APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 25, no. 1 (2024), 97-115
doi:10.4995/agt.2024.20026

Remarks on fixed point assertions in digital topology, 7

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Communicated by J. Rodríguez-López

Abstract

This paper continues a series discussing flaws in published assertions concerning fixed points in digital images.

2020 MSC: *54H30; 54H25.*

Keywords: *digital image; fixed point; metric space.*

1. Introduction

As stated in [6]:

> The topic of fixed points in digital topology has drawn much attention in recent papers. The quality of discussion among these papers is uneven; while some assertions have been correct and interesting, others have been incorrect, incorrectly proven, or reducible to triviality.

Here, we continue the work of [12, 6, 7, 9, 10, 11], discussing many shortcomings in earlier papers and offering corrections and improvements.

The topic of freezing sets [8] belongs to the fixed point theory of digital topology and is central to the paper [1]. We show that the latter paper contains no original results.

Other papers studied herein contain assertions of fixed points in digital metric spaces. Quoting and paraphrasing [10]:

Received 11 July 2023 – Accepted 7 September 2023

Authors of many weak papers concerning fixed points in digital topology seek to obtain results in a "digital metric space" (see section 2.2 for its definition). This seems to be a bad idea. We slightly paraphrase [9]:

- Nearly all correct nontrivial published assertions concerning digital metric spaces use the metric and do not use the adjacency. As a result, the digital metric space seems to be an artificial notion, not really concerned with digital images.
- If X is finite (as in a "real world" digital image) or the metric d is a common metric such as any ℓ_p metric, then (X, d) is uniformly discrete as a topological space, hence not very interesting.
- Many published assertions concerning digital metric spaces mimic analogues for subsets of Euclidean $\mathbb{R}^n$. Often, the authors neglect important differences between the topological space $\mathbb{R}^n$ and digital images, resulting in assertions that are incorrect or incorrectly "proven," trivial, or trivial when restricted to conditions that many regard as essential. E.g., in many cases, functions that satisfy fixed point assertions must be constant or fail to be digitally continuous [12, 6, 7].

Since acceptance for publication of [11], additional highly flawed papers rooted in digital metric spaces have come to our attention. These are [15, 20, 21, 25, 28, 31, 32].

2. Preliminaries

Much of the material in this section is quoted or paraphrased from [9].

We use $\mathbb{N}$ to represent the natural numbers, $\mathbb{Z}$ to represent the integers, $\mathbb{R}$ to represent the reals, and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

A *digital image* is a pair (X, κ), where $X \subset \mathbb{Z}^n$ for some positive integer n, and κ is an adjacency relation on X. Thus, a digital image is a graph. In order to model the "real world," we usually take X to be finite, although there are several papers that consider infinite digital images. The points of X may be thought of as the "black points" or foreground of a binary, monochrome "digital picture," and the points of $\mathbb{Z}^n \setminus X$ as the "white points" or background of the digital picture.

2.1. **Adjacencies, continuity, fixed point.** In a digital image (X, κ), if $x, y \in X$, we use the notation $x \leftrightarrow_\kappa y$ to mean x and y are κ-adjacent; we may write $x \leftrightarrow y$ when κ can be understood. We write $x \leftrightarroweq_\kappa y$, or $x \leftrightarroweq y$ when κ can be understood, to mean $x \leftrightarrow_\kappa y$ or $x = y$.

The most commonly used adjacencies in the study of digital images are the c_u adjacencies. These are defined as follows.

Definition 2.1. Let $X \subset \mathbb{Z}^n$. Let $u \in \mathbb{Z}$, $1 \le u \le n$. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in X$. Then $x \leftrightarrow_{c_u} y$ if

- $x \neq y$,
- for at most u distinct indices i, $|x_i - y_i| = 1$, and
- for all indices j such that $|x_j - y_j| \neq 1$ we have $x_j = y_j$.

Definition 2.2. Let (X, κ) be a digital image. Let $x, y \in X$. Suppose there is a set $P = \{x_i\}_{i=0}^n \subset X$ such that $x = x_0$, $x_i \leftrightarrow_\kappa x_{i+1}$ for $0 \le i < n$, and $x_n = y$. Then P is a *κ-path* (or just a *path* when κ is understood) in X from x to y, and n is the *length* of this path.

Definition 2.3 ([27]). A digital image (X, κ) is *κ-connected*, or just *connected* when κ is understood, if given $x, y \in X$ there is a κ-path in X from x to y.

Definition 2.4 ([27, 4]). Let (X, κ) and (Y, λ) be digital images. A function $f : X \to Y$ is *(κ, λ)-continuous*, or *κ-continuous* if $(X, \kappa) = (Y, \lambda)$, or *digitally continuous* when κ and λ are understood, if for every κ-connected subset X' of X, $f(X')$ is a λ-connected subset of Y.

Theorem 2.5 ([4]). *A function $f : X \to Y$ between digital images (X, κ) and (Y, λ) is (κ, λ)-continuous if and only if for every $x, y \in X$, if $x \leftrightarrow_\kappa y$ then $f(x) \leftrightarroweq_\lambda f(y)$.*

Remark 2.6. For $x, y \in X$, $P = \{x_i\}_{i=0}^n \subset X$ is a κ-path from x to y if and only if $f : [0, n]_\mathbb{Z} \to X$, given by $f(i) = x_i$, is (c_1, κ)-continuous. Therefore, we may also call such a function f a *(κ-)path* in X from x to y.

We use id_X to denote the identity function on X, and $C(X, \kappa)$ for the set of functions $f : X \to X$ that are κ-continuous.

A *fixed point* of a function $f : X \to X$ is a point $x \in X$ such that $f(x) = x$. We denote by $\mathrm{Fix}(f)$ the set of fixed points of $f : X \to X$.

Let $X = \Pi_{i=1}^n X_i$. The *projection to the j^{th} coordinate* function $p_j : X \to X_j$ is the function defined for $x = (x_1, \ldots, x_n) \in X$, $x_i \in X_i$, by $p_j(x) = x_j$.

As a convenience, if x is a point in the domain of a function f, we will often abbreviate "$f(x)$" as "fx".

2.2. **Digital metric spaces.** A *digital metric space* [18] is a triple (X, d, κ), where (X, κ) is a digital image and d is a metric on X. The metric is usually taken to be the Euclidean metric or some other ℓ_p metric; alternately, d might be taken to be the shortest path metric. These are defined as follows.

- Given $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{Z}^n$, $p > 0$, d is the ℓ_p metric if
$$d(x, y) = \left(\sum_{i=1}^n \mid x_i - y_i \mid^p \right)^{1/p}.$$
Note the special cases: if $p = 1$ we have the *Manhattan metric*; if $p = 2$ we have the *Euclidean metric*.

- [13] If (X,κ) is a connected digital image, d is the *shortest path metric* if for $x,y \in X$, $d(x,y)$ is the length of a shortest κ-path in X from x to y.

Remark 2.7. If X is finite or

- [7] d is an ℓ_p metric, or
- (X,κ) is connected and d is the shortest path metric,

then (X,d) is *uniformly discrete*, i.e., there exists $\varepsilon > 0$ such that $d(x,y) < \varepsilon$ implies $x = y$.

For an example of a digital metric space that is not uniformly discrete, see Example 2.10 of [9].

We say a sequence $\{x_n\}_{n=0}^{\infty}$ is *eventually constant* if for some $m > 0$, $n > m$ implies $x_n = x_m$. The notions of convergent sequence and complete digital metric space are often trivial, e.g., if the digital image is uniformly discrete, as noted in the following, a minor generalization of results of [23, 12].

Proposition 2.8 ([9])**.** *Let (X,d) be a metric space. If (X,d) is uniformly discrete, then any Cauchy sequence in X is eventually constant, and (X,d) is a complete metric space.*

We will use the following.

Theorem 2.9. *Let (X,d,κ) be a connected digital metric space in which*

- *d is the shortest path metric, or*
- *d is any ℓ_p metric and $\kappa = c_1$.*

Let $f,g : X \to X$ be such that

$$d(fx,fy) < d(gx,gy) \text{ for all } x,y \in X.$$

If $g \in C(X,\kappa)$, then f is a constant function.

Proof. Let $x \leftrightarrow_\kappa y$ in X. Then our choices of d and κ, and the continuity of g, imply

$$d(fx,fy) < d(gx,gy) \leq 1,$$

so $d(fx,fy) = 0$, i.e., $fx = fy$.

Now let $a,b \in X$. Since X is connected, there is a κ-path $\{x_i\}_{i=0}^{n}$ in X from a to b. By the above, $f(x_i) = f(x_{i+1})$ for $i \in \{1,\ldots,n-1\}$. Thus $f(a) = f(b)$. Hence f is constant. □

Given a bounded metric space (X,d), the *diameter of X* is

$$diam(X) = \max\{\, d(x,y) \mid x,y \in X \,\}.$$

3. [1] and freezing sets

Freezing sets are defined as follows.

Definition 3.1 ([8])**.** Let (X,κ) be a digital image. We say $A \subset X$ is a *freezing set for X* if given $g \in C(X,\kappa)$, $A \subset \text{Fix}(g)$ implies $g = \text{id}_X$.

Several papers subsequent to [8] have further developed our knowledge of freezing sets. Such a claim cannot be made for [1]. Below, we quote verbatim (with some corrections noted) each assertion presented as new in [1], with the assertion it mimics from [8]. Since [8] is cited in [1], the authors of [1] should have known better.

Note [1] uses "D.I" to abbreviate "digital image".

3.1. **Theorem 2.4 of** [1]**.** Theorem 2.4 of [1] reads as follows (note "V is a freezing subset" should be "A is a freezing subset").

> If (U, κ) is a D.I and V is a freezing subset for U and $f : (U, \kappa) \to (V, \lambda)$ is an isomorphism, then $f(A)$ is a freezing set for (V, λ).

Compare with Theorem 5.3 of [8]:

> Let A be a freezing set for the digital image (X, κ) and let $F : (X, \kappa) \to (Y, \lambda)$ be an isomorphism. Then $F(A)$ is a freezing set for (Y, λ).

3.2. **Theorem 2.5 of** [1]**.** Theorem 2.5 of [1] reads as follows.

> If $(U, C_{\mathbb{Z}}) \subset \mathbb{Z}^n$ is a D.I for $z \in [1, n]$, $f \in C(U, c_z\})$, $\alpha, \alpha' \in U$: $\alpha \leftrightarrow_{c_z} \alpha'$ and $p_i(f(\alpha)) \leq p_i(\alpha) \leq p_i(\alpha')$, then $p_i(f(\alpha)) \leq p_i(\alpha')$.

Notes on Theorem 2.5 of [1]:

- "$(U, C_{\mathbb{Z}})$" should be "(U, c_z)".
- Each instance of "$\leq$" should be "$<$". It is easy to construct examples for which the stated conclusion is not obtained if we allow "$\leq$" instead of "$<$".
- Other errors appear in the "proof" of this assertion.

Compare with Lemma 5.5 of [8]:

> Let $(X, c_u) \subset \mathbb{Z}^n$ be a digital image, $1 \leq u \leq n$. Let $q, q' \in X$ be such that $q \leftrightarrow_{c_u} q'$. Let $f \in C(X, c_u)$.
> (1) If $p_i(f(q)) > p_i(q) > p_i(q')$ then $p_i(f(q')) > p_i(q')$.
> (2) If $p_i(f(q)) < p_i(q) < p_i(q')$ then $p_i(f(q')) < p_i(q')$.

3.3. **Theorem 2.6 of** [1]**.** A digital image (X, κ) is reducible [22] if and only if id_X is homotopic in 1 step to a non-surjective map f, in which case $x \in X \setminus f(X)$ is a reduction point.

Theorem 2.6 of [1] reads as follows (note the "α" in ii. should be "a").

> i. If (U, κ) is a D.I and $A \subset U$ is a retract of U, then (A, κ) has no freezing sets for (U, κ).
> ii. If (U, κ) is a reducible digital image and A is a freezing subset for U, then if $a \in U$ is a reduction point of u, $\alpha \in A$.

Since U is a retract of U via the identity function, item i. is incorrect as stated. If we focus on proper subsets that are retracts, we can compare item i) with Theorem 5.6 of [8]:

> Let (X,κ) be a digital image. Let X' be a proper subset of X that is a retract of X. Then X' does not contain a freezing set for (X,κ).

Compare item ii. with Corollary 5.7 of [8]:

> Let (X,κ) be a reducible digital image. Let x be a reduction point for X. Let A be a freezing set for X. Then $x \in A$.

3.4. **Theorem 3.2 of [1].** The *boundary of* $X \subset Z^n$ [26] is

$$Bd(X) = \{x \in X \mid \text{there exists } y \in \mathbb{Z}^n \setminus X \text{ such that } y \leftrightarrow_{c_1} x\}.$$

Theorem 3.2 of [1] reads as follows (note "$\forall z \in [1,n]$" should be "for some $z \in [1,n]_{\mathbb{Z}}$").

> Let $U \subset \mathbb{Z}^n$ be finite, A is a subset of U, $f \in C(U,c_z)$ $\forall z \in [1,n]$. If $Bd(A) \subset \text{Fix}(f)$ and $Bd(A)$ is a freezing set for (U,c_z), then $A \subset \text{Fix}(f)$.

Since a superset of a freezing set is a freezing set, the conclusion of this assertion as written is immediate.

Proposition 5.12 of [8], which has a stronger hypothesis in not requiring $Bd(A)$ to be a freezing set, states the following.

> Let $X \subset \mathbb{Z}^n$ be finite. Let $1 \leq u \leq n$. Let $A \subset X$. Let $f \in C(X,c_u)$. If $Bd(A) \subset \text{Fix}(f)$, then $A \subset \text{Fix}(f)$.

3.5. **Theorem 3.3 of [1].** Theorem 3.3 of [1] reads as follows.

> If $\Pi_{j=1}^n [0,m_j]_{\mathbb{Z}} \subset \mathbb{Z}^n$ such that $m_j > 1$ $\forall j$ then $Bd(U)$ is a minimal freezing set for (U,c_n).

Presumably, $U = \Pi_{j=1}^n [0,m_j]_{\mathbb{Z}}$. Also, the authors of [1] fail to prove the claim of minimality.

Compare with Theorem 5.17 of [8]:

> Let $X = \prod_{i=1}^n [0,m_i]_{\mathbb{Z}} \subset \mathbb{Z}^n$, where $m_i > 1$ for all i. Then $Bd(X)$ is a minimal freezing set for (X,c_n).

3.6. **Theorem 3.4 of [1].** Theorem 3.4 of [1] reads as follows, where "κ^1, κ^2" should be "κ_1, κ_2".

> If (U_i,κ_i) is a set of D.I $\forall i \in [1,v]_{\mathbb{Z}}$, $U = \Pi_{i=1}^v U_i$ and a subset A of U is a freezing set for $(U, NP_v(\kappa^1,\kappa^2,\ldots,\kappa_v))$, then we have $p_i(A)$ is a freezing set for (U_i,κ_i) $\forall i \in [1,v]_{\mathbb{Z}}$.

Compare with Theorem 5.18 of [8]:

> Let (X_i,κ_i) be a digital image, $i \in [1,v]_{\mathbb{Z}}$. Let $X = \prod_{i=1}^v X_i$. Let $A \subset X$. Suppose A is a freezing set for $(X, NP_v(\kappa_1,\ldots,\kappa_v))$. Then for each $i \in [1,v]_{\mathbb{Z}}$, $p_i(A)$ is a freezing set for (X_i,κ_i).

4. [15]'s common fixed point results

S. Dalal is the author or coauthor of three papers with the title "Common Fixed Point Results for Weakly Compatible Map in Digital Metric Spaces" [15, 16, 17]. We have discussed flaws and improvements of [16] in [7], and those of [17] in [6]. In this section, we discuss flaws and improvements of [15].

Definition 4.1 ([17]). Suppose S and T are self-maps on a digital metric space (X, d, κ). Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that

$$lim_{n\to\infty} S(x_n) = lim_{n\to\infty} T(x_n) = t \text{ for some } t \in X. \tag{4.1}$$

We have the following.

- S and T are called *compatible* if $lim_{n\to\infty} d(S(T(x_n)), T(S(x_n))) = 0$ for all sequences $\{x_n\}_{n=1}^{\infty} \subset X$ that satisfy statement (4.1).
- S and T are called *compatible of type (A)* if
$$lim_{n\to\infty} d(S(T(x_n)), T(T(x_n))) = 0 =$$
$$lim_{n\to\infty} d(T(S(x_n)), S(S(x_n)))$$
for all sequences $\{x_n\}_{n=1}^{\infty} \subset X$ that satisfy statement (4.1).
- S and T are called *compatible of type (P)* if
$$lim_{n\to\infty} d(S(S(x_n)), T(T(x_n))) = 0$$
for all sequences $\{x_n\}_{n=1}^{\infty} \subset X$ that satisfy statement (4.1).

Proposition 4.2 ([17]). *Let S and T be compatible maps of type (A) on a digital metric space (X, d, ρ). If one of S and T is continuous, then S and T are compatible.*

The continuity assumption of Proposition 4.2 is of the $\varepsilon - \delta$ type of analysis. It is often unnecessary. Thus, we have the following.

Theorem 4.3 ([6]). *Let (X, d, κ) be a digital metric space, where either X is finite or d is an ℓ_p metric. Let S and T be self-maps on X. Then the following are equivalent.*

- *S and T are compatible.*
- *S and T are compatible of type (A).*
- *S and T are compatible of type (P).*

Indeed, other notions defined as variants on compatibility are also equivalent to compatibility (see Theorem 5.6 of [11]).

Proposition 4.4 ([15]). *Let S and T be compatible maps on a digital metric space (X, d, ρ) into itself. Suppose $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. Then*

(a) $\lim_{n\to\infty} STx_n = Tt$ *if T is continuous at t.*
(b) $\lim_{n\to\infty} TSx_n = St$ *if S is continuous at t.*

As above, the continuity assumption of Proposition 4.4 is of the $\varepsilon - \delta$ type of analysis and is often unnecessary. Thus, we have the following.

Proposition 4.5. *Let S and T be compatible maps on a digital metric space (X, d, ρ) into itself. Let (X, d) be uniformly discrete. Suppose* $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ *for some* $t \in X$. *Then*
(a) $\lim_{n\to\infty} STx_n = Tt$.
(b) $\lim_{n\to\infty} TSx_n = St$.

Proof. By uniform discreteness, we have, for almost all n, $Sx_n = Tx_n = t$. Thus for almost all n,

$$STx_n = \text{ (by compatibility) } TSx_n = Tt,$$

which proves (a). Assertion (b) follows similarly. □

The following appears as Theorem 3.1 of [15].

Assertion 4.6. Let A, B, S and T be four self-mappings of a complete digital metric space (X, d, ρ) satisfying the following conditions.
(a) $S(X) \subset B(X)$ and $T(X) \subset A(X)$;
(b) the pairs (A, S) and (B, T) are compatible;
(c) one of S, T, A, and B is continuous;
(d) $d(Sx, Tx) \le \phi(\max\{d(Ax, By), d(Sx, Ax), d(Sx, By)\})$ for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is continuous, monotone increasing, and $\phi(t) < t$ for $t > 0$. Then A, B, S, and T have a unique common fixed point in X.

However, the argument offered as a proof for Assertion 4.6 in [15] is marred by an error that also appears in [16]: A sequence $\{y_n\}_{n=0}^{\infty} \subset X$ of points is constructed such that $lim_{n\to\infty} d(y_{2n}, y_{2n+1}) = 0$. It is wrongly concluded that $\{y_n\}_{n=0}^{\infty}$ is a Cauchy sequence; a counterexample is given at Example 6.2 of [7].
We must conclude that Assertion 4.6 is unproven.

5. [20]'s path-length metric assertions

In this section, we discuss flaws in the paper [20].

5.1. Unoriginal assertions.

Several of the assertions presented as original appear in earlier literature.

Theorems 3.1, 3.2, and 3.3 of [20] duplicate results of [23], a paper cited in [20]. While [23] uses the Euclidean metric, [20] should have noted that the proofs of [23] also work for their respective analogs using the shortest-path metric.

Theorem 3.1 of [20] states the following.

> In a digital metric space (Y^*, d), if a sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then $x_n = x_m$ for all $m, n > \alpha$, where $\alpha \in \mathbb{N}$.

But this duplicates Proposition 3.5 of [23].

Theorem 3.2 of [20] states the following.

> In a digital metric space (Y^*, d), if a sequence $\{x_n\}_{n=1}^{\infty}$ converges to a limit $L \in Y^*$, then there is an $\alpha \in \mathbb{N}$ such that for

all $m, n > \alpha$, $x_n = L$ i.e. $x_n = x_{n+1} = x_{n+2} = \cdots = L$ for $n \geq \alpha$.

But this duplicates Proposition 3.9 of [23].

Theorem 3.3 of [20] states the following.

A digital metric space (Y^*, d) is complete.

But this duplicates Theorem 3.11 of [23].

5.2. **Contractions.** The following definition is not attributed to a source in [20]. It appears in [19], which was cited in [20].

Definition 5.1. A self mapping S on a digital metric space (Y, d) is said to be a *digital contraction mapping* if and only if there exists a non-negative number $q < 1$ such that

$$d(Sx, Sy) \leq qd(x, y) \ \forall\ x, y \in Y.$$

The following is stated as Corollary 3.1 of [20].

Assertion 5.2. Let (Y, d) be a digital metric space endowed with graph G, where d is the path length metric. Let $S : Y \to Y$ be a digital contraction map on Y. Then S has a unique fixed point.

Remark 5.3. Without the assumption of connectedness, the path length metric is undefined, so Assertion 5.2 as written is, at best, misleading.

Proposition 5.4. *If in Assertion 5.2 G is connected, then S must be a constant map.*

Proof. Since d is the path length metric, for $x \leftrightarrow y$, we have

$$d(Sx, Sy) \leq qd(x, y) = q < 1 = d(\mathrm{id}_Y(x), \mathrm{id}_Y(y)).$$

The assertion follows from Theorem 2.9. □

5.3. **Quasi-contractions.**

Definition 5.5 ([14])**.** A self mapping S on a metric space (Y, d) is said to be a *quasi-contraction* if and only if there exists a non-negative number $q < 1$ such that

$$d(Sx, Sy) \leq q \max\{d(x, y), d(x, Sx), d(y, Sy), d(x, Sy), d(y, Sx)\} \\ \forall\ x, y \in Y$$

Theorem 3.4 of [20] is stated as follows.

Theorem 5.6. *Let (Y, d, κ) be a digital metric space, where d is the shortest path metric. Let $S : Y \to Y$ be a quasi-contraction. Then*
(a) for all $x \in Y$, $\lim_{i\to\infty} S^i x = u_1 \in Y$;
(b) u_1 is the unique fixed point of S, and
(c) $d(S^i x, u_1) \leq \frac{q^i}{1-q} d(x, Sx)$ for all $x \in Y$, $i \in \mathbb{N}$, where q is as in Definition 5.5.

As noted above, Theorem 5.6 is only valid when (X, κ) is connected. With the inclusion of such a hypothesis, we show below how Theorem 5.6 can be strengthened.

Theorem 5.7. *Let (Y, d, κ) be a connected digital metric space, where d is the shortest path metric. Let $S : Y \to Y$ be a quasi-contraction. Then there exists $u \in Y$ such that*
(a) for every $x \in Y$ there exists $n_0 \in \mathbb{N}$ such that $i \geq n_0$ implies $S^i x = u$; and
(b) u is the unique fixed point of S.

Proof. Since (Y, d) is uniformly discrete, (a) of Theorem 5.6 implies $S^i x = u$ for almost all i. (b) follows immediately. □

6. θ-contraction assertion of [21]

The following set of functions Θ is defined in [21]. $\theta \in \Theta$ if $\theta : [0, \infty) \to [0, \infty)$ and

- θ is increasing;
- $\theta(0) = 0$; and
- $t > 0$ implies $0 < \theta(t) < \sqrt{t}$.

Definition 6.1 ([21])**.** Let (X, d) be a metric space. Let $T : X \to X$. Let $\theta \in \Theta$. If $d(Tx, Ty) \leq \theta(d(x, y))$ for all $x, y \in X$, T is a *digital θ-contraction*.

The following is stated as the main result, Theorem 3.1, of [21].

Assertion 6.2. Suppose (X, d, ℓ) is a digital metric space, $\theta \in \Theta$, and $T : X \to X$ is a digital θ-contraction. Then T has a unique fixed point.

However, the argument offered as a proof of this assertion is flawed as follows. A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is constructed such that $\{d(x_{n+1}, x_n)\}_{n=1}^{\infty}$ is a strictly decreasing sequence. The authors conclude that $x_n = x_{n+1}$ for large n. But this does not follow, since it has not been shown that $\{d(x_{n+1}, x_n)\}_{n=1}^{\infty}$ decreases to 0.

We must conclude that Assertion 6.2 is unproven.

We note the following case, in which Assertion 6.2 reduces to triviality.

Proposition 6.3. *Let (X, d, κ) be a connected digital metric space in which*

- *d is the shortest path metric, or*
- *d is any ℓ_p metric and $\kappa = c_1$.*

Then every θ-contraction on (X, d) is a constant map.

Proof. Suppose T is a θ-contraction on (X, d). Given κ-adjacent $x_0, y_0 \in X$,

$$d(Tx_0, Ty_0) \leq \theta(d(x_0, y_0)) < \sqrt{d(x_0, y_0)} = 1.$$

Hence $d(Tx_0, Ty_0) = 0$. Since X is κ-connected, the assertion follows. □

7. [25]'s contractions

We find the following stated as Theorem 3.1 of [25].

Assertion 7.1. Let (X, d, k) be a complete digital metric space with k-adjacency where d is usual Euclidean metric for $\mathbb{Z}^n$ and let f and g be self-mappings on X satisfying the following conditions:

- $f(X) \subseteq g(X)$;
- g is continuous; and
- for some q such that

$$0 < q < 1, \text{ and for every } x, y \in X, \quad d(fx, fy) \le qd(gx, gy). \tag{7.1}$$

Then f and g have a unique common fixed point in X provided f and g commute.

Notes:

- There is an error, perhaps a typo, in the argument given as a proof for this assertion:

  "$d(y_n, y_m) \to 1$" should be "$d(y_n, y_m) \to 0$".
- The continuity assumed in (3.2) is of the $\varepsilon - \delta$ variety of analysis. Since d is the Euclidean metric in $\mathbb{Z}^n$, all self-maps on the uniformly discrete $(\mathbb{Z}^n, d)$ or any of its subsets have this continuity. Therefore, we can improve on the assertion of [25] as follows.

Theorem 7.2. *Let (X, d) be a uniformly discrete metric space and let f and g be self-mappings on X such that:*

- *$f(X) \subseteq g(X)$;*
- *For some q such that $0 < q < 1$ and every $x, y \in X$, $d(fx, fy) \le qd(gx, gy)$.*
- *f and g commute.*

Then f and g have a unique common fixed point in X.

Proof. We use ideas from [25]. Let $x_0 \in X$. Since $f(X) \subseteq g(X)$, take x_1 such that $fx_0 = gx_1$, and inductively, $fx_n = gx_{n+1}$. Then

$$d(fx_n, fx_{n+1}) \le qd(gx_n, gx_{n+1}) = qd(fx_{n-1}, fx_n).$$

An easy induction yields that

$$d(fx_n, fx_{n+1}) \le q^n d(fx_0, fx_1) \to_{n \to \infty} 0.$$

Thus, there exists $z \in X$ such that for almost all m, n,

$$gx_{m+1} = fx_m = fx_n = gx_{n+1} = z.$$

Using the uniformly discrete and commutative properties, for almost all n,

$$d(gz, z) = d(gfx_n, fx_n) = d(fgx_n, fx_n) \le qd(ggx_n, gx_n) = qd(gz, z).$$

Thus, $d(gz, z) = 0$, so z is a fixed point of g.

Then

$$d(fz, z) = d(fz, fx_n) \le qd(gz, gx_n) = qd(gz, z) = 0,$$

so z is a common fixed point of f and g.

To show the uniqueness of z, suppose z_1 is a common fixed point of f and g. Then

$$d(z, z_1) = d(fz, fz_1) \leq qd(gz, gz_1) = qd(z, z_1),$$

which implies $d(z, z_1) = 0$, i.e., $z = z_1$. □

The following provides important cases in which Theorem 7.2, and therefore Assertion 7.1, reduce to triviality.

Proposition 7.3. *Let (X, d, κ) be a connected digital metric space, where*

- *d is the shortest path metric, or*
- *$\kappa = c_1$ and d is an ℓ_p metric.*

Suppose f and g are maps satisfying (7.1). If g is κ-continuous, then f is a constant function.

Proof. Let $x \leftrightarrow_\kappa y$ in X. By (7.1), continuity, and our choices of d and κ,

$$d(fx, fy) \leq qd(gx, gy) \leq q < 1.$$

Hence $fx = fy$. Since X is connected, it follows as in the proof of Theorem 2.9 that f is constant. □

8. Weakly compatible mappings in [28]

Several papers, including [28], attribute the following definition to alleged sources that do not contain it or, by virtue of their own citations, clearly are not the source.

Definition 8.1. Let (X, d) be a metric space and $f, g : X \to X$. We say f and g are *weakly compatible* if they commute at coincidence points, i.e., $f(x) = g(x)$ implies $f(g(x)) = g(f(x))$.

The following is stated as Proposition 2.15 of [28].

Assertion 8.2. Let $J, K : F \to F$ be weakly compatible maps. If a point η is a unique point of coincidence of mappings J and K, i.e., $J(\sigma) = K(\sigma) = \eta$, then η is the unique common fixed point of J and K.

The argument given to prove Assertion 8.2 claims that $J(\sigma) = K(\sigma) = \eta$ implies $J(\sigma) = J(K(\sigma))$, and $K(J(\sigma)) = K(\sigma)$. No reason is given to support the latter equations, and there is no obvious reason to accept them. Therefore, we must regard Assertion 8.2 as unproven.

The following is stated as Theorem 3.1 of [28].

Assertion 8.3. Let (F, d, Y) be a digital metric space, where Y is an adjacency and d is the Euclidean metric on $\mathbb{Z}^n$. Let $J, K : F \to F$ such that

- $J(F) \subset K(F)$, and
- for some ξ such that $0 < \xi < 1/4$,

$$d(Ju, Jq) \leq \xi[d(Ju, Kq) + d(Jq, Ku) + d(Ju, Ku) + d(Jq, Kq)] \; \forall u, q \in F.$$

If $K(F)$ is complete and J and F are weakly compatible, then there exists a unique common fixed point in F for J and K.

The argument offered as proof of Assertion 8.3 depends on Assertion 8.2, which, we have shown above, is unproven. Thus, we must regard Assertion 8.3 as unproven.

Remark 8.4. Example 3.2 of [28] asks us to consider as a digital metric space (F, ϕ, Y) where $F = [0, 1]$, and the function $J : F \to F$ given by $J(u) = \frac{1}{1+u}$. But F clearly is not a subset of any $\mathbb{Z}^n$, and J is not integer-valued.

9. [31]'s common fixed point assertions

9.1. [31]**'s Theorem 3.1.** The following is stated as Theorem 3.1 of [31].

Assertion 9.1. Let (X, ℓ, d) be a digital metric space. Let $A, B : X \to X$ with $B(X) \subset A(X)$. Let γ be a right continuous real function such that $\gamma(a) < a$ for $a > 0$. Suppose for all $x, y \in X$ we have

$$d(Bx, By) \le \gamma(d(Ax, Ay)).$$

Then A and B have a unique common fixed point.

That Assertion 9.1 is incorrect is shown by the following.

Example 9.2. Let $X = \mathbb{N}$ and let $d(x, y) = \mid x - y \mid$. Let $A(x) = x + 1$, $B(x) = 2$, $\gamma(x) = x/2$ for all $x \in \mathbb{N}$. Clearly, the hypotheses of Assertion 9.1 are satisfied, but A has no fixed point.

9.2. [31]**'s Theorem 3.2.**

Definition 9.3. (Incorrectly attributed in [31] to [26]; found in [2])
Let $f, g : X \to X$ be functions. If there is a coincidence point x_0 of f and g at which f and g commute (i.e., $f(x_0) = g(x_0)$ and $f(g(x_0)) = g(f(x_0))$, then f and g are *occasionally weakly compatible*.

Let Φ be the set of functions $\phi : [0, \infty) \to [0, \infty)$ such that ϕ is increasing, $\phi(t) < t$ for $t > 0$, and $\phi(0) = 0$.

Definition 9.4 ([30]). Let (X, d) be a metric space. Let $\alpha : X \times X \to [0, \infty)$. Let $\phi, \psi \in \Phi$. Let $T : X \to X$. If for all $x, y \in X$ we have

$$\alpha(x, y)\psi(d(Tx, Ty)) \le \psi(d(Tx, Ty)) - \phi(d(Tx, Ty))$$

then T is an $\alpha - \psi - \phi$ *contractive mapping*.

Definition 9.5 ([29]). Let (X, d) be a metric space. Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say T is α*-admissible* if $\alpha(x, y) \ge 1$ implies $\alpha(Tx, Ty) \ge 1$.

The following is stated as Theorem 3.2 of [31].

Assertion 9.6. Let (X, ℓ, d) be a digital metric space. Let S, T, A, and B be $\alpha - \psi - \phi$ contractive mappings of (X, d). Let the pairs (A, S) and (B, T) be occasionally weakly compatible. Suppose for all $x, y \in X$,

$$\alpha(x, y)\psi(d(Ax, By)) \le \psi(M(x, y)) - \gamma(M(x, y))$$

where

$$M(x,y) = \max \left\{ \begin{array}{c} d(Sx,Ty),\ d(By,Sx),\ d(Sx,Ax),\ d(By,Ty),\ d(Ax,Ty), \\ \frac{2d(Sx,Ax)}{1+d(By,Ty)} \end{array} \right\}$$

Then there is a unique fixed point of S, T, A, and B.

The argument offered as proof of Assertion 9.6 in [31] is marred by the following flaws.

- The "γ" in the inequality should be "ϕ".
- The argument starts:

  Since A, B are α-admissible then for all $x, y \in X$, $\alpha(x,y) \geq 1$.

  Notice it was not hypothesized that A and B are α-admissible. Even if this is a mere omission, the conclusion would be unsupported; it does not follow from Definition 9.5. The unproven allegation that $\alpha(x,y) \geq 1$ is part of the argument that $Ax = Sx = By = Ty$, which in turn is an important part of the uniqueness argument.
- No proof is offered for the claim of a fixed point for any of S, T, A, and B.

We must conclude that Assertion 9.6 is unproven.

9.3. [31]**'s Example 3.2.** Example 3.2 of [31] wants us to consider the digital metric space $(\mathbb{N}, d, 4)$, where $d(x,y) = \mid x - y \mid$. The paper fails to define 4-adjacency on $\mathbb{N}$; perhaps 2-adjacency was intended.

Further flaws:

- It is claimed that functions A, B, S, and T have a common fixed point, where

$$Ax = x + 1, \quad By = y + 1, \quad Sx = x - 1, \quad Ty = y - 1.$$

  But clearly none of these functions has a fixed point.
- We are asked to consider an inequality that uses functions ψ, α, and φ that are not defined.

9.4. [31]**'s M_6.** In [31], M_6 is defined as the set of real-valued continuous functions $\phi : [0,1]^6 \to \mathbb{R}$ such that

(A) $\int_0^{\phi(u.u.0,0,u,u)} \varphi(t)dt \leq 0$ implies $u \geq 0$.

(B) $\int_0^{\phi(u.u.0,0,u,0)} \varphi(t)dt \leq 0$ implies $u \geq 0$.

(C) $\int_0^{\phi(0,u.u.0,0,u)} \varphi(t)dt \leq 0$ implies $u \geq 0$.

Notice the use of two different "phi" symbols, "ϕ" and "φ".

- If this is intended, "φ" is undefined.
- If it is intended that "φ" should be "ϕ", then every continuous function $\phi : [0,1]^6 \to \mathbb{R}$ belongs to M_6, since the domain of such a function requires u, as a parameter of ϕ, to be nonnegative.

Example 3.3 of [31] claims a certain function ϕ satisfies (A) and (B) of the definition of M_6 and therefore belongs to M_6, although no claim is made that ϕ satisfies (C).

10. [32]'s common fixed point assertions

The paper [32] presents five assertions concerning pairs of various types of expansive self-mappings on digital images. Each of these assertions concludes that the maps of the pair have common fixed points. We show below that all of these assertions, "usually" or always, reduce to triviality: further, three of them must be regarded as unproven in full generality due to errors in their "proofs".

We note that [32] uses "α-adjacent" for what we have been calling "c_α-adjacent".

10.1. [32] Theorem 3.1.

The assertion labeled in [32] as Theorem 3.3.1 is clearly intended to be labeled Theorem 3.1. It is stated as follows.

Theorem 10.1 ([32]). *Let (X, d, k) be a complete digital metric space and suppose $T_1, T_2 : X \to X$ are continuous, onto mappings satisfying*

$$d(T_1x, T_2y) \geq \alpha d(x, y) + \beta[d(x, T_1x) + d(y, T_2y)] \tag{10.1}$$

for all $x, y \in X$, where $\alpha > 0$, $1/2 \leq \beta \leq 1$, and $\alpha + \beta > 1$. Then T_1 and T_2 have a common fixed point in X.

See section 10.6 for discussion of the triviality of this assertion.

10.2. [32] Theorem 3.2.

Theorem 3.2 of [32] is stated as follows.

Theorem 10.2 ([32]). *Let (X, d, k) be a complete digital metric space and suppose $T_1, T_2 : X \to X$ are continuous, onto mappings satisfying*

$$d(T_1x, T_2y) \geq \alpha d(x, y) + \beta[d(x, T_2y) + d(y, T_1x)] \tag{10.2}$$

for all $x, y \in X$, where $\alpha > 0$, $1/2 \leq \beta \leq 1$, and $\alpha + \beta > 1$. Then T_1 and T_2 have a common fixed point in X.

See section 10.6 for discussion of the triviality of this assertion.

10.3. [32] "Theorem" 3.3.

The following is stated as Theorem 3.3 of [32].

Assertion 10.3 ([32]). Let (X, d, k) be a complete digital metric space and suppose $T_1, T_2 : X \to X$ are continuous, onto mappings satisfying

$$d(T_1x, T_2y) \geq \alpha d(x, y) + \beta d(x, T_1x) + \gamma d(y, T_2y) + \eta[d(x, T_1x) + d(y, T_2y)] \tag{10.3}$$

for all $x, y \in X$, where

$$\alpha \geq -1, \quad \beta > 0, \quad \gamma \leq 1/2, \quad 1/2 < \eta \leq 1, \quad \text{and} \quad \alpha + \beta + \gamma + \eta > 1.$$

Then T_1 and T_2 have a common fixed point in X.

The argument offered as proof of Assertion 10.3 creates a sequence $\{x_n\}_{n=1}^{\infty}$, reaches an inequality

$$[1 - (\beta + \eta)]d(x_{2n}, x_{2n+1}) \geq (\alpha + \gamma + \eta)d(x_{2n+1}, x_{2n+2})$$

and claims to derive that

$$d(x_{2n}, x_{2n+1}) \geq \frac{\alpha + \gamma + \eta}{1 - (\beta + \eta)} d(x_{2n+1}, x_{2n+2}).$$

This reasoning would be correct if we knew that $1 - (\beta + \eta) > 0$; however, we don't have such knowledge, since the hypotheses allow $1 - (\beta + \eta) \leq 0$.

Thus, we must consider Assertion 10.3, as written, unproven.

See section 10.6 for discussion of the triviality of this assertion.

10.4. [32] **"Theorem" 3.4.** The following is stated as Theorem 3.4 of [32].

Assertion 10.4 ([32]). Let (X, d, k) be a complete digital metric space and suppose $T_1, T_2 : X \to X$ are continuous, onto mappings satisfying

$$d(T_1x, T_2y) \geq \alpha[d(x,y) + d(x, T_1x) + d(y, T_2y)] + [\beta d(x, T_2y) + d(y, T_1x)] \quad (10.4)$$

for all $x, y \in X$, where

$$\alpha \geq 0, \quad \beta < 1, \quad \alpha + \beta > 1.$$

Then T_1 and T_2 have a common fixed point.

It seems likely that "$[\beta d(x, T_2y)$" is intended to be "$\beta[d(x, T_2y)$".

The "proof" of Assertion 10.4 in [32] contains errors similar to those in the "proof" of Assertion 10.3. We discuss one of these errors.

The argument creates a sequence $\{x_n\}_{n=1}^{\infty}$, reaches the inequality

$$[1 - (\alpha + \beta)]d(x_{2n}, x_{2n+1}) \geq (2\alpha + 2\beta)d(x_{2n+1}, x_{2n+2})$$

and claims to derive from it

$$d(x_{2n}, x_{2n+1}) \geq \frac{2\alpha + 2\beta}{1 - (\alpha + \beta)} d(x_{2n+1}, x_{2n+2})$$

which does not follow, since by hypothesis, the denominator of the fraction is negative.

Thus we must regard Assertion 10.4 as unproven.

See section 10.6 for discussion of the triviality of this assertion.

10.5. [32] **"Theorem" 3.5.** The following is stated as Theorem 3.5 of [32].

Assertion 10.5. Let (X, d) be a complete metric space and suppose $T_1, T_2 : X \to X$ are continuous onto mappings satisfying

$$d(T_1x, T_2y) \geq \alpha \max\{d(x,y), d(x, T_1x), d(y, T_2y)\} + \beta \max\{d(x, T_2y), d(x,y)\} + \gamma d(x,y) \quad (10.5)$$

where

$$\alpha \geq 0, \quad \beta > 0, \quad \gamma \leq 1, \quad \alpha + \beta + \gamma > 1.$$

Then T_1 and T_2 have a common fixed point.

The argument given as "proof" of Assertion 10.5 in [32] is flawed as follows. A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is created and the following inequality is reached:

$$d(x_{2n+1}, x_{2n+2}) \le hd(x_{2n}, x_{2n+1}), \quad \text{where} \quad h = \frac{1}{\alpha + \beta + \gamma} \tag{10.6}$$

An implicit induction is then used to claim that (10.5) implies

$$d(x_{2n}, x_{2n+1}) \le h^{2n} d(x_0, x_1)$$

However, the reasoning is incorrect, since the left side of (10.6) requires the smaller index to be odd, and there is no analog for the smaller index being even.

See section 10.6 for discussion of the triviality of this assertion.

10.6. **On triviality of assertions of [32].** We consider conditions under which the assertions of [32] reduce to triviality. In the following, we require all of the constants α, β, γ, and η to be positive (perhaps this was intended by the authors of [32], but as written their assertions occasionally permit negative values). In other ways, our hypotheses have greater generality, in that we omit certain hypotheses of [32].

Proposition 10.6. *Let (X, d) be a metric space and suppose $T_1, T_2 : X \to X$ are mappings such that T_2 is onto and for all $x, y \in X$, T_1 and T_2 satisfy any of (10.1), (10.2), (10.3), (10.4), or (10.5) where all of α, β, γ, and η are positive. Then $T_1 = T_2 = \mathrm{id}_X$. Further, if $\alpha > 1$ then X has only one point.*

Proof. Given $x_0 \in X$, T_2 being onto implies there exists $y_0 \in X$ such that $T_1 x_0 = T_2 y_0$. Thus, for the pair (x_0, y_0), the left side each of (10.1), (10.2), (10.3), (10.4), or (10.5) is 0, so each term of the right side is 0. Hence

$$x_0 = y_0 \quad \text{and} \quad d(x_0, T_1 x_0) = 0 = d(y_0, T_2 y_0).$$

Since x_0 is an arbitrary member of X, it follows that $T_1 = T_2 = \mathrm{id}_X$.

But then each of (10.1), (10.2), (10.3), (10.4), or (10.5) implies

$$d(x, y) = d(T_1 x, T_2 y) \ge \alpha d(x, y),$$

which is impossible if $x \ne y$ and $\alpha > 1$. Thus $\alpha > 1$ implies X has a single point. □

11. Further remarks

We have continued the work of [12, 6, 7, 9, 10, 11] in discussing flaws in some papers claiming fixed point results in digital topology. The literature of digital topology includes many fixed point assertions that are correct, correctly proven, and beautiful. However, papers we have considered here have many errors and assertions that turn out to be trivial.

Although authors are responsible for their errors and other shortcomings, it is clear that many of the papers studied in the current paper were reviewed inadequately, and should have been rejected.

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