)(x_k)}\right)^{1/\alpha}, \quad k = 0,1,2,\ldots,$$

where

$$y_k = a + \left((x_k - a)^\alpha - \alpha\frac{f(x_k)}{(T_\alpha^a f)(x_k)}\right)^{1/\alpha}, \quad k = 0,1,2,\ldots,$$

is at least 4, and the error equation is

$$e_{k+1} = \left(C_2^3 - C_2C_3 + \frac{1}{2}\frac{(1-\alpha)(C_2^2 - C_3)}{\bar{x}-a} + \frac{1}{12}\frac{(1-\alpha^2)^2C_2}{(\bar{x}-a)^2} + \frac{1}{24}\frac{(1-\alpha)(1-\alpha^2)}{(\bar{x}-a)^3}\right)e_k^4 + O\left(e_k^5\right),$$

being $C_j = \dfrac{f^{(j)}(\bar{x})}{j!f'(\bar{x})}$, for $j \geq 2$, such that $a < x_k$, $\forall k$.

Proof. Taking profit of the previous expansion, up to order four, we can expand the different factors that compose the iterative expression of the OeCO scheme. Then,

$$\begin{aligned}
\frac{f(y_k)}{(T_\alpha^a f)(x_k)} = \; & \frac{1}{2}(\bar{x}-a)^{\alpha-2}(1-\alpha-2(\bar{x}-a)C_2)e_k^2 \\
& + \frac{1}{6}(\bar{x}-a)^{\alpha-3}((\alpha-1)(5\alpha-7) + 18(1-\alpha)(\bar{x}-a)C_2 + 24(\bar{x}-a)^2C_2^2 - 12(\bar{x}-a)^2C_3)e_k^3 \\
& + \frac{1}{24}(\bar{x}-a)^{\alpha-4}\Big((\alpha-1)(5\alpha-7)(4\alpha-7) + 2(53\alpha^2 - 126\alpha + 73)(\bar{x}-a)C_2 \\
& + (1-\alpha)(\bar{x}-a)^2(276C_2^2 - 132C_3) + (\bar{x}-a)^3(312C_2^3 - 336C_2C_3) \\
& - 72(a^3 - 3a^2\bar{x} + 3a\bar{x}^2 - \bar{x}^3)C_4\Big)e_k^4 + O\left(e_k^5\right),
\end{aligned}$$

and

$$\begin{aligned}\left[\frac{f(x_k)}{f(x_k)-2f(y_k)}\right] &= 1+\left(2C_2+\frac{1-\alpha}{\bar{x}-a}\right)e_k+\left(4C_3-2C_2^2+\frac{(1-\alpha)C_2}{\bar{x}-a}+\frac{\alpha^2-1}{3(\bar{x}-a)^2}\right)e_k^2\\ &+\Bigg(-4C_2C_3-\frac{(\alpha-1)(3C_3-2C_2^2)}{\bar{x}-a}-\frac{(1-\alpha^2)C_2}{2(\bar{x}-a)^2}-\frac{(\alpha-1)(\alpha+1)(2\alpha+1)}{12(\bar{x}-a)^3}\\ &-\frac{6(a^3-3a^2\bar{x}+3a\bar{x}^2-\bar{x}^3)C_4}{(\bar{x}-a)^3}\Bigg)e_k^3+O\left(e_k^4\right).\end{aligned}$$

Hence,

$$\begin{aligned}\alpha\left[\frac{f(x_k)}{f(x_k)-2f(y_k)}\right]\frac{f(y_k)}{(T_\alpha^a f)(x_k)} &= \alpha\Bigg[\frac{1}{2}(\bar{x}-a)^{\alpha-2}(1-\alpha-2(\bar{x}-a)C_2)e_k^2\\ &-\frac{1}{3}(\bar{x}-a)^{\alpha-3}((\alpha-1)(\alpha-2)+3(1-\alpha)(\bar{x}-a)C_2+6(\bar{x}-a)^2(C_2^2-C_3))e_k^3\\ &+\frac{1}{24}(\bar{x}-a)^{\alpha-4}\Big((\alpha-1)(4\alpha^2-15\alpha+17)+36(\alpha-1)(\bar{x}-a)^2(C_2^2-C_3)\\ &-72(\bar{x}-a)^3C_2^3+2(\alpha-1)(7\alpha-11)(\bar{x}-a)C_2+144(\bar{x}-a)^3C_2C_3\\ &+72(a^3-3a^2\bar{x}+3a\bar{x}^2-\bar{x}^3)C_4\Big)e_k^4\Bigg]+O\left(e_k^5\right),\end{aligned}$$

Using the generalized binomial theorem, the expansion of $(y_k-a)^\alpha$ results in

$$\begin{aligned}(y_k-a)^\alpha &= (\bar{x}-a)^\alpha+\frac{\alpha}{2}(\bar{x}-a)^{\alpha-2}(1-\alpha-2(\bar{x}-a)C_2)e_k^2\\ &+\frac{\alpha}{3}(\bar{x}-a)^{\alpha-3}((\alpha-1)(\alpha-2)+3(1-\alpha)(\bar{x}-a)C_2+6(\bar{x}-a)^2C_2^2-6(\bar{x}-a)^2C_3)e_k^3\\ &-\frac{\alpha}{8}(\bar{x}-a)^{\alpha-4}((\alpha-1)(\alpha-2)(\alpha-3)+16(\alpha-1)(\bar{x}-a)^2C_2^2-32(\bar{x}-a)^3C_2^3\\ &+16(1-\alpha)(\bar{x}-a)^2C_3-4(\alpha^2-3\alpha+2)(\bar{x}-a)C_2+56(\bar{x}-a)^3C_2C_3\\ &+24(a^3-3a^2\bar{x}+3a\bar{x}^2-\bar{x}^3)C_4)e_k^4+O\left(e_k^5\right).\end{aligned}$$

Therefore,

$$\begin{aligned}x_{k+1} &= a+\left((y_k-a)^\alpha-\alpha\left[\frac{f(x_k)}{f(x_k)-2f(y_k)}\right]\frac{f(y_k)}{(T_\alpha^a f)(x_k)}\right)^{1/\alpha}\\ &= \bar{x}+\left(C_2^3-C_2C_3+\frac{1}{2}\frac{(1-\alpha)(C_2^2-C_3)}{\bar{x}-a}+\frac{1}{12}\frac{(1-\alpha^2)^2C_2}{(\bar{x}-a)^2}+\frac{1}{24}\frac{(1-\alpha)(1-\alpha^2)}{(\bar{x}-a)^3}\right)e_k^4+O\left(e_k^5\right).\end{aligned}$$

Finally,

$$e_{k+1}=\left(C_2^3-C_2C_3+\frac{1}{2}\frac{(1-\alpha)(C_2^2-C_3)}{\bar{x}-a}+\frac{1}{12}\frac{(1-\alpha^2)^2C_2}{(\bar{x}-a)^2}+\frac{1}{24}\frac{(1-\alpha)(1-\alpha^2)}{(\bar{x}-a)^3}\right)e_k^4+O\left(e_k^5\right).$$

This completes the proof. □

Now, we analyze the convergence of the conformable Chun method in the following result.

Theorem 10. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function in the interval I containing the zero $\bar{x}$ of $f(x)$. Let $(T_\alpha^a f)(x)$ be the conformable derivative of $f(x)$ starting from a, with order α, for any $\alpha \in (0,1]$. Let us suppose that $(T_\alpha^a f)(x)$ is continuous and not null at $\bar{x}$. If an initial* *approximation x_0 is sufficiently close to $\bar{x}$, then the local order of convergence of the conformable Chun-type method (CeCO)*

$$x_{k+1} = a + \left((y_k - a)^\alpha - \alpha \left[\frac{f(x_k) + 2f(y_k)}{f(x_k)} \right] \frac{f(y_k)}{(T_\alpha^a f)(x_k)} \right)^{1/\alpha}, \quad k = 0, 1, 2, \ldots,$$

where

$$y_k = a + \left((x_k - a)^\alpha - \alpha \frac{f(x_k)}{(T_\alpha^a f)(x_k)} \right)^{1/\alpha}, \quad k = 0, 1, 2, \ldots,$$

is at least 4, and the error equation is

$$\begin{aligned} e_{k+1} &= \left(5C_2^3 - C_2C_3 + \frac{(1-\alpha)(13C_2^2 - C_3)}{2(\bar{x}-a)} + \frac{(24(1-\alpha)^2 + (1-\alpha)(13-11\alpha))C_2}{12(\bar{x}-a)^2} + \frac{(1-\alpha)^2(13-11\alpha)}{24(\bar{x}-a)^3} \right) e_k^4 \\ &+ O\left(e_k^5\right), \end{aligned}$$

being $C_j = \dfrac{f^{(j)}(\bar{x})}{j! f'(\bar{x})}$, for $j \geq 2$, such that $a < x_k$, $\forall k$.

Proof. We can make use of the Taylor expansions presented in Theorem 9 so that the calculations of some elements of the iterative expressions are the same. So,

$$\begin{aligned} \frac{f(x_k) + 2f(y_k)}{f(x_k)} &= 1 + \left(2C_2 + \frac{1-\alpha}{\bar{x}-a} \right) e_k + \left(4C_3 - 6C_2^2 - \frac{3(1-\alpha)C_2}{\bar{x}-a} - \frac{2(\alpha-1)(\alpha-2)}{3(\bar{x}-a)^2} \right) e_k^2 \\ &+ \left(16C_2^3 - 20C_2C_3 + 6C_4 + \frac{(\alpha-1)(5C_3 - 10C_2^2)}{\bar{x}-a} - \frac{(\alpha-1)C_2}{6(\bar{x}-a)^2} \right. \\ &\left. - \frac{(\alpha-1)(7 + \alpha(2\alpha-7))}{4(\bar{x}-a)^3} \right) e_k^3 + O\left(e_k^4\right), \end{aligned}$$

and

$$\begin{aligned} \alpha \left[\frac{f(x_k) + 2f(y_k)}{f(x_k)} \right] \frac{f(y_k)}{(T_\alpha^a f)(x_k)} &= \alpha \left[\frac{1}{2} (\bar{x}-a)^{\alpha-2} (1 - \alpha - 2(\bar{x}-a)C_2) e_k^2 \right. \\ &- \frac{1}{3} (\bar{x}-a)^{\alpha-3} ((\alpha-1)(\alpha-2) + 3(1-\alpha)(\bar{x}-a)C_2 + 6(\bar{x}-a)^2(C_2^2 - C_3)) e_k^3 \\ &+ \frac{1}{24} (\bar{x}-a)^{\alpha-4} \left(8\alpha^3 - 17\alpha^2 + 4\alpha + 5 + (\alpha-1)(\bar{x}-a)^2(108C_2^2 - 36C_3) \right. \\ &- 24(\bar{x}-a)^3 C_2^3 - 2(29\alpha^2 - 54\alpha + 25)(\bar{x}-a)C_2 - 144(\bar{x}-a)^3 C_2C_3 \\ &\left. \left. - 72(a^3 - 3a^2\bar{x} + 3a\bar{x}^2 - \bar{x}^3)C_4 \right) e_k^4 \right] + O\left(e_k^5\right), \end{aligned}$$

Using the generalized binomial theorem again,

$$\begin{aligned} x_{k+1} &= a + \left((y_k - a)^\alpha - \alpha \left[\frac{f(x_k) + 2f(y_k)}{f(x_k)} \right] \frac{f(y_k)}{(T_\alpha^a f)(x_k)} \right)^{1/\alpha} = \bar{x} \\ &+ \left(5C_2^3 - C_2C_3 + \frac{(1-\alpha)(13C_2^2 - C_3)}{2(\bar{x}-a)} + \frac{(24(1-\alpha)^2 + (1-\alpha)(13-11\alpha))C_2}{12(\bar{x}-a)^2} + \frac{(1-\alpha)^2(13-11\alpha)}{24(\bar{x}-a)^3} \right) e_k^4 \\ &+ O\left(e_k^5\right), \end{aligned}$$

and finally,

$$e_{k+1} = \left(5C_2^3 - C_2C_3 + \frac{(1-\alpha)(13C_2^2 - C_3)}{2(\bar{x}-a)} + \frac{(24(1-\alpha)^2 + (1-\alpha)(13-11\alpha))C_2}{12(\bar{x}-a)^2} + \frac{(1-\alpha)^2(13-11\alpha)}{24(\bar{x}-a)^3}\right)e_k^4 + O\left(e_k^5\right).$$

This completes the proof. □

Since the error equations of the classical Ostrowski and Chun schemes are

$$e_{k+1} = \left(C_2^3 - C_2C_3\right)e_k^4 + O\left(e_k^5\right),$$

and

$$e_{k+1} = \left(5C_2^3 - C_2C_3\right)e_k^4 + O\left(e_k^5\right),$$

respectively, Theorem 3 and our proposed technique are again confirmed for these fourth-order conformable schemes. Moreover, let us remark that OeCO and CeCO are the first optimal multipoint conformable procedures, according to Kung and Traub's conjecture [13].

In next section, we perform some numerical tests with some nonlinear equations, and we study the stability of these methods proposed; in the case of multipoint methods, a comparison with the classical version (when $\alpha = 1$) is made.

4. Numerical Tests

The following results were obtained by using Matlab R2020a with the double-precision arithmetic, $|f(x_{k+1})| < 10^{-8}$ or $|x_{k+1} - x_k| < 10^{-8}$, as the stopping criterion, and at most 500 iterations. We also used the Approximated Computational Order of Convergence (ACOC), denoted by ρ,

$$\rho = \frac{\ln(|x_{k+1} - x_k| / |x_k - x_{k-1}|)}{\ln(|x_k - x_{k-1}| / |x_{k-1} - x_{k-2}|)}, \quad k = 2, 3, \dots,$$

and defined in [22], in order to check that the theoretical order of convergence is attained in practice.

We are going to test four nonlinear functions to draw a comparison between one-point and multipoint procedures; we also compare these results with the Matlab Toolbox *fsolve*. In the case of methods whose a is not computed in each iteration (NeL3, TeCO, CKeCO, OeCO, and CeCO) we choose $a = -10$ to ensure that $a < x_k\ \forall k$.

For each test function, we calculate, in addition to its respective conformable derivative, its classical derivatives of orders one, two, and three because they are necessary for schemes NeL3, NeA3, and NeLA4.

For each test function we show two tables: first, the results of one-point procedures (NeA3, NeL3, and NeLA4) and *fsolve*, then, the results of multipoint methods (TeCO, CKeCO, OeCO, and CeCO). We remark that α is not used by *fsolve*, and this tool provides $\bar{x}$, $|f(x_{k+1})|$ and the number of iterations (iter). For each pair of tables, the same initial estimate x_0 is used, as well as equispaced values of $\alpha \in (0, 1]$ for schemes whose α is not computed in each iteration (NeA3, TeCO, CKeCO, OeCO, and CeCO).

Our first test function is $f_1(x) = -12.84x^6 - 25.6x^5 + 16.55x^4 - 2.21x^3 + 26.71x^2 - 4.29x - 15.21$, with real and complex roots $\bar{x}_1 = 0.82366 + 0.24769i$, $\bar{x}_2 = 0.82366 - 0.24769i$, $\bar{x}_3 = -2.62297$, $\bar{x}_4 = -0.584$, $\bar{x}_5 = -0.21705 + 0.99911i$, and $\bar{x}_6 = -0.21705 - 0.99911i$. The necessary derivatives for this function are

$$\begin{aligned} f_1'(x) &= -77.04x^5 - 128x^4 + 66.2x^3 - 6.63x^2 + 53.42x - 4.29, \\ f_1''(x) &= -385.2x^4 - 512x^3 + 198.6x^2 - 13.26x + 53.42, \\ f_1'''(x) &= -1540.8x^3 - 1536x^2 + 397.2x - 13.26, \\ (T_\alpha^a f_1)(x) &= (x-a)^{1-\alpha} f_1'(x). \end{aligned}$$

In Table 1, NeA3 can require fewer (but also more) iterations when $\alpha \neq 1$ than in the case that $\alpha = 1$, and the computational order of convergence is around the theoretical third-order of convergence. It may be slightly greater for low values of α. When $\alpha = 1$, ρ is close to 2 as in this case the scheme is reduced to the classical Newton procedure (see Theorem 5). We also see that NeL3 and NeLA4 improve the classical Newton method as well in the number of iterations required and in the computational order of convergence.

In Table 2, TeCO requires fewer iterations than the classical Traub scheme (when $\alpha = 0.8$ and $\alpha = 0.9$ regarding the case that $\alpha = 1$). ρ is around 3 in most of cases, although it reaches the maximum number of iterations for low values of α. We also observe that the classical Chun–Kim procedure does not find any solution, whereas CKeCO converges for some values of α, and the computational order of convergence is around 3 in most cases. No results are shown when $\alpha = 0.1$ because it converges to a point which is not a solution of $f_1(x)$. OeCO presents a similar behavior compared to the classical Ostrowski method, even though ρ is not expected. CeCO can require fewer iterations than the classical Chun scheme, and the computational order of convergence tends to be 4; again, no results are shown when $\alpha = 0.1$ because it converges to a point which is not a solution of $f_1(x)$.

Table 1. Results of one-point methods and *fsolve* for $f_1(x)$, with initial estimate $x_0 = 1$.

		NeA3 Method			
α	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Iter**	ρ
1	$\bar{x}_4$	4.36×10^{-11}	6.85×10^{-7}	13	2.01
0.9	$\bar{x}_4$	3.55×10^{-15}	1.02×10^{-9}	7	2.92
0.8	$\bar{x}_3$	3.91×10^{-13}	3.58×10^{-8}	15	2.89
0.7	$\bar{x}_6$	5.33×10^{-15}	5.35×10^{-10}	38	3.01
0.6	$\bar{x}_4$	1.71×10^{-13}	8.73×10^{-6}	22	3.37
0.5	$\bar{x}_3$	6.18×10^{-13}	6.70×10^{-10}	54	2.96
0.4	$\bar{x}_3$	3.91×10^{-13}	1.69×10^{-7}	31	2.91
0.3	$\bar{x}_4$	1.03×10^{-9}	1.88×10^{-4}	19	3.57
0.2	$\bar{x}_3$	4.02×10^{-9}	8.18×10^{-5}	35	2.81
0.1	$\bar{x}_4$	9.07×10^{-10}	1.94×10^{-4}	168	3.53
		NeL3 Method			
α_k	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Iter**	ρ
-	$\bar{x}_3$	4.13×10^{-13}	4.10×10^{-6}	11	2.89
		NeLA4 Method			
α_k	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Iter**	ρ
-	$\bar{x}_1$	6.53×10^{-15}	1.51×10^{-7}	3	4.00
		fsolve			
-	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	iter	ρ
-	-	-	-	-	-

Table 2. Results of multipoint methods for $f_1(x)$, with initial estimate $x_0 = 1$.

	TeCO Method						CKeCO Method			
α	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1}-x_k\rvert$	**Iter**	ρ	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1}-x_k\rvert$	**Iter**	ρ
1	$\bar{x}_3$	6.18×10^{-13}	4.49×10^{-7}	115	2.89	-	-	-	>500	-
0.9	$\bar{x}_4$	4.04×10^{-9}	2.43×10^{-4}	69	2.80	-	-	-	>500	-
0.8	$\bar{x}_4$	5.30×10^{-11}	5.73×10^{-5}	61	2.83	$\bar{x}_4$	9.95×10^{-14}	9.91×10^{-10}	190	3.00
0.7	$\bar{x}_2$	1.16×10^{-13}	3.57×10^{-6}	329	0.00	-	-	-	>500	-
0.6	$\bar{x}_4$	1.07×10^{-14}	1.12×10^{-6}	119	2.90	-	-	-	>500	-
0.5	$\bar{x}_4$	3.24×10^{-10}	1.05×10^{-4}	213	2.82	-	-	-	>500	-
0.4	$\bar{x}_4$	1.17×10^{-13}	7.31×10^{-9}	104	2.95	$\bar{x}_5$	1.41×10^{-11}	1.88×10^{-5}	484	2.80
0.3	-	-	-	>500	-	$\bar{x}_4$	2.11×10^{-13}	1.11×10^{-5}	490	0.00
0.2	-	-	-	>500	-	-	-	-	>500	-
0.1	-	-	-	>500	-	-	-	-	-	-
	OeCO Method						CeCO Method			
α	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1}-x_k\rvert$	**Iter**	ρ	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1}-x_k\rvert$	**Iter**	ρ
1	$\bar{x}_4$	1.07×10^{-14}	2.77×10^{-5}	3	2.41	$\bar{x}_3$	6.18×10^{-13}	4.76×10^{-9}	47	3.73
0.9	$\bar{x}_4$	9.95×10^{-14}	1.97×10^{-5}	3	2.43	$\bar{x}_3$	3.11×10^{-10}	2.76×10^{-4}	75	3.21
0.8	$\bar{x}_4$	9.95×10^{-14}	1.36×10^{-5}	3	2.44	$\bar{x}_3$	6.18×10^{-13}	9.77×10^{-5}	66	3.28
0.7	$\bar{x}_4$	1.17×10^{-13}	9.14×10^{-6}	3	2.46	$\bar{x}_3$	2.26×10^{-12}	1.89×10^{-10}	87	3.81
0.6	$\bar{x}_4$	9.95×10^{-14}	5.93×10^{-6}	3	2.48	$\bar{x}_3$	1.81×10^{-10}	2.42×10^{-4}	108	3.22
0.5	$\bar{x}_4$	1.17×10^{-13}	3.68×10^{-6}	3	2.49	$\bar{x}_3$	4.92×10^{-11}	1.76×10^{-4}	87	3.24
0.4	$\bar{x}_4$	1.17×10^{-13}	2.16×10^{-6}	3	2.51	$\bar{x}_4$	1.17×10^{-13}	1.08×10^{-7}	212	3.68
0.3	$\bar{x}_4$	1.07×10^{-14}	1.18×10^{-6}	3	2.53	$\bar{x}_4$	1.07×10^{-14}	1.75×10^{-7}	53	3.67
0.2	$\bar{x}_4$	2.24×10^{-13}	5.90×10^{-7}	3	2.55	$\bar{x}_4$	2.24×10^{-13}	7.13×10^{-7}	45	3.61
0.1	$\bar{x}_4$	4.21×10^{-13}	2.58×10^{-7}	3	2.57	-	-	-	-	-

We can see that in Tables 1 and 2 a solution is found in most of cases, whereas *fsolve* does not find any solution with this initial estimate. Of course, if we change the initial estimation to a complex one, *fsolve* will find any root.

The second test function is $f_2(x) = ix^{1.8} - x^{0.9} - 16$, with complex roots $\bar{x}_1 = 2.90807 - 4.24908i$, and $\bar{x}_2 = -3.85126 + 1.74602i$. The necessary derivatives for this function are

$$\begin{aligned} f_2'(x) &= 1.8ix^{0.8} - 0.9x^{-0.1}, \\ f_2''(x) &= 1.44x^{-0.2} + 0.09x^{-1.1}, \\ f_2'''(x) &= -0.288x^{-1.2} - 0.099x^{-2.1}, \\ (T_\alpha^a f_2)(x) &= (x-a)^{1-\alpha} f_2'(x). \end{aligned}$$

In Table 3, NeA3 can require fewer iterations when $\alpha \neq 1$, ρ is around 3, and the computational order of convergence is 2 when $\alpha = 1$, supporting the theory. We can observe that NeL3 and NeLA4 improve the classical Newton procedure in the number of iterations required and the estimated order of convergence ρ.

In Table 4, TeCO and CKeCO require fewer iterations than the classical Traub method and Chun–Kim's scheme, respectively (in some cases), and the computational order of convergence is around 3 for any α in both procedures. OeCO does not improve the classical Ostrowski method in terms of the number of iterations and ρ. CeCO presents a similar behavior compared to the classical Chun scheme in some cases, and in other cases the computational order of convergence can be slightly greater.

We can observe that in Tables 3 and 4 our methods converge in fewer iterations than *fsolve* in many cases.

Our third test function is $f_3(x) = e^x - 1$, with real root $\bar{x}_1 = 0$. The necessary derivatives for this function are

$$\begin{aligned} f_3'(x) &= f_3''(x) = f_3'''(x) = e^x, \\ (T_\alpha^a f_3)(x) &= (x-a)^{1-\alpha} f_3'(x). \end{aligned}$$

Table 3. Results of one-point methods and *fsolve* for $f_2(x)$, with initial estimate $x_0 = 0.5$.

		NeA3 Method			
α	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Iter**	ρ
1	$\bar{x}_1$	3.31×10^{-14}	2.63×10^{-7}	8	2.00
0.9	$\bar{x}_1$	7.16×10^{-15}	4.79×10^{-8}	8	3.01
0.8	$\bar{x}_1$	2.22×10^{-15}	2.36×10^{-9}	8	2.93
0.7	$\bar{x}_1$	3.70×10^{-9}	0.003	7	2.46
0.6	$\bar{x}_1$	7.16×10^{-15}	1.97×10^{-7}	8	2.89
0.5	$\bar{x}_1$	3.11×10^{-15}	1.13×10^{-7}	10	2.90
0.4	$\bar{x}_1$	7.23×10^{-15}	2.63×10^{-9}	9	3.02
0.3	$\bar{x}_1$	4.96×10^{-13}	1.84×10^{-4}	10	2.72
0.2	$\bar{x}_2$	7.12×10^{-15}	4.59×10^{-8}	13	2.93
0.1	$\bar{x}_1$	7.89×10^{-14}	1.97×10^{-5}	13	3.34
		NeL3 Method			
α_k	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Iter**	ρ
-	$\bar{x}_2$	5.44×10^{-10}	0.0021	4	2.76
		NeLA4 Method			
α_k	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Iter**	ρ
-	$\bar{x}_1$	1.07×10^{-14}	1.09×10^{-8}	4	3.01
		fsolve			
-	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	iter	ρ
-	$\bar{x}_1$	7.58×10^{-8}	-	7	-

Table 4. Results of multipoint methods for $f_2(x)$, with initial estimate $x_0 = 0.5$.

		TeCo Method					**CKeCO Method**			
α	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Iter**	ρ	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Iter**	ρ
1	$\bar{x}_1$	7.16×10^{-15}	6.43×10^{-7}	7	3.18	$\bar{x}_1$	2.22×10^{-15}	2.34×10^{-8}	7	3.09
0.9	$\bar{x}_1$	2.42×10^{-14}	6.19×10^{-5}	11	3.46	$\bar{x}_1$	1.14×10^{-11}	4.94×10^{-4}	8	2.76
0.8	$\bar{x}_2$	9.57×10^{-15}	2.49×10^{-7}	10	2.90	$\bar{x}_2$	8.40×10^{-9}	0.005	7	3.87
0.7	$\bar{x}_2$	4.72×10^{-15}	4.02×10^{-6}	7	2.75	$\bar{x}_2$	3.97×10^{-15}	2.37×10^{-5}	8	2.76
0.6	$\bar{x}_2$	1.45×10^{-13}	1.37×10^{-4}	8	2.95	$\bar{x}_2$	3.97×10^{-15}	5.55×10^{-6}	9	3.22
0.5	$\bar{x}_2$	1.78×10^{-15}	6.41×10^{-8}	7	2.86	$\bar{x}_2$	3.66×10^{-15}	2.49×10^{-5}	6	2.94
0.4	$\bar{x}_2$	1.26×10^{-14}	3.59×10^{-5}	6	2.97	$\bar{x}_2$	9.27×10^{-15}	4.86×10^{-6}	5	2.97
0.3	$\bar{x}_2$	5.33×10^{-15}	3.10×10^{-7}	5	3.07	$\bar{x}_2$	1.65×10^{-14}	1.51×10^{-7}	7	2.96
0.2	$\bar{x}_2$	1.65×10^{-14}	2.47×10^{-6}	5	3.03	$\bar{x}_2$	1.65×10^{-14}	5.19×10^{-9}	19	3.00
0.1	$\bar{x}_1$	2.14×10^{-14}	4.53×10^{-9}	30	2.98	$\bar{x}_2$	2.31×10^{-11}	6.61×10^{-4}	9	2.97
		OeCO Method					**CeCO Method**			
α	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Iter**	ρ	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Iter**	ρ
1	$\bar{x}_1$	2.04×10^{-9}	0.026	4	5.51	$\bar{x}_1$	2.89×10^{-13}	0.002	7	2.74
0.9	$\bar{x}_1$	4.29×10^{-14}	8.34×10^{-9}	5	3.72	$\bar{x}_1$	1.61×10^{-14}	2.34×10^{-4}	8	3.77
0.8	$\bar{x}_1$	1.39×10^{-14}	2.05×10^{-7}	5	3.65	$\bar{x}_1$	1.90×10^{-14}	2.93×10^{-7}	13	4.31
0.7	$\bar{x}_1$	3.31×10^{-14}	4.74×10^{-6}	5	3.57	$\bar{x}_2$	3.31×10^{-9}	0.0227	8	3.06
0.6	$\bar{x}_1$	3.30×10^{-14}	9.49×10^{-5}	5	3.52	$\bar{x}_2$	6.03×10^{-11}	0.0087	9	2.85
0.5	$\bar{x}_1$	2.25×10^{-14}	0.0015	5	3.59	$\bar{x}_2$	1.99×10^{-15}	4.90×10^{-8}	10	3.80
0.4	$\bar{x}_1$	1.00×10^{-9}	0.0206	5	4.07	$\bar{x}_1$	6.52×10^{-10}	0.0118	12	2.67
0.3	$\bar{x}_1$	2.22×10^{-15}	5.59×10^{-6}	6	3.43	$\bar{x}_1$	1.11×10^{-14}	5.65×10^{-4}	12	4.19
0.2	$\bar{x}_1$	1.38×10^{-9}	0.0219	6	3.47	$\bar{x}_2$	4.80×10^{-14}	5.66×10^{-6}	95	3.89
0.1	$\bar{x}_1$	1.90×10^{-14}	1.94×10^{-8}	8	4.27	$\bar{x}_2$	8.42×10^{-14}	5.83×10^{-7}	383	4.04

In Table 5, NeA3 requires fewer iterations when $\alpha \neq 1$, and ρ is around 3. When $\alpha = 1$, the computational order of convergence is 2, as expected. We observe that NeL3 improves NeA3 in the number of iterations required, with ρ being around 3. NeLA4 does not work for this function because $f_3'(x) = f_3''(x) = f_3'''(x)$, and this involves singularities in (16) and (17).

In Table 6, TeCO, CKeCO, OeCO, and CeCO present a very similar behaviors compared to their classical versions, respectively, and the computational order of convergence is as expected in each case.

Table 5. Results of one-point methods and *fsolve* for $f_3(x)$, with initial estimate $x_0 = 1.5$.

		NeA3 Method			
α	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Iter**	ρ
1	$\bar{x}_1$	2.22×10^{-16}	2.08×10^{-8}	6	2.00
0.9	$\bar{x}_1$	1.23×10^{-9}	9.05×10^{-4}	4	2.70
0.8	$\bar{x}_1$	1.65×10^{-12}	1.25×10^{-4}	4	2.83
0.7	$\bar{x}_1$	9.55×10^{-15}	2.57×10^{-5}	4	2.89
0.6	$\bar{x}_1$	1.39×10^{-17}	6.67×10^{-6}	4	2.93
0.5	$\bar{x}_1$	1.11×10^{-16}	2.05×10^{-6}	4	2.95
0.4	$\bar{x}_1$	2.78×10^{-16}	7.13×10^{-7}	4	2.96
0.3	$\bar{x}_1$	3.34×10^{-16}	2.74×10^{-7}	4	2.97
0.2	$\bar{x}_1$	4.00×10^{-16}	1.14×10^{-7}	4	2.98
0.1	$\bar{x}_1$	1.24×10^{-16}	5.06×10^{-8}	4	2.99
		NeL3 Method			
α_k	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Iter**	ρ
-	$\bar{x}_1$	0	1.21×10^{-5}	3	3.11
		NeLA4 Method			
α_k	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Iter**	ρ
-	-	-	-	-	-
		fsolve			
-	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	iter	ρ
-	$\bar{x}_1$	2.08×10^{-8}	-	5	-

Table 6. Results of multipoint methods for $f_3(x)$, with initial estimate $x_0 = 1.5$.

		TeCO Method					**CKeCO Method**			
α	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Iter**	ρ	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Iter**	ρ
1	$\bar{x}_1$	5.33×10^{-15}	2.20×10^{-5}	4	2.92	$\bar{x}_1$	8.17×10^{-14}	5.18×10^{-5}	4	2.90
0.9	$\bar{x}_1$	8.88×10^{-15}	2.46×10^{-5}	4	2.92	$\bar{x}_1$	1.19×10^{-13}	5.80×10^{-5}	4	2.90
0.8	$\bar{x}_1$	1.24×10^{-14}	2.74×10^{-5}	4	2.91	$\bar{x}_1$	1.67×10^{-13}	6.48×10^{-5}	4	2.90
0.7	$\bar{x}_1$	1.60×10^{-14}	3.06×10^{-5}	4	2.91	$\bar{x}_1$	2.31×10^{-13}	7.22×10^{-5}	4	2.90
0.6	$\bar{x}_1$	2.13×10^{-14}	3.41×10^{-5}	4	2.91	$\bar{x}_1$	3.29×10^{-13}	8.04×10^{-5}	4	2.90
0.5	$\bar{x}_1$	2.84×10^{-14}	3.78×10^{-5}	4	2.91	$\bar{x}_1$	4.58×10^{-13}	8.93×10^{-5}	4	2.90
0.4	$\bar{x}_1$	4.44×10^{-14}	4.20×10^{-5}	4	2.91	$\bar{x}_1$	6.41×10^{-13}	9.90×10^{-5}	4	2.89
0.3	$\bar{x}_1$	5.86×10^{-14}	4.65×10^{-5}	4	2.91	$\bar{x}_1$	8.86×10^{-13}	1.10×10^{-4}	4	2.89
0.2	$\bar{x}_1$	8.17×10^{-14}	5.14×10^{-5}	4	2.90	$\bar{x}_1$	1.22×10^{-12}	1.21×10^{-4}	4	2.89
0.1	$\bar{x}_1$	1.12×10^{-13}	5.68×10^{-5}	4	2.90	$\bar{x}_1$	1.66×10^{-12}	1.34×10^{-4}	4	2.89
		OeCO Method					**CeCO Method**			
α	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Iter**	ρ	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Iter**	ρ
1	$\bar{x}_1$	0	7.53×10^{-6}	3	3.91	$\bar{x}_1$	1.54×10^{-10}	0.0041	3	3.58
0.9	$\bar{x}_1$	1.18×10^{-15}	7.76×10^{-6}	3	3.91	$\bar{x}_1$	1.97×10^{-10}	0.0043	3	3.58
0.8	$\bar{x}_1$	1.78×10^{-15}	7.99×10^{-6}	3	3.91	$\bar{x}_1$	2.52×10^{-10}	0.0046	3	3.58
0.7	$\bar{x}_1$	1.78×10^{-15}	8.22×10^{-6}	3	3.90	$\bar{x}_1$	3.20×10^{-10}	0.0048	3	3.57
0.6	$\bar{x}_1$	0	8.44×10^{-6}	3	3.90	$\bar{x}_1$	4.05×10^{-10}	0.0051	3	3.57
0.5	$\bar{x}_1$	1.78×10^{-15}	8.67×10^{-6}	3	3.90	$\bar{x}_1$	5.11×10^{-10}	0.0054	3	3.57
0.4	$\bar{x}_1$	0	8.89×10^{-6}	3	3.90	$\bar{x}_1$	6.42×10^{-10}	0.0056	3	3.57
0.3	$\bar{x}_1$	0	9.10×10^{-6}	3	3.90	$\bar{x}_1$	8.04×10^{-10}	0.0059	3	3.56
0.2	$\bar{x}_1$	3.55×10^{-15}	9.31×10^{-6}	3	3.89	$\bar{x}_1$	1.00×10^{-9}	0.0062	3	3.56
0.1	$\bar{x}_1$	5.33×10^{-15}	9.52×10^{-6}	3	3.89	$\bar{x}_1$	1.25×10^{-9}	0.0065	3	3.56

We can note that in Tables 5 and 6 our methods converge in fewer iterations than *fsolve* in all cases.

Finally, the fourth test function is $f_4(x) = \sin(10x) - 0.5x + 0.2$, with real roots $\bar{x}_1 = -1.4523$, $\bar{x}_2 = -1.3647$, $\bar{x}_3 = -0.87345$, $\bar{x}_4 = -0.6857$, $\bar{x}_5 = -0.27949$, $\bar{x}_6 = -0.021219$, $\bar{x}_7 = 0.31824$, $\bar{x}_8 = 0.64036$, $\bar{x}_9 = 0.91636$, $\bar{x}_{10} = 1.3035$, $\bar{x}_{11} = 1.5118$, $\bar{x}_{12} = 1.9756$, and $\bar{x}_{13} = 2.0977$. The necessary derivatives for this function are

$$\begin{aligned} f_4'(x) &= 10\cos(10x) - 0.5, \\ f_4''(x) &= -100\sin(10x), \\ f_4'''(x) &= -1000\cos(10x), \\ (T_\alpha^a f_4)(x) &= (x-a)^{1-\alpha} f_4'(x). \end{aligned}$$

In Table 7, NeA3 requires fewer iterations for most of values when $\alpha \neq 1$, and ρ is around 3 or greater. When $\alpha = 1$, the computational order of convergence is around 2. We observe that NeLA4 improves the classical Newton procedure as well in the number of iterations and in terms of the computational order of convergence, ρ. No results are shown for NeL3 because it converges to a point which is not solution of $f_4(x)$.

In Table 8, TeCO, CKeCO, OeCO, and CeCO present very similar behavior compared to their classical versions, respectively, and the computational order of convergence is greater than expected by using TeCO, CKeCO, and CeCO. ρ cannot be provided when using OeCO because at least three iterations are needed to compute it.

We can see that in Tables 7 and 8 our methods converge in fewer iterations than *fsolve* in most of cases.

Table 7. Results of one-point methods and *fsolve* for $f_4(x)$, with initial estimate $x_0 = 2$.

			NeA3 Method		
α	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Iter**	ρ
1	$\bar{x}_{12}$	1.94×10^{-11}	7.01×10^{-7}	4	1.99
0.9	$\bar{x}_{12}$	2.78×10^{-16}	2.31×10^{-8}	4	2.80
0.8	$\bar{x}_{12}$	2.78×10^{-16}	5.39×10^{-8}	4	2.92
0.7	$\bar{x}_{12}$	2.78×10^{-16}	5.32×10^{-8}	4	3.14
0.6	$\bar{x}_{12}$	3.01×10^{-9}	1.59×10^{-4}	3	7.91
0.5	$\bar{x}_{12}$	8.16×10^{-12}	2.32×10^{-5}	3	4.38
0.4	$\bar{x}_{12}$	4.60×10^{-13}	9.21×10^{-6}	3	3.93
0.3	$\bar{x}_{12}$	6.80×10^{-14}	4.97×10^{-6}	3	3.74
0.2	$\bar{x}_{12}$	1.58×10^{-14}	3.15×10^{-6}	3	3.63
0.1	$\bar{x}_{12}$	3.61×10^{-15}	2.20×10^{-6}	3	3.55
			NeL3 Method		
α_k	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Iter**	ρ
-	-	-	-	-	-
			NeLA4 Method		
α_k	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	**Iter**	ρ
-	$\bar{x}_{13}$	1.72×10^{-15}	1.15×10^{-5}	3	4.33
			fsolve		
-	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	iter	ρ
-	$\bar{x}_{12}$	1.85×10^{-11}	-	4	-

Table 8. Results of multipoint methods for $f_4(x)$, with initial estimate $x_0 = 2$.

	TeCO Method					**CKeCO Method**				
α	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1}-x_k\rvert$	**Iter**	ρ	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1}-x_k\rvert$	**Iter**	ρ
1	$\bar{x}_{12}$	1.13×10^{-12}	1.27×10^{-5}	3	4.18	$\bar{x}_{12}$	3.37×10^{-13}	8.77×10^{-6}	3	4.07
0.9	$\bar{x}_{12}$	1.13×10^{-12}	1.27×10^{-5}	3	4.18	$\bar{x}_{12}$	3.37×10^{-13}	8.72×10^{-6}	3	4.07
0.8	$\bar{x}_{12}$	1.11×10^{-12}	1.26×10^{-5}	3	4.18	$\bar{x}_{12}$	3.27×10^{-13}	8.66×10^{-6}	3	4.07
0.7	$\bar{x}_{12}$	1.06×10^{-12}	1.25×10^{-5}	3	4.18	$\bar{x}_{12}$	2.97×10^{-13}	8.60×10^{-6}	3	4.07
0.6	$\bar{x}_{12}$	1.06×10^{-12}	1.25×10^{-5}	3	4.17	$\bar{x}_{12}$	3.17×10^{-13}	8.54×10^{-6}	3	4.07
0.5	$\bar{x}_{12}$	1.04×10^{-12}	1.24×10^{-5}	3	4.17	$\bar{x}_{12}$	3.07×10^{-13}	8.49×10^{-6}	3	4.07
0.4	$\bar{x}_{12}$	1.04×10^{-12}	1.24×10^{-5}	3	4.17	$\bar{x}_{12}$	3.17×10^{-13}	8.43×10^{-6}	3	4.06
0.3	$\bar{x}_{12}$	1.02×10^{-12}	1.23×10^{-5}	3	4.17	$\bar{x}_{12}$	3.07×10^{-13}	8.37×10^{-6}	3	4.06
0.2	$\bar{x}_{12}$	1.01×10^{-12}	1.22×10^{-5}	3	4.17	$\bar{x}_{12}$	2.77×10^{-13}	8.32×10^{-6}	3	4.06
0.1	$\bar{x}_{12}$	9.81×10^{-13}	1.22×10^{-5}	3	4.17	$\bar{x}_{12}$	2.77×10^{-13}	8.26×10^{-6}	3	4.06
	OeCO Method					**CeCO Method**				
α	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1}-x_k\rvert$	**Iter**	ρ	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1}-x_k\rvert$	**Iter**	ρ
1	$\bar{x}_{12}$	7.85×10^{-11}	4.17×10^{-4}	2	-	$\bar{x}_{12}$	4.27×10^{-15}	5.00×10^{-7}	3	4.81
0.9	$\bar{x}_{12}$	7.83×10^{-11}	4.17×10^{-4}	2	-	$\bar{x}_{12}$	1.58×10^{-14}	4.96×10^{-7}	3	4.81
0.8	$\bar{x}_{12}$	7.80×10^{-11}	4.17×10^{-4}	2	-	$\bar{x}_{12}$	1.58×10^{-14}	4.92×10^{-7}	3	4.81
0.7	$\bar{x}_{12}$	7.79×10^{-11}	4.17×10^{-4}	2	-	$\bar{x}_{12}$	4.27×10^{-15}	4.89×10^{-7}	3	4.81
0.6	$\bar{x}_{12}$	7.76×10^{-11}	4.17×10^{-4}	2	-	$\bar{x}_{12}$	4.27×10^{-15}	4.85×10^{-7}	3	4.81
0.5	$\bar{x}_{12}$	7.74×10^{-11}	4.16×10^{-4}	2	-	$\bar{x}_{12}$	4.27×10^{-15}	4.81×10^{-7}	3	4.81
0.4	$\bar{x}_{12}$	7.72×10^{-11}	4.16×10^{-4}	2	-	$\bar{x}_{12}$	1.44×10^{-14}	4.77×10^{-7}	3	4.81
0.3	$\bar{x}_{12}$	7.70×10^{-11}	4.16×10^{-4}	2	-	$\bar{x}_{12}$	2.58×10^{-14}	4.74×10^{-7}	3	4.81
0.2	$\bar{x}_{12}$	7.67×10^{-11}	4.16×10^{-4}	2	-	$\bar{x}_{12}$	5.72×10^{-15}	4.70×10^{-7}	3	4.81
0.1	$\bar{x}_{12}$	7.66×10^{-11}	4.15×10^{-4}	2	-	$\bar{x}_{12}$	4.59×10^{-14}	4.67×10^{-7}	3	4.81

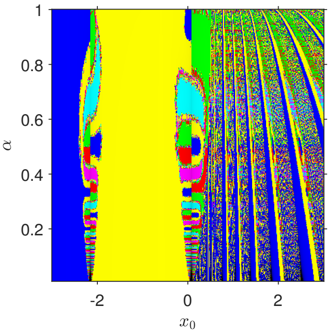
Numerical Stability

We are now going to analyze the dependence on the initial estimates of the methods proposed in this work. For NeA3, TeCO, CKeCO, OeCO, and CeCO, whose α is fixed in each iteration, we use convergence planes, defined in [23]. In these planes, we use a 400×400 grid, where the horizontal axis corresponds to initial estimate x_0, and the vertical axis corresponds to $\alpha \in (0, 1]$. Each color represents a different solution found with a tolerance of 10^{-3}, and it is painted in black when no solution is found in 500 iterations. Moreover, for each convergence plane, we calculate the percentage of converging pairs (x_0, α) in order to compare the efficiency of the schemes.

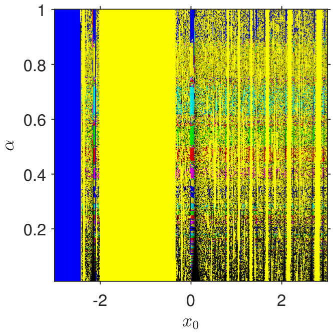
In Figure 1, we observe that NeA3 and OeCO almost reach a 100% convergence, TeCO and CeCO attain around 94% convergence, and CKeCO obtains around 77% convergence; all roots are found in each plane.

In Figure 2, we see that every procedure almost reaches 100% convergence, and all roots are found in each plane. In Figure 3, we note that NeA3, TeCO, CKeCO, and CeCO attain between 53% and 59% convergence, whereas OeCO obtains around 74% convergence. In this case, considering that $-10 \leq x_0 \leq 10$, and $a = -10$, we point out that stability could be slightly improved if we choose $a < -10$.

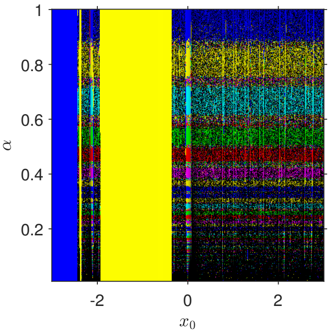
In Figure 4, we can see that NeA3 and CKeCO reach between 42% and 48% convergence, TeCO and CeCO attain between 62% and 66% convergence, and OeCO obtains around 74% convergence; again, all roots are found in each plane.



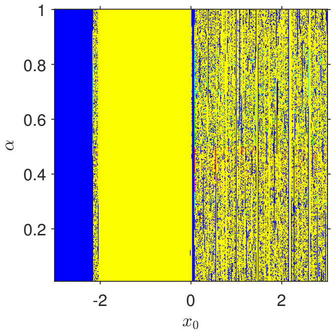
(**a**) NeA3, 99.76% convergence



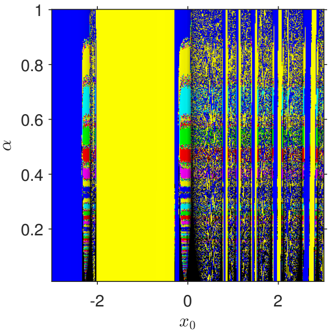
(**b**) TeCO, 94.63% convergence



(**c**) CKeCO, 76.76% convergence

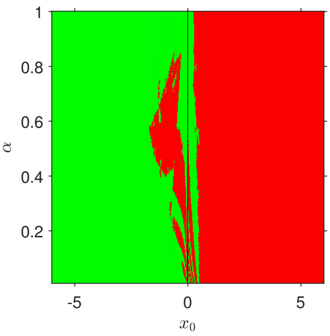


(**d**) OeCO, 99.46% convergence

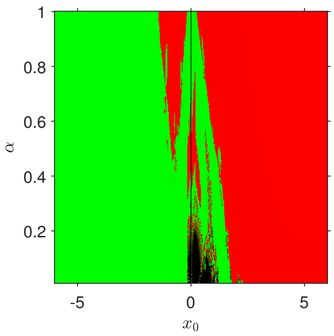


(**e**) CeCO, 94.04% convergence

Figure 1. Convergence planes for $f_1(x)$.

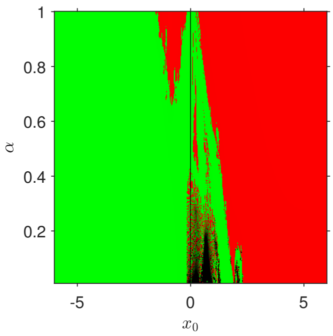


(**a**) NeA3, 99.75% convergence

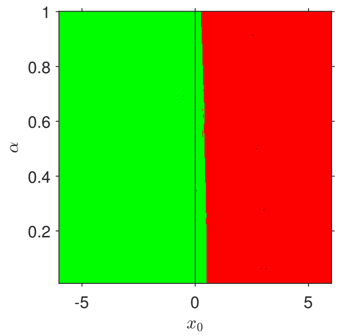


(**b**) TeCO, 98.91% convergence

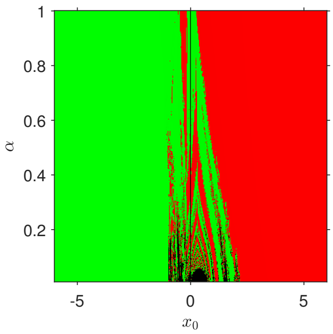
Figure 2. *Cont.*



(**c**) CKeCO, 99.01% convergence

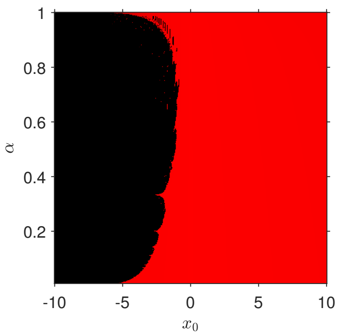


(**d**) OeCO, 99.75% convergence

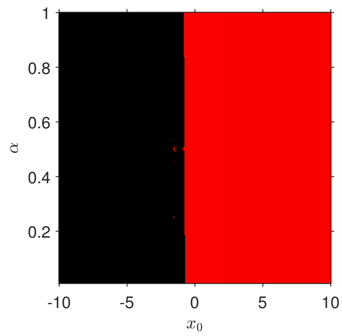


(**e**) CeCO, 98.72% convergence

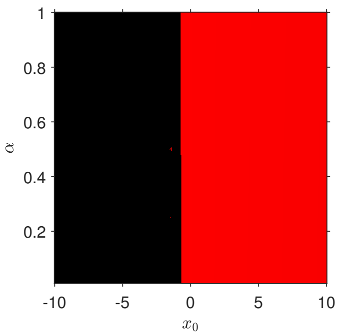
Figure 2. Convergence planes for $f_2(x)$.



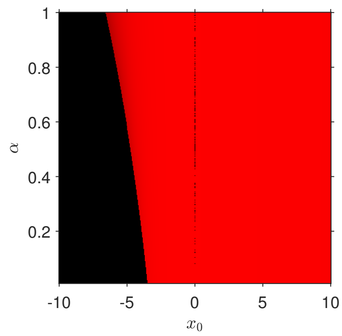
(**a**) NeA3, 59.26% convergence



(**b**) TeCO, 53.88% convergence

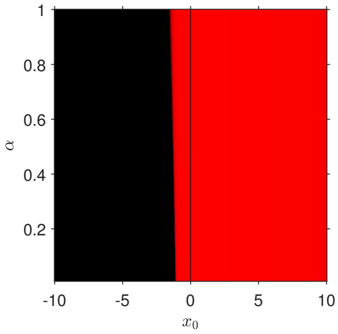


(**c**) CKeCO, 53.50% convergence



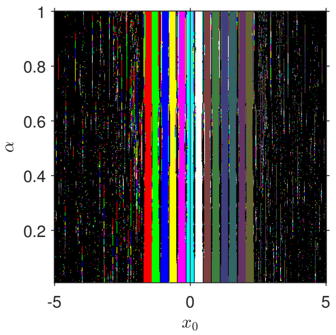
(**d**) OeCO, 74.04% convergence

Figure 3. *Cont.*

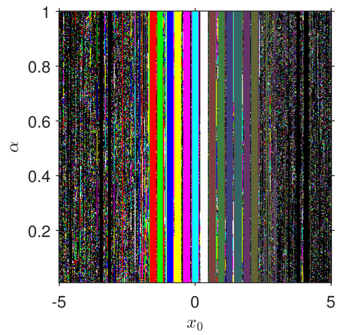


(**e**) CeCO, 56.31% convergence

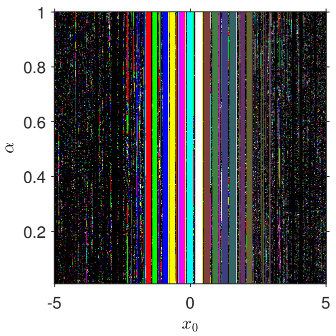
Figure 3. Convergence planes for $f_3(x)$.



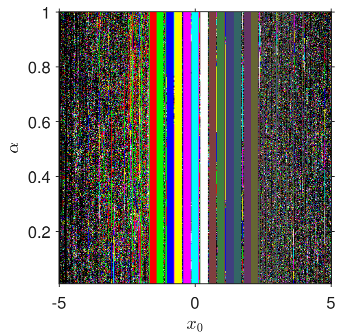
(**a**) NeA3, 42.96% convergence



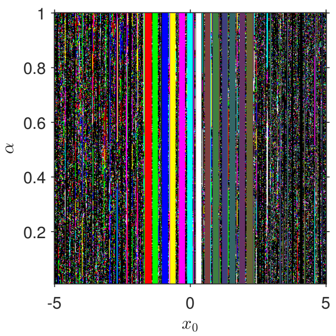
(**b**) TeCo, 66.19% convergence



(**c**) CKeCO, 48.23% convergence



(**d**) OeCO, 73.67% convergence



(**e**) CeCO, 62.55% convergence

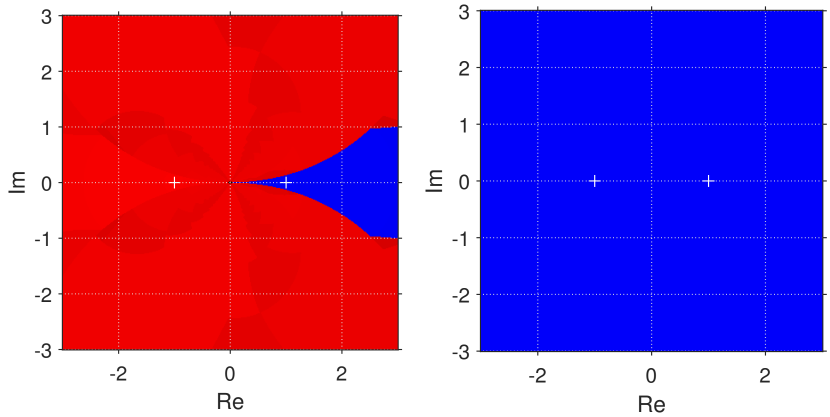
Figure 4. Convergence planes for $f_4(x)$.

In order to visualize the dependence on the initial estimates of NeL3 and NeLA4, with α_k calculated in each iteration, we use dynamical planes. These planes are constructed with the real part of the initial estimate x_0 in the horizontal axis, and the imaginary part in the vertical axis. As in convergence planes, we use a 400 × 400 grid, a tolerance of 10^{-3}, and at most 500 iterations. In these cases, we use real and complex polynomials whose roots are known, which are represented with "white crosses" on each plane.

Our polynomials with their respective roots are

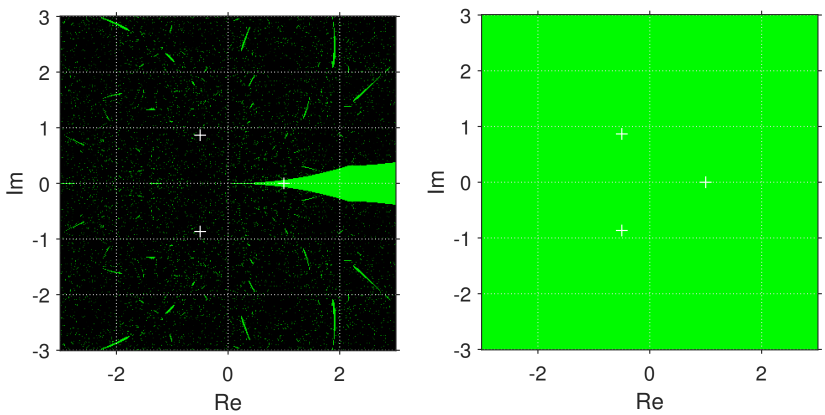
- $p_1(z) = z^2 - 1$, with $\bar{z}_1 = -1$ and $\bar{z}_2 = 1$.
- $p_2(z) = z^3 - 1$, with $\bar{z}_1 = 1$, $\bar{z}_2 = -1/2 + (\sqrt{3}/2)i$ and $\bar{z}_3 = -1/2 - (\sqrt{3}/2)i$.
- $p_3(z) = z^2 - i$, with $\bar{z}_1 = \sqrt{2}/2 + (\sqrt{2}/2)i$ and $\bar{z}_2 = -\sqrt{2}/2 - (\sqrt{2}/2)i$.
- $p_4(z) = z^3 + 4z^2 - 10$, with $\bar{z}_1 \approx 1.3652$, $\bar{z}_2 \approx -2.6826 + 0.3583i$ and $\bar{z}_3 \approx -2.6826 - 0.3583i$.

In general, we observe that NeL3 and NeLA4 tend to converge to one root in most planes. NeL3 converges to both roots in Figure 5, but convergence is not guaranteed in most planes because it obtains many nonconverging complex initial estimations $(Re(x_0), Im(x_0))$, whereas in NeLA4 most initial guesses converge to one root. Performance of methods NeL3 and NeLA4 on $p_i(z)$, $i = 2, 3, 4$ is similar (see Figures 6–8).



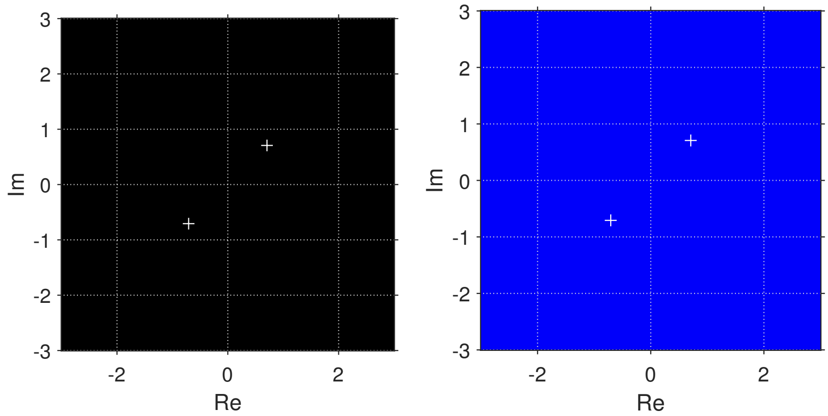
(**a**) NeL3 (**b**) NeLA4

Figure 5. Dynamical planes for $p_1(z)$.



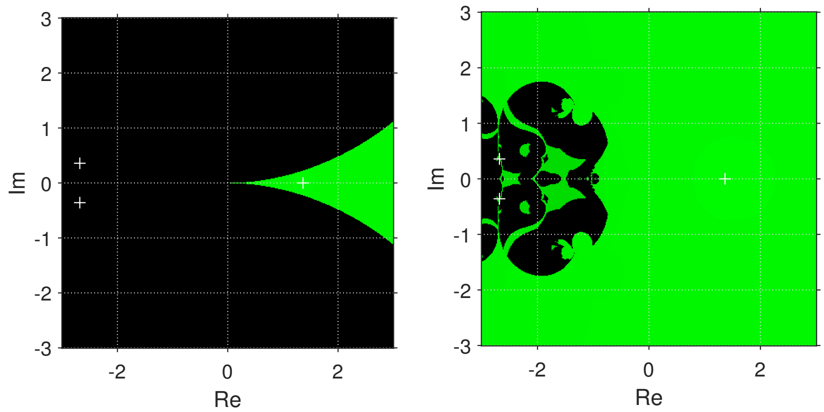
(**a**) NeL3 (**b**) NeLA4

Figure 6. Dynamical planes for $p_2(z)$.



(**a**) NeL3 (**b**) NeLA4

Figure 7. Dynamical planes for $p_3(z)$.



(**a**) NeL3 (**b**) NeLA4

Figure 8. Dynamical planes for $p_4(z)$.

5. Conclusions

In this manuscript, three one-point conformable Newton-type methods have been designed (NeL3, NeA3, and NeLA4), and their convergence analysis has been performed, improving the quadratic order of the starting scheme. None of these procedures are optimal because they require the evaluation of higher-order derivatives. On the other hand, a general technique has been provided so as to obtain the conformable version of any iterative method; later, this technique was used to derive the conformable version of four multipoint classical schemes (TeCO, CKeCO, OeCO, and CeCO), where two of them are the first optimal multipoint conformable procedures in the literature (OeCO and CeCO), holding an order of convergence equal to that of the classical case. So, the technique is supported by the results obtained. Numerical tests were performed, and the dependence on initial estimates was analyzed by visualizing convergence and dynamical planes. We observed that, in general, these methods can converge when the classical ones fail, or in fewer iterations in some cases. Additionally, the theoretical order of convergence is obtained in practice, tending to equal or improve that of classical multipoint schemes. We also see that it is possible to obtain real or complex roots with real initial estimates, and that we can obtain different roots by choosing a different value of α and the same initial estimation. These methods found a solution when *fsolve* failed, and required fewer iterations than the Matlab Toolbox in most cases. Finally, most of the planes in this work confirm that these procedures have, in general, good stability properties in terms of the wideness of basins of attraction of the roots, and all roots are found in most of such planes.

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