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Partial actions of groups on profinite spaces

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Abstract

We show that for a nice partial action η with closed domain of a compact group G on a profinite space X the orbit space $X/\sim_G$ is profinite, this leads to the fact that when G is profinite the enveloping space X_G is also profinite. Moreover, we provide conditions for the induced quotient map $\pi_G : X \to X/\sim_G$ of η to have a continuous section. Relations between continuous sections of π_G and continuous sections of the quotient map induced by the enveloping action of η are also considered. At the end of this work, we prove that the category of separately continuous actions on profinite spaces is reflective in the category of separately continuous actions on compact Hausdorff spaces.

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1. Introduction

A topological space X is called profinite if there exists an inverse system of finite discrete spaces for which its inverse limit is homeomorphic to X, equivalently, X is profinite if it is compact, Hausdorff and zero-dimensional (that

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is, X has a basis of clopen subsets). A topological group is profinite if it is profinite as a topological space. Important examples of profinite groups and profinite spaces are the group of p-adic integers, where p is a prime number, the Galois group on an arbitrary Galois extension, fundamental groups of connected schemes, and the set of connected components of a compact Hausdorff space. For details about profinite spaces and profinite groups, the interested reader may consult [14] or [18].

On the other hand, partial actions of groups appeared in the context of C^*-algebras in [5] and [12], in which C^*-algebraic crossed products by partial automorphisms were introduced and studied by analyzing their internal structure. Partial actions of groups have also appeared in other contexts, such as the theory of operator algebras, Galois cohomology, Hopf algebras, Polish spaces, the theory of $\mathbb{R}$-trees and model theory (see [4] and [8] for a detailed account of recent developments about partial actions).

A relevant question is whether a partial action is obtained as restriction of a corresponding collection of total maps on some superspace. In the topological context this is known as the globalization problem, and was studied in [2] and [10]. They showed that for any partial action $\eta = \{\eta_g\}_{g\in G}$ of a topological group G on a topological space X, there is a topological space Y and a continuous action μ of G on Y, such that X is embedded in Y and η is the restriction of μ to X. Such a space is called a globalization of X. It is also shown that there is a minimal globalization X_G called the enveloping space of X. However, structural properties of X are not in general inherited by X_G, for instance, in [9], it is shown that the enveloping space of a partial action of a countable group on a compact metric space is Hausdorff if and only if the domain of each η_g is clopen for all $g \in G$. While assuming that both X and G are Polish spaces, sufficient conditions for X_G to be Polish are established in [15]. Likewise, in [19] it was proven that the globalization of a partial action on a connected 2-complex may result in a complex which is not connected. Therefore, a natural problem is to establish which properties of the space X are also satisfied by the space X_G.

The present work is structured as follows: After the introduction, we provide in Section 2 the necessary background and notations about (set theoretic) partial and topological partial actions, and their corresponding globalization. We give some preliminary results that will be needed in the work. Also, we introduce the category of continuous partial actions of a group G on topological spaces, which we will denote as $\mathcal{PA}_G$, and we will show that $\mathcal{PA}_G$ is a category with products. In Section 3 we work with partial actions on a profinite space X and we present in Theorem 3.1 a sufficient condition for the space $X/\sim_G$ to be profinite, where $\sim_G$ is the orbit equivalence relation determined by the partial action, as given in equation (2.8). Later, we treat the problem of finding a continuous section to the quotient map $\pi_G : X \to X/\sim_G$ and we show the existence of such a section when the group G is profinite (see Theorem 3.6), extending the classical result on continuous and free (global) actions. It is important to notice that continuous sections have been relevant in the context of partial actions, for instance, in [6] continuous sections of Banach bundles play a crucial role in the characterization of continuous twisted partial actions of locally compact groups on C^*-algebras.

The goal of the second part of Section 3 is to find relations between continuous sections of the quotient map π_G and Π_G, respectively (being Π_G the corresponding quotient map induced by the globalization). We show how to find a continuous section of Π_G having a continuous section of π_G. It is important to remark that the converse does not seem to be true, that is, having a continuous section of Π_G does not seem to imply that π_G has a continuous section; ítems (ii) and (iii) of Proposition 3.9 deal with this problem (see also Proposition 3.10). We finish this work with Section 4, which has a categorial flavor. Indeed, we show in Proposition 4.5 that $\mathcal{PA}_G$ is a category with inverse limits. Since the globalization problem is closely related to a reflectivity property (see Proposition 4.4), being the globalization a reflector, in [11] the authors studied the problem of when the corresponding reflector of the category of partial group actions on sets with an algebraic structure to a subcategory of global actions, is a globalization. In our case, we deal with some categories related to global actions of G on compact Hausdorff and profinite spaces, and we show in Proposition 4.8 that the category of separately continuous actions on profinite spaces is reflective in the category of separately continuous actions on compact Hausdorff spaces.

2. Preliminaries on partial actions

In this section we establish our conventions about partial actions and we prove some results that will be useful throughout the work. We start with the following definitions:

Definition 2.1. Let A, B, X be sets. We say that $f : A \rightharpoonup B$ is a partial function if there exists $C \subseteq A$ such that the restriction of f to C is a function. A partial set action of A on X is a partial function $A \times X \rightharpoonup X$ given by $(a, x) \mapsto a \cdot x$, for all $a \in A$ and $x \in X$, such that $a \cdot x$ is defined, which we denote by $\exists a \cdot x$.

Definition 2.2 ([10, p. 87-88]). Let G be a group with identity element 1 and X be a set. A partial action of G on X is a partial set action η of G on X such that for each $g, h \in G$ and $x \in X$ the following assertions hold:

(PA1) If $\exists g \cdot x$, then $\exists g^{-1} \cdot (g \cdot x)$ and $g^{-1} \cdot (g \cdot x) = x$,
(PA2) If $\exists g \cdot (h \cdot x)$, then $\exists (gh) \cdot x$ and $g \cdot (h \cdot x) = (gh) \cdot x$,
(PA3) $\exists 1 \cdot x$ and $1 \cdot x = x$.

It is said that G *acts* (globally) on X if $\exists g \cdot x$, for all $(g, x) \in G \times X$.

Given a partial action η of G on X, $g \in G, x \in X$ and $U \subseteq X$; we set:

- $G * U = \{(g, u) \in G \times U \mid \exists g \cdot u\}$. In particular, $G * X$ is the domain of η,
- $X_g = \{x \in X \mid \exists\, g^{-1} \cdot x\}$.

Then η induces a family of bijections $\{\eta_g \colon X_{g^{-1}} \ni x \mapsto g \cdot x \in X_g\}_{g\in G}$, such that η_1 is the identity of X and $\eta_{g^{-1}} = \eta_g^{-1}$. We also denote this family by η. The following result characterizes partial actions in terms of a family of bijections:

Proposition 2.3 ([17, Lemma 1.2]). *A partial action η of G on X is a family of bijections $\eta = \{\eta_g \colon X_{g^{-1}} \to X_g\}_{g\in G}$, where $X_g \subseteq X$, for all $g \in G$; such that:*

(i) $X_1 = X$ *and* $\eta_1 = \mathrm{id_X}$,
(ii) $\eta_g(X_{g^{-1}} \cap X_h) = X_g \cap X_{gh}$,
(iii) $\eta_g\eta_h \colon X_{h^{-1}} \cap X_{h^{-1}g^{-1}} \to X_g \cap X_{gh}$, *and* $\eta_g\eta_h = \eta_{gh}$ *in* $X_{h^{-1}} \cap X_{h^{-1}g^{-1}}$,

for all $g, h \in G$.

Given $x \in X$ and $U \subseteq X$ we set:

$$G^x = \{g \in G \mid \exists g \cdot x\}, \quad G^U = \bigcup_{u\in U} G^u, \tag{2.1}$$

and

$$G^U \cdot U = \{g \cdot u \mid u \in U, g \in G^u\}. \tag{2.2}$$

Then U is called *G-invariant* if $G^U \cdot U \subseteq U$.

The next lemma will be useful in our work:

Lemma 2.4. *Let η be a partial action of G on X and U a nonempty subset of X, then the following statements are true:*

(i) $\eta(G * U) = G^U \cdot U$,
(ii) $G^U \cdot U$ *and its complement are G-invariant,*
(iii) *If U is G-invariant, then the restriction $\eta \restriction_{G*U}$ is a partial action of G on U.*

Proof. Statements (i) and (iii) are straightforward. To show (ii) Let $A := G^U \cdot U$ and $B := X \setminus A$. Given $a \in A$, there are $u \in U$ and $g \in G^u$ such that $a = g \cdot u$; thus, for all $h \in G^a$, one gets by (PA2) that $hg \in G^u$ and $h \cdot a = (hg) \cdot u \in A$, then $G^A \cdot A \subseteq A \subseteq G^A \cdot A$ and A is G-invariant. On the other hand, given $b \in B$ and $g \in G^b$ such that $g \cdot b = a \in A$, it follows by (PA1) that $g^{-1} \in G^a$ and $b = g^{-1} \cdot a \in G^A \cdot A = A$, hence $(G^B \cdot B) \cap A = \varnothing$, so that $G^B \cdot B \subseteq (X \setminus A) = B \subseteq G^B \cdot B$, and B is G-invariant.

□

Remark 2.5. As a consequence of ítem (i) in Lemma 2.4, the following formula holds:

$$G^U \cdot U = \bigcup_{g\in G^U} \eta_g(U \cap X_{g^{-1}}).$$

From now on in this work, G will denote a topological group and X a topological space. We endow $G \times X$ with the product topology and $G * X$ with the subspace topology. Moreover, $\eta : G * X \to X$ will denote a partial action. It is said that η is:

- *topological,* if X_g is open and η_g is a homeomorphism, for all $g \in G$;

- *continuous,* if η is a continuous map;
- *nice,* if η is continuous and $G * X$ is open in $G \times X$.

We proceed with the following lemma:

Lemma 2.6. *Let η be a partial action of G on X. The following assertions hold:*

(i) *If η is topological and U is an open subset of X, then the set $G^U \cdot U$ defined in Remark (2.5) is open,*

(ii) *If $G * X$ is clopen, then X_g is clopen, for all $g \in G$.*

Proof. (i) It follows from the fact that for every $g \in G$ the set X_g is open, η_g is a homeomorphism and $G^U \cdot U = \bigcup_{g \in G^U} \eta_g(U \cap X_{g^{-1}})$ as observed in Remark 2.5. For (ii) take $g \in G$, first let us prove that X_g is closed. Indeed, if $X_g = X$ this is clear. Otherwise, take $x \in X \setminus X_g$ and since $G * X$ is closed, there exist open sets $U \subseteq G$ and $V \subseteq X$ such that $(g^{-1}, x) \in U \times V \subseteq (G * X)^c$. Moreover, if $y \in V$ then $(g^{-1}, y) \notin G * X$ and $y \notin X_g$. This shows that X_g is closed. To prove that X_g is open take $x \in X_g$, then $(g^{-1}, x) \in G * X$ and there are open sets $U \subseteq G$ and $V \subseteq X$ such that $(g^{-1}, x) \in U \times V \subseteq G * X$, from this we get $x \in V \subseteq X_g$ and X_g is open. □

Let X and Y be sets equipped with partial actions by a group G. A function $f : X \to Y$ is called a *G-map* if for each $g \in G$ and $x \in X$ such that $\exists g \cdot x$ then $\exists g \cdot f(x)$ and $f(g \cdot x) = g \cdot f(x)$. If G is a topological group, then we denote by $\mathcal{PA}_G$ the category of pairs (θ, X), where θ is a continuous partial action of G on the topological space X, and its morphisms are continuous G-maps. Also, we denote by $\mathcal{A}_G$ the full subcategory of $\mathcal{PA}_G$ whose objects are continuous actions of G on topological spaces. The next lemma tells us that $\mathcal{PA}_G$ is a category with products (see Proposition 2 and Theorem 3 of [3]). Other categorical properties of $\mathcal{PA}_G$ and $\mathcal{A}_G$ will be explored in Section 4.

Lemma 2.7. *Let G be a group and $\{X_j\}_{j \in J}$ be a collection of nonempty sets equipped with partial actions $\theta^j : G * X_j \to X_j$, for all $j \in J$. Put $X = \prod_{j \in J} X_j$ and set*

$$G * X = \{(g, (x_j)_{j \in J}) \in G \times X : (g, x_j) \in G * X_j, \ \forall j \in J\}.$$

We define:

$$\Delta(\theta) : G * X \ni (g, x) \mapsto (\theta^j(g, x_j))_{j \in J} \in X, \tag{2.3}$$

where $x = (x_j)_{j \in J} \in X$, then $\Delta(\theta)$ is a partial action of G on X. Moreover, if G is a topological group, $\{X_j\}_{j \in J}$ is a family of topological spaces and X is endowed with the product topology, then the following assertions hold:

(a) *For any $j \in J$, the projection $\rho_j : X \to X_j$ is a continuous G-map and $\Delta(\theta)$ is continuous, provided that θ^j is continuous for any $j \in J$,*

(b) *$G * X$ is closed in $G \times X$ provided that $G * X_j$ is closed in $G \times X_j$, for each $j \in J$,*

(c) *The triple $(X, \Delta(\theta), \{\rho_j\}_{j \in J})$ satisfies the universal property of product in $\mathcal{PA}_G$.*

Furthermore, if J is finite, then $\Delta(\theta)$ is topological as long as for all $j \in J$, each θ^j is topological.

Proof. It is not difficult to see that $\Delta(\theta)$ is a partial action of G on X. Take $j \in J$. It is easy to see that ρ_j is a continuous G-map. Now suppose that θ^j is continuous, to prove that $\Delta(\theta)$ is continuous, we only need to show that $\rho_j \circ \Delta(\theta) : G * X \to X_j$ is continuous. Set $\gamma_j : G * X \ni (g, x) \mapsto (g, \rho_j(x)) \in G * X_j$, then γ_j is continuous and $\rho_j \circ \Delta(\theta) = \theta^j \circ \gamma_j$ is continuous, as desired.

To prove (b), take $g \in G$ and $x \in X$ such that $(g, x) \notin G * X$. Then there exists $j \in J$ with $(g, x_j) \notin G * X_j$. Since $G * X_j$ is closed there are neighborhoods $U \subseteq G$ and $V \subseteq X_j$, of g and x_j, respectively for which $(U \times V) \cap G * X_j = \varnothing$. Note that $W = U \times \rho_j^{-1}(V)$ is a neighborhood of (g, x) in $G \times X$ and $W \cap (G * X) = \varnothing$. This shows that $G * X$ is closed in $G \times X$.

We now prove (c). Let (υ, Y) be an object in $\mathcal{PA}_G$, where $\upsilon : G * Y \to Y$ is a continuous partial action, and suppose that there exists $\{\zeta_j : Y \to X_j\}_{j \in J}$ a collection of continuous G-maps. We set $\zeta : Y \ni y \mapsto (\zeta_j(y))_{j \in J} \in X$. Observe that ζ is a continuous G-map and the following diagram is commutative for each $j \in J$:

$$\begin{tikzcd} Y \arrow[r, "\zeta"] \arrow[d, "\zeta_j"'] & X \arrow[dl, "\rho_j"] \\ X_j & \end{tikzcd}$$

Also note the map ζ is unique, and this finishes part (c). From the rest of the proof, we suppose that J is finite. Observe that:

$$X_g = \prod_{j \in J} (X_j)_g, \text{ for any } g \in G.$$

Since each θ^j is topological then for all $g \in G$ and $j \in J$, the set $(X_j)_g$ is open; besides, since J is finite, then the product topology coincides with the box topology on X, therefore X_g is open. The continuity of each bijection $\Delta(\theta)_g : X_{g^{-1}} \ni (x_j)_{j \in J} \mapsto (\theta^j(g, x_j))_{j \in J} \in X_g$, is clear, then $\Delta(\theta)$ is topological. $\square$

Given a topological group G, it is easy to see that $\rho : G \times G \ni (g, h) \mapsto gh^{-1} \in G$ is a continuous (global) action of G on itself. Then the following result follows from Lemma 2.7 and the action ρ defined above:

Corollary 2.8. Let η be a topological partial action of G on X, then the family of bijections $\hat{\eta} = \{\hat{\eta}_g : (G \times X)_{g^{-1}} \to (G \times X)_g\}_{g \in G}$, where $(G \times X)_g = G \times X_g$ and

$$\hat{\eta}_g : G \times X_{g^{-1}} \ni (h, x) \mapsto (hg^{-1}, \eta_g(x)) \in G \times X_g,$$

is a topological partial action of G on $G \times X$. In addition:

(a) $\hat{\eta}$ is continuous whenever η is,
(b) $\mathrm{dom}(\hat{\eta})$ is closed in $G \times (G \times X)$ whenever $\mathrm{dom}(\eta)$ is closed in $G \times X$.

2.1. **Induced partial actions and globalization.** Let u be a continuous action of G on a topological space Y and $X \subseteq Y$ be an open set. For $g \in G$, set

$$X_g = X \cap u_g(X) \quad \text{and} \quad \eta_g = u_g \restriction X_{g^{-1}}, \tag{2.4}$$

then $\eta\colon G * X \ni (g,x) \mapsto \eta_g(x) \in X$ is a nice partial action of G on X, and we say that η is *induced* by u or that η is the *restriction* of u to X. Observe that $G * X$ is open in $G \times X$ because:

$$G * X = \{(g,x) \in G \times X : u_g(x) \in X\} = (G \times X) \cap u^{-1}(X).$$

An important question in the study of partial actions is whether they can be induced by global actions. In the topological sense, this turns out to be affirmative, and a proof was given in [2, Theorem 1.1] and independently in [10, Section 3.1]. We recall their construction. Let $\eta = \{\eta_g : X_{g^{-1}} \to X_g\}_{g\in G}$ be a topological partial action of G on X. We define an equivalence relation R on $G \times X$ as follows:

$$(g,x)R(h,y) \Longleftrightarrow x \in X_{g^{-1}h} \quad \text{and} \quad \eta_{h^{-1}g}(x) = y, \tag{2.5}$$

and we denote by $[g,x]$ the equivalence class of the pair (g,x). Consider the *enveloping space* or the *globalization* $X_G = (G \times X)/R$ of X, endowed with the quotient topology. Then by [10, (iii) Proposition 3.9], the following action is continuous:

$$\mu\colon G \times X_G \ni (g,[h,x]) \to [gh,x] \in X_G. \tag{2.6}$$

The map μ is called the *enveloping action* of η. Further the following map:

$$\iota\colon X \ni x \mapsto [1,x] \in X_G, \tag{2.7}$$

is a continuous injection such that $G \cdot \iota(X) = X_G$. Moreover, it follows by [10, Proposition 3.12] that $\iota : X \to \iota(X)$ is an open map if and only if η is continuous. On the other hand, $\iota(X)$ is open in X_G if and only if $G * X$ is open in $G \times X$. Therefore, if η is a nice partial action, then $\iota : X \to X_G$ is an open map and we can identify X with the open subspace $\iota(X)$ of X_G.

2.2. **An alternative construction of the enveloping action.** Given a topological partial action of G on X, one defines the *orbit equivalence relation* $\sim_G$ on X as follows:

$$x \sim_G y \Longleftrightarrow \ \exists g \in G^x \text{ such that } g \cdot x = y, \tag{2.8}$$

for each $x, y \in X$. The elements of $X/\sim_G$ are the *orbits* $G^x \cdot x$, with $x \in X$, the set $X/\sim_G$ is called the *orbit space* and it is endowed with the quotient topology. It follows by [15, Lemma 3.2] that the next *induced quotient map:*

$$\pi_G : X \ni x \mapsto G^x \cdot x \in X/\sim_G, \tag{2.9}$$

is continuous and open.

We will use the fact that $\mathcal{PA}_G$ is a category with products to present an alternative construction of the enveloping action of a partial action. First we highlight the following auxiliary lemma:

Lemma 2.9. *Let $\eta : G * X \to X$ be a topological partial action and let $\gamma : G \times X \to X$ be a continuous global action, such that:*

$$\text{If } x \in X, g \in G^x \text{ and } h \in G, \text{ then } g \in G^{\gamma(h,x)}, \text{ and } \gamma(h,\eta(g,x)) = \eta(g,\gamma(h,x)). \tag{2.10}$$

Then γ induces a continuous global action of G on the orbit space $X/\sim_G$.

Proof. Consider the following map:

$$\begin{aligned}\Gamma_\gamma \colon G \times (X/\sim_G) &\longrightarrow X/\sim_G\\ (g, \pi_G(x)) &\mapsto \pi_G(\gamma(g,x)),\end{aligned}$$

where π_G is the induced quotient map of η, defined in (2.9). Then follows by (2.10) that Γ_γ is a well-defined map, moreover it is not difficult to check that Γ_γ is a global action. Observe that Γ_γ is continuous because the following diagram is commutative:

$$\begin{tikzcd} G \times X \arrow[r, "\gamma"] \arrow[d, "\mathrm{id}_G \times \pi_G"'] & X \arrow[d, "\pi_G"] \\ G \times X/\sim_G \arrow[r, "\Gamma_\gamma"'] & X/\sim_G \end{tikzcd}$$

□

Remark 2.10. Let $\eta : G * X \to X$ be a topological partial action of G on X and consider the topological partial action $\hat{\eta}$ given in Corollary 2.8. Denote by $\sim_{\hat{G}}$ the orbit equivalence relation on $G \times X$ determined by $\hat{\eta}$.

a) By [15, Theorem 3.3], we get

$$X_G = (G \times X)/\sim_{\hat{G}}. \tag{2.11}$$

b) On the other hand, if $\lambda : G \times G \to G$ is the binary operation on G and $\mathbf{1} : G \times X \ni (g,x) \mapsto x \in X$ is the trivial action, then it follows from Lemma 2.7 that the next function is a global action:

$$\begin{aligned}\lambda \times \mathbf{1} \colon G \times (G \times X) &\longrightarrow G \times X\\ (g,(h,x)) &\mapsto (gh,x).\end{aligned}$$

Observe that $\lambda \times \mathbf{1}$ and $\hat{\eta}$ satisfy (2.10), hence $\lambda \times \mathbf{1}$ induces a global action $\Gamma_{\lambda\times\mathbf{1}}$. It follows from (2.11) that $\Gamma_{\lambda\times\mathbf{1}} = \mu$.

3. Partial actions on profinite spaces

For the reader's convenience we recall that a topological space X is profinite if and only if X is compact, Hausdorff, and zero-dimensional (that is, X has a basis of clopen subsets). In addition, if X is compact (not necessarily Hausdorff), then follows by [14, Proposition 2.3] that X is Hausdorff and zero-dimensional if and only if any two different points in X can be separated. Recall that two points u and v in X can be *separated* if there are disjoint open subsets U and V of X such that $u \in U$, $v \in V$, and $U \cup V = X$.

Our next goal is to show that $X/\sim_G$ is profinite as long as X is profinite and G is compact. By the previous paragraph, it is enough to verify that any two points in $X/\sim_G$ can be separated.

Theorem 3.1. *Let $\eta : G * X \to X$ be a nice partial action of a compact group G on a profinite space X such that $G * X$ is closed, then $X/\sim_G$ is a profinite space.*

Proof. Note that $X/\sim_G$ is compact. Now we show that different points $G^x \cdot x$ and $G^y \cdot y$ in $X/\sim_G$ can be separated. Let $\mathcal{C} := \{U \subseteq X : U \text{ is clopen, } x \in U\}$, then $\mathcal{C} \neq \varnothing$. We claim that there exists $U \in \mathcal{C}$ such that $G^U \cdot U \cap G^y \cdot y = \varnothing$. Otherwise, for each $V \in \mathcal{C}$, the set $\tilde{F}_y(V) = \{(g, v) \in G * V : \eta_g(v) = y\}$ is nonempty. Since $\tilde{F}_y(V) = \eta^{-1}(y) \cap (G * V)$, then it is closed in $G * V$ and thus closed in $G * X$. Now, if $V_1, V_2 \in \mathcal{C}$ then $\tilde{F}_y(V_1 \cap V_2) \subseteq \tilde{F}_y(V_1) \cap \tilde{F}_y(V_2)$. In that sense $\{\tilde{F}_y(V)\}_{V \in \mathcal{C}}$ is a family in $G * X$ with the finite intersection property, thus there exists $(g, v) \in \bigcap\limits_{V \in \mathcal{C}} \tilde{F}_y(V)$, this gives $v = x$ and $\eta_g(x) = y$ which leads to a contradiction. Then there is $U \in \mathcal{C}$ such that $G^U \cdot U \cap G^y \cdot y = \varnothing$. Now we check that $G^U \cdot U$ is clopen. Indeed, it is open thanks to Lemma 2.6; moreover, given that $G * U$ is compact, X is Hausdorff and $\eta(G * U) = G^U \cdot U$, then we have that $G^U \cdot U$ is closed. Finally, let $A = G^U \cdot U$ and $B = X \setminus A$, then by Lemma 2.4, the sets A and B are G-invariant and clopen, so $\pi_G^{-1}(\pi_G(A)) = A$ and $\pi_G^{-1}(\pi_G(B)) = B$, thus $G^x \cdot x$ and $G^y \cdot y$ are separated by the sets $\pi_G(A)$ and $\pi_G(B)$, respectively. $\square$

Corollary 3.2. *Let η be a nice partial action of a profinite group G on a profinite space X. If $G * X$ is closed, then the enveloping space X_G is profinite.*

Proof. By part (a) of Remark 2.10 we have that $X_G = (G \times X)/\sim_{\hat{G}}$, where $\sim_{\hat{G}}$ is the orbit equivalence relation of the nice partial action $\hat{\eta}$ presented in Corollary 2.8. On the other hand, it follows by (b) in Corollary 2.8 that $G * (G \times X)$ is closed in $G \times G \times X$. Thus, X_G is profinite thanks to Theorem 3.1. $\square$

Example 3.3. [8, p. 22] **Partial Bernoulli action.** Let G be a discrete group and $X := \{0, 1\}^G$. There is a continuous global action $\beta = \{\beta_g\}_{g \in G}$, where for all $\omega \in X$, $\beta_g(\omega) = g\omega$. The topological partial Bernoulli action η is obtained by restricting β to the open set $\Omega_1 = \{\omega \in X : \omega(1) = 1\}$. Thus, by (2.4), $D_g := \Omega_1 \cap \beta_g(\Omega_1) = \{\omega \in X : \omega(1) = 1 = \omega(g)\}$, and $\eta_g = \beta_g \restriction D_{g^{-1}}, g \in G$. Let us show that $G * \Omega_1$ is clopen. Let $\{(n_i, x_i)\}_{i \in I}$ be a net in $G * \Omega_1$, convergent to (n, x). Since G is discrete then $(n_i)_{i \in I}$ is eventually constant, so $n_i = n$ for large $i \in I$. On the other hand, as $x_i \longrightarrow x$, then $1 = x_i(1) \longrightarrow x(1)$ and from this $x(1) = 1$. Similarly, it is obtained that $x(n^{-1}) = 1$ and from the above we conclude $(n, x) \in G * \Omega_1$. Now, if $(n, x) \in G * \Omega_1$, then $x \in V = (\eta_{n^{-1}})^{-1}(\{1\})$, and $(n, x) \in \{n\} \times V \subseteq G * \Omega_1$. This shows that $G * \Omega_1$ is clopen. Thus, if G is finite, then Theorem 3.1 implies that $X/\sim_G$ is a profinite space.

3.1. **On continuous sections of the quotient map π_G.** In this section, we are interested in providing conditions under which the quotient map π_G defined in (2.9), has a continuous section. For this, we start with the following lemma:

Lemma 3.4. *Let $\mu : G \times Y \ni (g, y) \mapsto g \cdot y \in Y$ be a continuous action of a topological group G on a profinite space Y. Suppose that X is a clopen subset of Y such that $G \cdot X = Y$. Then there exists a retraction $r : Y \to X$, such that the following items are true:*

(i) *The map*

$$\overline{r} : Y/\sim_G \to X/\sim_G, \ \ G \cdot y \mapsto G^{r(y)} \cdot r(y), \tag{3.1}$$

is continuous, where, for $x \in X$, the set $G^x \cdot x$ is the orbit of x given by the induced partial action of μ on X (see equation (2.10)).

(ii) *If $\overline{i} : X/\sim_G \to Y/\sim_G$, $G^x \cdot x \to G \cdot x$, then $\overline{r} \circ \overline{i} = \mathrm{id}_{X/\sim_G}$.*

Proof. By assumption $Y = \bigcup\limits_{g \in G} g \cdot X$. Since Y is compact there are $g_1 = 1, g_2, \cdots, g_n \in G$ such that $Y = \bigcup\limits_{i=1}^{n} g_i \cdot X$. For $1 \le j \le n$, set $X_j = g_j \cdot X \setminus \bigcup\limits_{i=1}^{j-1} g_i \cdot X$, thus the family $\{X_j\}_{j=1}^n$ is a partition of Y such that $X_1 = X$, and $g_j^{-1} \cdot X_j \subseteq X$. Further, the continuity of the action implies that the function $r_j : X_j \ni x \mapsto g_j^{-1} \cdot x \in X$, is continuous and thus $r = \bigcup\limits_{j=1}^{n} r_j : Y \to X$ is a retraction of Y in X.

(i) We first check that $\overline{r}$ is well defined. Take $x, y \in Y$ such that $G \cdot x = G \cdot y$, then there exist $g \in G$ and $1 \le i, j \le n$, for which $x = g \cdot y, x \in X_i$ and $y \in X_j$. Then $r(y) = g_j^{-1} \cdot y$, and

$$r(x) = g_i^{-1} \cdot x = (g_i^{-1} g) \cdot y = (g_i^{-1} g g_j) \cdot r(y),$$

but $r(x) \in (g_i^{-1} g g_j \cdot X) \cap X$, that is, $g_j^{-1} g^{-1} g_i \in G^{r(x)}$, where $G^{r(x)}$ is given by (2.10). Note that $(g_j^{-1} g^{-1} g_i) \cdot r(x) = r(y)$, then $G^{r(x)} \cdot r(x) = G^{r(y)} \cdot r(y)$ and $\overline{r}$ is well defined. To show that $\overline{r}$ is continuous, consider the quotient map $\Pi_G : Y \to Y/\sim_G$ induced by μ. Since $\overline{r} \circ \Pi_G = \pi_G \circ r$, we get that $\overline{r}$ is continuous.

(ii) It is clear that $\overline{i}$ is well-defined, moreover, the result follows from the fact that r is a retraction. □

Definition 3.5. Let $\eta : G * X \longrightarrow X$ be a topological partial action of a group G on a space X. We say that η is free if for each $(g, x) \in G * X$ such that $\eta(g, x) = x$, we have $g = 1$.

Given a free partial action η, a relevant fact is that its enveloping action μ of η is automatically free.

Theorem 3.6. Let η be a continuous and free partial action of a profinite group G on a profinite space X such that $G * X$ is clopen. Then $\pi_G : X \to X/\sim_G$ has a continuous section.

Proof. Let μ be the enveloping action of η given by (2.6). Since η is free, then so is μ, and it follows by [14, Proposition 2.9] that the corresponding quotient map Π_G has a continuous section $q : X_G/\sim_G \to X_G$. Let

$$\bar{\iota} : X/\sim_G \ni G^x \cdot x \mapsto G \cdot \iota(x) \in X_G/\sim_G,$$

where ι is the map defined in (2.7). Note that $\bar{\iota} \circ \pi_G = \Pi_G \circ \iota$, then $\bar{\iota}$ is continuous. Moreover, $G \cdot \iota(X) = X_G$ and $\iota(X)$ is clopen in X_G because ι is open, and $\iota(X)$ is compact in the profinite (in particular Hausdorff) space X_G. It follows from Lemma 3.4 that there is a retraction $r : X_G \to X$ such that the next diagram is commutative:

$$\begin{tikzcd}
X \arrow[r, "\iota"'] \arrow[d, "\pi_G"] & X_G \arrow[l, bend right, "r"'] \arrow[d, "\Pi_G"'] \\
X/\sim_G \arrow[r, "\bar{\iota}"'] \arrow[u, bend left, "r \circ q \circ \bar{\iota}"] & X_G/\sim_G \arrow[l, bend left, "\bar{r}"] \arrow[u, bend right, "q,"']
\end{tikzcd}$$

in particular, $\pi_G \circ r = \bar{r} \circ \Pi_G$. Further, by the same lemma, $\bar{r} \circ \bar{\iota} = \mathrm{id}_{X/\sim_G}$, then the following equalities are valid:

$$\pi_G \circ (r \circ q \circ \bar{\iota}) = \bar{r} \circ (\Pi_G \circ q) \circ \bar{\iota} = \bar{r} \circ \bar{\iota} = \mathrm{id}_{X/\sim_G},$$

and the map $r \circ q \circ \bar{\iota}$ is continuous, thus π_G has a continuous section. $\square$

Remark 3.7. In general the assumption that η acts freely on X, cannot be omitted even when η acts globally, see for instance [18, Example 5.6.8].

Example 3.8. Let G be a profinite group and μ be a free continuous action of G on a profinite space X. Take Y a clopen subset of X, then μ induces a free partial action $\eta : G * Y \to Y$. It is not difficult to show that $G * Y$ is closed in $G \times Y$. Moreover $G * Y$ is open in $G \times Y$ thanks to [10, ii) Theorem 3.13]. Then it follows by Theorem 3.6 that the quotient map π_G has a continuous section.

3.2. **Relations between continuous sections of π_G and Π_G.** Let μ be the globalization of η. We study relations between continuous sections of the maps π_G and Π_G, being Π_G the corresponding quotient map of the enveloping action μ.

Proposition 3.9. Let η be a nice partial action of a topological group G on a space X. Then the following statements hold:

(i) If π_G has a continuous section, so does Π_G,

(ii) If q is a continuous section of Π_G such that $\operatorname{im} q \subseteq \iota(X)$, then π_G has a continuous section,

(iii) If Π_G and $\hat{\Pi}_G$ have continuous sections, then π_G has a continuous section, where $\hat{\Pi}_G$ is the quotient map $G \times X \to X_G$ induced by the partial action $\hat{\eta}$ of Corollary 2.8.

Proof. (i) Suppose that $q : X/\sim_G \to X$ is a continuous section of π_G. Consider the following function:

$$s : (X_G/\sim_G) \ni G\cdot[1,x] \mapsto \iota(q(\pi_G(x))) \in X_G,$$

where ι is given by (2.7). We claim that s is a continuous section of Π_G. First, note that s is well defined. In fact, let $x, y \in X$ such that $[1,x] \sim_G [1,y]$, we have $\pi_G(x) = \pi_G(y)$. Set $z_x = q(\pi_G(x))$ and $z_y = q(\pi_G(y))$, then $\pi_G(z_x) = \pi_G(x) = \pi_G(y) = \pi_G(z_y)$. Thus the next equalities are valid:

$$s(G\cdot[1,x]) = [1,z_x] = [1,z_y] = s(G\cdot[1,y]),$$

and s is well-defined. Since q and ι are continuous, we get that s is continuous. To finish the proof, we take $x \in X$ and let $y_x \in X$, such that $q(\pi_G(x)) = y_x$. Since $\pi_G(y_x) = \pi_G(x)$, there is $g \in G^x$ such that $\eta_g(x) = y_x$. Thus, $\mu(g,[1,x]) = [1,y_x]$, and we have the following equalities:

$$(\Pi_G \circ s)(G\cdot[1,x]) = \Pi_G([1,y_x]) = G\cdot[1,y_x] = G\cdot[1,x].$$

This shows that s is a continuous section of Π_G.

(ii) Let $r : (X/\sim_G) \ni G^x\cdot x \mapsto \iota^{-1}(q(G\cdot[1,x])) \in X$, be a function. It is not difficult to check that r is well-defined, moreover, that $G*X$ is open implies that ι is open and thus r is continuous. Finally, we take $x, z_x \in X$ such that $q(G\cdot[1,x]) = [1,z_x]$, then $G\cdot[1,x] = G\cdot[1,z_x]$ which gives $\pi_G(x) = \pi_G(z_x)$, this implies that $\pi_G(r(G^x\cdot x)) = \pi_G(z_x) = \pi_G(x)$, and r is a continuous section of π_G.

(iii) Let $q : (X_G/\sim_G) \to X_G$ and $t : X_G \to G\times X$ be continuous sections of Π_G and $\hat{\Pi}_G$, respectively. Consider the next map:

$$p : X/\sim_G \ni G^x\cdot x \mapsto \mathrm{proj}_2(t(q(G\cdot[1,x]))) \in X.$$

We check that p is a continuous section of π_G. The map p is well-defined and continuous, since $X/\sim_G \ni G^x\cdot x \mapsto G\cdot[1,x] \in X_G/\sim_G$ is continuous. On the other hand, we take $x \in X$. If $h,k \in G$ and $y,z \in X$, such that $q(G\cdot[1,x]) = [h,y]$ and $t([h,y]) = (k,z)$, then $G\cdot[1,x] = G\cdot[h,y]$ and $[k,z] = [h,y]$. Thus, $\eta_{h^{-1}g}(x) = y$ and $\eta_{k^{-1}h}(y) = z$, for some $g \in G$, thus $G^x\cdot x = G^z\cdot z$, and we have that $\pi_G(p(G^x\cdot x)) = G^z\cdot z = G^x\cdot x$, as desired. □

It follows by [16, Lemma 4.2] that Π_G has a Borel section, provided that G and X are also a Polish group and a Polish space, respectively. Thus, by ítem (i) of Proposition 3.9, and the proof (iii) of the same proposition, we have the next result:

Proposition 3.10. Let η be a nice partial action of a profinite Polish group G on a profinite Polish space X such that $G*X$ is closed, then π_G has a Borel section.

4. Partial actions and reflective categories

Let's remember the definition of reflective subcategory.

Definition 4.1. Let $\mathcal{C}$ be a category. A subcategory $\mathcal{D}$ of $\mathcal{C}$ is reflective if the inclusion functor $\mathcal{D} \to \mathcal{C}$ has a left adjoint.

Remark 4.2. It is well-known that the category $\mathcal{P}$ of profinite spaces is a reflective subcategory of the category $\mathcal{K}$ of compact Hausdorff spaces, where the inclusion functor $\mathcal{P} \to \mathcal{K}$ has as a left adjoint the functor $\mathrm{Comp} : \mathcal{K} \to \mathcal{P}$, sending a compact space X to the space $\mathrm{Comp}(X)$ of connected components of X, equipped with the weakest topology making the projection $\pi_X : X \to \mathrm{Comp}(X)$ continuous, here the unit of the adjunction is precisely the projection π_X.

Proposition 4.3. [13, Theorem 2 (i), p. 83] A subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is reflective if and only if, for any object $C \in \mathcal{C}$ there exists an object $R_{\mathcal{D}}(C) \in \mathcal{D}$ and a morphism $\epsilon_{\mathcal{D}}(C) : C \to R_{\mathcal{D}}(C)$, such that for all $D \in \mathcal{D}$ and $\varphi : C \to D$ there is a unique $\psi : R_{\mathcal{D}}(C) \to D$ in $\mathcal{D}$ with $\psi \circ \epsilon_{\mathcal{D}}(C) = \varphi$.

The following fact follows from [2, Theorem 1.1] and Proposition 4.3:

Proposition 4.4. Given a topological group G, the category $\mathcal{A}_G$ is reflective in $\mathcal{PA}_G$. More precisely, the enveloping functor $E : \mathcal{PA}_G \to \mathcal{A}_G$ sends an object (η, X) in $\mathcal{PA}_G$ to (μ, X_G) (defined by (2.6)), and a morphism $f : X \to Y$ to $E(f) : X_G \ni [g, x] \mapsto [g, f(x)] \in Y_G$; is left adjoint to the inclusion functor.

We have the next result:

Proposition 4.5. Let G be a topological group. Then the category $\mathcal{PA}_G$ has inverse limits.

Proof. Let $(J, \leq)$ be a directed set. Given an inverse system $\theta = \{(\theta^j, X_j), f_i^j\}_{\substack{i,j \in J \\ i \leq j}}$ in $\mathcal{PA}_G$, consider $X = \prod\limits_{j \in J} X_j$, endowed with the product topology, then by Lemma 2.7 the map $\Delta(\theta) : G * X \to X$, defined in (2.3) is a continuous partial action of G on X. Set

$$\hat{X} = \{(x_j)_{j \in J} \in X : f_i^j(x_j) = x_i, \forall i, j \in J, i \leq j\},$$

and

$$G * \hat{X} = \{(g, (x_j)_{j \in J}) \in G \times \hat{X} : (g, x_j) \in G * X_j, \forall j \in J\},$$

both endowed with the subspace topology. Then $G * \hat{X} \subseteq G * X$ and $\hat{X}$ is G-invariant over $\Delta(\theta)$, thus by (iii) of Lemma 2.4 we have that $\hat{\theta} := \Delta(\theta)\restriction_{G * \hat{X}}$ is a continuous partial action of G on $\hat{X}$, and it's not difficult to see that $(\hat{\theta}, \hat{X})$, together with the family of projections $\{\rho_j \restriction_{\hat{X}}\}_{j \in J}$ (defined in (a) of Lemma 2.7), is the inverse limit of θ, that is, $\varprojlim (\theta^j, X_j) = (\hat{\theta}, \hat{X})$. $\square$

Corollary 4.6. *Let G be a profinite group, $(I, \leq)$ be a directed set and $\{(\eta_i, X_i), f_i^j\}_{\substack{i,j \in I \\ i \leq j}}$ be an inverse system in $\mathcal{PA}_G$ such that $\varprojlim (\eta_i, X_i) = (\eta, X)$ in $\mathcal{PA}_G$. If E is the enveloping functor defined in Proposition 4.4 and for all $i \in I$, the action $(\mu_i, (X_i)_G)$ is the enveloping action of (η_i, X_i), then*

$\{(\mu_i, (X_i)_G), E(f_i^j)\}_{\substack{i,j\in I \\ i\leq j}}$ is an inverse system in $\mathcal{A}_G$ and $\varprojlim(\mu_i, (X_i)_G) = (\mu, X_G)$, where (μ, X_G) is the enveloping action of (η, X). In particular, if every X_i is finite and discrete, then X_G is profinite provide that $\operatorname{dom}\eta$ is clopen or each $\operatorname{dom}\eta_i$ is clopen, for all $i \in I$.

Proof. The first assertion follows from Proposition 4.4 and the fact that left adjoint functors preserve inverse limits. For the last assertion, suppose first that $\operatorname{dom}\eta$ is closed. Now X is profinite because $\varprojlim X_i = X$, and the result follows from Corollary 3.2. Finally, if $\operatorname{dom}\eta_i$ is closed for any $i \in I$, we have by Corollary 3.2 that $(X_i)_G$ is profinite for any $i \in I$, thus X_G is profinite because $\varprojlim (X_i)_G = X_G$ is an inverse limit of profinite spaces. □

4.1. **A remark on separately continuous actions on topological spaces.** For the reader's convenience, we recall that an action $\beta : G \times X \to X$ of a topological group G on a topological space X is *separately continuous*, if for each pair $(g_0, x_0) \in G \times X$ the maps $\beta_{g_0} : X \ni x \mapsto \beta(g_0, x) \in X$ and

$$\beta^{x_0} : G \ni g \mapsto \beta(g, x_0) \in X, \tag{4.1}$$

are continuous.

We denote by $\mathcal{SA}_G$ the category of separately continuous actions of a topological group G on topological spaces and whose morphisms are continuous G-maps. Also, let $\mathcal{SAK}_G$ and $\mathcal{SAP}_G$ be the subcategories of $\mathcal{SA}_G$ whose objects are separately continuous actions of G on compact Hausdorff spaces and profinite spaces, respectively. We shall prove that $\mathcal{SAP}_G$ is reflective in $\mathcal{SAK}_G$. For this, it is necessary to introduce some notations and facts from categories related to topological spaces. To the group G we associate the category $\mathcal{C}_G$ having as object the unitary set $\{\bullet\}$ and as morphisms the group G; we denote also by Top the category of topological spaces and continuous maps. It is well-known that the category of functors $\operatorname{Fun}(\text{Top}, \mathcal{C}_G)$ is equivalent to the category $\mathcal{A}_G$ of continuous actions of G on topological spaces, and $\mathcal{SA}_G$ can be indentified with the full subcategory of $\operatorname{Fun}(\text{Top}, C_G)$ of those actions β such that the map β^{x_0} defined in (4.1) is continuous, for any $x_0 \in X$.

Lemma 4.7. *Let $\mathcal{D}$ be a reflective subcategory of a category $\mathcal{C}$. Given a small category $\mathcal{I}$, $\operatorname{Fun}(\mathcal{I}, \mathcal{D})$ is reflective in $\operatorname{Fun}(\mathcal{I}, \mathcal{C})$.*

We finish this work with the following fact:

Proposition 4.8. *$\mathcal{SAP}_G$ is a reflective subcategory of $\mathcal{SAK}_G$.*

Proof. We observed in Remark 4.2 that the category $\mathcal{P}$ of profinite spaces is a reflective subcategory of the category $\mathcal{K}$ of compact Hausdorff spaces, then follows by Lemma 4.7 that the inclusion functor $\operatorname{Fun}(\mathcal{C}_G, \mathcal{P}) \to \operatorname{Fun}(\mathcal{C}_G, \mathcal{K})$ has a left adjoint, which we denote by $\operatorname{Comp}_G : \operatorname{Fun}(\mathcal{C}_G, \mathcal{K}) \to \operatorname{Fun}(\mathcal{C}_G, \mathcal{P})$. We consider a separately continuous action $\beta : G \times X \to X$ in $\operatorname{Fun}(\mathcal{C}_G, \mathcal{K})$, we shall check that its reflection $\bar{\beta} = \operatorname{Comp}_G(\beta)$ belongs to $\mathcal{SAP}_G$. Indeed, if $\bar{x} \in \operatorname{Comp}(X)$, we set $\bar{\beta}^{\bar{x}} : G \ni g \mapsto \bar{\beta}_g(\bar{x}) \in \operatorname{Comp}(X)$. Consider $x \in X$ such that $\bar{x} = \pi_X(x)$, where $\pi_X : X \to Comp(X)$ is the projection map. Since β^x

is continuous we have that $\bar{\beta}^{\bar{x}} = \pi_X \circ \beta^x$ is continuous, and thus $\bar{\beta}$ belongs to $\mathcal{SAP}_G$, therefore $\mathcal{SAP}_G$ is reflective in $\mathcal{SAK}_G$. $\square$

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