Applied General Topology

Appl. Gen. Topol. 25, no. 1 (2024), 159-174
doi:10.4995/agt.2024.19798

Some existence and uniqueness results for a solution of a system of equations

Deepak Khantwal and Rajendra Pant

Department of Mathematics & Applied Mathematics, University of Johannesburg Kingsway Campus, Auckland Park 2006, South Africa (deepakkhantwal15@gmail.com, dkhantwal@uj.ac.za (D. Khantwal), pant.rajendra@gmail.com, rpant@uj.ac.za (R. Pant))

Communicated by I. Altun

Abstract

This paper presents some existence and uniqueness results for a solution of a system of equations. Our results extend and generalize the well-known and celebrated results of Boyd and Wong [Proc. Amer. Math. Soc. 20 (1969)], Matkowski [Dissertations Math. (Rozprawy Mat.) 127 (1975)], Proinov [Nonlinear Anal. 64 (2006)], Ri [Indag. Math. (N. S.) 27 (2016)] and many others. We also present some illustrative examples to validate our results.

2020 MSC: *47H10; 54H25.*

Keywords: *Matkowski's contraction; system of equations; control function; metric space.*

1. Introduction and Preliminaries

Let $(W_i, \rho_i),\ i = 1, 2, \dots, n$, be metric spaces and $W := W_1 \times \cdots \times W_n$. Assume that $T_i : W \to W_i,\ i = 1, \dots, n$, are mappings, $\mathbb{N}$ the set of natural numbers, $\mathbb{R}$ the set of real numbers and $(\omega^m) = (\omega_1^m, \dots, \omega_n^m)$, $m \in \mathbb{N}$, be a sequence in W. We denote $\Phi = \{\varphi : [0, \infty) \to [0, \infty) \mid \varphi(t) < t,\ \limsup_{t \to s^+} \varphi(t) < s$ for all $t > 0\}$.

In 1975, Matkowski [20] obtained an important generalization of the Banach contraction theorem (BCT) for a system of mappings $(T_1, \dots, T_n)$ on the finite

Received 28 May 2023 – Accepted 28 October 2023

product of metric spaces and established an existence and uniqueness result to demonstrate a solution of the following system of equations:

$$T_i(\omega_1, \ldots, \omega_n) = \omega_i, \; i = 1, 2, \ldots, n. \tag{1.1}$$

Using some slightly different conditions, Czerwik [7] generalized a certain fixed point result of Eldestein [8] and established the following existence and uniqueness result for a system of mappings.

Theorem 1.1 ([7]). *Let (W_i, ρ_i), $i = 1, 2, \ldots, n$, be compact metric spaces. Suppose that $T_i : W \to W_i$, $i = 1, 2, \ldots, n$, fulfill the following conditions:*

$$\rho_i\left(T_i\omega, T_i\bar{\omega}\right) < \sum_{k=1}^{n} a_{ik}\rho_k(\omega_k, \bar{\omega}_k) \text{ in } B = W \times W - \triangle,$$

$$|\lambda_i| \leq 1, \quad i = 1, 2, \ldots, n$$

where $\triangle = \{(\omega, \bar{\omega}) \in W \times W : \omega_i = \bar{\omega}_i, \; i = 1, 2, \ldots, n\}$, $a_{ik} > 0$, $i, k = 1, \ldots, n$, and λ_i, $i = 1, 2, \ldots, n$ are characteristic roots of the matrix (a_{ik}), $i, k = 1, \ldots, n$. Then the system of equations (1.1) has a unique solution.

These types of results are fruitful to study the existence solutions of the system of functional equations of the following form:

$$\phi_i\left(t\right) = h_i\left(t, \phi_1\left[f_{i1}\left(t\right)\right], ..., \phi_n\left[f_{in}\left(t\right)\right]\right) \quad \text{for } i = 1, 2, ..., n \tag{1.2}$$

where $f_{ik} : A \to A \subset X \neq \varnothing$, $h_i : X \times \mathbb{R}^n \to \mathbb{R}$, $i, k = 1, 2, \ldots, n$ and $\phi_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, ..., n$ are the unknown functions.

In 1981, Reddy and Subrahmanyam [26] generalized Krasnoselski's fixed point result [18] for two systems of mappings and applied it to find convex solutions of the system of functional equations (1.2). On the same line, Khantwal and Gairola [16] generalized the result of Matkowski to provide an existence result for bounded solutions of the system of functional equations (1.2). Due to applicability of finding a solution of the system of functional equations (1.2), many extensions and generalizations of Matkowski's result [19, 20] have appeared in the literature (see [1], [6], [9], [10], [11], [12], [15], [22], [27], [29], [30], [31] and references therein).

On the other hand, Proinov [25] generalized the BCT to more general class of mappings. He introduced a new class of mappings, which includes the contraction mappings of Boyd-Wong [3], Matkowski [20] and Meir-Keeler [23] type and established the following result.

Theorem 1.2 ([25]). *Let (Y, ρ) be a complete metric space. Assume that $g : Y \to Y$ is an asymptotically regular and continuous mapping. If there exists a function $\phi : [0, \infty) \to [0, \infty)$ such that for any $\varepsilon > 0$ there exists $\delta > \varepsilon$ such that $\varepsilon < t < \delta$ implies $\phi(t) \leq \varepsilon$ and the following conditions hold:*

(P1): *$\rho(g(u), g(v)) \leq \phi(L(u, v))$ for all $u, v \in Y$,*
(P2): *$\rho(g(u), g(v)) < L(u, v)$, whenever $L(u, v) \neq 0$,*

where $L(u,v) = \rho(u,v) + \eta[\rho(u,g(u)) + \rho(v,g(v))]$, $\eta \geq 0$, *then* g *has a fixed point* $w \in Y$.

Moreover, for $\eta = 1$, *the continuity of* g *can be dropped if the function* ϕ *is continuous and* $\phi(t) < t$ *for* $t > 0$.

This result generalizes or extends certain results of Ćirić [5], Jacimiski [14], Matkowski [21] and others. For recent developments along this direction one can refer to [2], [17], [24] and [32].

In 2016, Ri [28] obtained a generalization of the BCT and the Boyd and Wong's fixed point theorem by relaxing the requirement of upper semi-continuity of the control function ϕ used in Boyd and Wong's result [3].

Theorem 1.3 ([28])**.** *Let* (Y,ρ) *be a complete metric space and* $\varphi : [0,\infty) \to [0,\infty)$ *be a function such that* $\varphi(t) < t$ *and* $\limsup_{s\to t^+} \varphi(s) < t$ *for all* $t > 0$. *Assume that* $f : Y \to Y$ *is a mapping such that*

$$\rho(fu, fv) \leq \varphi(\rho(u,v)) \quad \text{for all } u,v \in Y. \tag{1.3}$$

Then f *has a unique fixed point.*

In this paper, we introduce the notion of a coordinatewise asymptotically regular mappings and show that the coordinatewise asymptotic regularity is not a sufficient condition for the existence of a solution for a system of equations (1.1). Further, motivated by the work of Matkowski [19, 20] and Czerwik [7], we generalize certain results from [3], [25], [28] to a system of mappings. We also show that the assumption of continuity of control function used in Theorem 1.2 for $\eta = 1$ can be weaken. Moreover, we prove an existence result for a new class of a system of mappings without using the assumption of continuity and present a generalization of [24, Theorem 7] to a system of mappings. We also present some illustrative examples to justify the validity of our results.

2. Main Results

Firstly, we define a new class of a system of mappings on the product of metric spaces.

Definition 2.1. Let (W_i, ρ_i), $i = 1,2,\ldots,n$, be metric spaces and $T_i : W \to W_i$, $i = 1,2,\ldots,n$ be mappings. Then, the system of mappings $(T_1,\ldots,T_n)$ is called *coordinatewise asymptotically regular* at some point $\omega^0 = (\omega_1^0,\ldots,\omega_n^0) \in W$, if the sequence of iterations (ω_i^m) defined by

$$\omega_i^1 = T_i\omega^0 \ \text{ and } \ \omega_i^{m+1} = T_i\omega^m \ \text{ for } \ m \in \mathbb{N}$$

satisfies

$$\lim_{m\to\infty} \rho_i(\omega_i^m, \omega_i^{m+1}) = 0 \ \text{ for } i = 1,2,\ldots,n.$$

If $(T_1,\ldots,T_n)$ is *coordinatewise asymptotically regular* at each point of W then we call the system $(T_1,\ldots,T_n)$ is *coordinatewise asymptotically regular* on W. For $n = 1$, the above definition coincides with the definition of the asymptotic regular mapping due to Browder and Petryshyn [4].

Definition 2.2. Let (Y, ρ) be a metric space. A mapping $g : Y \to Y$ is called asymptotically regular at some $u \in Y$ if $\lim_{n\to\infty} \rho(g^n u, g^{n+1}u) = 0$. In other words, the mapping g is asymptotically regular at point $u \in Y$ if the sequence of iterations $(g^n u)$ satisfies $\lim_{n\to\infty} \rho(g^n u, g^{n+1}u) = 0$. The mapping g is called asymptotically regular on Y if it is asymptotically regular at each point of Y.

Example 2.3. Let $W_i = [0, 1]$ be equipped with the usual metric ρ_i for $i = 1, 2$. Define $T_1 : W_1 \times W_2 \to W_i$ by

$$T_1(\omega_1, \omega_2) = \begin{cases} 1/(r+1), & \text{if } \omega_1 = 1/r,\ r \in \mathbb{N}, \\ 1/2, & \text{if} \omega_1 \neq 1/r,\ r \in \mathbb{N}, \end{cases}$$

and $T_2 : W_1 \times W_2 \to W_i$ by

$$T_2(\omega_1, \omega_2) = \begin{cases} 1/(s+1), & \text{if } \omega_2 = 1/s,\ s \in \mathbb{N}, \\ 1/2, & \text{if } \omega_2 \neq 1/s,\ s \in \mathbb{N}. \end{cases}$$

We consider the following three cases:

Case 1 Let $\omega_1 = 1/s$ and $\omega_2 = 1/r$. Then for $\omega^0 = (\omega_1, \omega_2)$, we have $\omega_1^m = 1/(m + r)$, $\omega_2^m = 1/(m + s)$ and $\lim_{m\to\infty} \rho_i(\omega_i^m, \omega_i^{m+1}) = 0$, $i = 1, 2$.

Case 2 Let $\omega_1 = 1/r$ and $\omega_2 \neq 1/s$. Then for $\omega^0 = (\omega_1, \omega_2)$, we have $\omega_1^m = 1/(m + r)$, $\omega_2^m = 1/(m + 1)$ and $\lim_{m\to\infty} \rho_i(\omega_i^m, \omega_i^{m+1}) = 0$, $i = 1, 2$.

Case 3 Let $\omega_1 \neq 1/r$ and $\omega_2 \neq 1/s$. Then, for $\omega^0 = (\omega_1, \omega_2)$ we have $\omega_1^m = 1/(m + 1)$, $\omega_2^m = 1/(m + 1)$ and $\lim_{m\to\infty} \rho_i(\omega_i^m, \omega_i^{m+1}) = 0$, $i = 1, 2$.

Thus, the system (T_1, T_2) is coordinatewise asymptotically regular even though the system of equations

$$T_i(\omega_1, \omega_2) = \omega_i \quad \text{for} \quad i = 1, 2,$$

has no solution in $W_1 \times W_2$. This implies that the condition of coordinatewise asymptotic regularity is not sufficient enough to ensure the existence of a solution of such types of system of equations.

Now, we prove an existence result for a solution of the system of equations (1.1) under the certain conditions.

Theorem 2.4. *Let (W_i, ρ_i), $i = 1, 2, \ldots, n$, be complete metric spaces and $T_i : W \to W_i$, $i = 1, 2, \ldots, n$, be continuous mappings. Assume that the system of mappings $(T_1, \ldots, T_n)$ is coordinatewise asymptotically regular on W. If there exists $\varphi \in \Phi$ such that for all $\omega, \bar{\omega} \in W$ and $i = 1, 2, \ldots, n$, the following conditions hold:*

$$\rho_i(T_i\omega, T_i\bar{\omega}) \leq \varphi\left(D_i(\omega, \bar{\omega})\right) \quad \text{for all} \quad \omega_k, \bar{\omega}_k \in W_k; \tag{2.1}$$

$$|\lambda_i| \leq 1 \quad \text{for } i = 1, 2, \ldots, n, \tag{2.2}$$

where $D_i(\omega, \bar{\omega}) = \sum_{k=1}^{n} a_{ik}\rho_k(\omega_k, \bar{\omega}_k) + \eta \left\{\rho_i(\omega_i, T_i\omega) + \rho_i(\bar{\omega}_i, T_i\bar{\omega})\right\}$, $a_{ik} > 0$, $i, k = 1, \ldots, n$, *and* λ_i, $i = 1, 2, \ldots, n$ *are characteristics roots of matrix* (a_{ik}), $i, k = 1, 2, \ldots, n$. *Then the system of equations* (1.1) *has a unique solution* $(z_1, \ldots, z_n) \in$

W. Further, for arbitrarily fixed $\omega_i^1 \in W_i,\ i = 1, 2, \ldots, n$, the sequence of successive approximations

$$\omega_i^{m+1} = T_i \omega^m \quad \textit{for } i = 1, 2, \ldots n \textit{ and } \ m \in \mathbb{N}$$

converges such that

$$z_i = \lim_{n\to\infty} \omega_i^m \quad \textit{for } i = 1, 2, \ldots n.$$

Moreover, if $\eta = 1$ then the continuities of $T_i,\ i = 1, 2, \ldots, n$, are not required.

Proof. For each $i = 1, 2, \ldots, n$, pick $\omega_i^0 \in W_i$ and define

$$\omega_i^{m+1} = T_i \omega^m \quad \text{for } i = 1, 2, \ldots, n \text{ and } m \in \mathbb{N} \cup \{0\}.$$

Now, by coordinatewise asymptotic regularity of $(T_1, \ldots, T_n)$, we get

$$\lim_{m\to\infty} \rho_i(\omega_i^m, \omega_i^{m+1}) = 0 \quad \text{for } i = 1, 2, \ldots, n. \tag{2.3}$$

Then for each $\varepsilon_i > 0,\ i = 1, 2, \ldots, n$, there exists $r_i \in \mathbb{N}$ such that

$$\rho_i(\omega_i^{m_i}, \omega_i^{m_i+1}) < \varepsilon_i \quad \text{for } r_i \le m_i \in \mathbb{N}.$$

In the above inequalities, taking $r = \max\{r_i \ : \ i = 1, 2, \ldots, n\}$, we get

$$\rho_i(\omega_i^m, \omega_i^{m+1}) < \varepsilon_i \quad \text{for } i = 1, 2, \ldots, n \text{ and } m \ge r \in \mathbb{N}. \tag{2.4}$$

Now, we prove that (ω_i^m) is a Cauchy sequence for each $i = 1, 2, \ldots, n$. Assume that sequence (ω_i^m) is not a Cauchy in W_i. Then for each $i = 1, 2, \ldots, n$ and $r \in \mathbb{N}$, there exist $\varepsilon_i > 0$ and sequences of positive integers $(p_i(r)),\ (q_i(r))$ with $r \le p_i(r) < q_i(r)$ such that

$$\rho_i(\omega_i^{p_i(r)}, \omega_i^{q_i(r)}) \ge \varepsilon_i. \tag{2.5}$$

We may assume that $q_i(r)$ is the smallest positive integer greater than $p_i(r)$ such that the inequality (2.5) holds with the following inequality

$$\rho_i(\omega_i^{p_i(r)}, \omega_i^{q_i(r)-1}) < \varepsilon_i \quad \text{for } i = 1, 2, \ldots, n. \tag{2.6}$$

Then by the triangle inequality and using (2.6), we have

$$\begin{aligned} \rho_i(\omega_i^{p_i(r)}, \omega_i^{q_i(r)}) &\le \rho_i(\omega_i^{p_i(r)}, \omega_i^{q_i(r)-1}) + \rho(\omega_i^{q_i(r)-1}, \omega_i^{q_i(r)}) \\ &< \varepsilon_i + \rho_i(\omega_i^{q_i(r)}, \omega_i^{q_i(r)-1}) \quad \text{for } i = 1, 2, \ldots, n. \end{aligned}$$

Making $r \to \infty$ and using (2.3), we get

$$\lim_{r\to\infty} \rho_i(\omega_i^{p_i(r)}, \omega_i^{q_i(r)}) = \varepsilon_i \quad \text{for } \ i = 1, 2, \ldots, n. \tag{2.7}$$

Next, we observe that,

$$\begin{aligned} \varepsilon_i &\le \rho_i(\omega_i^{p_i(r)}, \omega_i^{q_i(r)}) \\ &\le \rho_i(\omega_i^{p_i(r)}, \omega_i^{p_i(r)+1}) + \rho_i(\omega_i^{p_i(r)+1}, \omega_i^{q_i(r)+1}) + \rho_i(\omega_i^{q_i(r)+1}, \omega_i^{q_i(r)}) \\ &\le \rho_i(\omega_i^{p_i(r)}, \omega_i^{p_i(r)+1}) + \rho_i(T_i\omega^{p_i(r)}, T_i\omega^{q_i(r)}) + \rho_i(\omega_i^{q_i(r)+1}, \omega_i^{q_i(r)}) \\ &\le \rho_i(\omega_i^{p_i(r)}, \omega_i^{p_i(r)+1}) + \varphi\left(D_i(\omega^{p_i(r)}, \omega^{q_i(r)})\right) + \rho_i(\omega_i^{q_i(r)+1}, \omega_i^{q_i(r)}) \end{aligned}$$

for $i = 1, 2, \ldots, n$. Making $r \to \infty$ and using (2.7), we get

$$\varepsilon_i \le \varphi\left(D_i(\omega^{p_i(r)}, \omega^{q_i(r)})\right) \text{ for } i = 1, 2, \ldots, n.$$

We note that

$$\lim_{r\to\infty} D_i(\omega^{p_i(r)}, \omega^{q_i(r)}) = \sum_{k=1}^{n} a_{ik}\epsilon_k \text{ for } i = 1, 2, \ldots, n$$

and let

$$\sum_{k=1}^{n} a_{ik}\epsilon_k = h_i \text{ for } i = 1, 2, \ldots, n.$$

Then $\limsup_{s\to t^+} \varphi(s) < t$ for all $t > 0$ implies

$$\varepsilon_i \le \lim_{r\to+\infty} \varphi(D_i(\omega^{p_i(r)}, \omega^{q_i(r)})) \le \lim_{\varepsilon'\to+0} \sup_{s\in(h_i,\ h_i+\varepsilon')} \varphi(s) < h_i$$

for $i = 1, 2, \ldots, n$. Hence we get

$$\varepsilon_i < \sum_{k=1}^{n} a_{ik}\varepsilon_k \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{2.8}$$

Then from (2.2) and Peron's theorem [13, page 53], there exist positive numbers $(t_1, \ldots, t_n)$ such

$$\sum_{k=1}^{n} a_{ik}t_k \le t_i \text{ for } i = 1, 2, \ldots, n. \tag{2.9}$$

Without loss of generality, we may assume that

$$\varepsilon_i \le t_i \text{ for } i = 1, 2, \ldots, n.$$

Then from (2.8) and (2.9), we have

$$\varepsilon_i < \sum_{k=1}^{n} a_{ik}\varepsilon_k \le \sum_{k=1}^{n} a_{ik}t_k \le t_i \text{ for } i = 1, 2, \ldots, n.$$

Since these inequalities are strict, there exists $h = \max\left\{\frac{\varepsilon_1}{t_1}, \frac{\varepsilon_2}{t_2}, \ldots, \frac{\varepsilon_n}{t_n}\right\} \in (0, 1)$ such that

$$\varepsilon_i \le ht_i \text{ for } i = 1, 2, \ldots, n.$$

Repeating this process m times, we get

$$\varepsilon_i \le h^m t_i \text{ for } i = 1, 2, \ldots, n.$$

Making $m \to \infty$, we get

$$\varepsilon_i \le 0 \text{ for } i = 1, 2, \ldots, n.$$

Hence (ω_i^m) is a Cauchy sequence for each $i = 1, 2, \ldots, n$. Since W_i is a complete metric space, there exists $z_i \in W_i$ such that $\lim_{m\to\infty} \omega_i^m = z_i$, $i = 1, 2, \ldots, n$ and $\omega^m = (\omega_1^m, \ldots, \omega_n^m) \to z = (z_1, \ldots, z_n)$. If T_i, $i = 1, 2, \ldots, n$, are continuous then $T_i\omega^m = \omega_i^{m+1} \to T_i z$ implies $T_i z = z_i$, $i = 1, 2, \ldots, n$.

Now suppose that $\eta = 1$ then from (2.1), we have

$$\rho_i(\omega_i^{m+1}, T_i z) = \rho_i(T_i\omega^m, T_i z) \le \varphi\left(D_i(\omega^m, z)\right) \text{ for } i = 1, 2, \ldots, n$$

where $D_i(\omega^m, z) = \sum\limits_{k=1}^{n} a_{ik}\rho_k(\omega_k^m, z_k) + \rho_i(\omega_i^m, \omega_i^{m+1}) + \rho_i(z_i, T_i z)$.
Making $m \to \infty$, we get

$$\rho_i(z_i, T_i z) \le \lim_{m\to\infty} \varphi\left(D_i(z_i, T_i z)\right) \quad \text{for } i = 1, 2, \ldots, n.$$

Also

$$\lim_{m\to\infty} D_i(\omega^m, z) = \sum_{k=1}^{n} a_{ik}\rho_k(z_k, T_k z) \text{ for } i = 1, 2, \ldots, n.$$

Let $\rho_i^* = \sum\limits_{k=1}^{n} a_{ik}\rho_k(z_k, T_k z)$, $i = 1, 2, \ldots, n$. Then by $\limsup_{s\to t^+}\varphi(s) < t$ for all $t > 0$, we obtain

$$\rho_i(z_i, T_i z) \le \lim_{m\to\infty} \varphi\left(D_i(\omega^m, z)\right) \le \lim_{\rho\to+0} \sup_{s\in(\rho_i^*, \rho_i^*+\rho)} \varphi(s) < \rho_i^* \text{ for } i = 1, 2, \ldots, n.$$

This implies

$$\rho_i(z_i, T_i z) < \sum_{k=1}^{n} a_{ik}\rho_k(z_k, T_k z) \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{2.10}$$

We may assume that

$$\rho_i(z_i, T_i z) \le t_i \text{ for } i = 1, 2, \ldots, n.$$

Then, taking into account of conditions (2.9), (2.10) and by Peron's theorem [13], we get

$$\rho_i(z_i, T_i z) < t_i \text{ for } i = 1, 2, \ldots, n.$$

Since these inequalities are strict, there exists an $\ell = \max\{\rho_i(z_i, T_i z)/t_i : i = 1, 2, \ldots, n\} \in (0, 1)$ such that

$$\rho_i(z_i, T_i z) \le \ell t_i \text{ for } i = 1, 2, \ldots, n.$$

Repeating the above process m times, we get

$$\rho_i(z_i, T_i z) \le \ell^m t_i \text{ for } i = 1, 2, \ldots, n.$$

Making $m \to \infty$, we get

$$\rho_i(z_i, T_i z) = 0 \text{ or } T_i z = z_i \text{ for } i = 1, 2, \ldots, n.$$

Hence the system of equations (1.1) has a solution in W.

For uniqueness of a solution of the system of equations (1.1), assume that $w = (w_1, \ldots, w_n)$ is another solution of the system (1.1) such that

$$\rho_i(z_i, w_i) \ne 0 \text{ for } i = 1, 2, \ldots, n.$$

Then from (2.1), we have

$$\begin{aligned}\rho_i(z_i, w_i) &\le \varphi\left(\sum_{k=1}^{n} a_{ik}\rho_k(z_k, w_k) + \eta\left\{\rho_i(z_i, T_i z) + \rho_i(w_i, T_i w)\right\}\right)\\ &< \sum_{k=1}^{n} a_{ik}\rho_k(z_k, w_k) \quad \text{for } i = 1, 2, \ldots, n. \qquad (2.11)\end{aligned}$$

We may assume that

$$\rho_i(z_i, w_i) \le t_i \quad \text{for } i = 1, 2, \ldots, n.$$

Then in view of Peron's theorem [13, page 53] and conditions (2.9), (2.11), we get

$$\rho_i(z_i, w_i) < t_i \quad \text{for } i = 1, 2, \ldots, n.$$

As the above inequalities are strict so there exists $\tau = \max\{\rho_i(z_i, w_i)/t_i : i = 1, 2, \ldots, n\} \in (0, 1)$ such that

$$\rho_i(z_i, w_i) \le \tau t_i \quad \text{for } i = 1, 2, \ldots, n.$$

Following this process m times, we get

$$\rho_i(z_i, w_i) \le \tau^m t_i \quad \text{for } i = 1, 2, \ldots, n.$$

Making $m \to \infty$, we get

$$\rho_i(z_i, w_i) = 0 \quad \text{or} \quad z_i = w_i \quad \text{for } i = 1, 2, \ldots, n.$$

This completes the proof. □

The following example illustrates the utility of our result.

Example 2.5. Let $W_i = \{0, 1, 2\},\ i = 1, 2$ and $(W_i, \rho_i),\ i = 1, 2$, be usual metric spaces. Define $T_1 : W_1 \times W_2 \to W_1$ by

$$T_1(\omega_1, \omega_2) = 4\omega_1 - 2\omega_1^2$$

and $T_2 : W_1 \times W_2 \to W_2$ by

$$T_2(\omega_1, \omega_2) = 4\omega_2 - 2\omega_2^2$$

for all $(\omega_1, \omega_2) \in W_1 \times W_2$.
Then, it is easy to see that $(W_i, \rho_i),\ i = 1, 2$ are complete metric spaces and $T_i,\ i = 1, 2$ are continuous mappings. Also, the system (T_1, T_2) is coordinate-wise asymptotically regular on $W_1 \times W_2$. Now, if we take

$$a_{11} = a_{12} = a_{21} = a_{22} = 1/2,\ \varphi(t) = t/2 \quad \text{and} \quad \eta = 4$$

then for all $\omega, \bar{\omega} \in W_1 \times W_2$, we have

$$\rho_i(T_i\omega, T_i\bar{\omega}) \le 2 \le \varphi(D_i(\omega, \bar{\omega})) \quad \text{for } i = 1, 2.$$

Hence, all the assumptions of Theorem 2.4 are verified and the system of equations (1.1) for $n = 2$, has a unique solution at $(0, 0)$. However for $\omega = (0, 0)$ and $\bar{\omega} = (1, 1)$, we have

$$\rho_i(T_i\omega, T_i\bar{\omega}) > \sum_{k=1}^{2} a_{ik}\rho_k(\omega_k, \bar{\omega}_k) \ \text{ for } i = 1, 2.$$

Thus, we cannot apply Theorem 1.1 and result of [20, Theorem 1.4].

Remark 2.6. By definition of ϕ, we know that for every $\epsilon > 0$ there exists $\delta > \epsilon$ such that $\epsilon < t < \epsilon + \delta$ implies $\phi(t) \leq \epsilon$. In other word, we can say $\phi(t) < t$ for all $t \in (\epsilon, \epsilon + \delta)$. This implies $\phi(t) < t$ for $t > 0$ and $\lim_{\delta \to 0} \sup_{s \in (\epsilon, \epsilon+\delta)} \phi(s) < s$. Hence $\phi \in \Phi$.

If we take $n = 1, T_i = g, a_{11} = 1, W_i = Y, \rho_i = \rho$ in Theorem 2.4, we get a generalized version of Theorem 1.2 which shows in case when $\eta = 1$, the assumption of continuity on the control function is weaken.

Corollary 2.7. *Let (Y, ρ) be a complete metric space. Assume that $g : Y \to Y$ is a continuous asymptotically regular mapping on Y which satisfies the following condition:*

$$\rho(gu, gv) \leq \varphi\left(D(u, v)\right)$$

where $D(u, v) = \rho(u, v) + \eta\left\{\rho(u, gu) + \rho(v, gv)\right\}, \ \eta \geq 0$ and $\varphi \in \Phi$. Then the mapping g has a unique fixed point in Y. Moreover, if we take $\eta = 1$ then continuity of g is not required.

Corollary 2.8. *Let (Z, ρ) be a complete metric space and $T : Z^n \to Z$ be a continuous asymptotically regular mapping on Z such that*

$$\rho\left(T(z, \ldots, z), T(\bar{z}, \ldots, \bar{z})\right) \leq \varphi\left(\rho(z, \bar{z}) + \eta\left\{\rho(z, Tz) + \rho(\bar{z}, T\bar{z})\right\}\right)$$

where $\varphi \in \Phi$. Then the system of equation $T(z, \ldots, z) = z$ has a unique solution. Moreover, if we take $\eta = 1$ then continuity of T need not be required.

Proof. The proof is obtained by taking $W_i = Z, \ T_i = T, \ \rho_i = \rho$ and $a_{ik} = q_k$ with $q_1 + \cdots + q_n = 1$ for each $i = 1, 2, \ldots, n$, in Theorem 2.4. □

If we take $D_i(\omega, \bar{\omega}) = \sum_{k=1}^{n} a_{ik}\rho_k(\omega_k, \bar{\omega}_k)$ for $i = 1, 2, \ldots, n$, in Theorem 2.4 then assumptions of continuity and coordinatewise asymptotic regularity remain redundant and we get an extension of [20, Theorem 1.4].

Theorem 2.9. *Let $(W_i, \rho_i), \ i = 1, 2, \ldots, n$, be complete metric spaces and $T_i : W \to W_i, \ i = 1, 2, \ldots, n$, be mappings. If there exists $\varphi \in \Phi$ such that for all $\omega, \bar{\omega} \in W$ and $i = 1, 2, \ldots, n$, the following condition hold:*

$$\rho_i(T_i\omega, T_i\bar{\omega}) \leq \varphi\left(\sum_{k=1}^{n} a_{ik}\rho_k(\omega_k, \bar{\omega}_k)\right) \tag{2.12}$$

where a_{ik}, $i,k = 1,2\ldots,n$ are defined in Theorem 2.4. Then, the system of equations (1.1) has a unique solution $(z_1,\ldots,z_n)$ in W. Moreover, for arbitrarily fixed $\omega_i^1 \in W_i$, $i = 1,2,\ldots,n$, the sequence of successive approximations $\omega_i^{m+1} = T_i\omega^m$ converges to $z_i = \lim_{m\to\infty} \omega_i^m$ for $i = 1,2,\ldots n$ and $m \in \mathbb{N}$.

Proof. For each $i = 1,2,\ldots,n$, pick $\omega_i^0 \in W_i$ and define

$$\omega_i^{m+1} = T_i\omega^m \quad \text{for } i = 1,2,\ldots,n \quad \text{and} \quad m \in \mathbb{N}\cup\{0\}.$$

Then from (2.2) and Peron's theorem [13, page 53], there exist positive numbers $(r_1,\ldots,r_n)$ such

$$\sum_{k=1}^{n} a_{ik}r_k \leq r_i \ \text{ for } \ i = 1,2,\ldots,n. \tag{2.13}$$

We may assume that

$$\rho_i(\omega_i^1,\omega_i^0) \leq r_i \ \text{ for } \ i = 1,2,\ldots,n.$$

Then from (2.12) and (2.13), we have

$$\begin{aligned}\rho_i(\omega_i^2,\omega_i^1) &= \rho_i(T_i\omega^1,T_i\omega^0)\\ &\leq \varphi\left(\sum_{k=1}^{n} a_{ik}\rho_k(\omega_k^1,\omega_k^0)\right)\\ &\leq \varphi\left(\sum_{k=1}^{n} a_{ik}r_k\right) < r_i \ \text{ for } \ i = 1,2,\ldots,n.\end{aligned}$$

Since these inequalities are strict, there exists an $h = \max\{\rho_i(\omega_i^2,\omega_i^1)/r_i : i = 1,2,\ldots,n\} \in (0,1)$ such that

$$\rho_i(\omega_i^2,\omega_i^1) \leq hr_i \ \text{ for } \ i = 1,2,\ldots,n.$$

Now using induction, we prove that the following inequalities are true for all $m \geq 1 \in \mathbb{N}$,

$$\rho_i(\omega_i^{m+1},\omega_i^m) \leq h^m r_i \ \text{ for } \ i = 1,2,\ldots,n \text{ and } m \in \mathbb{N}\cup\{0\}.$$

Assume that the above inequalities are true for some $m \in \mathbb{N}$. Then from (2.12), we have

$$\begin{aligned}\rho_i(\omega_i^{m+2},\omega_i^{m+1}) &= \rho_i(T_i\omega^{m+1},T_i\omega^m)\\ &\leq \varphi\left(\sum_{k=1}^{n} a_{ik}\rho_k(\omega_k^{m+1},\omega_k^m)\right)\\ &\leq \varphi\left(\sum_{k=1}^{n} a_{ik}h^m r_k\right) < h^m r_k \ \text{ for } \ i = 1,2,\ldots,n.\end{aligned}$$

Again, since the above inequalities are strict, we can find $h \in (0,1)$ such that

$$\rho_i(\omega_i^{m+2},\omega_i^{m+1}) \leq h^{m+1} r_i \ \text{ for } \ i = 1,2,\ldots,n.$$

Making $m \to \infty$, we get

$$\lim_{m\to\infty} \rho_i(\omega_i^{m+1}, \omega_i^m) = 0 \ \text{ for } \ i = 1, 2, \ldots, n.$$

Hence the system of mappings $(T_1, \ldots, T_n)$ is an asymptotically regular on W. Also, the condition (2.10) implies that the mappings $T_i,\ i = 1, 2, \ldots, n$ are continuous on W. Rest of the proof may be completed following the proof of Theorem 2.4. □

If we take $W_i = Z,\ T_i = T,\ a_{ik} = 1,\ \rho_i = \rho$ for each $i, k = 1, 2, \ldots, n$ in Theorem 2.9, we get the following result.

Corollary 2.10. *Let (Z, ρ) be a complete metric space and $T : Z^n \to Z$ be a mapping on Z such that*

$$\rho\left(T(z, \ldots, z), T(\bar{z}, \ldots, \bar{z})\right) \le \varphi\left(\rho(z, \bar{z})\right)$$

where $\varphi \in \Phi$. Then $T(z, \ldots, z) = z$ has a unique solution. Moreover, if we take $\eta = 1$ then continuity of T is not required.

If we take $n = 1$, $T_i = f$, $a_{11} = 1$, $W_i = Y$, and $\rho_i = \rho$ in Corollary 2.9, then we obtain Theorem 1.3 as a direct consequence of Corollary 2.9.

Now, we establish an existence and uniqueness result for a new class of system of mappings without using the assumption of continuity.

Theorem 2.11. *Let $(W_i, \rho_i),\ i = 1, 2, \ldots, n$, be complete metric spaces and $T_i : W \to W_i,\ i = 1, 2, \ldots, n$, be mappings. If the system of mappings $(T_1, \ldots, T_n)$ is coordinatewise asymptotically regular on W such that the following conditions hold:*

$$\rho_i(\omega_i, T_i\bar{\omega}) \le \sum_{k=1}^{n} a_{ik}\rho_k(\omega_k, \bar{\omega}_k) + \mu\{\rho_i(\omega_i, T_i\omega) + \rho_i(T_i\omega^j, T_i\omega^{j+1})\}; \tag{2.14}$$

$$|\lambda_i| < 1 \ \ \textit{for} \ \ i = 1, 2, \ldots, n \tag{2.15}$$

for all $\omega, \bar{\omega} \in W$, where $a_{ik} > 0,\ i, k = 1, \ldots, n,\ \mu \in [0, \infty),\ j \in \mathbb{N}$ and $\lambda_i,\ i = 1, \ldots, n$ are characteristics roots of matrix $(a_{ik}),\ i, k = 1, 2, \ldots, n$. Then, the system of equations (1.1) has a unique solution $(z_1, \ldots, z_n) \in W$ and for arbitrarily fixed $\omega_i^1 \in W_i,\ i = 1, 2, \ldots, n$ the sequence of successive approximations $\omega_i^{m+1} = T_i\omega^m$ for $i = 1, 2 \ldots n$ and $m \in \mathbb{N}$ converges such that $z_i = \lim_{n\to\infty} \omega_i^m$ for $i = 1, 2, \ldots n$.

Proof. For each $i = 1, 2, \ldots, n$, pick $\omega_i^0 \in W_i$ and define

$$\omega_i^{m+1} = T_i\omega^m \quad \text{for } m \in \mathbb{N} \quad \text{and} \quad i = 1, 2, \ldots, n.$$

Now, by coordinatewise asymptotic regularity of $(T_1, \ldots, T_n)$, we get

$$\lim_{m\to\infty} \rho_i(\omega_i^m, \omega_i^{m+1}) = 0 \quad \text{for} \ \ i = 1, 2, \ldots, n.$$

Then, for every $\varepsilon_i > 0,\ i = 1, 2, \ldots, n$ there exists an $r \in \mathbb{N}$ such that

$$\rho_i(\omega_i^m, \omega_i^{m+1}) < \varepsilon_i \ \text{ for } \ i = 1, 2, \ldots, n \text{ and } m \ge r \in \mathbb{N}. \tag{2.16}$$

Now, we assume that the sequence $(\omega_i^m) \in W_i$ is not Cauchy for each $i = 1, 2, \dots, n$. Then following the proof of Theorem 2.4 we get, there exist $\varepsilon_i > 0$ and two sequences of positive integers $(p_i(r))$, $(q_i(r))$ with $r \le p_i(r) < q_i(r)$ such that

$$\lim_{r\to\infty} \rho_i(\omega_i^{p_i(r)}, \omega_i^{q_i(r)}) = \varepsilon_i \ \text{ for } i = 1, 2, \dots, n \text{ and } r \in \mathbb{N}. \tag{2.17}$$

Next, we observe that,

$$\begin{aligned}
\varepsilon_i &\le \rho_i(\omega_i^{p_i(r)}, \omega_i^{q_i(r)}) \\
&\le \rho_i(\omega_i^{p_i(r)}, \omega_i^{p_i(r)+1}) + \rho_i(\omega_i^{p_i(r)+1}, \omega_i^{q_i(r)+1}) + \rho_i(\omega_i^{q_i(r)+1}, \omega_i^{q_i(r)}) \\
&\le \rho_i(\omega_i^{p_i(r)}, \omega_i^{p_i(r)+1}) + \rho_i(T_i\omega^{p_i(r)}, T_i\omega^{q_i(r)}) + \rho_i(\omega_i^{q_i(r)+1}, \omega_i^{q_i(r)}) \\
&\le \rho_i(\omega_i^{p_i(r)}, \omega_i^{p_i(r)+1}) + \sum_{k=1}^{n} a_{ik}\rho_k(\omega_k^{p_i(r)}, \omega_k^{q_i(r)}) \\
&\quad + \mu\left\{\rho_i(\omega_i^{p_i(r)}, T_i\omega^{p_i(r)}) + \rho_i(T_i^j\omega^{p_i(r)}, T_i^{j+1}\omega^{p_i(r)})\right\} + \rho_i(\omega_i^{q_i(r)+1}, \omega_i^{q_i(r)}) \\
&= \rho_i(\omega_i^{p_i(r)}, \omega_i^{p_i(r)+1}) + \sum_{k=1}^{n} a_{ik}\rho_k(\omega_k^{p_i(r)}, \omega_k^{q_i(r)}) + \mu\left\{\rho_i(\omega_i^{p_i(r)}, \omega_i^{p_i(r)+1})\right\} \\
&\quad + \mu\left\{\rho_i(\omega_i^{p_i(r)+j}, \omega_i^{p_i(r)+j+1})\right\} + \rho_i(\omega_i^{q_i(r)+1}, \omega_i^{q_i(r)})
\end{aligned}$$

for $i = 1, 2, \dots, n$. Making $r \to \infty$ and using (2.16), (2.17), we get

$$\varepsilon_i \le \sum_{k=1}^{n} a_{ik}\varepsilon_k \quad \text{for } i = 1, 2, \dots, n. \tag{2.18}$$

Now, from Peron's theorem [13, page 53] and condition (2.15) there exist positive numbers $(t_1, \dots, t_n)$ such that

$$\sum_{k=1}^{n} a_{ik}t_k < t_i \ \text{ for } \ i = 1, 2, \dots, n.$$

We may assume that

$$\varepsilon_i \le t_i \ \text{ for } \ i = 1, 2, \dots, n.$$

Further, if we put

$$h = \max_{1\le i\le n}\left(t_i^{-1}\sum_{k=1}^{n} a_{ik}t_k\right) \tag{2.19}$$

then $h \in (0, 1)$ and

$$\sum_{k=1}^{n} a_{ik}t_k \le ht_i \ \text{ for } \ i = 1, 2, \dots, n.$$

From (2.18), we have

$$\varepsilon_i \le \sum_{k=1}^{n} a_{ik}\varepsilon_k \le \sum_{k=1}^{n} a_{ik}t_k \le \sum_{k=1}^{n} a_{ik}ht_k < ht_i \ \text{ for } \ i = 1, 2, \dots, n.$$

Repeating this process m times, we get

$$\varepsilon_i \leq h^m t_i \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Making $m \to \infty$, we get the following contradictions

$$\varepsilon_i \leq 0 \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Hence, (ω_i^m) is a Cauchy sequence for each $i = 1, 2, \ldots, n$. Since W_i is a complete metric space, there exists $z_i \in W_i$ such that $\lim_{m\to\infty} \omega_i^m = z_i$ for $i = 1, 2, \ldots, n$. Now from (2.14), we have

$$\rho_i(\omega_i^m, T_i z) \leq \sum_{k=1}^{n} a_{ik} \rho_k(\omega_k^m, z_k) + \mu\{\rho_i(\omega_i^m, \omega_i^{m+1}) + \rho_i(\omega_i^{m+j}, \omega_i^{m+j+1})\}$$

for $i = 1, 2, \ldots, n$. Making $m \to \infty$, we get

$$\rho_i(z_i, T_i z) \leq 0 \text{ for } i = 1, 2, \ldots, n$$

which implies that $T_i z = z_i$ for $i = 1, 2, \ldots, n$. Hence the system of equations (1.1) has a solution in W. For uniqueness of the solution, assume that $w = (w_1, \ldots, w_n)$ is another solution of system of equations (1.1). Then

$$\begin{aligned} 0 < \rho_i(z_i, w_i) &= \rho_i(z_i, T_i w) \\ &\leq \sum_{k=1}^{n} a_{ik} \rho_k(z_k, w_k) + \mu\{\rho_i(z_i, T_i z) + \rho_i(T_i z^j, T_i z^{j+1})\} \\ &\leq \sum_{k=1}^{n} a_{ik} \rho_k(z_k, w_k) \quad \text{for } i = 1, 2, \ldots, n. \end{aligned}$$

We may assume that

$$\rho_i(z_i, w_i) \leq t_i \quad \text{for} \quad i = 1, 2, \ldots, n,$$

then

$$\rho_i(z_i, w_i) \leq \sum_{k=1}^{n} a_{ik} \rho_i(z_i, w_i) \leq \sum_{k=1}^{n} a_{ik} t_k < t_i \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Taking into account of (2.19), there exists $h \in (0, 1)$ such that

$$\rho_i(z_i, w_i) \leq \sum_{k=1}^{n} a_{ik} t_k \leq h t_i \quad \text{for } i = 1, 2, \ldots, n.$$

Continuing this process m times, we get

$$\rho_i(z_i, w_i) \leq h^m t_i \quad \text{for } i = 1, 2, \ldots, n.$$

Making $m \to \infty$, we get

$$\rho_i(z_i, w_i) = 0 \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Hence $z_i = w_i$ for $i = 1, 2, \ldots, n$. □

Example 2.12. Let $W_i = [0, 1]$ and ρ_i be usual metric on W_i for each $i = 1, 2$. Define $T_i : W_1 \times W_2 \to W_i$ for $i = 1, 2$ by

$$T_1(\omega_1, \omega_2) = \begin{cases} 0, & \text{when } 0 \le \omega_1 < 1, \\ 1/2, & \text{when } \omega_1 = 1, \end{cases} \quad \text{and}$$
$$T_2(\omega_1, \omega_2) = \begin{cases} 0, & \text{when } 0 \le \omega_2 < 1, \\ 1/2, & \text{when } \omega_2 = 1. \end{cases}$$

Then, it is easily seen that the system (T_1, T_2) is continuous and coordinatewise asymptotically regular on $W_1 \times W_2$. Now, for $\omega, \bar{\omega} \in [0, 1) \times [0, 1)$ or $\omega = \bar{\omega} = (1, 1)$, we have

$$\rho_i(\omega_i, T_i\bar{\omega}) = \omega_i \le \mu\rho_i(\omega_i, T_i\omega) \;\text{ for } i = 1, 2 \text{ and } \mu \ge 2.$$

If $\omega \in [0, 1)$ and $\bar{\omega} = (1, 1)$ then

$$\rho_i(\omega_i, T_i\bar{\omega}) = |\omega_i - \bar{\omega}_i| \le \mu\rho_i(\omega_i, T_i\omega) \text{ for } i = 1, 2 \text{ and } \mu \ge 2.$$

Thus the system (T_1, T_2) satisfies the condition (2.14) for $n = 2$. Hence all the assumptions of Theorem 2.9 are verified and $(\omega_1, \omega_2) = (0, 0)$ is a solution of the system of equations (1.1) for $n = 2$.

If we take $W_i = Z,\ T_i = T,\ a_{ik} = h,\ \rho_i = \rho$ for each $i, k = 1, 2, \ldots, n$ in Theorem 2.11, we get the following result.

Corollary 2.13. *Let (Z, ρ) be a complete metric space and $T : Z^n \to Z$ be a mapping on Z such that*

$$\rho\left((z, \ldots, z), T(\bar{z}, \ldots, \bar{z})\right) \quad \le \quad h\rho(z, \bar{z}) + \mu \left\{ \begin{array}{l} \rho(z, T(z, \ldots, z)) + \\ \rho(T^j(z, \ldots, z), T^{j+1}(z, \ldots, z)) \end{array} \right\}$$

where $\varphi \in \Phi,\ \mu \in [0, \infty),\ j \in \mathbb{N}$ and $h \in (0, 1)$. Then the equation $T(z, \ldots, z) = z$ has a unique solution.

If we take $n = 1,\ a_{11} = k,\ T_i = f,\ W_i = Y,\ \rho_i = \rho$, in Theorem 2.11 then we get following result of [24, Theorem 7].

Corollary 2.14. *Let (Y, ρ) be a complete metric space. Assume that $f : W \to W$ is an asymptotically regular mapping satisfying the following condition :*

$$\rho(u, fv) \le k\rho(u, v) + \mu\{\rho(u, fu) + \rho(f^j u, f^{j+1}u)\}$$

where $j \in \mathbb{N}$, $k \in (0, 1)$ and $\mu \in [0, \infty)$. Then there exists a unique fixed point $p \in Y$ for f and for any $\bar{\omega} \in Y$, we have $\lim_{n \to \infty} f^n(\omega) = p$.

Acknowledgements. *We are very thankful to the reviewers for their constructive comments and suggestions that have been useful for the improvement of this paper. The first author acknowledges the support from the URC/FRC fellowship, University of Johannesburg, South Africa.*

References

[1] J. B. Baillon and S. L. Singh, Nonlinear hybrid contractions on product spaces, Far East J. Math. Sci. 1, no. 2 (1993), 117–127.

[2] R. K. Bisht, R. P. Pant, and V. Rakočević, Proinov contractions and discontinuity at fixed point, Miskolc Math. Notes 20, no. 1 (2019), 131–137.

[3] D. W. Boyd and J. S. W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. 20 (1969), 458–464.

[4] F. E. Browder and W. V. Petryshyn, The solution by iteration of nonlinear functional equations in Banach spaces, Bull. Amer. Math. Soc. 72 (1966), 571–575.

[5] Lj. B. Ćirić, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc. 45 (1974), 267–273.

[6] S. Czerwik, A fixed point theorem for a system of multivalued transformations, Proc. Amer. Math. Soc. 55, no. 1 (1976), 136–139.

[7] S. Czerwik, Generalization of Edelstein's fixed point theorem, Demonstratio Math. 9, no. 2 (1976), 281–285.

[8] M. Edelstein, On fixed and periodic points under contractive mappings, J. London Math. Soc. 37 (1962), 74–79.

[9] U. C. Gairola and P. S. Jangwan, Co-ordinatewise R-weakly commuting maps and fixed point theorem on product spaces, Demonstratio Math. 36, no. 4 (2003), 939–949.

[10] U. C. Gairola and P. S. Jangwan, Coincidence theorem for multi-valued and single-valued systems of transformations, Demonstratio Math. 41, no. 1 (2008), 129–136.

[11] U. C. Gairola, S. N. Mishra, and S. L. Singh, Coincidence and fixed point theorems on product spaces, Demonstratio Math. 30, no. 1 (1997), 15–24.

[12] U. C. Gairola, S. L. Singh, and J. H. M. Whitfield, Fixed point theorems on product of compact metric spaces, Demonstratio Math. 28, no. 3 (1995), 541–548.

[13] F. R. Gantmakher, The theory of matrices, vol. 2, American Mathematical Soc., 2000.

[14] J. Jachymski, Equivalent conditions and the Meir-Keeler type theorems, J. Math. Anal. Appl. 194, no. 1 (1995), 293–303.

[15] D. Khantwal, S. Aneja, and U. C. Gairola, A generalization of Matkowski's and Suzuki's fixed point theorems, Asian-Eur. J. Math. 15, no. 9 (2022), Article ID 2250169, 12.

[16] D. Khantwal and U. C. Gairola, An extension of Matkowski's and Wardowski's fixed point theorems with applications to functional equations, Aequationes Math. 93, no. 2 (2019), 433–443.

[17] D. Khantwal, I. K. Letlhage, and R. Pant, Fixed point results for Suzuki type contractions in relational metric spaces with applications, Indian J. Math. 64, no. 3 (2022), 279–304.

[18] M. A. Krasnoselski, Two remarks on the method of successive approximations, Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) 10 (1956), no. 2 (17), 55–59.

[19] J. Matkowski, Some inequalities and a generalization of Banach's principle, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 21 (1973), 323–324.

[20] J. Matkowski, Integrable solutions of functional equations, Dissertationes Math. (Rozprawy Mat.) 127 (1975), 68.

[21] J. Matkowski, Fixed point theorems for contractive mappings in metric spaces, Časopis Pěst. Mat. 105, no. 4 (1980), 341–344.

[22] J. Matkowski, and S. L. Singh, Banach type fixed point theorems on product of spaces, Indian J. Math. 38, no. 1 (1996), 73–80.

[23] A. Meir and E. Keeler, A theorem on contraction mappings, J. Math. Anal. Appl. 28 (1969), 326–329.

[24] R. Pant and R. Shukla, New fixed point results for Proinov-Suzuki type contractions in metric spaces, Rend. Circ. Mat. Palermo (2) 71, no. 2 (2022), 633–645.

[25] P. D. Proinov, Fixed point theorems in metric spaces, Nonlinear Anal. 64, no. 3 (2006), 546–557.

[26] K. B. Reddy, and P. V. Subrahmanyam, Extensions of Krasnoselskii 's and Matkowski's fixed point theorems, Funkcial. Ekvac. 24, no. 1 (1981), 67–83.
[27] K. B. Reddy and P. V. Subrahmanyam, Altman's contractors and fixed points of multivalued mappings, Pacific J. Math. 99, no. 1 (1982), 127–136.
[28] S. Ri, A new fixed point theorem in the fractal space, Indag. Math. (N.S.) 27, no. 1 (2016), 85–93.
[29] S. L. Singh, and U. C. Gairola, Coordinatewise commuting and weakly commuting maps, and extension of Jungck and Matkowski contraction principles, J. Math. Phys. Sci. 25, no. 4 (1991), 305–318.
[30] S. L. Singh, and U. C. Gairola, A general fixed point theorem, Math. Japon. 36, no. 4 (1991), 791–801.
[31] S. L. Singh, S. N. Mishra, and V. Chadha, Round-off stability of iterations on product spaces, C. R. Math. Rep. Acad. Sci. Canada 16, no. 2-3 (1994), 105–109.
[32] S. L. Singh, S. N. Mishra, and R. Pant, New fixed point theorems for asymptotically regular multi-valued maps, Nonlinear Anal. 71, no. 7-8 (2009), 3299–3304.

