

Research Article

Devaney Chaos and Distributional Chaos in the Solution of Certain Partial Differential Equations

Xavier Barrachina and J. Alberto Conejero

*Institut Universitari de Matemàtica Pura i Aplicada, Universitat Politècnica de València, Edifici 8E,
46022 València, Spain*

Correspondence should be addressed to J. Alberto Conejero, aconejero@upv.es

Received 3 June 2012; Revised 13 September 2012; Accepted 17 October 2012

Academic Editor: Elena Litsyn

Copyright © 2012 X. Barrachina and J. A. Conejero. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The notion of distributional chaos has been recently added to the study of the linear dynamics of operators and C_0 -semigroups of operators. We will study this notion of chaos for some examples of C_0 -semigroups that are already known to be Devaney chaotic.

1. Introduction

During the last years, several notions have been introduced for describing the dynamical behavior of linear operators on infinite-dimensional spaces, such as hypercyclicity, chaos in the sense of Devaney, chaos in the sense of Li-Yorke, subchaos, mixing and weakly mixing properties, and frequent hypercyclicity, among others. These notions have been extended, as far as possible, to the setting of C_0 -semigroups of linear and continuous operators. The recent monograph by Grosse-Erdmann and Peris Manguillot [1] is a good reference for researchers interested in the study of linear dynamics. In particular, it contains a chapter dedicated to analyze the dynamics of C_0 -semigroups. See also [2], which contains additional information on further topics in the area.

In the sequel, let X be an infinite-dimensional separable Banach space. A C_0 -semigroup \mathcal{T} is a family of linear and continuous operators $\{T_t\}_{t \geq 0} \subset L(X)$ such that $T_0 = Id$, $T_t T_s = T_{t+s}$ for all $t, s \geq 0$, and for all $s \geq 0$, we have $\lim_{t \rightarrow s} T_t = T_s$ pointwise on X .

We say that a C_0 -semigroup \mathcal{T} is *Devaney chaotic* if it is transitive and it has a dense set of periodic points. On the one hand, a C_0 -semigroup \mathcal{T} is *transitive* if for any pair of nonempty open sets $U, V \subset X$ there is some $t > 0$ such that $T_t(U) \cap V \neq \emptyset$. In this setting, transitivity is equivalent to the existence of some $x \in X$ with dense orbit in X , that is, $\overline{\{T_t x, t \geq 0\}} = X$, see for instance [1, Th. 1.57]. This phenomenon is usually known in operator theory as *hypercyclicity*,

and such a vector x is said to be a *hypercyclic vector* for \mathcal{T} . On the other hand, a vector $x \in X$ is said to be a *periodic point* for \mathcal{T} if there is some $t > 0$ such that $T_t x = x$.

Other definitions of chaos, such as the one introduced by Li-Yorke and the one of distributional chaos introduced by Schweizer and Smítal, have been also considered. The relationships between these two notions in the Banach space setting have been recently studied in [3]. We recall that a C_0 -semigroup \mathcal{T} is said to be *Li-Yorke chaotic* if there exists an uncountable subset $\Gamma \subset X$, called the *scrambled set*, such that for every pair $x, y \in \Gamma$ of distinct points, we have that

$$\liminf_{t \rightarrow \infty} \|T_t x - T_t y\| = 0, \quad \limsup_{t \rightarrow \infty} \|T_t x - T_t y\| > 0. \quad (1.1)$$

Clearly, every hypercyclic C_0 -semigroup is Li-Yorke chaotic: we just have to fix a hypercyclic vector $x \in X$ and consider $\Gamma := \{\lambda x; |\lambda| \leq 1\}$ as a scrambled set, as it is indicated in [4, page 84].

Distributional chaos is inspired by the notion of Li-Yorke chaos. In order to define it, given a subset $B \subset \mathbb{R}_0^+$, we define its upper density as $\overline{\text{Dens}}(B) := \limsup_{t \rightarrow \infty} (1/t)\mu(B \cap [0, t])$, where μ stands for the Lebesgue measure on \mathbb{R}_0^+ .

Definition 1.1. A C_0 -semigroup \mathcal{T} in $L(X)$ is *distributionally chaotic* if there are an uncountable set $S \subset X$ and $\delta > 0$, so that for each $\varepsilon > 0$ and each pair $x, y \in S$ of distinct points, we have

$$\overline{\text{Dens}}(\{s \geq 0 : \|T_s x - T_s y\| \geq \delta\}) = 1, \quad \overline{\text{Dens}}(\{s \geq 0 : \|T_s x - T_s y\| < \varepsilon\}) = 1. \quad (1.2)$$

The set S is called the *scrambled set*. If S is dense in X , then \mathcal{T} is said to be *densely distributionally chaotic*.

A vector $x \in X$ is said to be *distributionally irregular* for \mathcal{T} if for every $\delta > 0$

$$\overline{\text{Dens}}(\{s \geq 0 : \|T_s x\| > \delta\}) = 1, \quad \overline{\text{Dens}}(\{s \geq 0 : \|T_s x\| < \delta\}) = 1. \quad (1.3)$$

Such vectors were considered in [5] so as to get a further insight into the phenomenon of distributional chaos, showing the equivalence between a distributionally chaotic operator and an operator having a distributionally irregular vector.

The first systematic approach to distributional chaos for linear operators was taken in [6], where this phenomenon was studied in detail for backward shift operators. Later, Peris and Barrachina proved that for translation C_0 -semigroups on weighted L^p spaces, $1 \leq p < \infty$, Devaney chaos implies distributional chaos. However, the converse does not hold. They also provide an example of a translation C_0 -semigroup that is distributionally chaotic but it is neither Devaney chaotic nor hypercyclic [7].

Hypercyclicity and Devaney chaos are hard to observe directly from the definition. The Hypercyclicity Criterion, in any of its forms, and the Desch-Schappacher-Webb Criterion have turn out to be powerful tools in order to verify these properties. Very recently, Albanese et al. have stated a criterion in order to show that a C_0 -semigroup is distributionally chaotic (and has a dense distributionally irregular manifold) [8]. Our goal is to study distributional chaos for some C_0 -semigroups that are already known to be Devaney chaotic. The dynamics exhibited by these C_0 -semigroups will motivate us to pose some open questions.

2. Criteria to Determine Devaney Chaos and Distributional Chaos

The following statement of the Hypercyclicity Criterion for C_0 -semigroups is inspired by its version for operators in [9].

Theorem 2.1 (Hypercyclicity Criterion for C_0 -semigroups; see [10, Th. 2.1], [11, Crit. 3.1], and [1, Th. 7.26]). *Let \mathcal{T} be a C_0 -semigroup in $L(X)$. If there are a sequence $\{t_n\}_n \subset \mathbb{R}^+$ with $\lim_{n \rightarrow \infty} t_n = \infty$, dense subsets $Y, Z \subset X$ and maps $S_{t_n} : Z \rightarrow X$, $n \in \mathbb{N}$ such that*

- (i) $\lim_{n \rightarrow \infty} T_{t_n} y = 0$ for all $y \in Y$,
- (ii) $\lim_{n \rightarrow \infty} S_{t_n} z = 0$ for all $z \in Z$, and
- (iii) $\lim_{n \rightarrow \infty} T_{t_n} S_{t_n} z = z$ for all $z \in Z$,

then \mathcal{T} is hypercyclic.

Sometimes the Hypercyclicity Criterion is hard to be applied, and in fact, it only provides one of the ingredients of Devaney chaos. Moreover, in many situations, we can have the infinitesimal generator of a C_0 -semigroup but we do not have the explicit representation of its operators. This is quite common when we deal with the solution C_0 -semigroups associated to certain partial differential equations. Desch et al. gave a criterion which permits us to state the Devaney chaos of a C_0 -semigroup in terms of the abundance of eigenvectors of the infinitesimal generator.

Theorem 2.2 (Desch-Schappacher-Webb Criterion; see [12, 13]). *Let X be a complex separable Banach space, and let \mathcal{T} be a C_0 -semigroup in $L(X)$ with infinitesimal generator $(A, D(A))$. Assume that there exist an open connected subset $U \subset \mathbb{C}$ and a weak holomorphic function $f : U \rightarrow X$, such that*

- (i) $U \cap i\mathbb{R} \neq \emptyset$,
- (ii) $f(\lambda) \in \ker(\lambda I - A)$ for every $\lambda \in U$, and
- (iii) for any $x^* \in X^*$, if $\langle f(\lambda), x^* \rangle = 0$ for all $\lambda \in U$, then $x^* = 0$.

Then the semigroup \mathcal{T} is chaotic.

For the case of distributional chaos, Albanese et al. obtained the following sufficient condition, inspired by the result for the discrete case given by Bermúdez et al. in [4].

Theorem 2.3 (Dense Distributionally Irregular Manifold Criterion; see [8, Cor. 2]). *Let \mathcal{T} be a C_0 -semigroup in $L(X)$. Assume that there exist*

- (i) a dense subset $X_0 \subset X$ such that $\lim_{t \rightarrow \infty} T_t x = 0$ for each $x \in X_0$, and
- (ii) a Lebesgue measurable subset $B \subset \mathbb{R}_0^+$ with $\overline{\text{Dens}}(B) = 1$ satisfying either $\int_B \|T_t\|^{-1} dt < \infty$, or X is a complex Hilbert space and $\int_B \|T_t\|^{-2} dt < \infty$.

Then \mathcal{T} has a dense manifold whose nonzero vectors are distributionally irregular vectors. (When this happens, one says that \mathcal{T} has a dense distributionally irregular manifold).

Furthermore, they also proved that a C_0 -semigroup \mathcal{T} is distributionally chaotic if, and only if, \mathcal{T} has a distributionally irregular vector [8, Th. 3.4]. Therefore, Theorem 2.3 can be also understood as a criterion for distributional chaos. In the sequel, we will apply this criterion several times in order to determine that certain C_0 -semigroups are distributionally chaotic.

3. Distributionally Chaotic C_0 -Semigroups

In this section, we consider several examples of C_0 -semigroups that are already known to be Devaney chaotic, and we will study when they exhibit distributional chaos. These examples will be considered on the following spaces:

$$L^p_\rho(I, \mathbb{C}) = \left\{ f \in \mathcal{M}(I, \mathbb{C}) : \|f\|_{p,\rho} = \left(\int_I |f(s)|^p \rho(s) ds \right)^{1/p} < \infty \right\}, \quad \text{with } 1 \leq p < \infty, \quad (3.1)$$

where I is an interval on \mathbb{R} and ρ a weight function. If $\rho(x) = 1$, then we will simply denote it as $L^p(I, \mathbb{C})$, $1 \leq p < \infty$. The hypothesis on ρ may be different on each example.

In [14], Takeo considered the following first order abstract Cauchy problem on $L^p(I, \mathbb{C})$, $1 \leq p < \infty$:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \zeta(x) \frac{\partial u}{\partial x} + h(x)u, \\ u(0, x) &= f(x), \quad x \in I, \end{aligned} \quad (3.2)$$

where ζ and h are bounded continuous functions defined on I . This ordinary differential equation has been used to model the dynamics of a population of cells under simultaneous proliferation and maturation [15]. When $\zeta(x)$ is constant and equal to 1 and $I = \mathbb{R}_0^+$, the solution C_0 -semigroup $\{T_t\}_{t \geq 0}$ of (3.2) is defined as

$$T_t f(x) = \exp\left(\int_x^{x+t} h(s) ds\right) f(x+t), \quad \forall x, t \geq 0, f \in L^p(\mathbb{R}_0^+, \mathbb{C}). \quad (3.3)$$

Theorem 3.1. *If $h(x)$ is a real function and there is a measurable set $B \subset \mathbb{R}_0^+$ such that $\overline{\text{Dens}(B)} = 1$ and $\int_B \exp(-p \int_0^x h(s) ds) dx < \infty$, then the C_0 -semigroup $\{T_t\}_{t \geq 0}$ defined in (3.3) is distributionally chaotic on $L^p(\mathbb{R}_0^+, \mathbb{C})$, $1 \leq p < \infty$.*

Proof. If we define $\rho(x) = \exp(-p \int_0^x h(s) ds)$, then the operators of $\{T_t\}_{t \geq 0}$ can be rewritten as

$$T_t f(x) = \left(\frac{\rho(x)}{\rho(x+t)} \right)^{1/p} f(x+t). \quad (3.4)$$

This function $\rho(x)$ is an admissible weight function in the sense of [12, Def. 4.1], which ensures that the left translation semigroup $\{\tau_t\}_{t \geq 0}$ defined as

$$\tau_t f(x) = f(x+t), \quad \text{for } x, t \geq 0, f \in L^p_\rho(\mathbb{R}_0^+, \mathbb{C}) \quad (3.5)$$

is a C_0 -semigroup on $L^p_\rho(\mathbb{R}_0^+, \mathbb{C})$.

Let us define $\phi(f)(x) = (\rho(x))^{1/p}f(x)$ and consider the following commutative diagram:

$$\begin{CD} L^p_\rho(\mathbb{R}_0^+, \mathbb{C}) @>\tau_t>> L^p_\rho(\mathbb{R}_0^+, \mathbb{C}) \\ @V\phi VV @VV\phi V \\ L^p(\mathbb{R}_0^+, \mathbb{C}) @>T_t>> L^p(\mathbb{R}_0^+, \mathbb{C}). \end{CD} \tag{3.6}$$

The hypothesis on B let us conclude that $\{\tau_t\}_{t \geq 0}$ is distributionally chaotic on $L^p_\rho(\mathbb{R}_0^+, \mathbb{C})$, see [7, Th. 2.3]. Therefore, the conclusion is obtained since distributional chaos is preserved under conjugacy [6, Th. 2]. \square

Remark 3.2. The previous result can be compared with the characterizations of hypercyclicity and Devaney chaos for the translation C_0 -semigroup on the spaces $L^p_\rho(\mathbb{R}_0^+, \mathbb{C})$, $1 \leq p < \infty$: The translation C_0 -semigroup $\{\tau_t\}_{t \geq 0}$ is hypercyclic on $L^p_\rho(\mathbb{R}_0^+, \mathbb{C})$ if and only if, $\liminf_{x \rightarrow \infty} \rho(x) = 0$ [12], and $\{\tau_t\}_{t \geq 0}$ is Devaney chaotic on it if, and only if, $\int_0^\infty \rho(x)dx < \infty$ [16, 17]. Using conjugacy, these results can be transferred to the C_0 -semigroup $\{T_t\}_{t \geq 0}$ [1, Ex. 7.5.2].

On the one hand, if $h(x)$ is constant and equal to 1, then we have that $\{T_t\}_{t \geq 0}$ is Devaney chaotic and distributionally chaotic on $L^p(\mathbb{R}_0^+, \mathbb{C})$. On the other hand, taking $B = [0, 2] \cup \bigcup_{n \in \mathbb{N}} [n^2 + 1, (n + 1)^2[$, $h(x) = 1$ if $x \in B$ and $h(x) = -1$ elsewhere, we have $\overline{\text{Dens}}(B) = 1$ and $\int_B \rho(x)dx < \infty$. Therefore $\{T_t\}_{t \geq 0}$ is distributionally chaotic on $L^p(\mathbb{R}_0^+, \mathbb{C})$. It is also hypercyclic since $\rho(n^2) = e^{-(n^2-2n+4)p}$ for $n \geq 2$, which yields that $\liminf_{x \rightarrow \infty} \rho(x) = 0$ [14, Th. 2.2]. However, it cannot be Devaney chaotic since $\int_{\mathbb{R}_0^+} \rho(x)dx = \infty$.

To sum up, we have an example of a C_0 -semigroup that is hypercyclic, distributionally chaotic, but it is not Devaney chaotic. This example can be compared with the example provided in [7, Ex. 2.] of a distributionally chaotic translation C_0 -semigroup that is neither hypercyclic nor chaotic.

Now, let us consider another example of a C_0 -semigroup whose dynamical behavior was already discussed in [14]: Let $\rho : [0, 1] \rightarrow \mathbb{R}^+$ be a continuous function such that there exist constants $M \geq 1$, $\omega \in \mathbb{R}$, and $\gamma < 0$ such that

$$\rho(x) \leq Me^{\omega t} \rho(e^{\gamma t} x), \quad \forall x \in [0, 1], t > 0. \tag{3.7}$$

With such a function ρ , we can consider the spaces $L^p_\rho([0, 1], \mathbb{C})$, for $1 \leq p < \infty$. The family of operators $\{S_t\}_{t \geq 0}$ with $S_t f(x) = f(e^{\gamma t} x)$, $t \geq 0$ defines a C_0 -semigroup on them [14].

Theorem 3.3. *If $\gamma < 0$, then the C_0 -semigroup $\{S_t\}_{t \geq 0}$ is distributionally chaotic on $L^p_\rho([0, 1], \mathbb{C})$, $1 \leq p < \infty$.*

Proof. Let us apply Theorem 2.3. Take $X_0 = \{f \in C([0, 1], \mathbb{C}) : f(0) = 0\}$. This set is dense in $L^p_\rho([0, 1], \mathbb{C})$ and, clearly, $\lim_{t \rightarrow \infty} S_t f = 0$ for every $f \in X_0$, which fulfills condition (i) in Theorem 2.3.

Let us prove that $\int_0^\infty \|S_t\|_{p,\rho}^{-1} dt$ is finite: Fix $t > 0$ and a continuous function g on $[0, 1]$ with $\|g\|_{p,\rho} = 1$, for instance $g(x) = 1/\rho(x)^{1/p}$.

There is some $t_0 > 0$ such that for $t > t_0$, we have $(\int_0^{e^{yt}} \rho(x) dx)^{1/p} \leq t^{-2} / \|g\|_\infty$. For these $t > t_0$, define

$$g_t(x) = \begin{cases} g(e^{-yt}x), & \text{if } 0 \leq x \leq e^{yt}, \\ 0, & \text{elsewhere.} \end{cases} \quad (3.8)$$

Since $\|g_t\|_{p,\rho} \leq t^{-2}$ and $S_t g_t = g$, then $\|S_t\|_{p,\rho} \geq t^2$ for $t \geq t_0$. So that $\int_{t_0}^\infty \|S_t\|_{p,\rho}^{-1} dt$ is convergent, which yields the conclusion. \square

Remark 3.4. The assumption $\gamma < 0$ forces $w > 0$: If not, take any $x \in [0, 1]$. Taking limits when $t \rightarrow \infty$ in the inequality $\rho(x)/\rho(e^{yt}x) \leq Me^{\omega t}$ we have $\rho(x)/\rho(0) \leq 0$, which is a contradiction because ρ is a positive continuous function.

Remark 3.5. An alternative proof provided by an anonymous referee is the following: If $\rho : [0, 1] \rightarrow \mathbb{R}^+$ is a continuous weight function which is admissible in the sense of (3.7), then $\psi : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ defined as $\psi(x) := \rho(e^{yx})e^{yx}$ is an admissible weight function in the sense of [12, Def. 4.1]. Therefore, taking $\phi : L_\psi^p(\mathbb{R}_0^+, \mathbb{C}) \rightarrow L_\rho^p([0, 1], \mathbb{C})$ defined as $\phi(f)(x) := f(\log(x)/\gamma)$, we have the following commutative diagram:

$$\begin{array}{ccc} L_\psi^p(\mathbb{R}_0^+, \mathbb{C}) & \xrightarrow{\tau_t} & L_\psi^p(\mathbb{R}_0^+, \mathbb{C}) \\ \phi \downarrow & & \downarrow \phi \\ L_\rho^p([0, 1], \mathbb{C}) & \xrightarrow{S_t} & L_\rho^p([0, 1], \mathbb{C}). \end{array} \quad (3.9)$$

If $\gamma < 0$, then $\int_0^\infty \psi(x) dx < \infty$. So that, by conjugacy, $\{S_t\}_{t \geq 0}$ is hypercyclic, Devaney and distributionally chaotic, see Remark 3.2.

We return to the initial value problem stated in (3.2). Consider the case when $I = [0, 1]$, $\zeta(x) := \gamma x$, $\gamma < 0$, and $h \in C([0, 1], \mathbb{C})$. Under these hypotheses, the C_0 -semigroup $\{\tilde{T}_t\}_{t \geq 0}$ defined as

$$\tilde{T}_t f(x) = \exp\left(\int_0^t h(e^{\gamma(t-r)}x) dr\right) f(e^{\gamma t}x) \quad \text{for } t \geq 0, x \in [0, 1], \quad (3.10)$$

gives the solution C_0 -semigroup to (3.2) on $L^p([0, 1], \mathbb{C})$, $1 \leq p < \infty$ [14, Th. 3.4]. The particular case when $\gamma = -1$ and $h(x) = -1/2$ was studied using the Wiener measure in [15].

Theorem 3.6. *If $\gamma < 0$ and $\min\{\Re(h(x)) : x \in [0, 1]\} > \gamma/p$, then the C_0 -semigroup $\{\tilde{T}_t\}_{t \geq 0}$ defined in (3.10) is distributionally chaotic on $L^p([0, 1], \mathbb{C})$, $1 \leq p < \infty$.*

Proof. We apply again Theorem 2.3: Condition (i) holds in the same way as in the proof of Theorem 3.3 taking $X_0 = \{f \in C([0, 1], \mathbb{C}) : f(0) = 0\}$.

In order to verify condition (ii), let $\alpha \in \mathbb{R}$ be such that $\min\{\Re(h(x)) : x \in [0, 1]\} > \alpha > \gamma/p$. For every $t > 0$, we define f_t as a function with $\|f_t\|_p = 1$ and $\text{supp}(f_t) \subset [0, e^{\gamma t}]$. Using it, we have the following estimations for $\|\tilde{T}_t\|_p$:

$$\|\tilde{T}_t\|_p \geq \|\tilde{T}_t f_t\|_p \geq e^{\alpha t} \left(\int_0^1 |f_t(e^{\gamma t} x)|^p dx \right)^{1/p} = e^{(\alpha-\gamma/p)t} \left(\int_0^{e^{\gamma t}} |f_t(y)|^p dy \right)^{1/p} = e^{(\alpha-\gamma/p)t} \tag{3.11}$$

so that $\int_0^\infty \|\tilde{T}_t\|_p^{-1} dt$ is finite, which yields the conclusion. □

Under the hypothesis of the last theorem, Takeo proved that $\{\tilde{T}_t\}_{t \geq 0}$ is Devaney chaotic by applying the Desch-Schappacher-Webb Criterion [14]. Independently, Brzeźniak and Dawidowicz also proved that $\{\tilde{T}_t\}_{t \geq 0}$ is Devaney chaotic when $\gamma = -1$ and $h(x) = \lambda \in \mathbb{R}$ with $\lambda > -1/p$, that is known as the von Foerster-Lasota equation [18, Theorems 8.3 and 8.4]. Furthermore, they also showed that for $\lambda \leq -1/p$ the orbits of all elements tend to 0, which makes chaos disappear. Therefore, we can affirm that Devaney chaos coincides exactly with distributional chaos for the same values of λ . As we will see later, this is due to the fact that Devaney chaos can be obtained here from the Desch-Schappacher-Webb Criterion. This can be easily seen if we reformulate Theorem 2.3 in terms of the infinitesimal generator of the C_0 -semigroup. The following result is a continuous version of [4, Cor. 31].

Theorem 3.7. *Let X be a complex Banach space and let \mathcal{T} be a C_0 -semigroup in $L(X)$ with infinitesimal generator $(A, D(A))$. If the following conditions hold:*

- (i) *there is a dense subset $X_0 \subset X$ with $\lim_{t \rightarrow \infty} T_t x = 0$, for each $x \in X_0$, and*
- (ii) *there is some $\lambda \in \sigma_p(A)$ with $\Re(\lambda) > 0$,*

then \mathcal{T} has a dense distributionally irregular manifold. In particular, \mathcal{T} is distributionally chaotic.

Proof. Fix $t > 0$. On the one hand, if condition (i) holds, then we have $\lim_{n \rightarrow \infty} T_t^n x = 0$ for every $x \in X_0$. On the other hand, by the point spectral mapping theorem for C_0 -semigroups, since $\lambda \in \sigma_p(A)$, then $e^{\lambda t} \in \sigma_p(T_t)$. Therefore $r(T_t) \geq |e^{\lambda t}| > 1$ and, by [4, Cor. 31], T_t admits a dense distributionally irregular manifold. By [8, Rem. 2], this is equivalent to say that \mathcal{T} admits a dense distributionally irregular manifold. Furthermore, \mathcal{T} is distributionally chaotic [8, Prop. 2]. □

Remark 3.8. Clearly, the conditions in Theorem 3.7 hold whenever the Desch-Schappacher-Webb Criterion can be applied. Therefore, among others, the following C_0 -semigroups that are known to be Devaney chaotic are also distributionally chaotic (and have a dense distributionally irregular manifold): [19], [20, Th. 3.1], [12, Ex. 4.12], [21], [22, Th. 1], and [23, Th. 2.1 and Th. 2.3]. See also [1, Ch. 7] for an improved version of the proof of these last two examples.

Finally, Brzeźniak and Dawidowicz also studied in [18] Devaney chaos for the case $\gamma = -1$ and $h(x) = \lambda \in \mathbb{R}$ in certain subspaces of Hölder continuous functions on $[0, 1]$. For $\alpha \in]0, 1]$, $0 < r \leq 1$, we define the space $C_r^\alpha([0, 1])$ of functions $f : [0, 1] \rightarrow \mathbb{R}$ such that

$$\|f\|_{\alpha,r} := \sup_{\substack{x,y \in [0,1] \\ 0 < |x-y| < r}} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty. \quad (3.12)$$

For $\alpha \in]0, 1[$, let us consider $V_\alpha([0, 1])$ the space of functions

$$\left\{ f \in C_1^\alpha([0, 1]) : \lim_{r \rightarrow 0^+} \|f\|_{\alpha,r} = 0, f(0) = 0 \right\}. \quad (3.13)$$

In [18], it is shown that $V_\alpha([0, 1])$ is a separable Banach space endowed with the norm $\|f\|_{\alpha,1}$. Furthermore, following a constructive approach, it is proved that if $\gamma = -1$ and $h(x) = \lambda > \alpha$, then $\{\tilde{T}_t\}_{t \geq 0}$ is Devaney chaotic there. We will prove that in this case $\{\tilde{T}_t\}_{t \geq 0}$ is also distributionally chaotic.

Theorem 3.9. *If $\gamma = -1$ and $h(x) = \lambda > \alpha$, then the C_0 -semigroup $\{\tilde{T}_t\}_{t \geq 0}$ defined in (3.10) is distributionally chaotic on $V_\alpha([0, 1])$, $\alpha \in]0, 1[$.*

Proof. We will apply Theorem 2.3 again. Since $\{\tilde{T}_t\}_{t \geq 0}$ is Devaney chaotic, then there is a dense set of points with bounded orbit. Therefore $\{\tilde{T}_t\}_{t \geq 0}$ is weakly mixing [1, Th. 7.23], and any non-trivial operator \tilde{T}_t is weakly mixing, too [11, Th. 2.4]. Fix $t > 0$. By [9, Th. 2.3], \tilde{T}_t satisfies the Hypercyclicity Criterion. So that, there is a dense set $X_0 \subset V_\alpha([0, 1])$ such that $\lim_{n \rightarrow \infty} \tilde{T}_t^n x = 0$ for all $x \in X_0$. Using the local equicontinuity of $\{\tilde{T}_t\}_{t \geq 0}$, we have $\lim_{t \rightarrow \infty} \tilde{T}_t x = 0$ for every $x \in X_0$ and condition (i) holds.

In order to verify condition (ii), take $0 < \varepsilon < (\lambda - \alpha)/2$ such that $\alpha + \varepsilon < 1$. Let us define $f_\varepsilon(x) = x^{\alpha+\varepsilon}$, $0 \leq x \leq 1$. Since $|x^{\alpha+\varepsilon} - y^{\alpha+\varepsilon}| \leq |(x - y)^{\alpha+\varepsilon}|$ for all $x, y \in [0, 1]$, then we can easily see that $\|f_\varepsilon\|_{\alpha,1} = 1$ and $f_\varepsilon \in V_\alpha([0, 1])$. We also get that $\|\tilde{T}_t f_\varepsilon\|_{\alpha,1} = e^{(\lambda - \alpha - \varepsilon)t}$ and hence $\int_0^\infty dt / \|\tilde{T}_t\|_{\alpha,1} \leq \int_0^\infty dt / \|\tilde{T}_t f_\varepsilon\|_{\alpha,1} < \infty$. \square

4. Discussion and Conclusions

Consider the initial value problem of (3.2) on $L^1(\mathbb{R}_0^+, \mathbb{C})$ with $\zeta(x) = 1$ and $h(x) = kx^{k-1}/(1 + x^k)$. Here, the solution C_0 -semigroup $\{T_t\}_{t \geq 0}$ is defined as

$$T_t f(x) = \frac{1 + (x + t)^k}{1 + x^k} f(x + t), \quad x, t \geq 0. \quad (4.1)$$

The C_0 -semigroup $\{T_t\}_{t \geq 0}$ defined in (4.1) is distributionally chaotic on $L^1(\mathbb{R}_0^+, \mathbb{C})$ by Theorem 3.1. The hypercyclicity of this C_0 -semigroup for $k = 2$ was obtained by El Mourchid in [24] and the Devaney chaos by Grosse-Erdmann and Peris in [1, Prop. 7.34]. In this case, the point spectrum of the infinitesimal generator is the closed left half plane. This inhibits the Desch-Schappacher-Webb Criterion to be applied in the way it has been formulated. Nevertheless, El Mourchid observed that the hypercyclic behavior of this C_0 -semigroup is

essentially due to the imaginary eigenvalues of its infinitesimal generator [24], see also [1, Ex. 7.5.1]. In fact, the Desch-Schappacher-Webb Criterion can be strengthened and reformulated as follows.

Theorem 4.1 (see [24, Th. 2.1] and [1, Th. 7.31]). *Let X be a complex separable Banach space, and let \mathcal{T} be a C_0 -semigroup on X with infinitesimal generator $(A, D(A))$. If there are $a < b$ and continuous functions $f_j : [a, b] \rightarrow X, j \in J$, with*

- (1) $f_j(s) \in \ker(isI - A)$ for every $s \in [a, b], j \in J$, and
- (2) $\text{span}\{f_j(s); s \in [a, b], j \in J\}$ is dense in X ,

then the semigroup \mathcal{T} is Devaney chaotic.

To sum up, we have seen that even when we apply this stronger version of the Desch-Schappacher-Webb Criterion for the C_0 -semigroup in (4.1), asking only for an abundance of eigenvalues of real part equal to zero, then we can also prove that there is a dense distributionally irregular manifold. Therefore, we can pose the following problem.

Problem 1. Do the hypothesis in Theorem 4.1 imply the existence of a dense distributionally irregular manifold for \mathcal{T} ? If not, is there at least a distributionally irregular vector for \mathcal{T} ?

By the equivalence between a C_0 -semigroup with a distributionally irregular vector and a distributionally chaotic C_0 -semigroup, [8, Th. 3.4], the former problem can also be presented as follows.

Problem 2. Do the hypothesis in Theorem 4.1 imply that \mathcal{T} is distributionally chaotic?

These questions could have a positive answer, but it is still unknown whether Devaney chaos implies distributional chaos on C_0 -semigroups.

Problem 3. Are there examples of Devaney chaotic C_0 -semigroups which are not distributionally chaotic?

A C_0 -semigroup is said to be *frequently hypercyclic* if there exists some $x \in X$ such that for every nonempty open set $U \subset X$, the set $U_x := \{s \geq 0 : T_s x \in U\}$ has positive lower density, that is $\liminf_{t \rightarrow \infty} (1/t)\mu(U_x \cap [0, t])$ is positive. In [25], Mangino and Peris observed that with the same arguments used in [12, 24], one can show that the Desch-Schappacher-Webb Criterion implies frequent hypercyclicity. They also provide the Frequent Hypercyclicity Criterion for C_0 -semigroups [25, Th. 2.2]. So that, one can raise the following question.

Problem 4. Do the hypothesis of the Frequent Hypercyclicity Criterion for C_0 -semigroups imply distributional chaos?

The hypothesis in Theorem 4.1 also yields the mixing property for the C_0 -semigroup \mathcal{T} , see [1]. We recall that a C_0 -semigroup is *topologically mixing* if for any pair of nonempty open sets $U, V \subset X$ there is some $t_0 > 0$ such that $T_t(U) \cap V \neq \emptyset$ for all $t \geq t_0$. Clearly, topological mixing implies transitivity (i.e., hypercyclicity), but it is strictly stronger than it. Topologically mixing translation C_0 -semigroups on the weighted L^p_ρ -spaces considered in this paper are characterized by the condition $\lim_{t \rightarrow \infty} \rho(t) = 0$ [26, Th. 4.3].

On the one hand, the aforementioned example of Peris and Barrachina [7, Ex. 2.7] provides an example of a distributionally chaotic C_0 -semigroup that it is not topologically mixing. On the other hand, in [27], there is an example of a backward shift operator on a weighted sequence space $\ell^p(v)$, $1 \leq p < \infty$, that is topologically mixing but it is not distributionally chaotic. This operator will provide us an analogous counterexample in the frame of C_0 -semigroups. We thank A. Peris for this counterexample.

Example 4.2. Consider the sequence $(n_k)_k$ defined as $n_k = (k!)^3$, $k \in \mathbb{N}$, and define the function $\rho : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ as $\rho(t) = 1$ if $0 \leq t$. This function is an admissible weight in the sense of [12, Def. 4.1] and makes the translation semigroup $\{\tau_t\}_{t \geq 0}$ to be a C_0 -semigroup. On the one hand, since $\lim_{t \rightarrow \infty} \rho(t) = 0$, then the translation C_0 -semigroup is topologically mixing. On the other hand, if the translation C_0 -semigroup was distributionally chaotic, by [7, Th. 2.10], the backward shift operator, defined as $B(x_1, x_2, \dots) = (x_2, x_3, \dots)$, would be distributionally chaotic on the space $\ell_1(v) := \{(x_n)_n : \sum_{j \in \mathbb{N}} |x_j| v_j < \infty\}$ with $(v_n)_n = (\rho(n))_n$, which is a contradiction as it is indicated in [27].

Acknowledgments

This work is supported in part by MEC and FEDER, Project MTM2010-14909, by Generalitat Valenciana, Project GV/2010/091, and by Universitat Politècnica de València, Project PAID-06-09-2932. X. Barrachina also wants to acknowledge the support of the Grant FPI-UPV 2009-04 from Programa de Ayudas de Investigación y Desarrollo de la Universitat Politècnica de València. The authors also thank the referees for helpful comments that improved the presentation of the paper.

References

- [1] K.-G. Grosse-Erdmann and A. Peris Manguillot, *Linear Chaos*, Springer, London, UK, 2011.
- [2] F. Bayart and É. Matheron, *Dynamics of Linear Operators*, vol. 179, Cambridge University Press, Cambridge, UK, 2009.
- [3] X. Wu and P. Zhu, "On the equivalence of four chaotic operators," *Applied Mathematics Letters*, vol. 25, no. 3, pp. 545–549, 2012.
- [4] T. Bermúdez, A. Bonilla, F. Martínez-Giménez, and A. Peris, "Li-Yorke and distributionally chaotic operators," *Journal of Mathematical Analysis and Applications*, vol. 373, no. 1, pp. 83–93, 2011.
- [5] N. C. Bernardes Jr., A. Bonilla, V. Müller, and A. Peris, "Distributional chaos for linear operators," Preprint.
- [6] F. Martínez-Giménez, P. Oprocha, and A. Peris, "Distributional chaos for backward shifts," *Journal of Mathematical Analysis and Applications*, vol. 351, no. 2, pp. 607–615, 2009.
- [7] X. Barrachina and A. Peris, "Distributionally chaotic translation semigroups," *Journal of Difference Equations and Applications*, vol. 18, no. 4, pp. 751–761, 2012.
- [8] A. A. Albanese, X. Barrachina, E. M. Mangino, and A. Peris, "Distributional chaos for strongly continuous semigroups of operators," *Communications on Pure and Applied Analysis*. In press.
- [9] J. Bès and A. Peris, "Hereditarily hypercyclic operators," *Journal of Functional Analysis*, vol. 167, no. 1, pp. 94–112, 1999.
- [10] J. A. Conejero and A. Peris, "Linear transitivity criteria," *Topology and its Applications*, vol. 153, no. 5-6, pp. 767–773, 2005.
- [11] J. A. Conejero and A. Peris, "Hypercyclic translation C_0 -semigroups on complex sectors," *Discrete and Continuous Dynamical Systems A*, vol. 25, no. 4, pp. 1195–1208, 2009.
- [12] W. Desch, W. Schappacher, and G. F. Webb, "Hypercyclic and chaotic semigroups of linear operators," *Ergodic Theory and Dynamical Systems*, vol. 17, no. 4, pp. 793–819, 1997.

- [13] J. Banasiak and M. Moszyński, "A generalization of Desch-Schappacher-Webb criteria for chaos," *Discrete and Continuous Dynamical Systems A*, vol. 12, no. 5, pp. 959–972, 2005.
- [14] F. Takeo, "Chaotic or hypercyclic semigroups on a function space $C_0(I, \mathbb{C})$ or $L^p(I, \mathbb{C})$," *SUT Journal of Mathematics*, vol. 41, no. 1, pp. 43–61, 2005.
- [15] A. Lasota and M. C. Mackey, *Chaos, Fractals, and Noise*, vol. 97, Springer, New York, NY, USA, 1994.
- [16] R. deLaubenfels and H. Emamirad, "Chaos for functions of discrete and continuous weighted shift operators," *Ergodic Theory and Dynamical Systems*, vol. 21, no. 5, pp. 1411–1427, 2001.
- [17] M. Matsui, M. Yamada, and F. Takeo, "Supercyclic and chaotic translation semigroups," *Proceedings of the American Mathematical Society*, vol. 131, no. 11, pp. 3535–3546, 2003.
- [18] Z. Brzeźniak and A. L. Dawidowicz, "On periodic solutions to the von Foerster-Lasota equation," *Semigroup Forum*, vol. 78, no. 1, pp. 118–137, 2009.
- [19] J. Banasiak and M. Moszyński, "Dynamics of birth-and-death processes with proliferation—stability and chaos," *Discrete and Continuous Dynamical Systems. Series A*, vol. 29, no. 1, pp. 67–79, 2011.
- [20] J. A. Conejero and E. M. Mangino, "Hypercyclic semigroups generated by Ornstein-Uhlenbeck operators," *Mediterranean Journal of Mathematics*, vol. 7, no. 1, pp. 101–109, 2010.
- [21] T. Kalmes, "On chaotic C_0 -semigroups and infinitely regular hypercyclic vectors," *Proceedings of the American Mathematical Society*, vol. 134, no. 10, pp. 2997–3002, 2006.
- [22] J. Banasiak and M. Lachowicz, "Chaos for a class of linear kinetic models," *Comptes Rendus de l'Académie des Sciences IIB*, vol. 329, no. 6, pp. 439–444, 2001.
- [23] J. A. Conejero, A. Peris, and M. Trujillo, "Chaotic asymptotic behavior of the hyperbolic heat transfer equation solutions," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 20, no. 9, pp. 2943–2947, 2010.
- [24] S. El Mourchid, "The imaginary point spectrum and hypercyclicity," *Semigroup Forum*, vol. 73, no. 2, pp. 313–316, 2006.
- [25] E. M. Mangino and A. Peris, "Frequently hypercyclic semigroups," *Studia Mathematica*, vol. 202, no. 3, pp. 227–242, 2011.
- [26] T. Bermúdez, A. Bonilla, J. A. Conejero, and A. Peris, "Hypercyclic, topologically mixing and chaotic semigroups on Banach spaces," *Studia Mathematica*, vol. 170, no. 1, pp. 57–75, 2005.
- [27] F. Martínez-Giménez, P. Oprocha, and P. A., "Peris, Distributional chaos for operators with full scrambled sets," *Mathematische Zeitschrift*. In press.