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ON TOTALLY SEMIPERMUTABLE PRODUCTS OF FINITE GROUPS

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Abstract. We say a group $G = AB$ is the totally semipermutable product of subgroups A and B if every Sylow subgroup P of A is totally permutable with every Sylow subgroup Q of B whenever $\gcd(|P|, |Q|) = 1$. Products of pairwise totally semipermutable subgroups are studied in this article. Let $\mathfrak{U}$ denote the class of supersoluble groups and $\mathfrak{D}$ denote the formation of all groups which have an ordered Sylow tower of supersoluble type. We obtain the $\mathfrak{F}$-residual of the product from the $\mathfrak{F}$-residuals of the pairwise totally semipermutable subgroups when $\mathfrak{F}$ is a subgroup-closed saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F} \subseteq \mathfrak{D}$.

1. Introduction

All groups considered in this article will be finite.

It is well known that the product of supersoluble subgroups need not be supersoluble, even when the subgroup factors are normal. This makes

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it interesting to study factorised groups whose subgroup factors have certain permutability properties. Recall that a group $G = AB$ is the totally permutable product of subgroups A and B if every subgroup of A permutes with every subgroup of B. Asaad and Shaalan [2] proved the following result about totally permutable products:

If $G = AB$ is the totally permutable product of supersoluble subgroups A and B, then G is supersoluble.

The study of totally permutable products grew, with many authors investigating this type of products of subgroups (see [3, Chapters 4 and 5]). In particular, Asaad and Shaalan's result above was extended to $\mathfrak{F}$-residuals of saturated formations containing $\mathfrak{U}$, the class of all finite supersoluble groups as the following result states. Recall that the $\mathfrak{F}$-residual of a group G, denoted by $G^{\mathfrak{F}}$ is the smallest normal subgroup such that $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$.

THEOREM 1.1 [4, Theorem 4]. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Let a group $G = G_1G_2 \cdots G_n$ be the pairwise totally permutable product of the subgroups $G_1, G_2, \ldots, G_n$. Then*

$$G^{\mathfrak{F}} = (G_1)^{\mathfrak{F}}(G_2)^{\mathfrak{F}} \cdots (G_n)^{\mathfrak{F}}.$$

In this article we study a new type of product of subgroups, introduced by Asaad in [1], that can be regarded as a local version of totally permutable products. We say a group $G = AB$ is the *totally semipermutable product* of subgroups A and B if every Sylow subgroup P of A is totally permutable with every Sylow subgroup Q of B whenever $\gcd(|P|, |Q|) = 1$.

Totally permutable products are totally semipermutable but the converse is not true in general as shown in [1, Example 1.1].

Asaad [1] proved the following result:

THEOREM 1.2 [1, Theorem 1.1]. *Let a group $G = G_1G_2 \cdots G_n$ be the pairwise totally semipermutable product of the subgroups $G_1, G_2, \ldots, G_n$. If $G_i \in \mathfrak{U}$ for all $i \in \{1, 2, \ldots, n\}$, then $G \in \mathfrak{U}$.*

We take note that Monakhov and Trofimuk in [8] also studied totally semipermutable products under a different name. Their work extended Theorem 1.2 when $n = 2$. Let $\mathfrak{D}$ denote the formation of all groups which have an ordered Sylow tower of supersoluble type.

THEOREM 1.3 [8, Theorem 4.1]. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F} \subseteq \mathfrak{D}$. Let the group $G = AB$ be the totally semipermutable product of subgroups A and B. If A and B belong to $\mathfrak{F}$, then G belongs to $\mathfrak{F}$.*

Our goal is to extend Theorems 1.2 and 1.3. In particular, we prove the following:

THEOREM 1.4. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F} \subseteq \mathfrak{D}$. Let $G = G_1G_2 \cdots G_n$ be the pairwise totally semipermutable product of the subgroups G_1, G_2, ..., G_n. Then*

$$G^{\mathfrak{F}} = (G_1)^{\mathfrak{F}}(G_2)^{\mathfrak{F}} \cdots (G_n)^{\mathfrak{F}}.$$

Moreover, $G_1^{\mathfrak{F}}$, $G_2^{\mathfrak{F}}$, ..., $G_n^{\mathfrak{F}}$ are normal subgroups of G.

Note that Theorem 1.4 also extends Theorem 1.1 to totally semipermutable products when $\mathfrak{F}$ is subgroup closed and is such that $\mathfrak{U} \subseteq \mathfrak{F} \subseteq \mathfrak{D}$.

2. Preliminary results

In this section we state some results that we need to prove our main results. We begin with some basic properties of totally semipermutable products.

LEMMA 2.1 [1, Lemma 2.8]. *Let a group $G = AB$ be the totally semipermutable product of the subgroups A and B. Then*:

(1) *If N is a normal subgroup of G, then $G/N = (AN/N)(BN/N)$ is the totally semipermutable product of the subgroups AN/N and BN/N.*

(2) *If P is a Sylow subgroup of A and Q is a Sylow subgroup of B with coprime orders*, *then PQ is supersoluble.*

(3) *$A \cap B$ is supersoluble.*

LEMMA 2.2 [8, Lemma 1.6]. *Let $G = [P]M$ be a primitive soluble group, where M is a primitivator of G and P is a Sylow p-subgroup of G. Let A and B be subgroups of M and $M = AB$. If $B \le N_G(X)$ for every subgroup X of P,*

(i) *B is a cyclic group of order dividing $p - 1$.*

(ii) $[A,B]$=1.

LEMMA 2.3 [5, Lemma 5]. *Let the group $G = AB$ be the product of the totally permutable subgroups A and B. Let $\mathfrak{F}$ be a saturated formation of soluble groups containing $\mathfrak{U}$. If B belongs to $\mathfrak{F}$, then $G^{\mathfrak{F}} = A^{\mathfrak{F}}$.*

Let π be a set of primes. A group G has property D_π if G has Hall π-subgroups and every π-subgroup is contained in a conjugate of a given Hall π-subgroup of G.

According to [7, Satz VI.4.6], if π is a set of primes, $G = AB$ is a product of the subgroups A and B and G has property D_π, there exist Hall π-subgroups A_π and B_π of A and B respectively such that $A_\pi B_\pi$ is a Hall π-subgroup of G.

This fact will be used in the paper without any further reference.

3. Main results

We start off this section with an extension of Lemma 2.3. Recall that if $\mathfrak{F}$ is a saturated formation, a subgroup U of a group G is called $\mathfrak{F}$-maximal in G provided that $U \in \mathfrak{F}$, and if $U \leq V \leq G$ and $V \in \mathfrak{F}$, then $U = V$. A subgroup U of a group G is called an $\mathfrak{F}$-projector of G if UK/K is $\mathfrak{F}$-maximal in G/K for all normal subgroup K of G. On the other hand, a chief factor H/K of a group G is called $\mathfrak{F}$-central in G, if the semidirect product $[H/K](G/C_G(H/K))$ belongs to $\mathfrak{F}$.

Theorem 3.1. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F} \subseteq \mathfrak{D}$. Let $G = AB$ be the totally semipermutable product of A and B. If B is an $\mathfrak{F}$-group, then $G^{\mathfrak{F}} = A^{\mathfrak{F}}$.*

Proof. Suppose the theorem is not true and let (G, A, B) be a counterexample with $|\,G\,| + |\,G : A\,|$ as small as possible. By Theorem 1.3, we may assume $A^{\mathfrak{F}} \neq 1$. We shall get a contradiction by the following steps.

(a) *$G^{\mathfrak{F}} = A^{\mathfrak{F}}N$ for each minimal normal subgroup N of G.* Let N be a minimal normal subgroup of G. By Lemma 2.1, G/N is the totally semipermutable product of AN/N and BN/N. Hence $(G/N)^{\mathfrak{F}} = (AN/N)^{\mathfrak{F}}$ by the choice of G. This implies that $G^{\mathfrak{F}}N = A^{\mathfrak{F}}N$. Then $G^{\mathfrak{F}} = A^{\mathfrak{F}}(N \cap G^{\mathfrak{F}})$. If $N \cap G^{\mathfrak{F}} = 1$ we get a contradiction. This means that every minimal normal subgroup of G is contained in $G^{\mathfrak{F}}$ and so $G^{\mathfrak{F}} = A^{\mathfrak{F}}N$ for each minimal normal subgroup N of G.

(b) *There exist a prime q dividing the order of B and a Sylow q-subgroup B_q of B such that $G = AB_q$.* Let q be a prime dividing $|B|$. There exist A_q, a Sylow q-subgroup of A, and B_q, a Sylow q-subgroup of B such that A_qB_q is a Sylow q-subgroup of G. Since A and B are totally semipermutable, we have that AB_q is a subgroup of G. If AB_q is a proper subgroup of G, then the choice of G implies that B_q normalises $A^{\mathfrak{F}}$. Therefore if for every prime q dividing the order of B, there exists a Sylow q-subgroup B_q of B such that AB_q is a proper subgroup of G, it follows that $A^{\mathfrak{F}}$ is a normal subgroup of G, a contradiction. Consequently we may assume there exist q dividing $|B|$ and a Sylow q-subgroup B_q of B such that $G = AB_q$.

(c) *A is a maximal subgroup of G.* Let M be a maximal subgroup of G containing A. Since $M = A(M \cap B_q)$ and A and $M \cap B_q$ are totally semipermutable, the choice of G implies $M^{\mathfrak{F}} = A^{\mathfrak{F}}$. Moreover $G = MB_q$. We prove that M and B_q are totally semipermutable subgroups. Let r be a prime dividing the order of M, $r \neq q$. Since $M = A(M \cap B_q)$, there exists M_r, a Sylow r-subgroup of M with $M_r \leq A$. Hence M_r and B_q are totally permutable by the hypotheses. We see that M_r^g and B_q are totally permutable for all $g \in M$. We have that $g = ab$ where $a \in A$ and $b \in B_q$. It is clear that $M_r^a \leq A$ and B_q are totally permutable. Hence $M_r^{ab} = M_r^g$ is totally permutable with $B_q^b = B_q$. By the choice of $|A|$, we obtain $G^{\mathfrak{F}} = M^{\mathfrak{F}} = A^{\mathfrak{F}}$, a contradiction. Hence we may assume that A is a maximal subgroup of G.

(d) $\mathrm{Core}_G(A) = 1$, *and G is a primitive group.* Assume, arguing by contradiction, that there exists a minimal normal subgroup N of G contained in A.

(i) *$N \le G^{\mathfrak{F}}$ and $G^{\mathfrak{F}}$ is an elementary abelian p-group for some prime p. In particular, G is soluble.* In this case $N \le N_G(A^{\mathfrak{F}})$ and so by (a), $A^{\mathfrak{F}}$ is normal in $G^{\mathfrak{F}}$. Assume N is non-abelian. Then N is the product of non-abelian simple groups. But $N/N \cap A^{\mathfrak{F}} \simeq A^{\mathfrak{F}}N/A^{\mathfrak{F}} \le A/A^{\mathfrak{F}} \in \mathfrak{F} \subseteq \mathfrak{S}$, which implies $N = N \cap A^{\mathfrak{F}}$, that is, $N \le A^{\mathfrak{F}}$ and the result is clear. Hence N is abelian, then N is a p-group for some prime p. Now we can assume that $G^{\mathfrak{F}}$ is a p-group. Otherwise, since $A^{\mathfrak{F}}$ is a normal subgroup of $G^{\mathfrak{F}}$ we have that $G^{\mathfrak{F}}/A^{\mathfrak{F}}$ is a p-group. Therefore $O^p(G^{\mathfrak{F}}) \le A^{\mathfrak{F}}$. If $O^p(G^{\mathfrak{F}}) \ne 1$, there would exist a minimal normal subgroup of G contained in $A^{\mathfrak{F}}$ and we would be done. If $\Phi(G^{\mathfrak{F}}) \ne 1$, then $G^{\mathfrak{F}} = A^{\mathfrak{F}}\Phi(G^{\mathfrak{F}}) = A^{\mathfrak{F}}$, because $\Phi(G^{\mathfrak{F}})$ is normal in G. Therefore $G^{\mathfrak{F}}$ is elementary abelian. Note that in this case the group G is soluble.

(ii) *p is the largest prime dividing $|G|$, and if P denotes a Sylow p-subgroup of G, then P is normal in G.* Let r denote the largest prime dividing $|G|$. Assume $r = q$. We have that B_q and A_s are totally permutable, where A_s denotes a Sylow s-subgroup of A, $s \ne r$ and then B_q is normalised by A_s for every A_s a Sylow s-subgroup of A with $s \ne q$ because A_sB_q is supersoluble by Lemma 2.1. Moreover there exists A_q a Sylow q-subgroup of A such that $Q = A_qB_q$ is a Sylow q-subgroup of G. Consequently $1 \ne B_q^G = B_q^A = B_q^{A_q} \le Q$. This implies $r = q = p$. Hence we may suppose that $r \ne q$. Let R denote a Sylow r-subgroup of G. Then $RG^{\mathfrak{F}}$ is normal in G and we may assume it is contained in A. Since $(RG^{\mathfrak{F}})^{\mathfrak{F}}$ is a normal subgroup of G contained in $A^{\mathfrak{F}}$, we obtain that $RG^{\mathfrak{F}}$ is an $\mathfrak{F}$-group by Step (a). Assume $r \ne p$. Then R is normal in $RG^{\mathfrak{F}}$. Hence R is normal in G, a contradiction. Therefore $r = p$ and p is the largest prime dividing the order of G. Let P denote a Sylow p-subgroup of G. Then $PG^{\mathfrak{F}} = P$ is normal in G.

(iii) $p \ne q$. Assume $p = q$. Then $G = AB_q = AP = AF(G)$. Since N is contained in A and $P \le C_G(N)$, we have that N is a minimal normal subgroup of A. This means that either $N \cap A^{\mathfrak{F}} \in \{1, N\}$. If $N \cap A^{\mathfrak{F}} = N$, then N is contained in $A^{\mathfrak{F}}$, a contradiction. Suppose that $N \cap A^{\mathfrak{F}} = 1$. Then $NA^{\mathfrak{F}}/A^{\mathfrak{F}}$ is a minimal normal subgroup of $A/A^{\mathfrak{F}}$. Since $A/A^{\mathfrak{F}} \in \mathfrak{F}$, we have N is $\mathfrak{F}$-central in A and hence N is also $\mathfrak{F}$-central in G. Then N is contained in every $\mathfrak{F}$-projector E of G by [6, Theorem IV.6.14]. Then $N \le G^{\mathfrak{F}} \cap E = 1$ by [6, Theorem IV.5.18], which is impossible.

(iv) *A contradiction.* Consider $A = A^{\mathfrak{F}}F$ where F is an $\mathfrak{F}$-projector of A. Since $G^{\mathfrak{F}}$ is abelian, $A^{\mathfrak{F}} \cap F = 1$ by [6, Theorem IV.5.18]. Then $G = AB_q = A^{\mathfrak{F}}FB_q$. Since $p \ne q$, we may assume that a Sylow q-subgroup

F_q of F is such that F_qB_q is a Sylow q-subgroup of G. Moreover B_q is permutable with all Sylow subgroups of A for primes different from q. Hence FB_q is a subgroup of G. Assume that $G = FB_q$. Then $A = F(A \cap B_q)$. Since $A^{\mathfrak{F}}$ is a p-group, we have that $A^{\mathfrak{F}} \le F$, a contradiction. Thus we may suppose that $T = FB_q$ is a proper subgroup of G. Moreover T is the totally semipermutable product of the $\mathfrak{F}$-subgroups F and B_q. The choice of G yields that T is an $\mathfrak{F}$-group. Since $G = F(G)T$, there exists an $\mathfrak{F}$-projector E of G such that $T \le E$ by [6, Lemma III.3.14]. Moreover $G^{\mathfrak{F}} \cap E = 1$ by [6, Theorem IV.5.18]. Consequently $G^{\mathfrak{F}} = A^{\mathfrak{F}}(G^{\mathfrak{F}} \cap FB_q) = A^{\mathfrak{F}}$, a contradiction.

Thus $\mathrm{Core}_G(A) = 1$. In particular, G is a primitive group.

(e) *q is the largest prime dividing the order of G. Moreover there exists a minimal normal subgroup of G which is a q-group and G is a primitive group of type 1. In particular, G is soluble.* Let p be the largest prime dividing the order of G and assume that $p \neq q$. Then p divides the order of A. If A_p denotes a Sylow p-subgroup of A, then A_pB_q is supersoluble by Lemma 2.1 and so B_q normalises A_p. Hence $1 \neq (A_p)^G = (A_p)^A$, contradicting the fact that $\mathrm{Core}_G(A) = 1$. Thus we may suppose that $p = q$. Now for every prime r dividing the order of A, with $r \neq q$, we obtain, arguing as above, that A_r normalises B_q. Hence $1 \neq (B_q)^G = (B_q)^{A_q} \le Q$, where A_q denotes a Sylow q-subgroup of A such that $A_qB_q = Q$ is a Sylow q-subgroup of G. Consequently there exists a minimal normal subgroup N of G which is a q-group. In particular, G is a primitive group of type 1. In particular, G is soluble. Thus $G = AN$, N abelian and $C_G(N) = N$.

(f) *$N = B_q$.* Assume B_q is not contained in N and let $b \in B_q \setminus N$. Since $G/N = (AN/N)(B_qN/N) = AN/N$ is a totally semipermutable product of AN/N and B_qN/N, B_qN/N is a q-group and $F(AN/N) = F(G/N)$ is a q'-group by [6, Lemma A.13.6], we have that for every $xN \in F(G/N)$, $\langle bN\rangle\langle xN\rangle$ is a totally permutable product of $\langle bN\rangle$ and $\langle xN\rangle$. On one hand, since $\langle xN\rangle \in F(G/N)$ it is subnormal in G/N and then in $\langle bN\rangle\langle xN\rangle$. But it is a Sylow subgroup of $\langle bN\rangle\langle xN\rangle$, thus $\langle xN\rangle$ is normal in $\langle bN\rangle\langle xN\rangle$. On the other hand since $\langle bN\rangle\langle xN\rangle$ is a totally permutable product of two supersoluble subgroups it is supersoluble and $\langle bN\rangle$ is normal in $\langle bN\rangle\langle xN\rangle$. Hence $\langle bN\rangle$ centralises $\langle xN\rangle$. In particular, $\langle bN\rangle \le C_{G/N}(F(G/N)) \le F(G/N)$ which is a q'-group, a contradiction. Hence $B_q \le N$ and then $N = B_q(N \cap A) = B_q$.

(g) *$A = A_qF(A)$.* By [6, Lemma A.13.6], we have that $F(A)$ is a q'-group. Denote $T = A_qF(A)B_q$, where A_q denotes a Sylow q-subgroup of A such that A_qB_q is a Sylow q-subgroup of G, and assume it is a proper subgroup of G. Observe that T is a normal subgroup of G. Since T is the totally semipermutable product of $A_qF(A)$ and B_q, we have that $T^{\mathfrak{F}} = (A_qF(A))^{\mathfrak{F}}$. If $T^{\mathfrak{F}} \neq 1$, we would have that there exists a minimal normal subgroup contained in A, a contradiction. Thus T is an $\mathfrak{F}$-group, in particular, $A_qF(A)$ is an $\mathfrak{F}$-group and A_q is normal in $A_qF(A)$. Therefore $A_q \le C_A(F(A)) \le F(A)$. Consequently A is a q'-group. In this case, every Sylow subgroup of A normalises every subgroup of N since the product of A and N is totally semipermutable. Therefore N is cyclic of order q and G is supersoluble, a contradiction. Hence $T = G$. In particular, $A = A_qF(A)(A \cap B_q) = A_qF(A)$.

(h) *The final contradiction.* Let $T = F(A)N$. Then T is the totally semipermutable product of the supersoluble subgroups $F(A)$ and N because $F(A)$ is a q'-group. Let X be a subgroup of N. Then $F(A)X$ is a subgroup of G and $N \cap F(A)X = (F(A) \cap N)X = X$ is a normal subgroup of $F(A)X$. This means that $F(A)$ acts on N as power automorphisms and $C_{F(A)}(N) = C_{F(A)}(X) = 1$. If X is of prime order, then $F(A)$ is isomorphic to a subgroup of $Aut(X)$ and then $F(A)$ is cyclic. In particular, every chief factor of A below $F(A)$ is cyclic and so it is $\mathfrak{F}$-central in A. Since $A/F(A)$ is an $\mathfrak{F}$-group, we conclude that $A \in \mathfrak{F}$, the final contradiction. □

We prove part of Theorem 1.4 in the next lemma.

Lemma 3.2. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F} \subseteq \mathfrak{D}$. Let $G = G_1G_2 \cdots G_n$ be the pairwise totally semipermutable product of the subgroups G_1, G_2, ..., G_n. Then $(G_i)^{\mathfrak{F}}$ is a normal subgroup of G for each $i \in \{1, 2, \ldots, n\}$.*

Proof. Fix $i \in \{1, \ldots, n\}$. We show that $(G_i)^{\mathfrak{F}}$ is a normal subgroup of G. Let q be a prime dividing $|G_j|$, with $i \neq j$. Then there exist $(G_i)_q$, a Sylow q-subgroup of G_i, and $(G_j)_q$, a Sylow q-subgroup of G_j, respectively, such that $(G_i)_q(G_j)_q$ is a Sylow q-subgroup of G_iG_j. Since G_i and G_j are totally semipermutable, $G_i(G_j)_q$ is a subgroup of G_iG_j. In particular, $G_i(G_j)_q$ is a totally semipermutable product of G_i and $(G_j)_q$, and $(G_j)_q \in \mathfrak{F}$. Hence $(G_i(G_j)_q)^{\mathfrak{F}} = (G_i)^{\mathfrak{F}}$ by Theorem 3.1, that is, $(G_j)_q$ normalizes $(G_i)^{\mathfrak{F}}$. Since $(G_j)_q$ is arbitrary, we have that G_j normalizes $(G_i)^{\mathfrak{F}}$. Therefore $(G_i)^{\mathfrak{F}}$ is a normal subgroup of G. □

Theorem 3.3. *Let a group $G = G_1G_2 \cdots G_n$ be the pairwise totally semipermutable product of the subgroups G_1, G_2, ..., G_n, and let $\mathfrak{F}$ be a subgroup-closed saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F} \subseteq \mathfrak{D}$. If $G_i \in \mathfrak{F}$ for all $i \in \{1, 2, \ldots, n\}$, then $G \in \mathfrak{F}$.*

Proof. Since every G_i has a Sylow tower of supersoluble type, by [8, Lemma 2.2(3)], each product G_iG_j with $i \neq j$ has a Sylow tower of supersoluble type. Applying [7, VI, Satz 10.2], we have G has a Sylow tower of supersoluble type. Assume the result is false, and let G be a counterexample with $|G| + |G_1| + \cdots + |G_n|$ minimal. By Theorem 1.3, we may assume that $n > 2$. Since $\mathfrak{F}$ is a saturated formation, G is a primitive soluble group, that is, G has a unique minimal normal subgroup N, $C_G(N) = N$, $G/N \in \mathfrak{F}$ and $G = NM$, where M is a core-free maximal subgroup of G. Since G has a Sylow tower of supersoluble type, we have that N is a Sylow p-subgroup of G and M is a p'-group. By Theorem 1.2, if all G_i are supersoluble, then $G \in \mathfrak{U} \subseteq \mathfrak{F}$. Consequently we may assume without loss of generality that G_1 is not supersoluble. Then $G_1^{\mathfrak{U}}$ is a non-trivial normal subgroup of G by Lemma 3.2. Thus N is contained in $G_1^{\mathfrak{U}}$. By the choice of $(G, G_1, \dots, G_n)$, we have that $Z = G_1 G_3 \cdots G_n \in \mathfrak{F}$. Then $G = G_2 Z$ and there exist a Hall p'-subgroup $(G_2)_{p'}$ of G_2 and a Hall p'-subgroup $Z_{p'}$ of Z such that $(G_2)_{p'} Z_{p'}$ is a Hall p'-subgroup of G. We may suppose that $M = (G_2)_{p'} Z_{p'}$. Since $Z \in \mathfrak{F}$, $Z/O_{p'p}(Z) \simeq Z/N \in f(p)$, where f denotes a local definition of $\mathfrak{F}$. Thus a Hall p'-subgroup of Z is an $f(p)$-group. Analogously, since $G_1 G_2 \in \mathfrak{F}$ by Theorem 1.3, we have that $(G_2)_{p'} \in f(p)$. Observe that NG_2 is a totally semipermutable product of N and G_2. It implies that $(G_2)_{p'}$ normalises every subgroup of N. Applying Lemma 2.2, we have that $(G_2)_{p'}$ centralises $Z_{p'}$. Hence $M \in f(p)$ and then $G \in \mathfrak{F}$, the final contradiction. □

We are in a position to prove Theorem 1.4.

Proof of Theorem 1.4. We argue by induction on $|G|$. Let N be a minimal normal subgroup of G. By Lemma 2.1, G/N is the pairwise totally semipermutable product of the subgroups G_1N/N, G_2N/N, …, G_nN/N. The induction hypothesis yields

$$(G/N)^{\mathfrak{F}} = (G_1N/N)^{\mathfrak{F}} (G_2N/N)^{\mathfrak{F}} \cdots (G_nN/N)^{\mathfrak{F}}.$$

It follows that $G^{\mathfrak{F}} N = (G_1)^{\mathfrak{F}} (G_2)^{\mathfrak{F}} \cdots (G_n)^{\mathfrak{F}} N$. By Theorem 3.3, $(G_i)^{\mathfrak{F}} \neq 1$ for some $i \in \{1, 2, \dots, n\}$. We may assume that $(G_1)^{\mathfrak{F}} \neq 1$. Since $(G_1)^{\mathfrak{F}}$ is a normal subgroup of G by Lemma 3.2, we may assume that N is contained in $(G_1)^{\mathfrak{F}} \leq G^{\mathfrak{F}}$. Consequently, $G^{\mathfrak{F}} = (G_1)^{\mathfrak{F}} (G_2)^{\mathfrak{F}} \cdots (G_n)^{\mathfrak{F}}$, as desired. □

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