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Fixed point approximations via generalized MR-Kannan mappings in Banach spaces

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Abstract

In this paper, we introduce generalizations of the concept of MR-Kannan type contractions and utilize those conditions to derive new fixed point theorems under both contractive and non-contractive conditions. Our approach enhances various existing results related to enriched mappings.

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1. Known results

The study of fixed points in general poses two significant problems [8]:

(1) What conditions on the structure of the ambient space and/or on the properties of the mapping must be added to guarantee the existence of at least one fixed point for the mapping?

(2) How can one effectively locate and approximate such a fixed point?

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In a recent work, Anjum et al. [1] addressed the aforementioned problems by introducing the notion of MR-Kannan type contractions and providing a characterization of normed spaces using MR-Kannan type contractions with a fixed point. Additionally, they studied the Ulam-Hyers stability and well-posedness results for the introduced mappings. It is worth noting that the notion of MR-Kannan type contractions encompasses the concept of enriched Kannan contractions introduced in [5]. The concept of enriched contraction covers a wide range of mappings, including both contractive and non-contractive classes. For further insights into enriched contractions, we refer the reader to Berinde [2], [3] Berinde and Păcuar [4] and references therein.

Throughout this paper, $(X, \|\cdot\|)$ denotes the normed space over the field $\mathbb{R}$, which is the set of all real numbers. In [1], Anjum et al. defined the following:

$$\Omega = \{\lambda : X \to \mathbb{R}\colon \lambda(x) \neq 0 \ \forall\ x \in X\},$$

and

$$\mho = \{\psi : X \to \mathbb{R} : \psi(x) \neq -1 \ \forall\ x \in X\}.$$

Let $T : X \to X$. For each fixed $\lambda \in \Omega$, an operator $T_\lambda : X \to X$ is said to be a generalized averaged mapping if

$$T_\lambda(x) = (1 - \lambda(x))x + \lambda(x)Tx, \quad \forall x \in X. \tag{1.1}$$

It is important to note that the class of generalized averaged mappings was studied in [9]. Indeed, if we choose $\gamma \in (0, 1)$ and set $\lambda(x) = \gamma$ for all $x \in X$, then (1.1) simplifies to an averaged mapping, given by

$$T_\lambda(x) = (1 - \gamma)x + \gamma Tx.$$

The following definition is essentially introduced in [1]:

Definition 1.1. A mapping $T : X \to X$ is said to be a (ψ, a)-MR-Kannan type contraction, if there exist $\psi \in \mho$ and $a \in [0, \frac{1}{2})$ such that

$$\left\| \frac{x\psi(x) + Tx}{1 + \psi(x)} - \frac{y\psi(y) + Ty}{1 + \psi(y)} \right\| \leq a\left(\left| \frac{1}{1 + \psi(x)} \right| \|x - Tx\| + \left| \frac{1}{1 + \psi(y)} \right| \|y - Ty\| \right),$$

holds for all $x, y \in X$.

In the first step, we generalize the definition of a (ψ, a)-MR-Kannan type contraction by redefining the classes of functions Ω and $\mho$ as follows:

$$\Omega^* = \{\lambda : X \to (0, 1) \ \forall\ x \in X\},$$

and

$$\mho^* = \{\psi : X \to [0, \infty) : \ \forall x \in X\}.$$

In light of Ω^* and $\mho^*$, we now define a (ψ, a, k)-MRB-Kannan type contraction.

Definition 1.2. A mapping $T : X \to X$ is said to be a (ψ, a, k)-MRB-Kannan type contraction, if there exist $\psi \in \mho^*$, $k \in (0, \infty)$ and $a \in [0, \frac{1}{2})$ such that

$$\left\| \frac{x\psi(x) + kTx}{k + \psi(x)} - \frac{y\psi(y) + kTy}{k + \psi(y)} \right\| \leq a \left(\left| \frac{k}{k + \psi(x)} \right| \|x - Tx\| + \left| \frac{k}{k + \psi(y)} \right| \|y - Ty\| \right), \tag{1.2}$$

holds for all $x, y \in X$.

The next definition is a (ψ, α, a, k)-MRB-Ćirić-Reich-Rus type contraction:

Definition 1.3. A mapping $T : X \to X$ is said to be a (ψ, a, k)-MRB-Ćirić-Reich-Rus type contraction, if there exist $\psi \in \mho^*, k \in (0, \infty)$, $\alpha \in (0, 1]$ and $a \in [0, \frac{1}{2})$ such that

$$\left\| \frac{x\psi(x) + kTx}{k + \psi(x)} - \frac{y\psi(y) + kTy}{k + \psi(y)} \right\| \leq \alpha(\|x - y\|) + a\left(\left| \frac{k}{k + \psi(x)} \right| \|x - Tx\| + \left| \frac{k}{k + \psi(y)} \right| \|y - Ty\| \right), \tag{1.3}$$

holds for all $x, y \in X$.

Remark 1.4. (i) If we put $a = 0$ in (1.3), then we get a (ψ, k)-MRB-Banach type contraction.

(ii) If we take $\psi(x) = b$ for all $x \in X$ and $k = 1$ in (1.3), then we get an enriched Ćirić-Reich-Rus type contraction [3].

2. Main results

We begin with the following result:

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a (ψ, a, k)-MRB-Kannan type contraction. Then*

(i) *Fix(T)= $\{x^*\}$;*

(ii) *there exists $\lambda \in \Omega^*$ such that the generalized Krasnoselskii iteration associated to T, that is, the sequence $\{x_n\}_{n=0}^{\infty}$, given by*

$$x_{n+1} = (1 - \lambda(x_n))x_n + \lambda(x_n)Tx_n, \quad n \geq 0, \tag{2.1}$$

converges to x^* for any initial guess $x_0 \in X$.

Proof. Let $\lambda(x) = \frac{k}{k+\psi(x)}$ for all $x \in X$. Taking $\psi(x) = 0$, the proof is straightforward. Therefore, considering $\psi(x) > 0$, it is clear that $\lambda \in \Omega^*$. Utilizing (1.2), we have:

$$\left\| \frac{\lambda(x)}{k}\left(k\left(\frac{1}{\lambda(x)}-1\right)x + kTx\right) - \frac{\lambda(y)}{k}\left(k\left(\frac{1}{\lambda(y)}-1\right)y + kTy\right)\right\| \\ \leq a\big(\, \|\lambda(x)(x-Tx)\| + \|\lambda(y)(y-Ty)\|\, \big),$$

which can be expressed equivalently as

$$\|T_\lambda x - T_\lambda y\| \leq a\bigg(\, \|x - T_\lambda x\| + \|y - T_\lambda y\| \,\bigg), \quad \forall x, y \in X, \tag{2.2}$$

where T_λ is the generalized averaged operator defined in (1.1). Since $a \in [0, \frac{1}{2})$, inequality (2.2) implies that T_λ is a Kannan contraction.

The generalized Krasnoselskii iteration process $\{x_n\}_{n=0}^{\infty}$, defined by (2.1), is precisely the Picard iteration associated with T_λ (1.1), i.e.,

$$x_{n+1} = T_\lambda x_n, \quad n \geq 0.$$

The remaining part of the proof follows a similar approach to the proof of Theorem 2.0.3 in [1]. □

The proof for the next fixed point theorem follows the same line of reasoning as presented in the proofs of Theorem 2.1.

Theorem 2.2. *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a (ψ, α, a, k)-MRB-Ćirić-Reich-Rus type contraction. Then conclusion of Theorem 2.1 holds.*

The local version of Theorem 2.1 can be proven using a similar approach as outlined in Theorem 2.0.5 in [1].

Theorem 2.3. *Let $(X, \|\cdot\|)$ be a Banach space and $B(\mathring{v}, r) = \{x \in X : \|\mathring{v} - x\| \leq r\}$, where $\mathring{v} \in X$ and $r > 0$. Let $T : B(\mathring{v}, r) \to X$ be a (ψ, a, k)-MRB-Kannan type contraction mapping. Further, assume that*

$$\|\mathring{v} - T\mathring{v}\| \leq \left|\frac{k + \psi(\mathring{v})}{k}\right| \left(1 - \frac{3a}{1+a}\right) r.$$

Then T has a unique fixed point in $B(\mathring{v}, r)$.

The characterization of a normed space presented below can be established using a similar approach as outlined in Corollary 3.0.2 of [1] (also see [10]).

Corollary 2.4. *Let $(X, \|\cdot\|)$ be a normed space and $T : X \to X$ be a (ψ, a, k)-MRB-Kannan type contraction mapping such that T has a unique fixed point. Then $(X, \|\cdot\|)$ is a Banach space.*

3. Generalized (ψ, a, k)-MRB-Kannan and (ψ, α, a, k)-MRB-Ćirić-Reich-Rus mappings

We now extend the criteria $a \in [0, \frac{1}{2})$ assumed in the (ψ, a, k)-MRB-Kannan type contraction mapping to $0 \leq a < \infty$ and introduce the notion of a generalized (ψ, a, k)-MRB-Kannan mapping.

Definition 3.1. A mapping $T : X \to X$ is said to be a generalized (ψ, a, k)-MRB-Kannan mapping, if there exist $\psi \in \mho^*, k \in (0, \infty)$ and $a \in [0, \infty)$ such that

$$\left\| \frac{x\psi(x) + kTx}{k + \psi(x)} - \frac{y\psi(y) + kTy}{k + \psi(y)} \right\| \leq a \left(\left| \frac{k}{k + \psi(x)} \right| \|x - Tx\| + \left| \frac{k}{k + \psi(y)} \right| \|y - Ty\| \right), \tag{3.1}$$

holds for all $x, y \in X$.

Similarly, we can define generalized (ψ, α, a, k)-MRB-Ćirić-Reich-Rus mapping:

Definition 3.2. A mapping $T : X \to X$ is said to be a generalized (ψ, α, a, k)-MRB-Ćirić-Reich-Rus mapping, if there exist $\psi \in \mho^*, k \in (0, \infty), \alpha \in [0, 1)$ and $a \in [0, \infty)$ such that

$$\begin{aligned} \left\| \frac{x\psi(x) + kTx}{k + \psi(x)} - \frac{y\psi(y) + kTy}{k + \psi(y)} \right\| \leq & \alpha(\|x - y\|) + a \left(\left| \frac{k}{k + \psi(x)} \right| \|x - Tx\| \right. \\ & \left. + \left| \frac{k}{k + \psi(y)} \right| \|y - Ty\| \right), \end{aligned} \tag{3.2}$$

holds for all $x, y \in X$

In 1966, Browder and Petryshyn [6] introduced the notion of asymptotic regularity.

Definition 3.3. A mapping T is said to be asymptotically regular on X if for each $x \in X$, $T^{n+1}x - T^n x \to 0$ as $n \to \infty$.

Now, we present a new result where T satisfies a generalized (ψ, a, k)-MRB-Kannan mapping under the assumption of asymptotic regularity of the same mapping.

Theorem 3.4. *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a generalized (ψ, a, k)-MRB-Kannan continuous mapping. Suppose $T_\lambda(x)$ is asymptotically regular mapping. Then*

(i) *Fix(T) $= \{x^*\}$;*
(ii) *for any initial guess $x_0 \in X$, a sequence $\{x_n\}_{n=0}^{\infty}$, given by*

$$x_{n+1} = (1 - \lambda(x_n))x_n + \lambda(x_n)Tx_n, \quad n \geq 0,$$

converges to x^*.

Proof. Let $\lambda(x) = \frac{k}{k+\psi(x)}$ for all $x \in X$. Taking $\psi(x) = 0$, the proof is straightforward. Therefore, considering $\psi(x) > 0$, it is clear that $\lambda \in \Omega^*$. Utilizing (3.1), we have:

$$\left\| \frac{\lambda(x)}{k} \left(k \left(\frac{1}{\lambda(x)} - 1 \right) x + kTx \right) - \frac{\lambda(y)}{k} \left(k \left(\frac{1}{\lambda(y)} - 1 \right) y + kTy \right) \right\| \\ \leq a\Big(\|\lambda(x)(x - Tx)\| + \|\lambda(y)(y - Ty)\| \Big),$$

which can be written in an equivalent form as

$$\|T_\lambda x - T_\lambda y\| \leq a\Bigg(\|x - T_\lambda x\| + \|y - T_\lambda y\| \Bigg), \quad \forall x, y \in X. \tag{3.3}$$

As $a \in [0, \infty)$, by (3.3) T_λ turns out to be a generalized Kannan contraction.

The generalized Krasnoselskii iteration process $\{x_n\}_{n=0}^{\infty}$, defined by (ii) is the Picard iteration associated with T_λ, that is,

$$x_{n+1} = T_\lambda x_n, \quad n \geq 0.$$

The proof up to the establishment of the Cauchy sequence of $\{x_n\}_{n=0}^{\infty}$ follows along similar lines as given in the proof of Theorem 2.6 of Górnicki [7]. Since X is a Banach space, we have $x^* = \lim_{n\to\infty} x_n$. Using the continuity of T_λ, we immediately obtain $x^* = T_\lambda x^*$, so by Fix(T)= Fix(T_λ), we have $Tx^* = x^*$. Uniqueness of the fixed point of the mapping follows easily. □

Theorem 3.5. *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a generalized (ψ, α, a, k)-MRB-Ćirić-Reich-Rus mapping. Then conclusion of Theorem 3.4 holds.*

Proof. The proof is similar to the proof of Theorem 3.4. □

Remark 3.6. The Ulam-Hyers stability and well-posedness results for the mappings considered here can be investigated following a similar approach as presented in [1].

4. Conclusion

In this paper, we have extended the scope of the study of MR-Kannan type contraction mappings in the context of the generalized averaged operator. Additionally, we have introduced the notion of a generalized MRB-Kannan type mapping, which further extends the concepts of MRB-Kannan type contractions and enriched contractions. Along similar lines, we have defined a generalized MRB-Ćirić-Reich-Rus mapping and proven the existence of a fixed point by incorporating the generalized averaged operator, asymptotic regularity, and continuity of the mapping.

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