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Common fixed point theorems on complete and weak G-complete fuzzy metric spaces

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Abstract

Motivated by Gopal and Vetro, we introduce a symmetric pair of β-admissible mappings and obtain common fixed point theorems for such a pair in complete and weak G-complete fuzzy metric spaces. In particular, we rectify, generalize and improve the common fixed point theorem obtained by Turkoglu and Sangurlu for two fuzzy ψ-contractive mappings. We include non-trivial examples to exhibit the generality and demonstrate our results.

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1. Introduction

Finding an appropriate analogue for metric spaces in fuzzy setting was a long standing problem. In 1975, motivated by the idea of Menger spaces [11], Kramosil and Michalek [10] gave a solution to this problem and introduced fuzzy metric spaces. Grabiec [5], in 1988, defined Cauchy sequences in such spaces. However, while modifying the definition of fuzzy metric, George and Veeramani [3] strengthened Grabiec's definition of Cauchy sequences which is now widely accepted as standard for fuzzy metric spaces.

In fact, Grabiec's original definition of Cauchy sequence (now known as G-Cauchy sequence [7]) was so weak that even a compact fuzzy metric space fails

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to be complete (now known as G-complete [7]) in the Grabiec's sense. Due to this drawback, in [7], Gregori et. al. introduced a new form of completeness. It is called weak G-completeness for fuzzy metric spaces. Weak G-completeness has been further studied in [1] and [2]. In this paper, we establish certain fixed point theorems in weak G-complete fuzzy metric spaces.

It is known that, alike metric spaces, fixed point theory is a rich subfield of fuzzy metric spaces where contractive and contractive-type mappings play important roles for obtaining fixed points theorems. The first fuzzy version of Banach Contraction Principle was established in 1988 by Grabiec for G-complete fuzzy metric spaces [5]. In 2002, Gregori and Sapena introduced fuzzy contractive mappings and obtained several fixed point theorems for complete fuzzy metric spaces [8]. Mihet enlarged this class of contractive mappings and introduced the notion of fuzzy ψ-contractive mappings. This new class of mappings was utilized to establish a new version of fuzzy Banach contraction theorem for complete non-Archimedean fuzzy metric spaces [12] which was further generalized for weak G-complete fuzzy metric spaces [7].

The above class of contractive mappings, introduced by Mihet, has been extensively used to obtain fixed point theorems in fuzzy metric spaces. In 2014, Turkoglu and Sangurlu obtained a common fixed point theorem for a pair of fuzzy ψ-contractive mappings in G-complete fuzzy metric spaces [15].

In this paper, motivated by the work of Gopal and Vetro [4], we introduce a symmetric pair of β-admissible mappings and a pair of β-ψ-fuzzy contractive mappings. These new families are utilized here to establish common fixed point theorems in complete and weak G-complete fuzzy metric spaces, both in the senses of [3] and [10]. In particular, we rectify the fixed point theorem obtained by Turkoglu et. al. [15] and substantially generalize and improve it. Our theory is supported and illustrated by appropriate examples.

2. Preliminaries

In this section, we recall some basic definitions and facts which are referred subsequently.

Definition 2.1 ([14])**.** A mapping $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if (i) $*$ is associative and commutative, (ii) $*$ is continuous, (iii) $a*1 = a$, $\forall\, a \in [0,1]$, and (iv) for $a,b,c,d \in [0,1]$, $a \leq c, b \leq d \implies a*b \leq c*d$.

It is easy to note that, the followings are examples of continuous t-norms:
(i) $a * b = \min(a,b)$, and
(ii) $a * b = ab$
for any $a, b \in [0,1]$.

Definition 2.2 (Kramosil and Michalek [10])**.** Given a nonempty set X, a continuous t-norm $*$ and a mapping $M : X \times X \times [0,\infty) \to [0,1]$, the ordered triple $(X, M, *)$ is called a KM fuzzy metric space if, for all $x, y, z \in X$ and $s, t > 0$, the following conditions hold:

a) $M(x,y,0) = 0$,

b) $M(x,y,t)=1,\ \forall\ t>0 \iff x=y$,
c) $M(x,y,t)=M(y,x,t)$,
d) $M(x,y,t)*M(y,z,s)\leq M(x,z,t+s)$,
e) $M(x,y,.):[0,\infty)\to[0,1]$ is left continuous.

Definition 2.3 (George and Veeramani [3])**.** Given a nonempty set X, a continuous t-norm $*$ and a mapping $M: X\times X\times(0,\infty)\to[0,1]$, the ordered triple $(X,M,*)$ is called a GV fuzzy metric space if, for all $x,y,z\in X$ and $s,t>0$, the following conditions hold:
a) $M(x,y,t)>0$,
b) $M(x,y,t)=1 \iff x=y$,
c) $M(x,y,t)=M(y,x,t)$,
d) $M(x,y,t)*M(y,z,s)\leq M(x,z,t+s)$,
e) $M(x,y,.):(0,\infty)\to[0,1]$ is continuous.

Unless otherwise specified, by a fuzzy metric space we refer to the GV fuzzy metric space.

Definition 2.4 ([9])**.** A (KM) fuzzy metric space $(X,M,*)$ is said to be non-Archimedean if $M(x,y,t)*M(y,z,s)\leq M(x,z,\max\{t,s\})$, for all $x,y,z\in X$ and $s,t>0$.

Lemma 2.5 ([3], [5])**.** *Given a (KM) fuzzy metric space $(X,M,*)$, $M(x,y,\cdot)$ defines a nondecreasing map, $\forall\ x,y\in X$.*

Let $(X,M,*)$ be a (KM) fuzzy metric space. It is well-known ([3], [7]) that, $\{B(x,r,t): x\in X, r\in(0,1), t>0\}$ forms a base for some topology τ_M on X, where $B(x,r,t)=\{y\in X: M(x,y,t)>1-r\}$, $\forall\ x\in X, r\in(0,1), t>0$. The topological behaviour of $(X,M,*)$ is defined with respect to the topology τ_M. In particular, given two (KM) fuzzy metric spaces $(X,M,*)$ and $(Y,N,\star)$, a mapping $f: X\to Y$ is called continuous if f is continuous as a mapping from (X,τ_M) to (Y,τ_N).

Similarly, sequential convergence is defined as follows: A sequence (x_n) in a (KM) fuzzy metric space $(X,M,*)$ is said to be convergent to some $x\in X$ (*resp.* clusters), if it does so in (X,τ_M).

It is easy to note that, if f is a continuous mapping from a (KM) fuzzy metric space $(X,M,*)$ to a (KM) fuzzy metric space $(Y,N,\star)$, and (x_n) is a sequence in X converging to $x\in X$, then $f(x_n)\to f(x)$ as $n\to\infty$.

The following is an easy consequence that has been shown in [3] for GV fuzzy metric spaces. The case for KM fuzzy metric spaces is similar as stated next.

Theorem 2.6. A sequence (x_n) in a (KM) fuzzy metric space $(X,M,*)$ converges to $x\in X$ if and only if $\lim\limits_{n\to\infty} M(x_n,x,t)=1,\ \forall\ t>0$.

Definition 2.7 ([3], [13])**.** Let $(X,M,*)$ be a (KM) fuzzy metric space. A sequence (x_n) in X is called Cauchy if for $\epsilon\in(0,1), t>0$, there exists $k\in\mathbb{N}$ such that $M(x_m,x_n,t)>1-\epsilon,\ \forall\ m,n\geq k$. Clearly every convergent sequence in $(X,M,*)$ is Cauchy. $(X,M,*)$ is complete if every Cauchy sequence in it converges.

Definition 2.8 ([5], [7]). Let $(X, M, *)$ be a (KM) fuzzy metric space. A sequence (x_n) in X is called G-Cauchy if $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1, \ \forall\ t > 0$. If, in $(X, M, *)$, every G-Cauchy sequence converges, then $(X, M, *)$ is said to be G-complete.

Though G-completeness necessarily imply completeness in (KM) fuzzy metric spaces, a compact (KM) fuzzy metric space may not be G-complete. To overcome this drawback, the following weaker version of completeness has been introduced in [7] .

A (KM) fuzzy metric space in which every G-Cauchy sequence clusters is called a weak G-complete (KM) fuzzy metric space.

We finish this section by recalling the definitions of β-ψ-fuzzy contractive mapping and β-admissible mapping introduced by Gopal and Vetro in [4].

Following [12], we denote by Ψ the family of all mappings $\psi : [0, 1] \to [0, 1]$ such that,

(i) ψ is non-decreasing and continuous,

(ii) $\psi(t) > t, \ \forall\ t \in (0, 1)$.

It is easy to check that $\psi(1) = 1$ and $\lim_{n\to\infty} \psi^n(r) = 1, \ \forall\ \psi \in \Psi, r \in (0, 1)$ (e.g. consult [16]).

Definition 2.9. Let $(X, M, *)$ be a (KM) fuzzy metric space and f be a self-mapping on X. For some $\psi \in \Psi$, and a mapping $\beta : X^2 \times (0, \infty) \to (0, \infty)$, f is called a β-ψ-fuzzy contractive mapping if $\forall\ x, y \in X$ with $x \neq y$ and $t > 0$,

$$M(x, y, t) > 0 \implies \beta(x, y, t)M(fx, fy, t) \geq \psi(M(x, y, t)).$$

Definition 2.10. Let $(X, M, *)$ be a (KM) fuzzy metric space and f be a self-mapping on X. For a mapping $\beta : X^2 \times (0, \infty) \to (0, \infty)$, f is called β-admissible if $\forall\ x, y \in X, t > 0$,

$$\beta(x, y, t) \leq 1 \implies \beta(fx, fy, t) \leq 1.$$

3. Main Results

We begin this section by introducing a pair of β-ψ-fuzzy contractive mappings and a symmetric pair of β-admissible mappings that extend respectively the class of β-ψ-fuzzy contractive mappings and β-admissible mappings in a (KM) fuzzy metric space.

Definition 3.1 ([12], [16]). Let $(X, M, *)$ be a (KM) fuzzy metric space and $\psi \in \Psi$.

a) A mapping $f : X \to X$ is called fuzzy ψ-contractive if $\forall\ x, y \in X, t > 0$,

$$M(x, y, t) > 0 \implies M(fx, fy, t) \geq \psi(M(x, y, t)).$$

b) Given a pair of mappings $f, g : X \to X$, (f, g) is called a pair of fuzzy ψ-contractive mappings if $\forall\ x, y \in X, t > 0$,

$$M(x, y, t) > 0 \implies M(fx, gy, t) \geq \psi(\min\{M(x, y, t), M(x, fx, t), M(y, gy, t)\}).$$

Definition 3.2. Let $(X, M, *)$ be a (KM) fuzzy metric space and f, g be self-mappings on X.

a) For a mapping $\beta : X^2 \times (0, \infty) \to (0, \infty)$, the pair (f, g) is called a symmetric pair of β-admissible mappings if $\forall\ x, y \in X, t > 0$,

$\beta(x, y, t) \leq 1 \implies \max\{\beta(fx, gy, t), \beta(gy, fx, t), \beta(gx, fy, t), \beta(fy, gx, t)\} \leq 1.$

b) For some $\psi \in \Psi$, and a mapping $\beta : X^2 \times (0, \infty) \to (0, \infty)$, the pair (f, g) is called a pair of β-ψ-fuzzy contractive mappings if $\forall\ x, y \in X, t > 0$, $M(x, y, t) > 0 \implies$

$$\beta(x, y, t)M(fx, gy, t) \geq \psi(\min\{M(x, y, t), M(x, fx, t), M(y, gy, t)\}).$$

We note that if $\beta(x, y, t) \equiv 1$, then a pair of β-ψ-fuzzy contractive mappings is a pair of fuzzy ψ-contractive mappings.

The subsequent discussion explores common fixed point theorems for symmetric pairs of β-admissible mappings on weak G-complete (KM) fuzzy metric spaces. Hereafter, $(X, M, *)$ denotes a weak G-complete (KM) fuzzy metric space, with f and g as self-mappings on X, ψ as a member of Ψ, and β as a mapping from $X^2 \times (0, \infty)$ to $(0, \infty)$ such that (f, g) forms a symmetric pair of β-admissible mappings, unless specified otherwise.

Theorem 3.3. Let f, g be continuous mappings such that

(i) $M(x, fx, t)$, $M(x, gx, t) > 0$, $\forall\ x \in X, t > 0$,

(ii) (f, g) is a pair of β-ψ-fuzzy contractive mappings,

(iii) for some $x_0 \in X$, $\beta(x_0, fx_0, t) \leq 1$ and $M(x_0, fx_0, t) > 0$, $\forall\ t > 0$.

Then f and g have a common fixed point in X.

Moreover, if $\beta(x, y, t) \leq 1,\ \forall\ x, y \in X, t > 0$ and for $x, y\ (x \neq y) \in X$, $M(x, y, t) > 0,\ \forall\ t > 0$, then the fixed point is unique.

Proof. Define a sequence (x_n) as follows: $x_1 = fx_0, x_2 = gx_1, \cdots, x_{2n+1} = fx_{2n}, x_{2n+2} = gx_{2n+1}, \cdots$.

We have $\beta(x_0, x_1, t) = \beta(x_0, fx_0, t) \leq 1,\ \forall\ t > 0$. Since $\beta(x_{n-1}, x_n, t) \leq 1 \implies \beta(x_n, x_{n+1}, t) \leq \max\{\beta(fx_{n-1}, gx_n, t), \beta(gx_{n-1}, fx_n, t)\} \leq 1,\ \forall\ n \in \mathbb{N}, t > 0$, so by applying induction, we obtain $\beta(x_n, x_{n+1}, t) \leq 1,\ \forall\ n = 0, 1, 2, \cdots$ and $t > 0$.

Consequently, $\beta(x_{n+1}, x_n, t) \leq 1,\ \forall\ n \in \mathbb{N}, t > 0$.

Let n be a positive integer.

If n is odd, then for chosen $t > 0$,

$M(x_{n-1}, x_n, t) > 0 \implies M(x_n, x_{n+1}, t) = M(fx_{n-1}, gx_n, t)$
$\geq \beta(x_{n-1}, x_n, t)M(fx_{n-1}, gx_n, t)$
$\geq \psi(\min\{M(x_{n-1}, x_n, t), M(x_{n-1}, fx_{n-1}, t), M(x_n, gx_n, t)\})$
$= \psi(M(x_{n-1}, x_n, t)) > 0.$

Again if n is even, then for chosen $t > 0$,

$M(x_{n-1}, x_n, t) > 0 \implies M(x_n, x_{n+1}, t) = M(gx_{n-1}, fx_n, t)$
$= M(fx_n, gx_{n-1}, t)$
$\geq \beta(x_n, x_{n-1}, t)M(fx_n, gx_{n-1}, t)$
$\geq \psi(\min\{M(x_n, x_{n-1}, t), M(x_n, fx_n, t), M(x_{n-1}, gx_{n-1}, t)\})$
$= \psi(M(x_{n-1}, x_n, t)) > 0.$

Thus, by using induction, we have $M(x_n, x_{n+1}, t) \geq \psi(M(x_{n-1}, x_n, t))$, $\forall\, n \in \mathbb{N}, t > 0$ and hence, $M(x_n, x_{n+1}, t) \geq \psi^n(M(x_0, x_1, t))$, $\forall\, n \in \mathbb{N}, t > 0$.

Thus $\lim\limits_{n\to\infty} M(x_n, x_{n+1}, t) = 1,\ \forall\, t > 0$. Hence (x_n) is a G-Cauchy sequence in X.

Since $(X, M, *)$ is weak G-complete, (x_n) has a cluster point x in X.

So there exists a subsequence (x_{r_n}) of (x_n) such that (x_{r_n}) converges to x where r_n's are either all even or all odd.

Without loss of generality, suppose all the r_n's are even. Then $\forall\, n \in \mathbb{N}, t > 0$,
$M(x, fx, t) \geq M\left(x, x_{r_n}, \frac{t}{3}\right) * M\left(x_{r_n}, x_{r_n+1}, \frac{t}{3}\right) * M\left(x_{r_n+1}, fx, \frac{t}{3}\right)$
$= M\left(x, x_{r_n}, \frac{t}{3}\right) * M\left(x_{r_n}, x_{r_n+1}, \frac{t}{3}\right) * M\left(fx_{r_n}, fx, \frac{t}{3}\right)$.

Taking limit as $n \to \infty$, we have $M(x, fx, t) = 1,\ \forall\, t > 0$, since f is continuous and (x_n) is G-Cauchy. Hence $fx = x$.

Again, since $\forall\, t > 0$, $M(x_{r_n-1}, x, t) \geq M\left(x_{r_n-1}, x_{r_n}, \frac{t}{2}\right) * M\left(x_{r_n}, x, \frac{t}{2}\right) \to 1 * 1 = 1$ as $n \to \infty$, so $\forall\, t > 0$, $M(x_{r_n-1}, x, t) \to 1$ as $n \to \infty$, and consequently, $\forall\, t > 0$, $M(x, gx, t) \geq M\left(x, x_{r_n}, \frac{t}{2}\right) * M\left(x_{r_n}, gx, \frac{t}{2}\right) = M\left(x, x_{r_n}, \frac{t}{2}\right) * M\left(gx_{r_n-1}, gx, \frac{t}{2}\right) \to 1 * 1 = 1$ as $n \to \infty$, since g is continuous.

Thus $M(x, gx, t) = 1,\ \forall\, t > 0$, and hence $gx = x$.

Again, if all the r_n's are odd, it can be similarly shown that $fx = gx = x$.

Thus, in general, x is a common fixed point of both f and g.

Let us now assume, in addition to condition (i) and (ii), that $\beta(x, y, t) \leq 1,\ \forall\, x, y \in X, t > 0$ and for $x, y\ (x \neq y) \in X$, $M(x, y, t) > 0$, $\forall\, t > 0$.

If possible, let x, y be two common fixed points of f and g such that $x \neq y$. Then there exists $t_0 > 0$ such that $0 < M(x, y, t_0) < 1$.

Now $M(x, y, t_0)$
$\geq \beta(x, y, t_0) M(fx, gy, t_0)$
$\geq \psi(\min\{M(x, y, t_0), M(x, fx, t_0), M(y, gy, t_0)\})$
$= \psi(M(x, y, t_0))$
$> M(x, y, t_0)$, a contradiction.

Hence the common fixed point is unique. □

By putting $\beta \equiv 1$ in Theorem 3.3, we obtain the following:

Corollary 3.4. Let f, g be continuous mappings with $M(x, fx, t), M(x, gx, t) > 0,\ \forall\, x \in X, t > 0$. Suppose for some $x_0 \in X$, $M(x_0, fx_0, t) > 0,\ \forall\, t > 0$ and $M(fx, gy, t) \geq \psi(\min\{M(x, y, t), M(x, fx, t), M(y, gy, t)\}),\ \forall\, x, y \in X, t > 0$. Then f and g have a common fixed point in X.

Moreover, if for $x, y\ (x \neq y) \in X$, $M(x, y, t) > 0,\ \forall\, t > 0$, then the fixed point is unique.

The last corollary shows that in the following common fixed point result obtained by Turkoglu and Sangurlu, we do not require to specify the continuous t-norm $*$ or assume f, g to be fuzzy ψ-contractive mappings.

Theorem 3.5 ([15], *Common fixed point theorem by Turkoglu and Sangurlu*)**.** *Let $(X, M, *)$ be a G-complete KM fuzzy metric space such that $a*b = \min\{a, b\}$. Let there be a $x_0 \in X$ such that $M(x_0, fx_0, t) > 0,\ \forall\, t > 0$. If f and g are* *continuous self-mappings on* X *satisfying* $M(x, fx, t), M(x, gx, t) > 0$, $\forall\ x \in X, t > 0$ *such that*

(i) f, g *are fuzzy* ψ*-contractive mappings,*

(ii) $M(fx, gy, t) \geq \psi(\min\{M(x, y, t), M(x, fx, t), M(y, gy, t)\})$, $\forall\ x, y \in X$, $t > 0$,

then f *and* g *have a common fixed point in* X.

Note 3.6. It should be noted, in [15], Turkoglu et. al. included the following additional condition in the hypothesis:

(x_n) *is a sequence in* X *such that* $x_1 = fx_0, x_2 = gx_1, \cdots, x_{2n+1} = f(x_{2n})$, $x_{2n+2} = g(x_{2n+1}), \cdots$.

However, noting that such sequence always exists, we remove it from the statement of Theorem 3.5.

Moreover, we have included the additional condition:

$M(x, fx, t), M(x, gx, t) > 0$, $\forall\ x \in X, t > 0$

in the hypothesis of Theorem 3.5 noting that it is essential even for the proof provided by Turkoglu et. al. [15].

Note 3.7. In [15], Turkoglu et. al. concluded that under the hypothesis of Theorem 3.5, f and g have a *unique* common fixed point. However, this is not true as is exhibited next.

Consider the KM fuzzy metric space $(X, M, *)$, where $X = [0, 1]$, $a * b = \min\{a, b\}$, $\forall\ a, b \in [0, 1]$, and for $x, y \in X$,

$$M(x, y, t) = \begin{cases} 0 & \text{if } x \neq y, t \geq 0 \\ 0 & \text{if } x = y, t = 0 \\ 1 & \text{if } x = y, t > 0 \end{cases}$$

Then X is a G-complete KM fuzzy metric space.

Then by setting $\psi(t) = \sqrt{t}$, $\forall\ t \in [0, 1]$, for $f, g : X \to X$ given by $f(x) = g(x) = x$, $\forall\ x \in X$ the hypothesis of Theorem 3.5 gets satisfied. However every point of X is a common fixed point of f and g and hence the common fixed point is not unique.

In what follows, we exhibit the applicability of our theorems, namely, Theorem 3.3 over Theorem 3.5 (due to [15]) in the following aspects:

(a) Theorem 3.3 is applicable even for a weak G-complete (KM) fuzzy metric space $(X, M, *)$ which is not G-complete;

(b) Theorem 3.3 is applicable also for $\beta \not\equiv 1$;

(c) Theorem 3.3 is applicable even when $a * b$ is not defined as $\min\{a, b\}$.

Example 3.8. Consider the fuzzy metric space $(X, M, \cdot)$, where $X = \left\{\frac{1}{2^n} : n \geq 2\right\} \bigcup \left[\frac{1}{2}, 1\right]$, $M(x, y, t) = \frac{\min\{x,y\}}{\max\{x,y\}}$, $\forall\ x, y \in X, t > 0$ and $\cdot$ is the usual product of reals. It is known that $(X, M, \cdot)$ is a weak G-complete fuzzy metric space which is not G-complete [7]. Further, τ_M defines the usual topology of $\mathbb{R}$ restricted to the set X [6].

Let f, g be two self-mappings on X such that $fx = x$, $gx = 1$, $\forall\ x \in X$. Then f, g are continuous on X.

Now by setting $\beta : X^2 \times (0, \infty) \to (0, \infty)$ as

$$\beta(x, y, t) = \begin{cases} \frac{1}{x} & \text{if } x \le y \\ \frac{1+x}{2x} & \text{if } y < x < 1 \\ 2 & \text{if } y < x = 1 \end{cases}$$

for all $x, y \in X, t > 0$, we see that (f, g) defines a symmetric pair of β-admissible mappings and $\forall\ t > 0$, $\beta(x_0, fx_0, t) \le 1, M(x_0, fx_0, t) > 0$ hold for $x_0 = 1$.

Let $\psi : [0, 1] \to [0, 1]$ be defined by $\psi(x) = \frac{1+x}{2}$, $\forall\ x \in [0, 1]$. Clearly, $\psi \in \Psi$.

We now show that, (f, g) is a pair of β-ψ-fuzzy contractive mappings.

Choose $x, y \in X$ and $t > 0$.

Case I: $x \le y$. Then $\beta(x, y, t)M(fx, gy, t) = \frac{1}{x} \times M(x, 1, t) = 1$
$\ge \psi(\min\{M(x, y, t), M(x, fx, t), M(y, gy, t)\})$.

Case II: $y < x < 1$. Then $\beta(x, y, t)M(fx, gy, t) = \frac{1+x}{2}$ and
$\psi(\min\{M(x, y, t), M(x, fx, t), M(y, gy, t)\}) = \psi(\min\{\frac{y}{x}, 1, y\}) = \psi(y) = \frac{1+y}{2}$.

Since $x > y$, we obtain
$\beta(x, y, t)M(fx, gy, t) \ge \psi(\min\{M(x, y, t), M(x, fx, t), M(y, gy, t)\})$.

Case III: $y < x = 1$. Then $\beta(x, y, t)M(fx, gy, t) = 2 \times M(x, 1, t) \ge \psi(\min\{M(x, y, t), M(x, fx, t), M(y, gy, t)\})$.

Thus, in view of Theorem 3.3, f, g have a common fixed point.

In what follows we show that, in Theorem 3.3, the last condition is essential to ensure the uniqueness of the common fixed point.

Example 3.9. Consider the fuzzy metric space $(X, M, \cdot)$ of Example 3.8. Let f, g be self-mappings on X such that

$$fx = gx = \begin{cases} \frac{1}{4} & \text{if } x = \frac{1}{4} \\ 1 & \text{otherwise} \end{cases}.$$

Clearly, f, g are continuous on X.

If $\beta : X^2 \times (0, \infty) \to (0, \infty)$ is defined by

$$\beta(x, y, t) = \begin{cases} 1 & \text{if } x = y = 1 \\ 4 & \text{otherwise} \end{cases}$$

then (f, g) is a pair of β-ψ-fuzzy contractive mappings, $\forall\ \psi \in \Psi$.

It is easy to see that conditions (i) and (ii) of Theorem 3.3 hold immediately.

So, in view of Theorem 3.3, f, g have a common fixed point.

However the fixed point is not unique. Indeed $x = 1$ and $x = \frac{1}{4}$ are two such points.

Theorem 3.10. Let f, g be continuous mappings such that
(i) $M(x, fx, t)$, $M(x, gx, t) > 0$, $\forall\ x \in X, t > 0$,
(ii) for some $\psi_1, \psi_2, \psi_3 \in \Psi$, $M(x, y, t) > 0 \implies$
$\beta(x, y, t)M(fx, gy, t) \geq \psi_1(M(x, y, t)) + \psi_2(M(x, fx, t)) + \psi_3(M(y, gy, t))$,
$\forall\ x, y \in X, t > 0$,
(iii) for some $x_0 \in X$, $\beta(x_0, fx_0, t) \leq 1$ and $M(x_0, fx_0, t) > 0$, $\forall\ t > 0$.
Then f and g have a common fixed point in X.

Proof. Set $\psi = \min\{\psi_1, \psi_2, \psi_3\}$. Then it is easy to see that $\psi \in \Psi$.
Further $\forall\ x, y \in X, t > 0$,
$M(x, y, t) > 0 \implies \beta(x, y, t)M(fx, gy, t)$
$\geq \psi_1(M(x, y, t)) + \psi_2(M(x, fx, t)) + \psi_3(M(y, gy, t))$
$\geq \psi(\min\{M(x, y, t), M(x, fx, t), M(y, gy, t)\})$.
Thus the conclusion follows from Theorem 3.3. □

Corollary 3.11. Let f, g be continuous mappings such that
(i) $M(x, fx, t)$, $M(x, gx, t) > 0$, $\forall\ x \in X, t > 0$,
(ii) for some $a, b, c > 1$, $M(x, y, t) > 0 \implies$

$$\beta(x, y, t)M(fx, gy, t) \geq aM(x, y, t) + bM(x, fx, t) + cM(y, gy, t),$$

$\forall\ x, y \in X, t > 0$,
(iii) for some $x_0 \in X$, $\beta(x_0, fx_0, t) \leq 1$ and $M(x_0, fx_0, t) > 0$, $\forall\ t > 0$.
Then f and g have a common fixed point in X.

Proof. Immediate from Theorem 3.10 by setting

$$\psi_1(x) = \min\{ax, \sqrt{x}\},$$
$$\psi_2(x) = \min\{bx, \sqrt{x}\},$$
$$\psi_3(x) = \min\{cx, \sqrt{x}\},$$

$\forall\ x \in [0, 1]$. □

In the next theorem, we omit the continuity hypothesis of f, g in Theorem 3.3 for fuzzy metric spaces defined by George and Veeramani [3].

Theorem 3.12. Let $(X, M, *)$ be a non-Archimedean weak G-complete fuzzy metric space and (f, g), a pair of β-ψ-fuzzy contractive mappings such that
(i) for some $x_0 \in X$, $\beta(x_0, fx_0, t) \leq 1\ \forall\ t > 0$,
(ii) for each sequence (x_n) in X with $\beta(x_n, x_{n+1}, t) \leq 1,\ \forall\ n \in \mathbb{N}, t > 0$ and a subsequence (x_{r_n}) of (x_n) with $\lim_{n\to\infty} x_{r_n} = x$, we have $\beta(x_{r_n}, x, t) \leq 1,\ \forall\ n \in \mathbb{N}, t > 0$.
Then f and g have a common fixed point in X.

Proof. Proceeding as in Theorem 3.3, we obtain the G-Cauchy sequence (x_n) in X given by $x_1 = fx_0, x_2 = gx_1, \cdots, x_{2n+1} = fx_{2n}, x_{2n+2} = gx_{2n+1}, \cdots$ such that $\beta(x_n, x_{n+1}, t) \leq 1,\ \forall\ n = 0, 1, 2, \cdots$ and $t > 0$.

Since $(X, M, *)$ is weak G-complete, there exists a subsequence (x_{r_n}) of (x_n) such that (x_{r_n}) converges to some $x \in X$ where r_n's are either all even or all odd.

Without loss of generality, suppose all the r_n's are even.

If possible, let $gx \neq x$. Then $0 < M(x, gx, t_0) < 1$, for some $t_0 > 0$.

Since X is non-Archimedean, $\forall\ n \in \mathbb{N}$ we have

$M(x, gx, t_0) \geq M(x, x_{r_n+1}, t_0) * M(x_{r_n+1}, gx, t_0)$
$= M(x, x_{r_n+1}, t_0) * M(fx_{r_n}, gx, t_0)$
$\geq M(x, x_{r_n+1}, t_0) * (\beta(x_{r_n}, x, t_0) M(fx_{r_n}, gx, t_0))$(by (iii))
$\geq M(x, x_{r_n+1}, t_0) * \psi(\min\{M(x_{r_n}, x, t_0), M(x_{r_n}, fx_{r_n}, t_0), M(x, gx, t_0)\})$
$\geq M(x, x_{r_n}, t_0) * M(x_{r_n}, x_{r_n+1}, t_0) * \psi(\min\{M(x_{r_n}, x, t_0), M(x_{r_n}, x_{r_n+1}, t_0), M(x, gx, t_0)\})$
$\to 1 * 1 * \psi(M(x, gx, t_0)) = \psi(M(x, gx, t_0))$ as $n \to \infty$, since $*$ is continuous and (x_n) is G-Cauchy.

Thus $M(x, gx, t_0) \geq \psi(M(x, gx, t_0)) > M(x, gx, t_0)$, a contradiction.

Consequently $gx = x$.

Again if $fx \neq x$, $0 < M(x, fx, t_1) < 1$, for some $t_1 > 0$.

Since X is non-Archimedean, $\forall\ n \in \mathbb{N}$ we have

$M(x, fx, t_1) \geq M(x, x_{r_n+2}, t_1) * M(x_{r_n+2}, fx, t_1)$
$= M(x, x_{r_n+2}, t_1) * M(fx, gx_{r_n+1}, t_1)$
$\geq M(x, x_{r_n+1}, t_1) * M(x_{r_n+1}, x_{r_n+2}, t_1) * M(fx, gx_{r_n+1}, t_1)$.

Since $\beta(x_{r_n}, x, t_1) \leq 1,\ \forall\ n \in \mathbb{N}$ we have $\beta(x, x_{r_n+1}, t_1) \leq 1,\ \forall\ n \in \mathbb{N}$ by (i).

Thus $\forall\ n \in \mathbb{N}$,

$M(x, fx, t_1) \geq M(x, x_{r_n+1}, t_1) * M(x_{r_n+1}, x_{r_n+2}, t_1) * (\beta(x, x_{r_n+1}, t_1) M(fx, gx_{r_n+1}, t_1))$
$\geq M(x, x_{r_n+1}, t_1) * M(x_{r_n+1}, x_{r_n+2}, t_1) * \psi(\min\{M(x, x_{r_n+1}, t_1), M(x, fx, t_1), M(x_{r_n+1}, gx_{r_n+1}, t_1)\})$
$\geq M(x, x_{r_n+1}, t_1) * M(x_{r_n+1}, x_{r_n+2}, t_1) * \psi(\min\{M(x, x_{r_n}, t_1) * M(x_{r_n}, x_{r_n+1}, t_1), M(x, fx, t_1), M(x_{r_n+1}, gx_{r_n+1}, t_1)\})$
$\geq M(x, x_{r_n+1}, t_1) * M(x_{r_n+1}, x_{r_n+2}, t_1) * \psi(\min\{M(x, x_{r_n}, t_1) * M(x_{r_n}, x_{r_n+1}, t_1), M(x, fx, t_1), M(x_{r_n+1}, x_{r_n+2}, t_1)\}) \to \psi(M(x, fx, t_1))$ as $n \to \infty$, since $*$ is continuous and (x_n) is G-Cauchy.

Thus $M(x, fx, t_1) \geq \psi(M(x, fx, t_1)) > M(x, fx, t_1)$, a contradiction.

Consequently $fx = x$.

Thus f, g have a common fixed point x.

Again, if all the r_n's are odd, it can be similarly shown that $fx = gx = x$. □

Example 3.13. Consider the fuzzy metric space $(X, M, \cdot)$ of Example 3.8. Let f, g be self-mappings on X such that

$$fx = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ \frac{1}{\sqrt{2}} & \text{otherwise} \end{cases}$$

and

$$gx = \begin{cases} 1 & \text{if } x = 1 \\ \frac{1}{\sqrt{2}} & \text{if } x = \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

If $\beta : X^2 \times (0,\infty) \to (0,\infty)$ is defined by

$$\beta(x,y,t) = \begin{cases} 1 & \text{if } x = y = 1 \\ 4 & \text{otherwise} \end{cases}$$

then (f,g) is a pair of β-ψ-fuzzy contractive mappings, $\forall\ \psi \in \Psi$.

It is easy to see that conditions (i)−(ii) of Theorem 3.12 hold immediately.

So, in view of Theorem 3.12, f, g have a common fixed point.

However the fixed point is not unique. Indeed $x = 1$ and $x = \frac{1}{\sqrt{2}}$ are two such points.

By putting $\beta \equiv 1$ in Theorem 3.12 and proceeding as in Theorem 3.3 for uniqueness of fixed point, we see that for a GV fuzzy metric space, non-Archimedeaness replaces the continuity of f, g in Corollary 3.4:

Corollary 3.14. Let $(X, M, *)$ be a non-Archimedean weak G-complete fuzzy metric space. Suppose

$$M(fx, gy, t) \geq \psi(\min\{M(x,y,t), M(x,fx,t), M(y,gy,t)\}),$$

$\forall\ x, y \in X, t > 0$. Then f and g have a unique common fixed point in X.

We will later see that the above conclusion of the last corollary holds, in fact, in a complete GV fuzzy metric space.

In what follows, we show that the additional hypothesis:

for each sequence (x_n) in X satisfying $\beta(x_n, x_{n+1}, t) \leq 1,\ \forall\ n \in \mathbb{N},\ t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\beta(x_m, x_n, t) \leq 1,\ \forall\ m, n \in \mathbb{N}$ with $m \geq n > k_0$ and $t > 0$

enables the conclusion of Theorem 3.12 to work in a complete GV fuzzy metric space.

Theorem 3.15. Let $(X, M, *)$ be a non-Archimedean complete fuzzy metric space and (f,g), a pair of β-ψ-fuzzy contractive mappings such that

(i) for some $x_0 \in X$, $\beta(x_0, fx_0, t) \leq 1$, $\forall\ t > 0$,

(ii) for each sequence (x_n) in X with $\beta(x_n, x_{n+1}, t) \leq 1,\ \forall\ n \in \mathbb{N}, t > 0$ and $\lim_{n\to\infty} x_n = x$, we have $\beta(x_n, x, t) \leq 1,\ \forall\ n \in \mathbb{N}, t > 0$,

(iii) for each sequence (x_n) in X satisfying $\beta(x_n, x_{n+1}, t) \leq 1$, $\forall\, n \in \mathbb{N},\ t > 0$, there exists $k' \in \mathbb{N}$ such that $\beta(x_m, x_n, t) \leq 1,\ \forall\ m, n \in \mathbb{N}$ with $m \geq n > k'$ and $t > 0$.

Then f and g have a common fixed point in X.

Proof. Consider the G-Cauchy sequence (x_n) obtained in Theorem 3.12. Then by condition (iv), there exists $k_1 \in \mathbb{N}$ such that $\beta(x_m, x_n, t) \leq 1,\ \forall\ m, n \in \mathbb{N}$ with $m \geq n > k_1$ and $t > 0$.

We show that (x_n) is a Cauchy sequence. If not, there exist $\epsilon \in (0,1), t > 0$ and $k_0 \in \mathbb{N}$ with $k_0 > k_1$ such that for $k \in \mathbb{N}$ with $k \geq k_0$, we have $m'(k), n(k) \in \mathbb{N}$ with $m'(k) > n(k) \geq k$ satisfying $M(x_{m'(k)}, x_{n(k)}, t) \leq 1 - \epsilon$.

For each $k \geq k_0$, we set $m(k)$ to be the smallest positive integer exceeding $n(k)$ such that $M(x_{m(k)}, x_{n(k)}, t) \leq 1 - \epsilon$ and $M(x_{m(k)-1}, x_{n(k)}, t) > 1 - \epsilon$.

Therefore $\forall\ k \geq k_0$,
$1-\epsilon \geq M(x_{m(k)}, x_{n(k)}, t)$
$\geq M(x_{m(k)-1}, x_{n(k)}, t) * M(x_{m(k)-1}, x_{m(k)}, t)$
$\geq (1-\epsilon) * M(x_{m(k)-1}, x_{m(k)}, t)$.

Taking limit as $k \to \infty$, we obtain $\lim\limits_{k\to\infty} M(x_{m(k)}, x_{n(k)}, t) = 1-\epsilon$.

Now for each $k \in \mathbb{N}$ with $k \geq k_0$, we set
$u(k) = M(x_{m(k)}, x_{m(k)+2}, t) * M(x_{m(k)+2}, x_{n(k)+2}, t) * M(x_{n(k)}, x_{n(k)+2}, t)$,
if $m(k) =$ odd, $n(k) =$ even
$= M(x_{m(k)}, x_{m(k)+2}, t) * M(x_{m(k)+2}, x_{n(k)+1}, t) * M(x_{n(k)}, x_{n(k)+1}, t)$,
if $m(k) =$ odd, $n(k) =$ odd
$= M(x_{m(k)}, x_{m(k)+1}, t) * M(x_{m(k)+1}, x_{n(k)+2}, t) * M(x_{n(k)}, x_{n(k)+2}, t)$,
if $m(k) =$ even, $n(k) =$ even
$= M(x_{m(k)}, x_{m(k)+1}, t) * M(x_{m(k)+1}, x_{n(k)+1}, t) * M(x_{n(k)}, x_{n(k)+1}, t)$,
if $m(k) =$ even, $n(k) =$ odd
and
$v(k) = M(x_{m(k)}, x_{m(k)+2}, t) * \beta(x_{m(k)+1}, x_{n(k)+1}, t) M(x_{m(k)+2}, x_{n(k)+2}, t) * M(x_{n(k)}, x_{n(k)+2}, t)$, if $m(k) =$ odd, $n(k) =$ even
$= M(x_{m(k)}, x_{m(k)+2}, t) * \beta(x_{m(k)+1}, x_{n(k)}, t) M(x_{m(k)+2}, x_{n(k)+1}, t) * M(x_{n(k)}, x_{n(k)+1}, t)$, if $m(k) =$ odd, $n(k) =$ odd
$= M(x_{m(k)}, x_{m(k)+1}, t) * \beta(x_{m(k)}, x_{n(k)+1}, t) M(x_{m(k)+1}, x_{n(k)+2}, t) * M(x_{n(k)}, x_{n(k)+2}, t)$, if $m(k) =$ even, $n(k) =$ even
$= M(x_{m(k)}, x_{m(k)+1}, t) * \beta(x_{m(k)}, x_{n(k)}, t) M(x_{m(k)+1}, x_{n(k)+1}, t) * M(x_{n(k)}, x_{n(k)+1}, t)$, if $m(k) =$ even, $n(k) =$ odd

Since $\beta(x_m, x_n, t) \leq 1,\ \forall\ m, n \in \mathbb{N}$ with $m \geq n > k_0$, we have $\forall\ k \geq k_0$,
$1-\epsilon \geq M(x_{m(k)}, x_{n(k)}, t) \geq u(k) \geq v(k) \cdots (*)$.

Now for each $k \in \mathbb{N}$ with $k \geq k_0$, we also set
$w(k) = \psi(\min\{M(x_{m(k)+1}, x_{n(k)+1}, t), M(x_{m(k)+1}, x_{m(k)+2}, t), M(x_{n(k)+1}, x_{n(k)+2}, t)\})$,
if $m(k) =$ odd, $n(k) =$ even
$= \psi(\min\{M(x_{m(k)+1}, x_{n(k)}, t), M(x_{m(k)+1}, x_{m(k)+2}, t), M(x_{n(k)}, x_{n(k)+1}, t)\})$,
if $m(k) =$ odd, $n(k) =$ odd
$= \psi(\min\{M(x_{m(k)}, x_{n(k)+1}, t), M(x_{m(k)}, x_{m(k)+1}, t), M(x_{n(k)+1}, x_{n(k)+2}, t)\})$,
if $m(k) =$ even, $n(k) =$ even
$= \psi(\min\{M(x_{m(k)}, x_{n(k)}, t), M(x_{m(k)}, x_{m(k)+1}, t), M(x_{n(k)}, x_{n(k)+1}, t)\})$,
if $m(k) =$ even, $n(k) =$ odd
and
$w'(k) = \psi(\min\{M(x_{m(k)}, x_{m(k)+1}, t) * M(x_{m(k)}, x_{n(k)}, t) * M(x_{n(k)}, x_{n(k)+1}, t), M(x_{m(k)+1}, x_{m(k)+2}, t), M(x_{n(k)+1}, x_{n(k)+2}, t)\})$, if $m(k) =$ odd, $n(k) =$ even
$= \psi(\min\{M(x_{m(k)}, x_{m(k)+1}, t) * M(x_{m(k)}, x_{n(k)}, t), M(x_{m(k)+1}, x_{m(k)+2}, t), M(x_{n(k)}, x_{n(k)+1}, t)\})$, if $m(k) =$ odd, $n(k) =$ odd
$= \psi(\min\{M(x_{m(k)}, x_{n(k)}, t) * M(x_{n(k)}, x_{n(k)+1}, t), M(x_{m(k)}, x_{m(k)+1}, t), M(x_{n(k)+1}, x_{n(k)+2}, t)\})$, if $m(k) =$ even, $n(k) =$ even
$= \psi(\min\{M(x_{m(k)}, x_{n(k)}, t), M(x_{m(k)}, x_{m(k)+1}, t), M(x_{n(k)}, x_{n(k)+1}, t)\})$,
if $m(k) =$ even, $n(k) =$ odd.

For each $k \in \mathbb{N}$ with $k \geq k_0$, we further set

$r(k) = M(x_{m(k)}, x_{m(k)+2}, t)*w(k)*M(x_{n(k)}, x_{n(k)+2}, t)$, if $m(k)$ = odd, $n(k)$ = even
$= M(x_{m(k)}, x_{m(k)+2}, t)*w(k)*M(x_{n(k)}, x_{n(k)+1}, t)$, if $m(k)$ = odd, $n(k)$ = odd
$= M(x_{m(k)}, x_{m(k)+1}, t)*w(k)*M(x_{n(k)}, x_{n(k)+2}, t)$, if $m(k)$ = even, $n(k)$ = even
$= M(x_{m(k)}, x_{m(k)+1}, t)*w(k)*M(x_{n(k)}, x_{n(k)+1}, t)$, if $m(k)$ = even, $n(k)$ = odd
and
$r'(k) = M(x_{m(k)}, x_{m(k)+2}, t)*w'(k)*M(x_{n(k)}, x_{n(k)+2}, t)$, if $m(k)$ = odd, $n(k)$ = even
$= M(x_{m(k)}, x_{m(k)+2}, t)*w'(k)*M(x_{n(k)}, x_{n(k)+1}, t)$, if $m(k)$ = odd, $n(k)$ = odd
$= M(x_{m(k)}, x_{m(k)+1}, t)*w'(k)*M(x_{n(k)}, x_{n(k)+2}, t)$, if $m(k)$ = even, $n(k)$ = even
$= M(x_{m(k)}, x_{m(k)+1}, t)*w'(k)*M(x_{n(k)}, x_{n(k)+1}, t)$, if $m(k)$ = even, $n(k)$ = odd.

Since $\lim_{k\to\infty} w'(k) = \psi(1-\epsilon)$, we have $\lim_{k\to\infty} r'(k) = \psi(1-\epsilon)$.

Using $(*)$, we have $1-\epsilon \geq M(x_{m(k)}, x_{n(k)}, t) \geq u(k) \geq v(k) \geq r(k) \geq r'(k)$, $\forall\, k \geq k_0$. Thus by taking limit as $k \to \infty$, we have $1-\epsilon \geq \psi(1-\epsilon) > 1-\epsilon$, which leads to a contradiction.

Thus (x_n) is a Cauchy sequence and hence converges to some $x \in X$.

We now show that $fx = gx = x$.

If $gx \neq x$, then there exists $t_0 > 0$ such that $0 < M(x, gx, t_0) < 1$.

Since $M(x, gx, t_0) \geq M(x, x_{2n+1}, t_0) * M(fx_{2n}, gx, t_0)$
$\geq M(x, x_{2n+1}, t_0) * \beta(x_{2n}, x, t_0)M(fx_{2n}, gx, t_0)$
$\geq M(x, x_{2n+1}, t_0) * \psi(\min\{M(x, x_{2n}, t_0), M(x, gx, t_0), M(x_{2n+1}, x_{2n}, t_0)\})$

so by taking limit as $n \to \infty$, we have $M(x, gx, t_0) \geq \psi(M(x, gx, t_0) > M(x, gx, t_0)$, which leads to a contradiction.

Thus $gx = x$. Similarly, $fx = x$. Hence $fx = gx = x$. □

By putting $\beta \equiv 1$ in Theorem 3.15 and proceeding as in Theorem 3.3 for uniqueness of fixed point, we have the following:

Corollary 3.16. Let $(X, M, *)$ be a non-Archimedean complete fuzzy metric space. Suppose

$$M(fx, gy, t) \geq \psi(\min\{M(x, y, t), M(x, fx, t), M(y, gy, t)\}),\ \forall\, x, y \in X, t > 0.$$

Then f and g have a unique common fixed point in X.

We encourage the readers to compare the above result with Corollary 3.4 and Theorem 3.5.

Theorem 3.17. Let f, g be continuous mappings such that

(i) $M(x, y, t) > 0 \implies$
$\beta(x, y, t)M(gfx, fgy, t) \geq \psi(\min\{M(x, y, t), M(x, fx, t), M(y, gy, t)\})$,
$\forall\, x, y \in X, t > 0$,

(ii) for some $x_0 \in X$, $\beta(x_0, fx_0, t) \leq 1$ and $M(fx_0, gfx_0, t), M(gfx_0, fgfx_0, t) > 0$, $\forall\, t > 0$.

Then f and g have a common fixed point in X.

Proof. Define a sequence (x_n) as follows: $x_1 = fx_0, x_2 = gx_1, \cdots, x_{2n+1} = fx_{2n}, x_{2n+2} = gx_{2n+1}, \cdots$.

Then proceeding as Theorem 3.3, we obtain

$$\beta(x_n, x_{n+1}, t), \beta(x_{n+1}, x_n, t) \leq 1,\ \forall\, n \in \mathbb{N}, t > 0.$$

Let n be a positive integer bigger than 1.
If n is odd, then for chosen $t > 0$, $M(x_{n-1}, x_n, t) > 0 \implies$
$M(x_{n+1}, x_{n+2}, t) = M(gfx_{n-1}, fgx_n, t)$
$\geq \beta(x_{n-1}, x_n, t)M(gfx_{n-1}, fgx_n, t)$
$\geq \psi(\min\{M(x_{n-1}, x_n, t), M(x_{n-1}, fx_{n-1}, t), M(x_n, gx_n, t)\})$
$= \psi(\min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\})$
If n is even, then for chosen $t > 0$, $M(x_{n-1}, x_n, t) > 0 \implies$
$M(x_{n+1}, x_{n+2}, t) = M(gfx_n, fgx_{n-1}, t)$
$\geq \beta(x_n, x_{n-1}, t)M(gfx_n, fgx_{n-1}, t)$
$\geq \psi(\min\{M(x_n, x_{n-1}, t), M(x_n, fx_n, t), M(x_{n-1}, gx_{n-1}, t)\})$
$= \psi(\min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\})$.
Consequently, $\forall\, n \geq 2, t > 0$, $M(x_{n-1}, x_n, t) > 0 \implies$ either $M(x_{n+1}, x_{n+2}, t) \geq \psi(M(x_{n-1}, x_n, t)$ or $M(x_{n+1}, x_{n+2}, t) \geq \psi(M(x_n, x_{n+1}, t))$.

Since $M(x_1, x_2, t), M(x_2, x_3, t) > 0$, we conclude $M(x_{n-1}, x_n, t) > 0,\ \forall\, n \geq 2, t > 0$.

Let us set $S_t = \min\{M(x_1, x_2, t), M(x_2, x_3, t)\},\ \forall\, t > 0$.

Then $M(x_{2n-1}, x_{2n}, t) \geq \psi^{n-1}(S_t)$ and $M(x_{2n}, x_{2n+1}, t) \geq \psi^{n-1}(S_t),\ \forall\, n \geq 2, t > 0$.

Now $\forall\, t > 0, S_t > 0 \implies \lim\limits_{n\to\infty} \psi^{n-1}(S_t) = 1$.

Thus $\forall\, t > 0$, $\lim\limits_{n\to\infty} M(x_{2n-1}, x_{2n}, t) = \lim\limits_{n\to\infty} M(x_{2n}, x_{2n+1}, t) = 1$, and consequently, $\lim\limits_{n\to\infty} M(x_n, x_{n+1}, t) = 1$.

So (x_n) is a G-Cauchy sequence in X.
Hence the result follows. □

Example 3.18. We note that Example 3.8 satisfies the hypothesis of Theorem 3.17.

Let us now replace the value of β, in Example 3.8, with 1. It is clear that the modified example satisfies the hypothesis of Theorem 3.17 as well.

We now show that the modified example does not satisfy the hypothesis of Theorem 3.3.

Suppose otherwise. Then by putting $x = \frac{1}{2}, y = 1, t = 1$ in

$$\beta(x, y, t)M(fx, gy, t) \geq \psi(\min\{M(x, y, t), M(x, fx, t), M(y, gy, t)\}),$$

we obtain $M\left(\frac{1}{2}, 1, 1\right) \geq \psi\left(M\left(\frac{1}{2}, 1, 1\right)\right)$, which is a contradiction since $\psi(x) > x,\ \forall\, x \in (0, 1)$.

Corollary 3.19. Let f be a continuous mapping such that
(i) $M(x, y, t) > 0 \implies$

$$\beta(x, y, t)M(fx, fy, t) \geq \psi(\min\{M(x, y, t), M(x, fx, t)\},$$

$\forall\, x, y \in X, t > 0$,

(ii) (f, I) is a symmetric pair of β-admissible mappings (I being the identity self-mapping on X),

(iii) for some $x_0 \in X$, $\beta(x_0, fx_0, t) \leq 1$ and $M(fx_0, f^2x_0, t) > 0,\ \forall\, t > 0$.
Then f has a fixed point in X.

Proof. Immediate by setting $g = I$ in Theorem 3.17. □

In the next theorem, we omit the continuity hypothesis of one of f, g in Theorem 3.17.

Theorem 3.20. Suppose that f or g is continuous and

(i) $M(x, y, t) > 0 \implies$

$$\beta(x, y, t)M(gfx, fgy, t) \geq \psi(\min\{M(x, y, t), M(x, fx, t), M(y, gy, t)\}),$$

$\forall\ x, y \in X, t > 0$,

(ii) for some $x_0 \in X$, $\beta(x_0, fx_0, t) \leq 1$ and $M(fx_0, gfx_0, t), M(gfx_0, fgfx_0, t) > 0$, $\forall\ t > 0$,

(iii) for each sequence (x_n) in X with $\beta(x_n, x_{n+1}, t) \leq 1$, $\forall\ n \in \mathbb{N}, t > 0$ and a subsequence (x_{r_n}) of (x_n) with $\lim_{n\to\infty} x_{r_n} = x$, we have $\beta(x_{r_n}, x, t), \beta(x, x_{r_n}, t) \leq 1$, $\forall\ n \in \mathbb{N}, t > 0$.

Then f and g have a common fixed point in X.

Proof. Proceeding as in Theorem 3.17, we obtain the G-Cauchy sequence (x_n) in X given by $x_1 = fx_0, x_2 = gx_1, \cdots, x_{2n+1} = fx_{2n}, x_{2n+2} = gx_{2n+1}, \cdots$ such that $\beta(x_n, x_{n+1}, t), \beta(x_{n+1}, x_n, t) \leq 1$, $\forall\ n \in \mathbb{N}, t > 0$.

Since $(X, M, *)$ is weak G-complete, there exists a subsequence (x_{r_n}) of (x_n) such that (x_{r_n}) converges to some $x \in X$ where r_n's are either all even or all odd.

Without loss of generality, suppose all the r_n's are even.

Case I: Let f be continuous. Then $\forall\ t > 0, n \in \mathbb{N}$,

$M(x, fx, t) \geq M\left(x, x_{r_n+1}, \frac{t}{2}\right) * M\left(x_{r_n+1}, fx, \frac{t}{2}\right)$

$= M\left(x, x_{r_n+1}, \frac{t}{2}\right) * M\left(fx_{r_n}, fx, \frac{t}{2}\right)$

$= M\left(x, x_{r_n}, \frac{t}{4}\right) * M\left(x_{r_n}, x_{r_n+1}, \frac{t}{4}\right) * M\left(fx_{r_n}, fx, \frac{t}{2}\right)$.

Taking limit as $n \to \infty$, we see that the right hand side tends to 1.

Thus we have $M(x, fx, t) = 1$, $\forall\ t > 0$ and consequently, $fx = x$.

We now show that $gx = x$. Choose $t_1 > 0$.

Then for sufficiently large values of n, $M(x, gx, t_1)$

$\geq M(x, x_{r_n}, \frac{t_1}{5}) * M(x_{r_n}, x_{r_n+1}, \frac{t_1}{5}) * M(x_{r_n+1}, x_{r_n+2}, \frac{t_1}{5}) * M(x_{r_n+2}, x_{r_n+3}, \frac{t_1}{5}) * M(x_{r_n+3}, gx, \frac{t_1}{5})$

$\geq M(x, x_{r_n}, \frac{t_1}{5}) * M(x_{r_n}, x_{r_n+1}, \frac{t_1}{5}) * M(x_{r_n+1}, x_{r_n+2}, \frac{t_1}{5}) * M(x_{r_n+2}, x_{r_n+3}, \frac{t_1}{5}) * M(fgx_{r_n+1}, gfx, \frac{t_1}{5})$

$\geq M(x, x_{r_n}, \frac{t_1}{5}) * M(x_{r_n}, x_{r_n+1}, \frac{t_1}{5}) * M(x_{r_n+1}, x_{r_n+2}, \frac{t_1}{5}) * M(x_{r_n+2}, x_{r_n+3}, \frac{t_1}{5}) * (\beta(x, x_{r_n+1}, \frac{t_1}{5})M(gfx, fgx_{r_n+1}, \frac{t_1}{5}))$ (since $\lim_{n\to\infty} x_{r_n+1} = x$, $\beta(x, x_{r_n+1}, t) \leq 1$, $\forall\ n \in \mathbb{N}, t > 0$)

$\geq M(x, x_{r_n}, \frac{t_1}{5}) * M(x_{r_n}, x_{r_n+1}, \frac{t_1}{5}) * M(x_{r_n+1}, x_{r_n+2}, \frac{t_1}{5}) * M(x_{r_n+2}, x_{r_n+3}, \frac{t_1}{5}) * \psi(\min\{M(x_{r_n+1}, x, \frac{t_1}{5}), M(x_{r_n+1}, gx_{r_n+1}, \frac{t_1}{5}), M(x, fx, \frac{t_1}{5})\})$

$\geq M(x, x_{r_n}, \frac{t_1}{5}) * M(x_{r_n}, x_{r_n+1}, \frac{t_1}{5}) * M(x_{r_n+1}, x_{r_n+2}, \frac{t_1}{5}) * M(x_{r_n+2}, x_{r_n+3}, \frac{t_1}{5}) * \psi(\min\{M(x_{r_n+1}, x, \frac{t_1}{5}), M(x_{r_n+1}, x_{r_n+2}, \frac{t_1}{5}), M(x, fx, \frac{t_1}{5})\})$.

Taking limit as $n \to \infty$, we see that the right hand side tends to 1.

Thus $M(x, gx, t_1) = 1$, $\forall\ t_1 > 0 \implies x = gx$.

Case II: Let g be continuous. Then we can similiarly show that x is a common fixed point of f, g.

Again, if all the r_n's are odd, it can be similarly shown that $fx = gx = x$. □

By putting $\beta \equiv 1$ in Theorem 3.20 we have the following:

Corollary 3.21. Let one of f, g is continuous such that
(i) $M(x, y, t) > 0 \implies$
$$M(gfx, fgy, t) \geq \psi(\min\{M(x, y, t), M(x, fx, t), M(y, gy, t)\}),$$
$\forall\ x, y \in X, t > 0$,
(ii) for some $x_0 \in X$, $M(fx_0, gfx_0, t), M(gfx_0, fgfx_0, t) > 0$, $\forall\ t > 0$.
Then f and g have a common fixed point in X.

We encourage the readers to compare the last result with Corollary 3.4 and Theorem 3.5.

Example 3.22. Consider the fuzzy metric space $(X, M, \cdot)$ of Example 3.8. Let f, g be self-mappings on X such that
$$fx = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ \frac{1}{2} & \text{otherwise} \end{cases}$$
and $gx = 1,\ \forall\ x \in X$.
Clearly, g is continuous on X but f is not.
Since $gfx = fgy = 1,\ \forall\ x, y \in X$ we have
$$M(gfx, fgy, t) \geq \psi(\min\{M(x, y, t), M(x, fx, t), M(y, gy, t)\}),$$
$\forall\ x, y \in X, t > 0$ and $\psi \in \Psi$.
Thus by setting $\beta \equiv 1$ we see that f, g satisfy the hypothesis of Theorem 3.20.
Here 1 is a common fixed point of f and g which is clearly unique.

The following question is open:

Question 3.23. Can the continuity hypothesis of both f and g in Theorem 3.20 be omitted?

In what follows, we show that the additional hypothesis:
for each sequence (x_n) in X satisfying $\beta(x_n, x_{n+1}, t) \leq 1,\ \forall\ n \in \mathbb{N},\ t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\beta(x_m, x_n, t) \leq 1,\ \forall\ m, n \in \mathbb{N}$ with $m \geq n > k_0$ and $t > 0$
enables the conclusion of Theorem 3.20 to work in a non-Archimedean complete (KM) fuzzy metric space.

Theorem 3.24. Let $(X, M, *)$ be a non-Archimedean complete (KM) fuzzy metric space and one of f, g is continuous such that
(i) $M(x, y, t) > 0 \implies$
$$\beta(x, y, t)M(gfx, fgy, t) \geq \psi(\min\{M(x, y, t), M(x, fx, t), M(y, gy, t)\}),$$
$\forall\ x, y \in X, t > 0$,
(ii) for some $x_0 \in X$, $\beta(x_0, fx_0, t) \leq 1$ and $M(fx_0, gfx_0, t), M(gfx_0, fgfx_0, t) > 0$, $\forall\ t > 0$,

(iii) for each sequence (x_n) in X with $\beta(x_n, x_{n+1}, t) \leq 1$, $\forall\, n \in \mathbb{N}, t > 0$ and a subsequence (x_{r_n}) of (x_n) with $\lim_{n\to\infty} x_{r_n} = x$, we have $\beta(x_{r_n}, x, t), \beta(x, x_{r_n}, t) \leq 1$, $\forall\, n \in \mathbb{N}, t > 0$,

(iv) for each sequence (x_n) in X satisfying $\beta(x_n, x_{n+1}, t) \leq 1$, $\forall\, n \in \mathbb{N}$, $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\beta(x_m, x_n, t) \leq 1$, $\forall\, m, n \in \mathbb{N}$ with $m \geq n > k_0$ and $t > 0$.

Then f and g have a common fixed point in X.

Proof. Proceeding as in Theorem 3.17, we obtain the G-Cauchy sequence (x_n) in X given by $x_1 = fx_0, x_2 = gx_1, \cdots, x_{2n+1} = fx_{2n}, x_{2n+2} = gx_{2n+1}, \cdots$ that will turn out to be Cauchy following an argument similar to Theorem 3.15. Since X is complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Similar to Theorem 3.20, we can show that $fx = gx = x$. □

Remark 3.25. By substituting $g = f$ into the aforementioned results, we obtain related fixed point theorems for self-mappings f on $(X, M, *)$.

Remark 3.26. Proceeding as in Theorem 3.3, it can be shown that the additional hypothesis: $\beta(x, y, t) \leq 1$, $\forall\, x, y \in X, t > 0$ and for $x, y\ (x \neq y) \in X$, $M(x, y, t) > 0$, $\forall\, t > 0$, leads to the uniqueness of the corresponding fixed points in all the aforementioned results.

Remark 3.27. We note that Example 3.9 satisfies the hypothesis of Theorem 3.17. Since f, g in the said Example have more than one common fixed points, the additional condition mentioned in Remark 3.26 is essential to ensure the uniqueness of the common fixed point in Theorem 3.17.

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